University of Technology

Computer and Information Technology

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Where does fuzzy logic come from?

Fuzzy logic was introduced by Prof. Dr. Lotfi Zadeh in 1965. Not in the least degree trying to undermine his achievements, this theory has its roots in the previous history of science, particularly in logic science. Although logic as a branch of western science had been developing as binary logic, there were some famous paradoxes that could not be solved by binary logic. These paradoxes are as follows:

Falakors

Pluck a hair from a man's head and he does not suddenly become bald. Pull out another, and a third, and a fourth, and he still is not bald. Keep plucking and eventually the wincing man will have no hair at all on his head, yet he is not bald.

The paradox of the millet seeds (Zeno the Eleatic)

Drop a millet seed on the ground and it makes no sound. But why is that dropping a bushel of millet seeds make a sound, since it contains only millet seeds?

Theseus' ship.

When theseus returned from slaying the minotaur, says Plutarch, the Athenians preserved his ship, and as planks rotted, they replaced them with ones. When the first plank was replaced, everyone agreed it was still the same ship. Adding a second plank made no difference either. At some point the Athenians may have replaced every plank in the ship. Was it a different ship? At what point did it become one?

Wang's paradox (Mathematician Hao Wang).

If a number x is small, then x+1 is also small. If x+1 is small then x+1+1 is also small. Therefore five trillion is a small number and so is infinity.

Woodger's paradox (Biologist John Woodger).

An animal can belong to only one taxonomic family. Therefore, at many points in evolution a child must have belonged to a completely different family from its parents. But genetically, this feat is basically impossible.

Plato (427 – 347 BC).

Saw degrees of truth everywhere and recoiled from them. "No chair is perfect; it is only a chair to a certain degree".

Charles Sanders Peirce (1839 - 1914).

"Who split the world into true and false, all that exist is continuous and such continuums govern knowledge".

Bertrand Russell (1872 - 1970).

"Both vagueness and precision are features of language, not reality. Vagueness clearly is a matter of degree".

Jan Lukasiewicz (1878 - 1956).

Proposed a formal model of vagueness, a logic based on more values than true and false, 1 stands for true, 0 stands for false, and 1/2 stands for possible (three-valued logic).

Albert Einstein (1879 - 1955).

"so far as the laws of mathematics refer to reality, the are not certain. And, so far as they are certain they do not refer to reality".

Lotfi Zadeh (1923 -).

Introduced fuzz sets and fuzzy logic theory. "As the complexity of a system increases, our ability to make precise and significant statements about its behavior diminishes until a threshold is reached beyond whichprecision and significanance (or relevance) become almost mutually exclusive characteristics. A corollary principle may be stated succinctly as, "the closer one looks at a real-world problem, the fuzzier becomes its solution".

Brief History of Fuzzy Technology

1965 Concepts of fuzzy sets theory by Lotfi Zadeh (USA).

1972 First working group on fuzzy systems in Japan by Toshiro Terano.

1973 Paper about fuzzy algorithms by Zadeh (USA).

1974 Steam Engine control by Ebrahim Mamdani (UK).

1977 First Fuzzy Expert System for loan applicant evaluation by Hans Zimmermann (Germany).

1980 Cement Kiln Control by F. L. Smidth, Co. Lauritz, and P. Holmblad (Denmark).
Fuzzy logic chess and backgammon program – Hans Berliner (USA).

1984 Water treatment (chemical Injection) control (Japan).
Subway Sendi transportation System control (Japan).

1985 First fuzzy chip developed by Masaki Togai Hiroyuke and Watanbe in Bell Labs (USA).

1986 Fuzzy expert system for diagnosing illness in Omron (Japan).

1987 Container Crank Control, Tunnel Excavation, Soldering robot, Automated aircraft vehicle landing, Togai Infralogic Inc. – first fuzzy company in Irvine (USA).

1988 Kiln Control by Yokogawa, first dedicated fuzzy controller sold – Omron (Japan).

1989 Creation of Laboratory for International Fuzzy Engineering Research (LIFE) in Japan.

1990 Fuzzy TV set by Sony (Japan), Fuzzy electronic eye by fujistu (Japan), Fuzzy Logic systems institute (FLSI) by Takeshi Yamakawa (Japan), Intelligent Systems Control Laboratory in Siemens (Germany).

1991 Fuzzy AI Promotion Center (Japan), Educational Kit by Motorola (USA).

After 1991 fuzzy technology came out of scientific laboratories and became an industrial tool. In what follows a small number of successful projects and intended to demonstrate a huge diversity of possible application:

- Automatic control of Dam gates hydroelectric power plants (Tokyo Electric power).
- Simplified control of robots (Hirota, Fuji Electric, Toshiba, Omron).
- Camera-aiming for the telecast of sporting events (Omron).
- Efficient and stable control of car engines (Nissan).
- Cruise-control for automobiles (Nissan, Subaru).
- Substitution of an expert for the assessment of stock exchange activities (Yamaichi, Hitachi).
- Optimized planning of timetables (Toshiba, Nippon-System, Keihan-Express).
- Archiving System of documents (Mitsubishi Elec.).
- Prediction system for early recognition of earthquakes (Seismology Bureau of metrology, Japan).
- Medicine Technology: Cancer Diagnosis (Kawasaki Medical School).
- Automatic motor-control for vacuum cleaners with recognition of a surface condition and a degree of soiling (Mitsushita).
- Back-Light Control for Camcorders (Sanyo).

Fuzzy Set and Membership Function

Def.(Fuzzy Set). If X is a collection of objects denoted by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \quad \text{eq.(1)}$$

where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X to the membership space M. (when M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of nonfuzzy set.)

The range of membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed [Zimmermann, 1985].

Representation of a Fuzzy Set:

In the literature the reader can find different ways of denoting a fuzzy set. In what follows three of these ways shall be presented:

The first way is by an ordered set of pairs, the first element of which denotes the object and the second the degree of membership [Zimmermann, 1985]. **Example:** The fuzzy set "Comfortable apartment for three persons" can be described as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \text{ is the number of bedrooms}\}$$
$$= \{(1, 0.4), (2, 0.7), (3, 1), (4, 0.7), (5, 0.4), (6, 0.1)\}$$

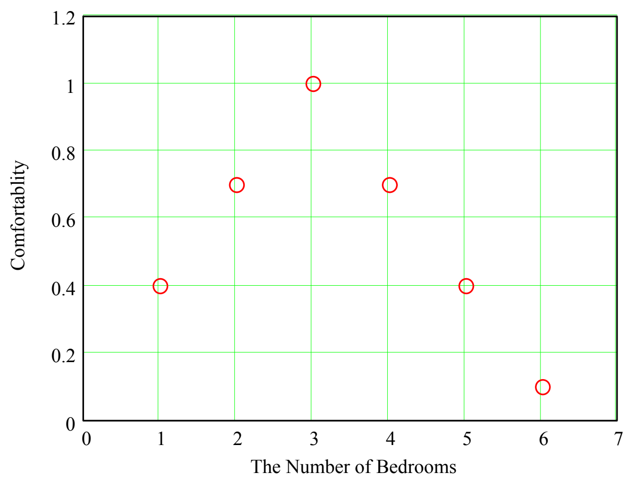

Fig (1): Comfortable apartment for three persons.

The second way is represented solely by stating its membership function [Zimmermann, 1985]. **Example:** The fuzzy set "Real numbers larger than 5" can be described as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \le 5 \\ \dfrac{1}{1+(x-5)^{-2}} & x > 5 \end{cases}$$

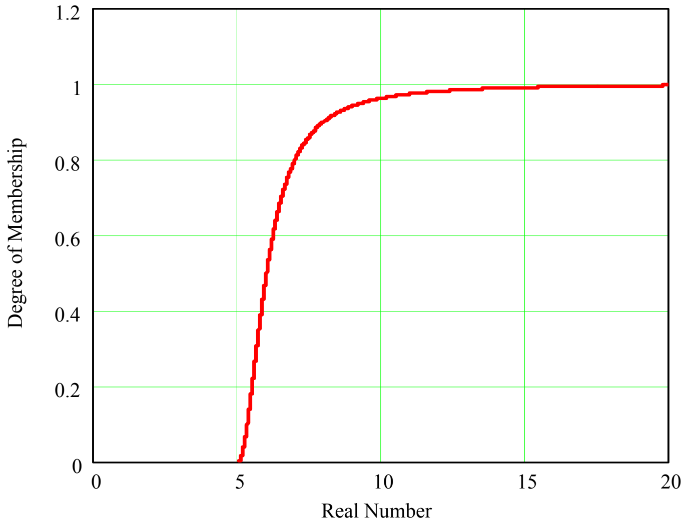

Fig.(2): Real numbers larger than 5

The third way is by the following:

a. Countable or discrete universe X [Zimmermann, 1985].

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + \cdots + \mu_{\tilde{A}}(x_n)/x_n = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i \qquad \text{eq.(2)}$$

Where + satisfies $a/u + b/u = \max(a,b)/u$, i.e., if the same element has two different degrees of membership then its membership degree becomes the largest of them [Dimiter, 1996]. The symbol $\sum$ here stands for successive + operation. **Example:** The fuzzy set "Integers close to 4" can be described as follows:

$$\tilde{A} = 0.1/1 + 0.4/2 + 0.8/3 + 1/4 + 0.8/5 + 0.4/6 + 0.1/7$$

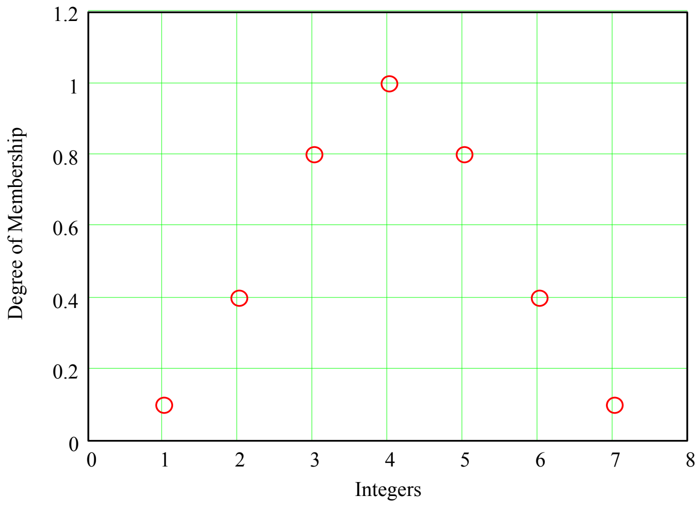

Fig.(3): Integers close to 4

b. Uncountable or Continuous universe X [Zimmermann, 1985].

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + \cdots + \mu_{\tilde{A}}(x_n)/x_n = \int_x \mu_{\tilde{A}}(x)/x \qquad \text{eq.(3)}$$

where the symbol $\int$ here stands for successive + operation. **Example:** The fuzzy set "Real numbers close to 4" can be described as follows:

$$\tilde{A} = \int_R \left(1 + (x-4)^2\right)^{-1} / x$$

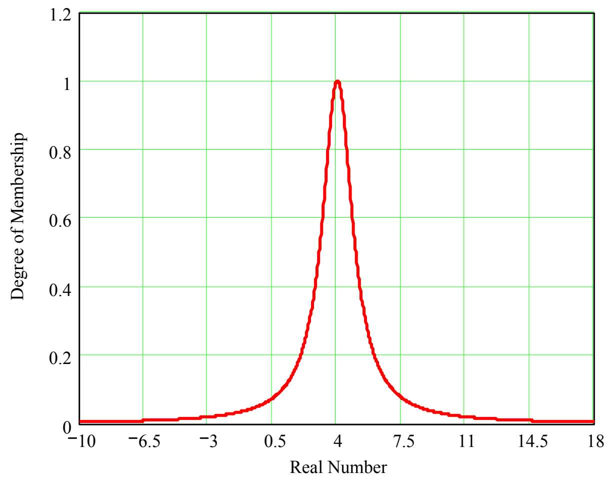

Fig.(4): Real numbers close to 4

Membership Function

1. The membership function is not limited to values between 0 and 1. If $\sup_x \mu_{\tilde{A}}(x) = 1$ the fuzzy set $\tilde{A}$ is called normal. A non-empty fuzzy set $\tilde{A}$ can always be normalized by dividing $\mu_{\tilde{A}}(x)$ by $\sup_x \mu_{\tilde{A}}(x)$ [Zimmermann, 1985].

2. There is a variety in the choice of the membership function depending on the mathematical background, mathematical model of the plant and designer expertise. In what follows we shall list some of this types:

a. The increasing membership functions with straight lines we will call $\Gamma-$function, because of the similarity of these functions with this character. This function is a function of one variable and two parameters defined as follows (fig.5) [Dimiter, 1996]:

$\Gamma -$ function : The function $\Gamma : U \to [0,1]$ is a function with two parameters defined as [Dimiter, 1996]:

$$\Gamma(u;\alpha,\beta) = \begin{cases} 0 & u < \alpha \\ (u-\alpha)/(\beta-\alpha) & \alpha \le u \le \beta \\ 1 & u > \beta \end{cases}$$

eq.(4)

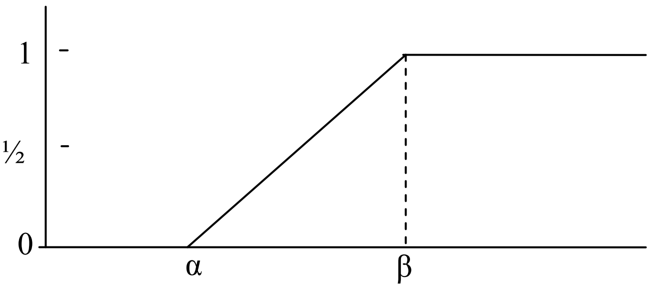

Fig.(5): Γ-Function

Zadeh's S-function [Dimiter, 1996], define as:

$$S(x;\alpha,\beta,\gamma) = \begin{cases} 0 & x \le \alpha \\ 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \alpha < x \le \beta \\ 1-2\left(\frac{x-\gamma}{\gamma-\alpha}\right)^2 & \beta < x \le \gamma \\ 1 & x > \gamma \end{cases}$$

eq.(5)

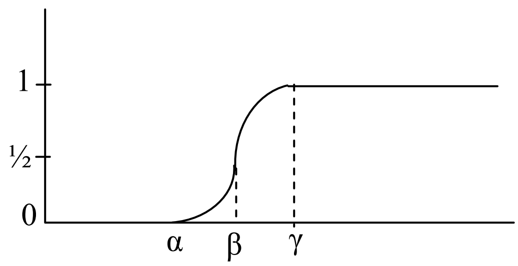

Fig.(6): Zadeh's S-function

where $\beta = (\alpha + \gamma)/2$, considered more fluent variant of the $\Gamma -$ functions (fig.6). frequently used in fuzzy logic, but only seldom in fuzzy control [Dimiter, 1996].

b. The decreasing membership functions with straight lines we will call $L -$ functions; they are defined as follows (fig.7) [Dimiter, 1996]:

$L -$ function : The function $L : U \to [0,1]$ is a function with two parameters defined as [Dimiter, 1996]:

$$L(u;\alpha,\beta) = \begin{cases} 1 & u < \alpha, \\ (\beta-u)/(\beta-\alpha) & \alpha \le u \le \beta \\ 0 & u > \beta \end{cases}$$

eq.(6)

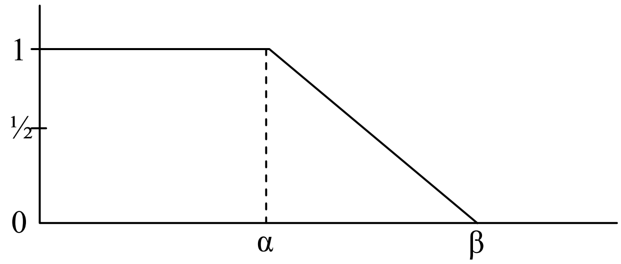

Fig.(7): L-function

c. Bell-shaped membership functions with straight lines we will call $\Lambda-$functions or triangular functions; they are defined as follow (fig.8) [Dimiter, 1996]:

$\Lambda-$functions: The function $\Lambda : U \rightarrow [0,1]$ is a function with three parameters defined as [Dimiter, 1996]:

$$\Lambda(u;\alpha,\beta,\gamma)=\begin{cases}0 & u\le\alpha\\ (u-\alpha)/(\beta-\alpha) & \alpha<u\le\beta\\ (\gamma-u)/(\gamma-\beta) & \beta<u\le\gamma\\ 0 & u>\gamma\end{cases}$$

eq.(7)

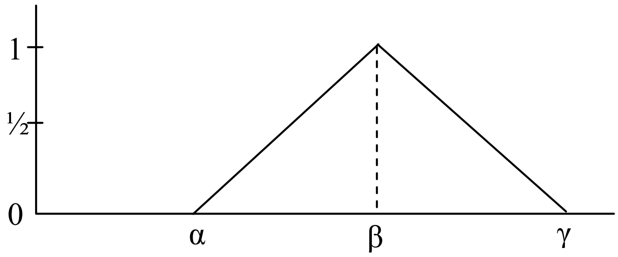

Fig.(8): Λ-function

Zadeh's bell-shaped $\pi-$function [Dimiter, 1996] defined as:

$$\pi(x;\beta,\gamma)=\begin{cases}S(x;\gamma-\beta,\gamma-\beta/2,\gamma) & x\ge\gamma\\ 1-S(x;\gamma,\gamma+\beta/2,\gamma+\beta) & x\le\gamma\end{cases}$$ eq.(8)

can be considered as a more fluent variant of the $\Lambda-$function. Like the $S-$function given above, it is of low practical use in fuzzy control [Dimiter, 1996].

d. Approximating membership functions with straight lines, where the top is not one point but an interval, we will call $\Pi-$functions; these are defined as follows (fig.9) [Dimiter, 1996]:

$\Pi-$function: The function $\Pi : U \rightarrow [0,1]$ is a function with four parameters defined as [Dimiter, 1996]:

$$\Pi(u;\alpha,\beta,\gamma,\delta)=\begin{cases}0 & u<\alpha\\ (u-\alpha)/(\beta-\alpha) & \alpha\le u<\beta\\ 1 & \beta\le u\le\gamma\\ (\delta-u)/(\delta-\gamma) & \gamma<u\le\delta\\ 0 & u>\delta\end{cases}$$

eq.(9)

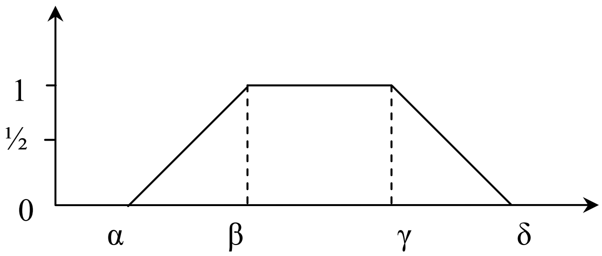

Fig.(9): Π-function

This $\Pi-$function can be used to describe the three functions mentioned above $(\Gamma, L \text{ and } \Lambda)$. Suppose that the underlying domain is $[-a,a]$, then the following equations hold [Dimiter, 1996], $\Gamma(x;\alpha,\beta)=\Pi(x;\alpha,\beta,a,a)$, $L(x;\gamma,\delta)=\Pi(x;-a,-a,\gamma,\delta)$, $\Lambda(x;\alpha,\beta,\delta)=\Pi(x;\alpha,\beta,\beta,\delta)$.

e. Gaussian Membership Functions

These membership functions are described by the following relationship:

$$A(x;\mu,\sigma) = \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right) \qquad \text{eq.(10)}$$

An example of the membership function is given by the following figure:

$$A(x;\mu,\sigma) = \exp\left(-\frac{(x-2)^2}{0.25}\right)$$

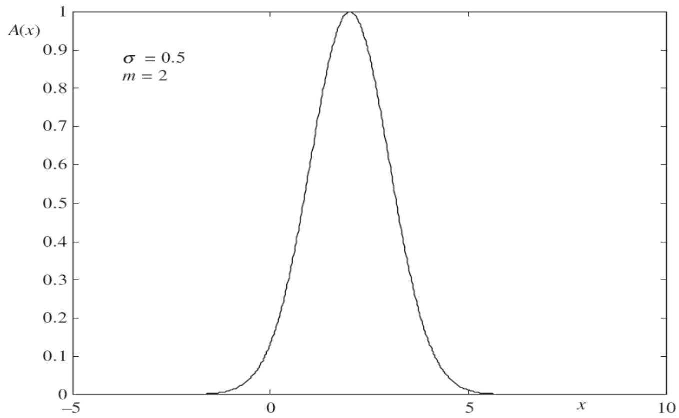

Fig.(10): Gaussian membership functions

Gaussian membership functions have two important parameters. The modal value μ represents the typical element of *A*, whereas σ denotes a spread of *A*. Higher values of σ correspond to larger spreads of the fuzzy sets [Witold, 2007].

f. Exponential-Like Membership Functions

They are described in the following form:

$$A(x;\mu,k) = \frac{1}{1+k\,(x-\mu)^2} \qquad \text{eq.(11)}$$

where k>0. The spread of the exponential-like membership function increases as the values of *k* get lower [Witold, 2007].

$$A(x;\mu,k) = \frac{1}{1+k\,(x-\mu)^2}$$

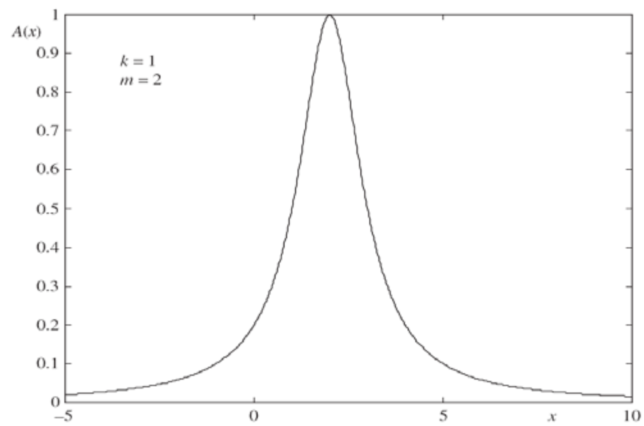

Fig.(11): Exponential-Like membership functions

Def.(Supremum).

The **Supremum** or **Least Upper Bound** of a set of real numbers S, denoted by *Sup(S)* is defined to be the smallest real number that is greater than or equal to every number in S.

Example.

- *Sup*(ϕ)= $-\infty$
- *Sup(of any not bounded above set such as Z, R)* = ∞
- *Sup {1, 2, 3} = 3*
- *Sup* { $x \in R: 0 < x < 1$ }=1

The **Supremum** or **Least Upper Bound** of a function *f*: $D \to R,\ D \neq \phi$, denoted **Sup *f***, is defined by **Sup *f*** = **Sup**{ $f(x)$: x$\in$D }, i.e., it is the Supremum of the range of *f*. **Example.**

- *f*:(0, 2) $\to$ *R*, $f(x) = x^2$, the range of f is (0, 4), Sup *f* = Sup(0, 4) = 4.
- *f*:(0, 1) $\to$ *R*, $f(x) = \frac{1}{x}$, the range of f is $(1, \infty)$, Sup *f* = ∞.
- *f*: *R* $\to$ *R*, *f (x) = sin(x)*, the range of *f* is $[-1, 1]$, Sup *f* = Sup$[-1, 1]$ = 1.

Def.(Triangular Norms and Conorms).

In a general, connectives $\wedge$ and $\vee$ are related to t-norm and t-conorm. A t-norm is a mapping T:[0, 1] × [0, 1] $\to$ [0, 1] satisfying:

(1). T(a, 1) = a (Unit Element)

(2). a ≥ b ⇒ T(a, c) ≥ T(b, c) (Monotonicity)

(3). T(a, b) = T(b, a) (Commutativity)

(4). T(a, T(b, c)) = T(T(a, b), c) (Associativity)

and similarly a t-conorm is a mapping S:[0, 1] × [0, 1] $\to$ [0, 1] satisfying:

(1). S(a, 0) = a (Unit Element)

(2). a ≥ b ⇒ S(a, c) ≥ S(b, c) (Monotonicity)

(3). S(a, b) = S(b, a) (Commutativity)

(4). S(a, S(b, c)) = S(S(a, b), c) (Associativity)

It can be seen that for any t-norm T and any co-t-norm S we have T(a, b) ≤ min(a, b) and S(a, b) ≥ max(a, b).

Operations and Important Concepts of Fuzzy Set

In what follows we will discuss some important concepts and operations that are necessary to deal with the fuzzy sets:

1. **Support of Fuzzy Set** [Dubios, 1980]**:** The support of fuzzy set $\tilde{A}$ is the ordinary subset of X:

$$\text{supp}(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\} \quad \text{eq.(12)}$$

2. **Height of Fuzzy Set** [Dubios, 1980]**:** The height of fuzzy set $\tilde{A}$ is supermum of the set, i.e., the least upper bound:

$$\text{hgt}(\tilde{A}) = \sup_x \mu_{\tilde{A}}(x) \quad \text{eq.(13)}$$

3. **Complement of Fuzzy Set** [Dubios, 1980]**:** The Complement $\tilde{A}^c$ of a fuzzy set $\tilde{A}$ is a fuzzy set defined by:

$$\tilde{A}^c = \{(x, \mu_{\tilde{A}^c}(x)) | \forall x \in X, \mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)\} \quad \text{eq.(14)}$$

4. **Union and Intersection of Fuzzy Sets:** The classical union $(\bigcup)$ and intersection $(\bigcap)$ of ordinary subsets of X can be extended to fuzzy sets by the following formulas:

$$\tilde{A} \cup \tilde{B} = \{(x, \mu_{\tilde{A}\cup\tilde{B}}(x)) | \forall x \in X, \mu_{\tilde{A}\cup\tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))\} \quad \text{eq.(15)}$$

$$\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) | \forall x \in X, \mu_{\tilde{A}\cap\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))\} \quad \text{eq.(16)}$$

where $\mu_{\tilde{A}\cup\tilde{B}}(x)$ and $\mu_{\tilde{A}\cap\tilde{B}}(x)$ are respectively the membership functions of $\tilde{A} \cup \tilde{B}$ and $\tilde{A} \cap \tilde{B}$ [Dubios, 1980].

5. **α-Cuts:** when we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set $\tilde{A}$, we may demand that its membership value be greater than some threshold $\alpha \in (0,1]$. The ordinary set of such elements is the α-cut A_α of the fuzzy set $\tilde{A}$ [Kandel, 1986]:

$$A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\} \quad \text{eq.(17)}$$

6. **Inclusion:** a fuzzy set $\tilde{A}$ is said to be included in fuzzy set $\tilde{B}$ denoted by $(\tilde{A} \subseteq \tilde{B})$ if and only if $\forall x \in X, \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$. Where the inequality is strict, then inclusion is said to be strict and is denoted $\tilde{A} \subset \tilde{B}$ [Dubios, 1980].