The Laplace Transform in Circuit Analysis

Reference:

Electric Circuits

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Objectives

- Be able to transform a circuit into the s domain using Laplace transforms.
- Know how to analyze a circuit in the s-domain and be able to transform an s-domain solution back to the time domain.
- Understand the definition and significance of the transfer function and be able to calculate the transfer function for a circuit using s-domain techniques.
- Know how to use a circuit’s transfer function to calculate the circuit’s unit impulse response, its unit step response, and its steady-state response to a sinusoidal input.

Outline

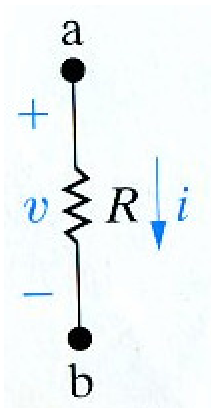
- Circuit elements in the s domain
- Circuit analysis in the s domain
- The transfer function
- The transfer function in partial fraction expansions
- The transfer function and the steady-state sinusoidal response

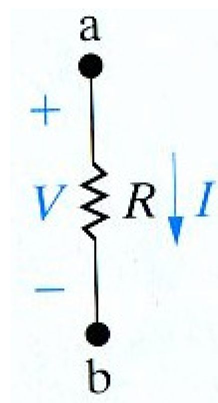
Circuit Elements in the s domain

Resistor in the s domain:

In time domain: $v = Ri$

In s domain: $V = RI$



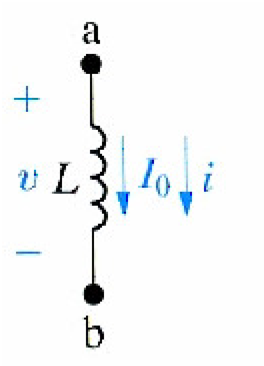


where $V = L\{v\}$ *and* $I = L\{i\}$

Circuit Elements in the s domain

Inductor in the s domain:

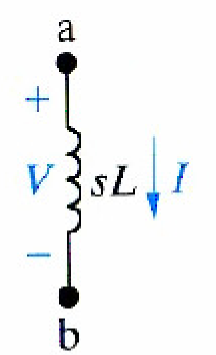
In time domain:



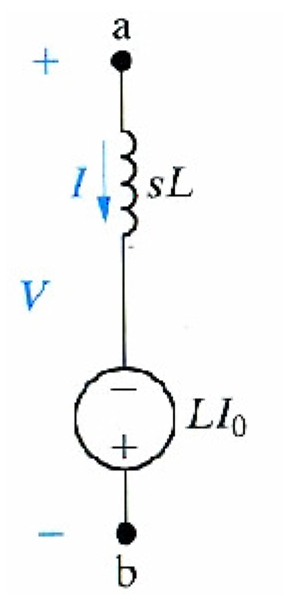
$$v = L\frac{di}{dt}$$

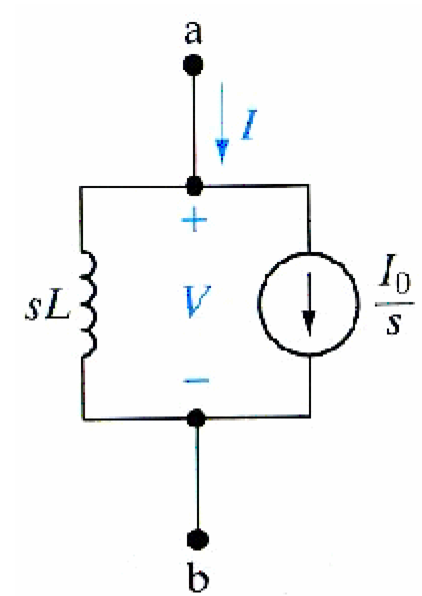
$$i = \frac{1}{L}\int_{0^-}^{t} v\,dx + I_0$$

In s domain:



(initial current is 0)





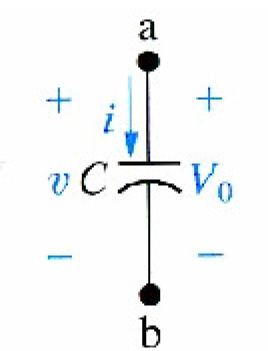
$$V = L[sI - i(0^-)] = sLI - LI_0$$

$$I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}$$

Circuit Elements in the s domain

Capacitor in the s domain:

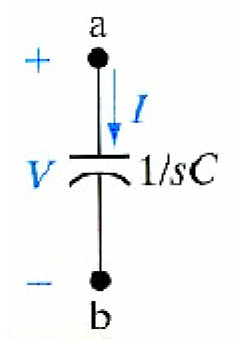
In time domain:



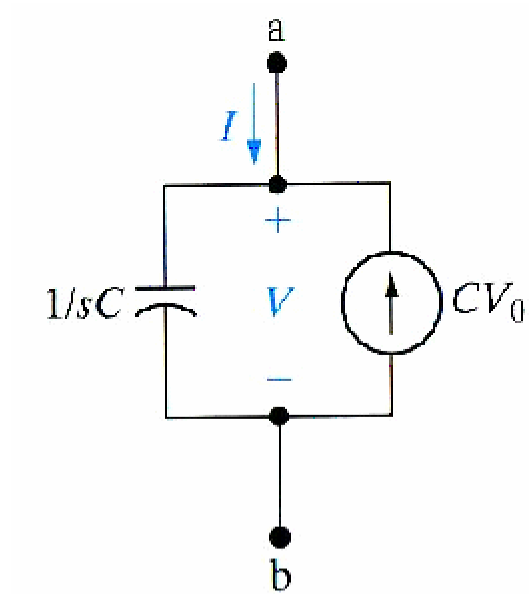
$$i = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int_{0^-}^{t} i\,dx + V_0$$

In s domain:



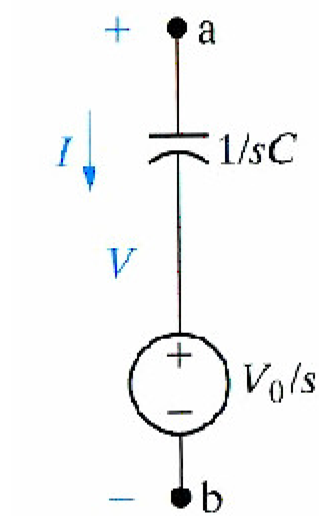
(initial voltage is 0)



$$I = C\left[sV - v(0^-)\right]$$

or

$$I = sCV - CV_0$$



$$V = \left(\frac{1}{sC}\right)I + \frac{V_0}{s}$$

Example: natural response de RC circuit

Circuit analysis in the s domain

Ohm's Law:

$$V = ZI$$

Where Z refers to the s-domain impedance of the element.

* Resistor has impedance of *R* ohms

* Inductor has impedance of *sL* ohms

* Capacitor has impedance of *1/sC* ohms

Kirchhoff's Law:

$$\text{Algebraic } \Sigma\, I = 0$$

$$\text{Algebraic } \Sigma\, V = 0$$

Transfer function

The transfer function is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source):

$$H(s) = \frac{Y(s)}{X(s)}$$

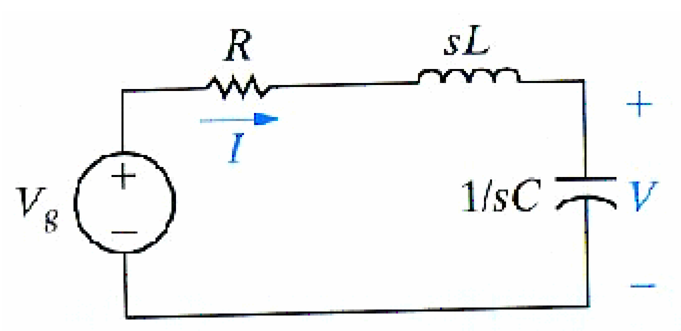
Y(s) is the Laplace transform of the output signal

X(s) is the Laplace transform of the input signal

Transfer function

- In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero.
- If a circuit has multiple independent sources, find the transfer function for each source and use superposition to find the response to all sources
- A single circuit can generate many transfer function

Example



$$H(s)=\frac{I}{V_g}=\frac{V_g}{R+sL+1/sC}\cdot\frac{1}{V_g}=\frac{sC}{s^2LC+sRC+1}$$

$$H(s)=\frac{V}{V_g}=\frac{1}{s^2LC+sRC+1}$$

Transfer function

The location of poles and zeros of H(s)

For a linear lumped-parameter circuits:

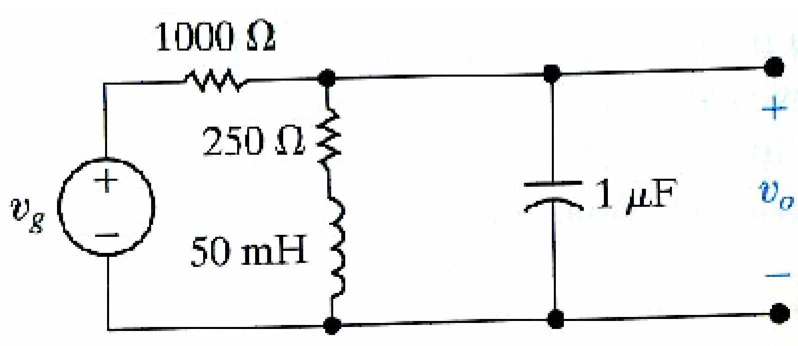
- H(s) is always a rational function of s.
- Complex poles and zeros always appear in conjugate pairs.
- The poles of H(s) must lie in the left half of the s plane.
- The zeros of H(s) may be lie in either the right half or the left half of the s plane

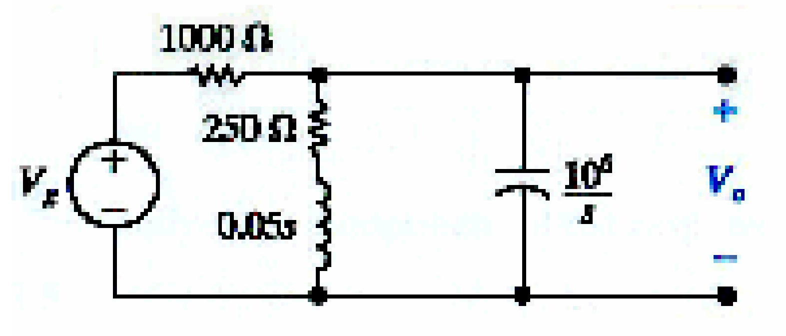
Transfer function in partial fraction expansions

$$Y(s) = H(s)X(s)$$

- Expanding the right-hand side into a sum of partial fractions produces a term for each pole of H(s) and X(s).
- The terms generated by the poles of H(s) give rise to the transient component of the total response.
- The terms generated by the poles of X(s) give rise to the steady-state component of the response.

Example





Transfer function:

$$H(s) = \frac{V_0}{V_g} = \frac{1000(s+5000)}{s^2 + 6000s + 25\times 10^6}$$

Transfer function and the steady-state response

Given sinusoidal source:

$$x(t) = A\cos(\omega t + \phi)$$

In s domain:

$$X(s) = \frac{A(s\cos\phi - \omega\sin\phi)}{s^2 + \omega^2} = \frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega}$$

$$K_1 = \frac{A}{2}\left|H(j\omega)\right| e^{j[\theta(\omega)+\phi]}$$

Transfer function:

$$H(j\omega) = \left|H(j\omega)\right| e^{j\theta(\omega)}$$

Transfer function and the steady-state response

<u>The steady state response:</u>

$$y_{ss}(t) = A\left|H(j\omega)\right|\cos[\omega t + \phi + \theta(\omega)]$$

- The amplitude of the response equals the amplitude of the source multiplies the magnitude of the transfer function.
- The phase angle of the response equals the phase angle of the source plus the phase angle of the transfer function.