The University of the West Indies
Mona Campus
Department of Physics

COURSE TITLE: **Electric Circuits**

COURSE CODE: **ECNG 1000**

LECTURES: **Three (3) per week**

Lecturer: **Dr. Leary Myers**

EVALUATION:

Course Test: **One sixty-minute paper**
(Weight - 10% of final grade).

Final Exam: **One three-hour paper at the end of the semester**
(Weight - 90% of final grade).

Syllabus for Circuit Theory - ECNG1000

1. Introduction to Circuit Theory

2. Techniques of Circuit Analysis

   (i) Nodal Analysis
   (ii) Mesh Analysis
   (iii) Linearity and Superposition
   (iv) Source Transformations
   (v) Thevenin's and Norton's Theorems
   (vi) Maximum Power Transfer Theorem

3. Natural response and complete response to source-free and dc excited RL and RC circuits. Source-free RLC circuits. Forced Response of RLC Circuits.

   (i) The simple RL circuit
   (ii) Properties of the exponential response
   (iii) The simple RC circuit
   (iv) The unit-step forcing function
   (v) The natural and forced responses of RL and RC circuits
   (vi) The source-free parallel circuit
   (vii) The overdamped parallel RLC circuit
   (viii) Critical Damping
   (ix) The underdamped parallel RLC circuit
   (x) The natural and forced responses of RLC circuits

4. AC Steady-State Analysis
   i. Sinusoidal and Complex Forcing Functions
   ii. Phasors
   iii. Impedance

5. The Laplace Transform

6. Steady-State Power Analysis
   i. Instantaneous and Average power in ac circuits
   ii. Effective or rms values
   iii. Real Power, Reactive Power, Complex Power
   iv. Power Factor correction in ac circuits

The University of the West Indies
Mona Campus
Department of Physics

COURSE TITLE: **Electric Circuits**

COURSE CODE: **ELET2470**

LECTURES: **TWO (2) per week**

Lab.: **Mondays and Wednesdays 1 – 5 pm**

Lecturer: **Dr. Leary Myers**

EVALUATION:

Laboratory exercises (practical work): Weight - 20% of the final grade.

Course Test: **One sixty-minute paper.**

(Weight - 20% of final grade).

Final Exam: **One two-hour paper at the end of the semester**

(Weight - 60% of final grade).

Syllabus for Circuit Theory - (P 24G) - ELET2470

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ELECTRIC CIRCUIT ANALYSIS

- Electric circuit analysis is the portal through which students of electric phenomena begin their journey.
- It is the first course taken in their majors by most electrical engineering and electrical technology students.
- It is the primary exposure to electrical engineering, sometimes the only exposure, for students in many related disciplines, such as computer, mechanical, and biomedical engineering.
- Virtually all electrical engineering specialty areas, including electronics, power systems, communications, and digital design, rely heavily on circuit analysis.
- The only study within the electrical disciplines that is arguably more fundamental than circuits is electromagnetic field (EM) field theory, which forms the scientific foundation upon which circuit analysis stands.

Definition: An electric circuit, or electric network, is a collection of electrical elements interconnected in some way.

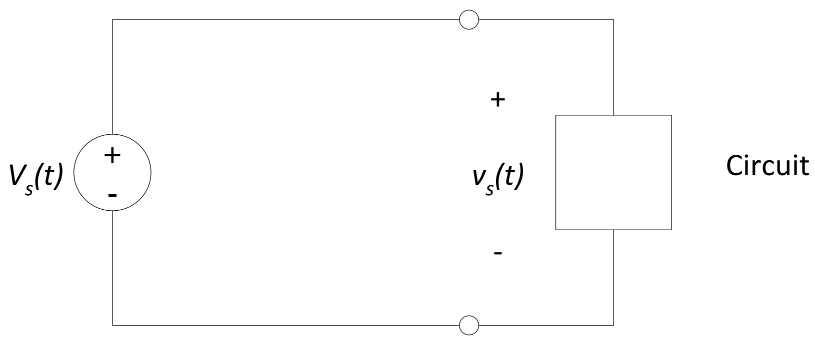
PASSIVE AND ACTIVE ELEMENTS

Circuit elements may be classified into two broad categories, passive elements and active elements by considering the energy delivered to or by them.

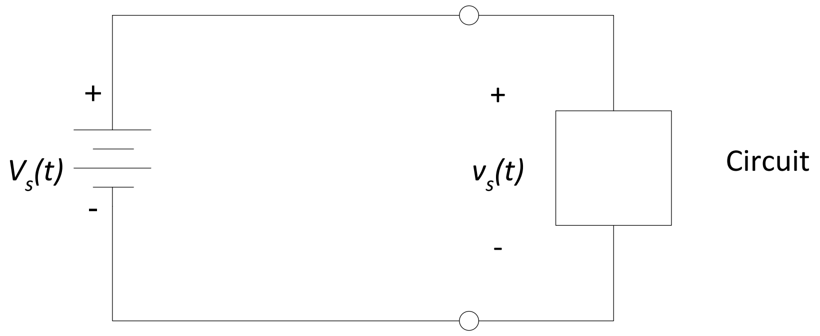
A circuit element is said to be passive if it cannot deliver more energy than has previously been supplied to it by the rest of the circuit.

An active element is any element that is not passive. Examples are generators, batteries, and electronic devices that require power supplies.

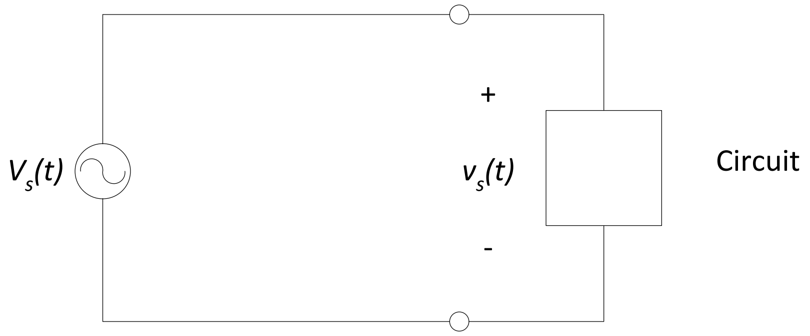
An **ideal voltage source** is an electric device that generates a prescribed voltage at its terminals irrespective of the current flowing through it. The amount of current supplied by the source is determined by the circuit connected to it.



General symbol for an ideal voltage source. $v_s(t)$ may be a constant (DC source)

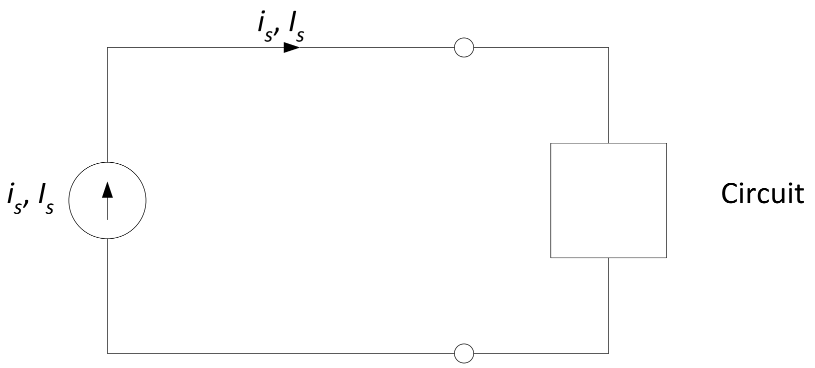


A special case: DC voltage source (ideal battery)



A special case: sinusoidal voltage source, $v_s(t) = V \cos \omega t$

An **ideal current source** provides a prescribed current to any circuit connected to it. The voltage generated by the source is determined by the circuit connected to it.



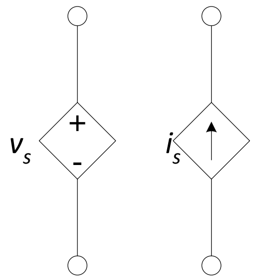
Symbol for an ideal current source

Summary

An **ideal/independent voltage source** is a two-terminal element, such as a battery or a generator that maintains a specified voltage between its terminals regardless of the rest of the circuit it is inserted into.

An **ideal/independent current source** is a two terminal element through which a specified current flows.

There exists another category of sources, however, whose output (current or voltage) is a function of some other voltage or current in a circuit. These are called **dependent** (or **controlled) sources**.



Source Type	Relationship
Voltage controlled voltage source (VCVS)	$v_s = \mu v_s$
Current controlled voltage source (CCVS)	$v_s = r i_s$
Voltage controlled current source (VCCS)	$is = g v_s$
Current controlled current source (CCCS)	$i_s = \beta\, i_x$

Summary

A **dependent or controlled voltage source** is a voltage source whose terminal voltage depends on, or is controlled by, a voltage or a current defined at some other location in the circuit. Controlled voltage sources are categorized by the type of controlling variable.
A **voltage-controlled voltage source** is controlled by a voltage and current-controlled voltage source by a current.
A **dependent or controlled current source** is a current source whose current depends on, or is controlled by, a voltage or a current defined at some other location in the circuit.

An **electrical network** is a collection of elements through which current flows.

The following definitions introduce some important elements of a network.

Branch

A **branch** is any portion of a circuit with two terminals connected to it. A branch may consist of one or more circuit elements.

Node

A point of connection of two or more circuit elements, together with all the connecting wires in unbroken contact with this point is called a **node**. Simply a **node** is the junction of two or more branches. (The junction of only two branches is usually referred to as a *trivial node*.)

It is sometimes convenient to use the concept of a **supernode**. A supernode is obtained by defining a region that encloses more than one node. Supernodes can be treated in exactly the same way as nodes.

Loop

A **loop** is any closed connection of branches.

Mesh

A **mesh** is a loop that does not contain other loops.

CIRCUIT THEORY

There are two branches of circuit theory, and they are closely linked to the fundamental concepts of input, circuit, and output.
Circuit Analysis – is the process of determining the output from a circuit for a given input.
Circuit Design (circuit synthesis) is the process of discovering a circuit that gives rise to that output when the input is applied to it. This is really a creative human activity.

Kirchhoff's Voltage Law (KVL)

The algebraic sum of voltage drops around any closed path is zero.

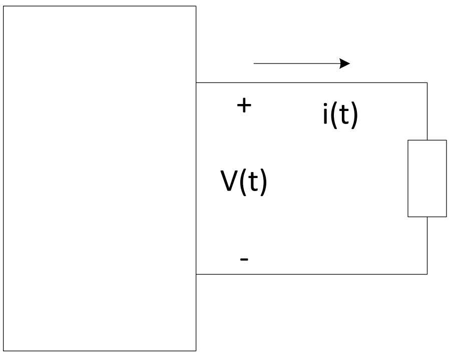
$$\sum_{n=1}^{N} v_n = 0$$

Kirchhoff's Current Law (KCL)

The sum of the currents entering any node equals the sum of the currents leaving the node.

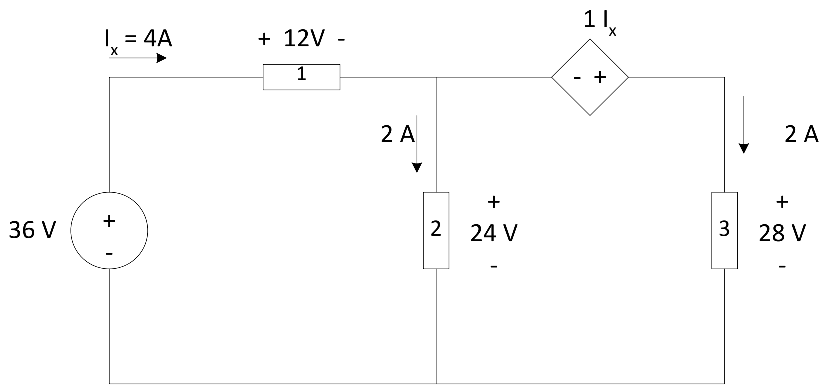
$$\sum_{n=1}^{N} i_n = 0$$

Passive Sign Convention



Voltage – Current Relationships for Energy Absorbed

Example:



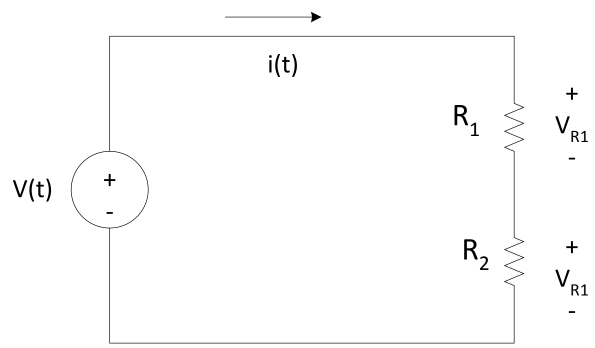
Compute the power that is absorbed or supplied by the elements in the network.

Solution:

If the positive current enters the positive terminal, the element is absorbing energy.

P_{36V} = (36)*(-4) = -144 W
P_1 = (12)*(4) = 48 W
P_2 = (24)*(2) = 48W
P_{DS} = (1*Ix)*(-2) = 1*4*(-2) = -8 W
P_3 = (28*(2) = 56 W

Voltage Division



By KVL; -v(t) + v_{R1} + v_{R2} = 0

But $v_{R1} = R_1$ i(t); $v_{R2} = R_2$ i(t), therefore v(t)= R_1 i(t) + R_2 i(t) or

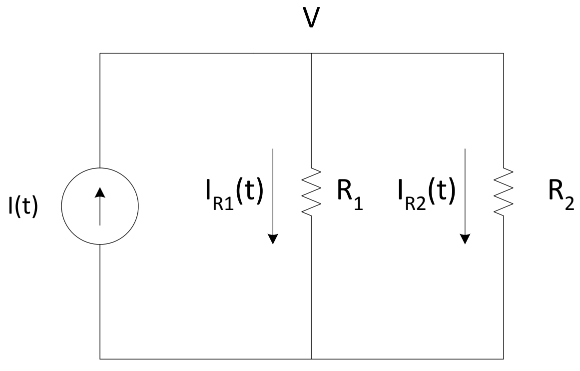
$$i(t) = \frac{v(t)}{R1 + R2}$$

So

$$V_{R1} = R_1 i(t) = R_1 \frac{v(t)}{R_1 + R_2} = \frac{R_1}{R_1 + R_2} v(t)$$

$$V_{R2} = R_2 i(t) = R_2 \frac{v(t)}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} v(t)$$

Current Division

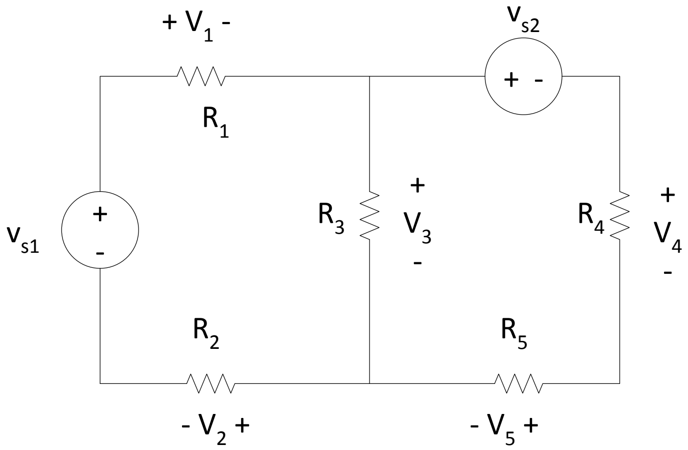


By KCL, $I(t) = I_{R1}(t) + I_{R2}(t) = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

Or

$$I = V\frac{R_1}{R_1 + R_2} + V\frac{R_2}{R_1 + R_2}$$

Loop Analysis



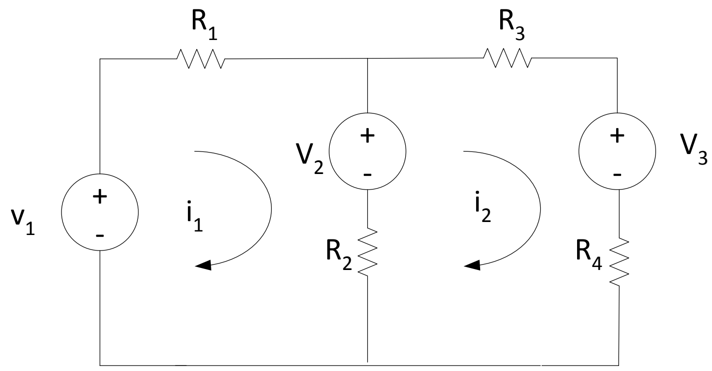
Loop 1: $V_1 + V_3 + V_2 - V_{s1} = 0$

Loop 2: $V_{s2} + V_4 + V_5 - V_3 = 0$

Mesh Analysis

A mesh is a special kind of loop that does not contain any loops within it.

Example Problem: Find the mesh currents in the circuit below given $V_1 = 10$ V; $V_2 = 9$ V; $V_3 = 1$ V; $R_1 = 5\ \Omega$; $R_2 = 10\ \Omega$; $R_3 = 5\ \Omega$; $R_4 = 5\ \Omega$.



By KVL;

Mesh 1: $-V_1 + R_1 i_1 + V_2 + R_2(i_1 - i_2) = 0$

Mesh 2: $-V_2 + R_3 i_2 + V_3 + R_4 i_2 + R_2(i_2 - i_1) = 0$

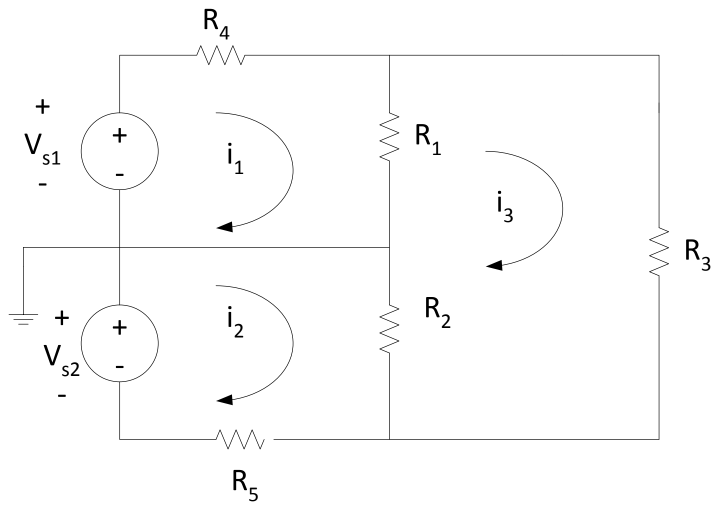
Rearranging the linear system of equations, we obtain

$15i_1 - 10i_2 = 1$

$-10i_1 + 20i_2 = 8$

Giving $i_1 = 0.5$ A and $i_2 = 0.65$ A

Example Problem: The circuit below is a simplified DC circuit model of a three-wire electrical distribution service to residential and commercial buildings. The two ideal sources and the resistances R_4 and R_5 represent the equivalent circuit of the distribution system; R_1 and R_2 represent 110-V lighting and utility loads of 800 W and 300 W respectively. Resistance R_3 represents a 220-V heating load of about 3 KW. Determine the voltages across the three loads.



Given that $V_{s1} = V_{s2} = 110$ V; $R_4 = R_5 = 1.3\ \Omega$; $R_1 = 15\ \Omega$; $R_2 = 40\ \Omega$; $R_3 = 16\ \Omega$.

Mesh 1: $-V_{s1} + R_4 i_1 + R_1(i_1 - i_3) = 0$

Mesh 2: $-V_{s2} + R_2(i_2 - i_3) + R_5 i_2 = 0$

Mesh 3: $R_3 i_3 + R_2(i_3 - i_2) + R_1(i_3 - i_1) = 0$

Rearranging, we obtain;

$-(R_1 + R_4)i_1 + R_1 i_3 = -V_{s1}$

$-(R_2 + R_5)i_2 + R_2 i_3 = -V_{s2}$

$R_1 i_1 + R_2 i_2 - (R_1 + R_2 + R_3)I_3 = 0$

In matrix form $$\begin{bmatrix} -16.3 & 0 & 15 \\ 0 & -41.3 & 40 \\ 15 & 40 & -71 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -110 \\ -110 \\ 0 \end{bmatrix}$$

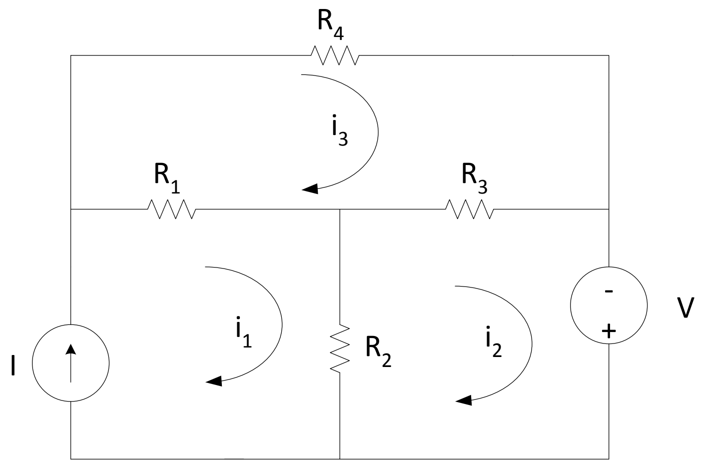
Which can be expressed as [R][I] = [V]

With a solution $[I] = [R]^{-1}[V]$

We find : $i_1 = 17.11$ A $i_2 = 13.57$ A $i_3 = 11.26$ A

Giving $V_{R1} = R_1(i_1 - i_3) = 87.75$ V
$V_{R2} = R_2(i_2 - i_3) = 92.4$ V
$V_{R3} = R_3 i_3 = 180.16$ V

Mesh Analysis with Current Sources



Find the mesh currents in the circuit above given I = 0.5 A; V = 6 V; R_1 = 3 Ω; R_2 = 8 Ω; R_3 = 6 Ω; R_4 = 4 Ω.

Mesh 1: The current source forces the mesh current to be equal to i_1.

$i_1 = I$

Mesh 2: $-V + R_2(i_2 - i_1) + R_3(i_2 - i_3) = 0$

Mesh 3: $R_4 i_3 + R_3(i_3 - i_2) + R_1(i_3 - i_1) = 0$

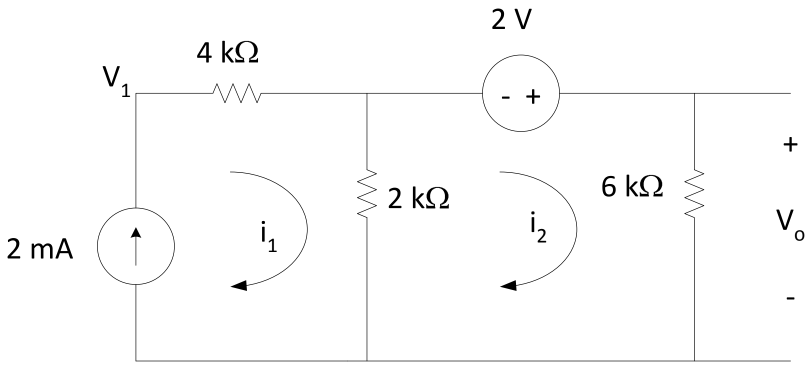
Rearranging the equations and substituting the known value of i_1, we obtain:

$14i_2 - 6i_3 = 10$

$-6i_2 + 13i_3 = 1.5$

Hence $i_2 = 0.95$ A and $i_3 = 0.55$ A

Example: Find V_1 and V_o.



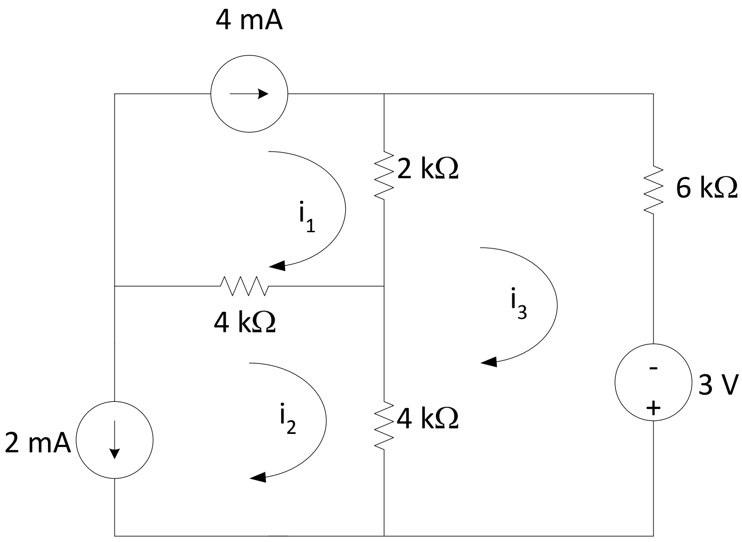
Mesh 1: $i_1 = 2 \text{ X } 10^{-3}$ A

Mesh 2: $2k(i_2 - i_1) - 2 + 6ki_2 = 0$

From above, $i_2 = 0.75$ mA and $V_o = 6ki_2 = 4.5$ V

Now by Ohms law, $V_1 = 4ki_1 + 2k(i_1 - i_2) = 10.5$ V

Example: Find V_o.



Mesh 1: $i_1 = 4$ mA

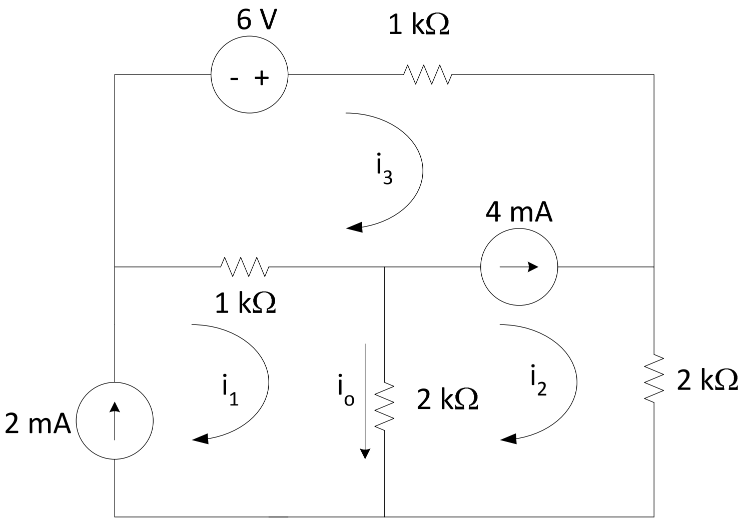
Mesh 2: $i_2 = -2$ mA

Mesh 3: $4k(i_3 - i_2) + 2k(i_3 - i_1) + 6ki_3 - 3 = 0$

Hence $i_3 = 0.25$ mA and $-V_o + 6ki_3 - 3 = 0$

$V_o = 6ki_3 - 3 = -1.5V$

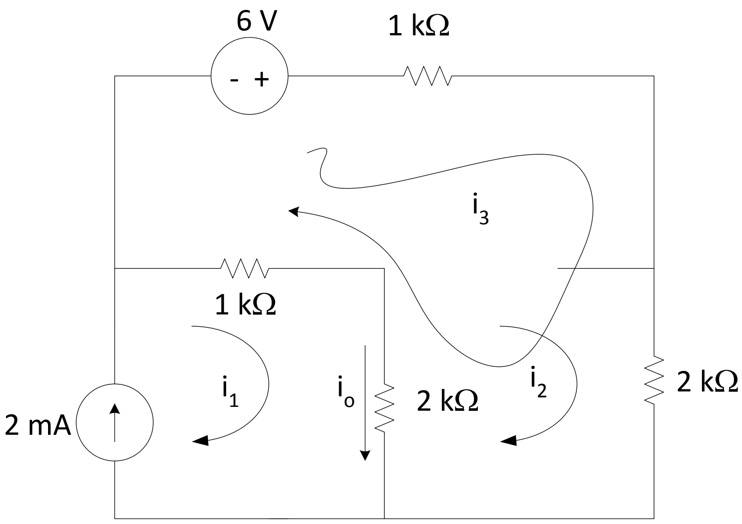
Example: Find i_0



Mesh 1: i_1 = 2 mA

Mesh 2: (Supernode approach) – Remove current source

$-6 + 1ki_3 + 2ki_2 + 2k(i_2 - i_1) + 1k(i_3 - i_1) = 0$

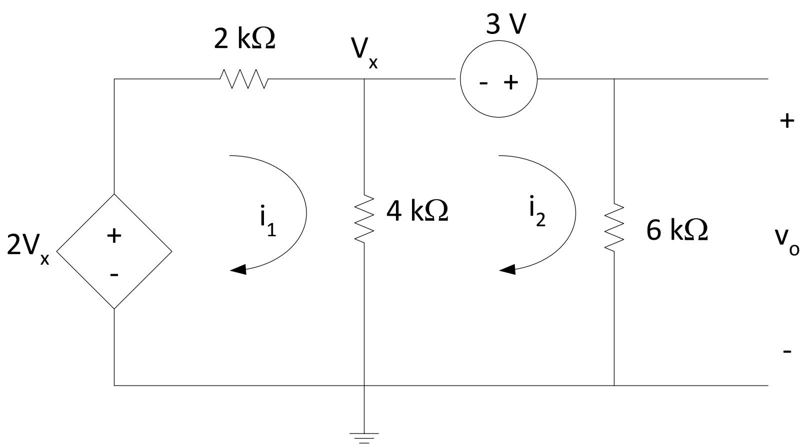


Subject to $i_2 - i_3 = 4mA$

From above we get $i_2 = \frac{10}{3}$ mA and $i_3 = -\frac{2}{3}$ mA. Therefore $i_o = i_1 - i_2 = -\frac{4}{3}mA$

Circuits Containing Dependent Sources

Example: Find v_o.



Mesh 1: $-2V_x + 2ki_1 + 4k(i_1 - i_2) = 0$ or $6ki_1 - 4ki_2 = 2V_x$

Mesh 2: $-3 + 6ki2 + 4k(i2 - i1) = 0$ or $-4ki_1 + 10k\, i_2 = 3$

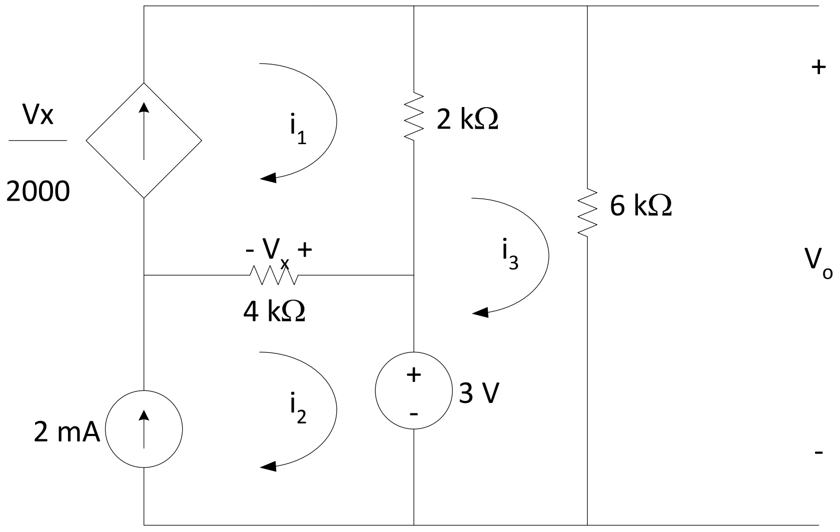
But $V_x = 4k(i_1 - i_2)$

Therefore $i_1 = 2i_2$;

$$i_2 = \frac{3}{2}mA$$

$V_o = 9$ V

Example: Find V_o.



Mesh 1: $i_1 = \frac{V_x}{2000}$

Mesh 2: $i_2 = 2$ mA

Mesh 3: $-3 + 2k(i_3 - i_1) + 6ki_3 = 0$

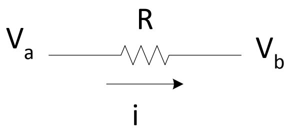
And $V_x = 4k(i_1 - i_2)$

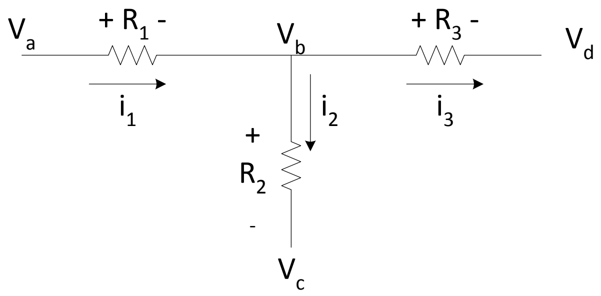
Solving, we get $i_3 = \frac{11}{8}\ mA$ and $V_o = \frac{33}{4}\ V$

Nodal Analysis

Node voltage analysis is the most general method for the analysis of electric circuits. The **node voltage method** is based on defining the voltage at each node as an independent variable. One of the nodes is selected as a **reference node** (usually – but not necessarily – ground) and each of the other node voltages is referenced to this node.

Once node voltages are defined, Ohm's law may be applied between any two adjacent nodes to determine the current flowing in each branch.





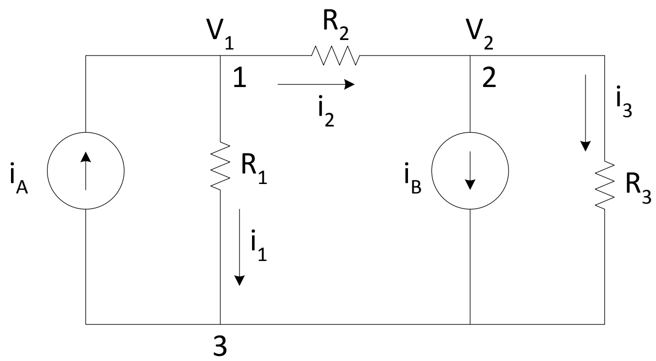
$$i = \frac{v_a - v_b}{R}$$

By KCL; $i_1 - i_2 - i_3 = 0$

$$\frac{v_a - v_b}{R_1} - \frac{v_b - v_c}{R_2} - \frac{v_b - v_d}{R_3} = 0$$

In a circuit containing *n* nodes, we can write, at most, *n – 1* independent equations.

Circuits containing only independent current sources



Let

$$R_1 = \frac{1}{G_1}$$

$$R_2 = \frac{1}{G_2}$$

$$R_3 = \frac{1}{G_3}$$

At node 1 by KCL; $i_A + i_1 + i_2 = 0$

$$-i_A + G_1(V_1 - 0) + G_2(V_1 - V_2) = 0$$

At node 2 by KCL; $-i_2 + i_B + i_3 = 0$

$$-G_2(V_1 - V_2) + i_B + G_3(V_2 - 0) = 0$$

So,

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} i_A \\ -i_B \end{bmatrix}$$

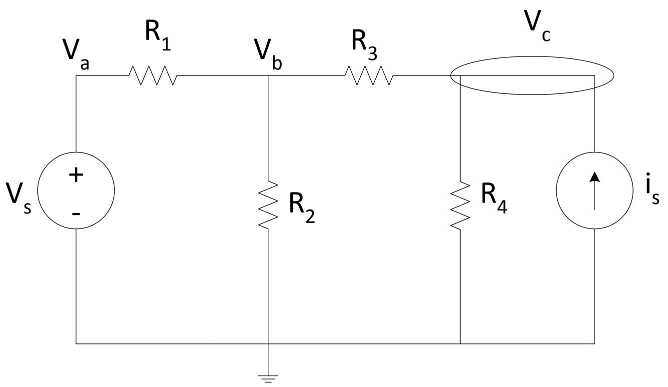
Summary

Step 1: Select a reference node (usually ground). This node usually has most elements tied to it. All other nodes will be referenced to this node.

Step 2. Define the remaining *n-1* node voltages as the independent or dependent variables.

Step 3. Apply KCL at each node labelled as an independent variable, expressing each current in terms of the adjacent node voltages.

Node Analysis with Voltage Sources



Apply KCL at the two nodes associated with the independent variables V_b and V_c.

At node b:

$$-\left(\frac{V_a - V_b}{R_1}\right) + \frac{V_b - 0}{R_2} + \frac{V_b - V_c}{R_3} = 0$$

Note $V_s = V_a$

At node c:

$$\frac{V_c - V_b}{R_3} + \frac{V_c}{R_4} - i_s = 0$$

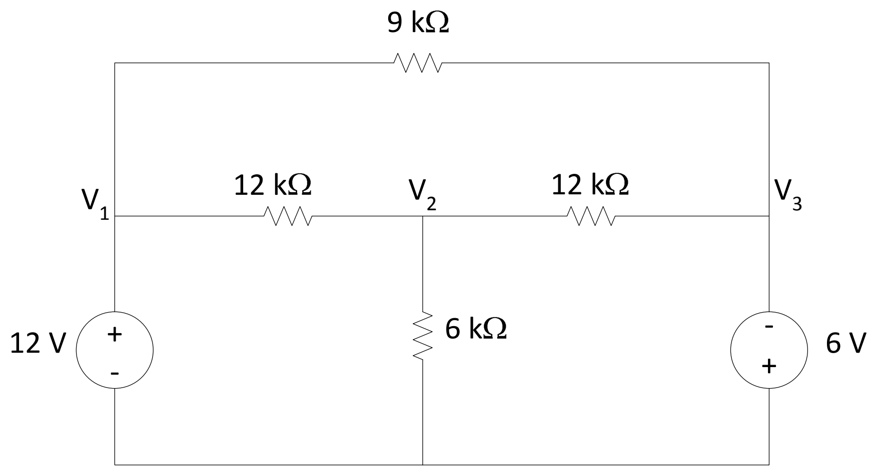
which can be rewritten as:

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_b + \left(-\frac{1}{R_3}\right) V_c = \frac{1}{R_1} V_s$$

$$\left(-\frac{1}{R_3}\right) V_b + \left(\frac{1}{R_3} + \frac{1}{R_4}\right) V_c = i_s$$

and solved.

Example: Find V_2.



At node 1: $V_1 = 12$ V

At node 3: $V_3 = -6$ V

At node 2:

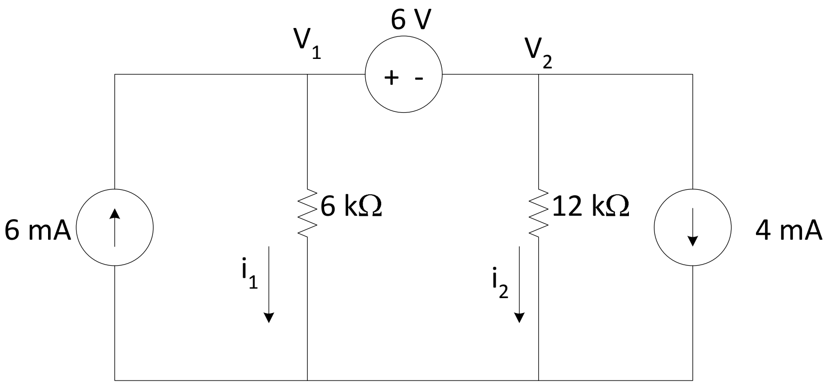
$$\frac{V_2 - V_1}{12k} + \frac{V_2}{6k} + \frac{V_2 - V_3}{12k} = 0$$

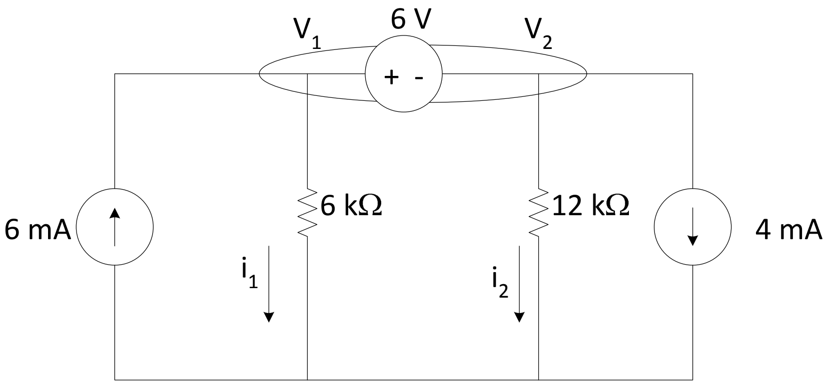
Solving,

$$V_2 = \frac{3}{2}\ V$$

Independent Voltage Source connected between two non-reference nodes.

Example: Determine the values of V_1 and V_2.





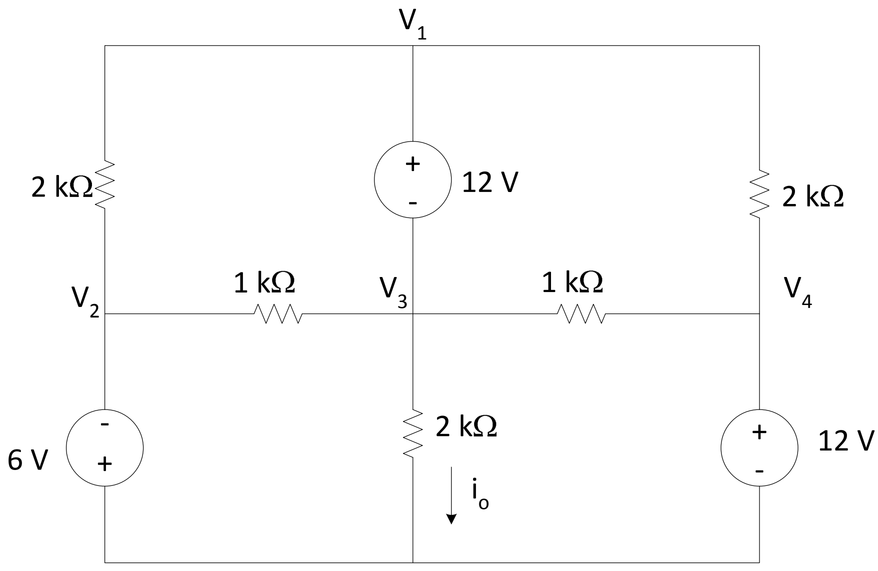
At node 1:

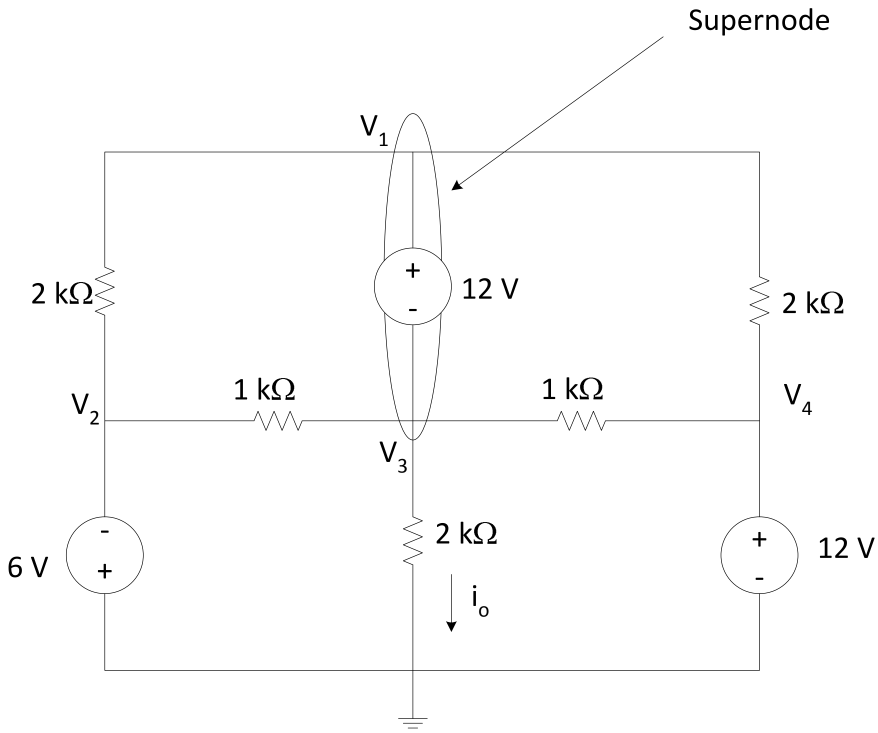
$$-6 \times 10^{-3} + \frac{V_1}{6k} + \frac{V_2}{12k} + 4 \times 10^{-3} = 0$$

Subject to: $V_1 - V_2 = 6$ V

Solution is $V_1 = 10$ V and $V_2 = 4$ V

Example: Find I_o.





At the supernode:

$$\frac{V_1 - V_2}{2k} + \frac{V_3 - V_2}{1k} + \frac{V_3 - V_4}{1k} + \frac{V_3}{2k} + \frac{V_1 - V_4}{2k} = 0$$

Subject to: $V_1 - V_3 = 12$ V

We observe that $V_2 = -6$ V and $V_4 = 12$ V

Solving for V_3, gives

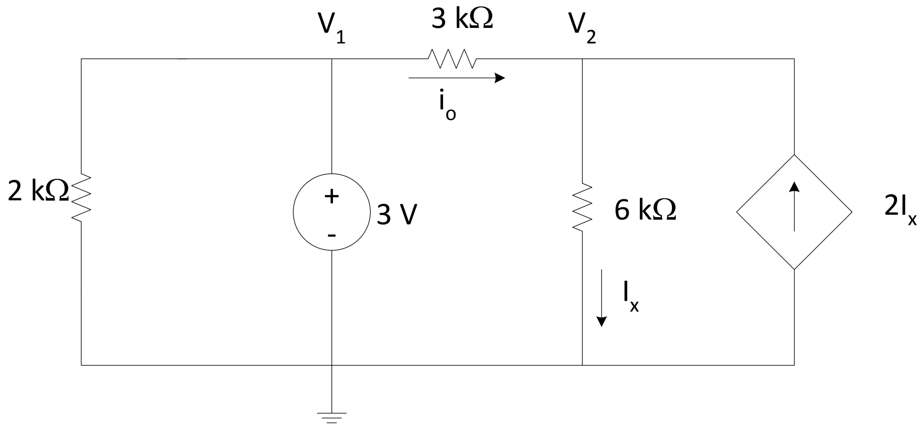
$$V_3 = -\frac{6}{7}\ V$$

and

$$i_o = \frac{V_3}{2k} = -\frac{3}{7}\ mA$$

Circuits containing Dependent Sources

Example: Determine the value of i_o.



Solution: We observe V_1 = 3 V and

$$I_x = \frac{V_2}{6k}$$

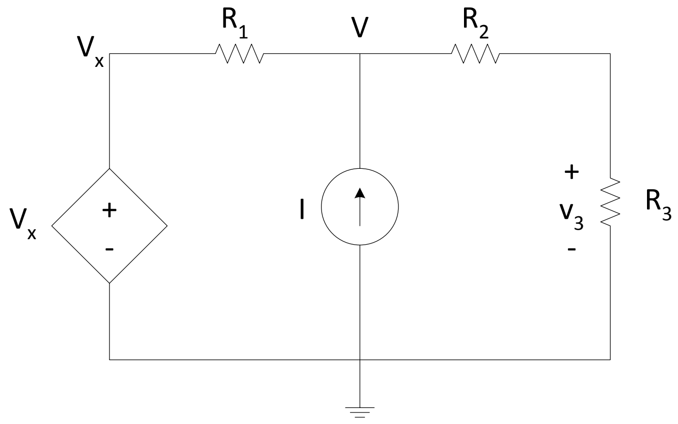
At node 2:

$$\frac{V_2 - V_1}{3k} + \frac{V_2}{6k} - 2I_x = 0$$

Solving, we get: V_2 = 6 V;

$$i_o = \frac{V_1 - V_2}{3k} = -1\ mA$$

Example: Find unknown node voltage V, given the following I = 0.5 A; $R_1 = 5\ \Omega$; $R_2 = 2\ \Omega$; $R_3 = 4\ \Omega$ and $v_x = 2 \times v_3$



At node with voltage V:

$$\frac{V - V_x}{R_1} - I + \frac{V - V_3}{R_2} = 0$$

At node with voltage v_3:

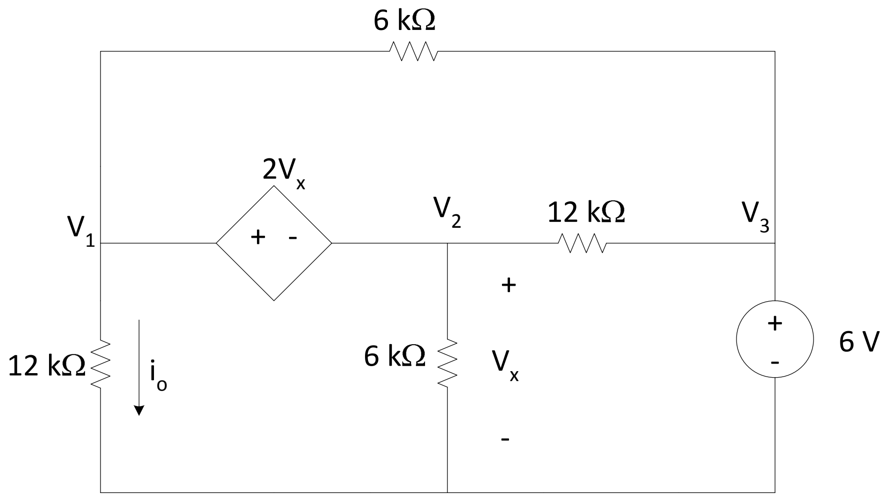
$$\frac{V_3 - V}{R_2} + \frac{v_3}{R_3} = 0$$

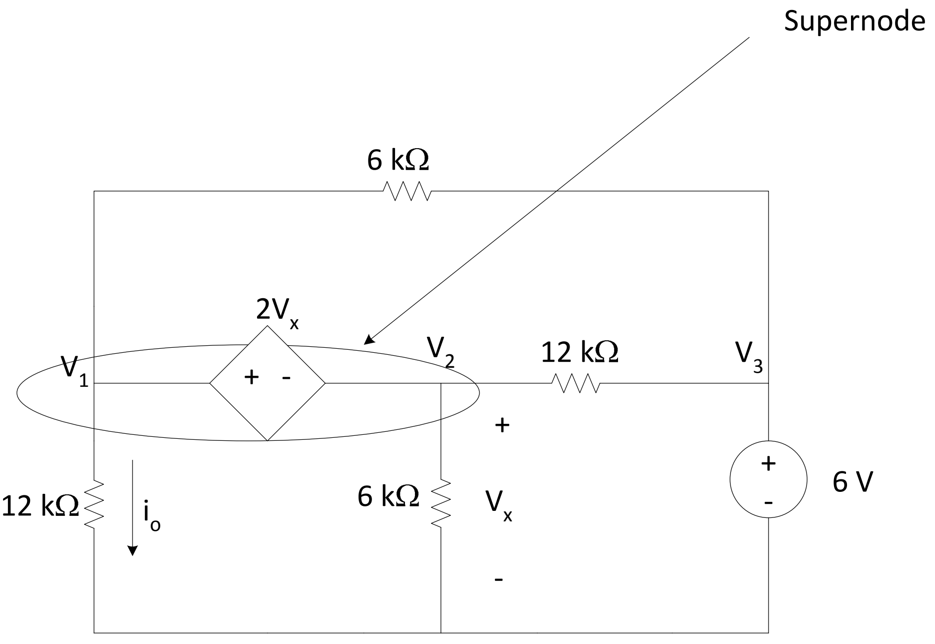
Substituting the dependent source relationship into the first equation, we obtain:

$$0.7V - 0.9v_3 = 0.5$$
$$-0.5V - 0.75v_3 = 0$$

Yielding V = 5V and V_3 = 3.33 V

Example: Find i_o.





At the supernode:

$$\frac{V_1 - V_3}{6k} + \frac{V_1}{12k} + \frac{V_2}{6k} + \frac{V_2 - V_3}{12k} = 0$$

Subject to: $V_1 - V_2 = 2V_x$

So,

$$V_1 = \frac{9}{2}\,V \;and\; i_o = \frac{V_1}{12k} = \frac{3}{8}\,mA$$

Linearity

A resistor is a linear element because its current voltage relationship has a linear characteristic. i.e. *v(t) = Ri(t)*

Linearity requires both additivity and homogeneity (scaling). In the case of a resistive element, if $i_1(t)$ is applied to a resistor, then the voltage across the resistor is $v_1(t) = Ri_1(t)$. Similarly if $i_2(t)$ is applied, the voltage across the resistor is $v_2(t) = Ri_2(t)$.

However if $i_1(t) + i_2(t)$ is applied, the voltage across the resistor is:

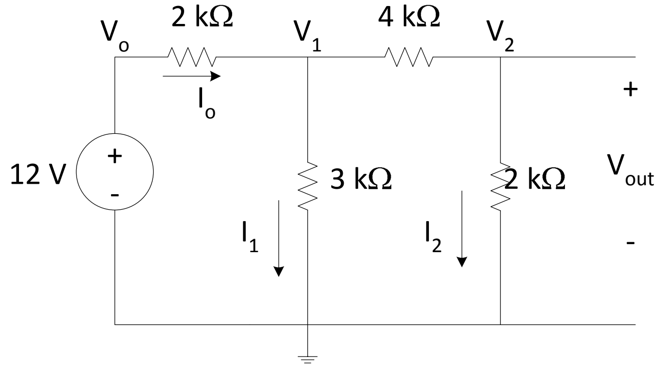
$$v(t) = Ri_1(t) + Ri_2(t) = v_1(t) + v_2(t)$$

This demonstrates the **additive** property.

If the current is scaled by a constant K, the voltage is also scaled by the constant K since R Ki(t) = K Ri(t) = K v(t). This demonstrates **homogeneity**.

A linear circuit is one only independent sources, linear dependent sources and linear elements. Capacitors and inductors are circuit elements that have a linear input-output relationship provided that their initial storage energy is zero.

Example to demonstrate Linearity



Assume V_{out} = 1 V = V_2, then

$$I_2 = \frac{1}{2k} = 0.5\ mA$$

$V_1 = 4kI_2 + V_2 = 3$ V and

$$I_1 = \frac{V_1}{3k} = 1\ mA$$

By KCL $I_o = I_1 + I_2 = 1.5$ mA and $V_o = 2kI_o + V_1 = 6$ V

The assumption that V_{out} = 1 V produced a source voltage V_o of 6 V.

BUT we know by observation V_o = 12 V, therefore the actual output voltage,

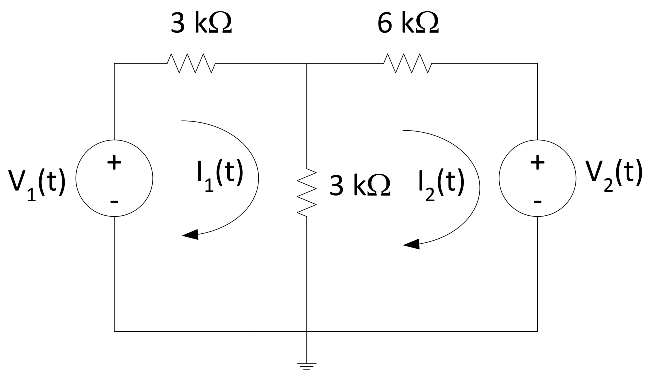
$$V_{out} = \left(\frac{12}{6}\right) 1V = 2\ V$$

Superposition Principle

In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as the algebraic sum of the individual contributions of each source acting alone.

N.B. When determining the contribution due to any independent source, all remaining voltage sources are made zero by replacing them with a short circuit and any remaining current sources are made zero by replacing them with an open circuit.

Use the circuit to examine the concept of superposition.



Mesh 1: $-V_1 + 3kI_1 + 3k(I_1 - I_2) = 0$ or $V_1 = 6kI_1 - 3kI_2$

Mesh 2: $6k\ I_2 + V_2 + 3k\ (I_2 - I_1) = 0$ or $V_2 = 3k\ I_1 - 9k\ I_2$

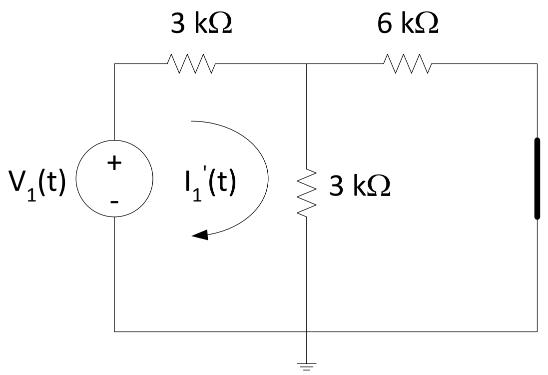
Solving for $I_1(t)$ yields:

$$I_1(t) = \frac{V_1}{5k} - \frac{V_2}{15k}$$

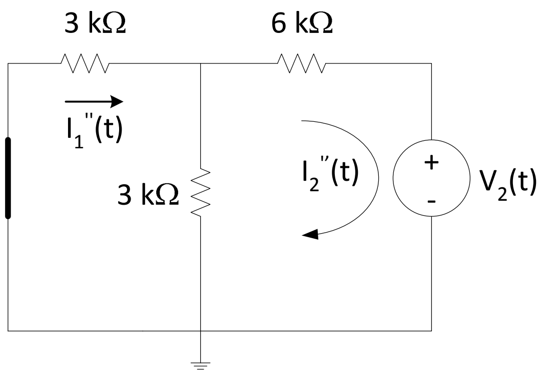
which implies that $I_1(t)$ has a component due to $V_1(t)$ and a component due to $V_2(t)$.

Each source acting alone would produce the following:

Set $V_2(t) = 0$



$$I_1'(t) = \frac{V_1(t)}{3k + 2k} = \frac{V_1(t)}{5k}$$



$$I_2''(t) = \frac{-V_2(t)}{6k + (9/_6)k} = \frac{-2V_2(t)}{15k}$$

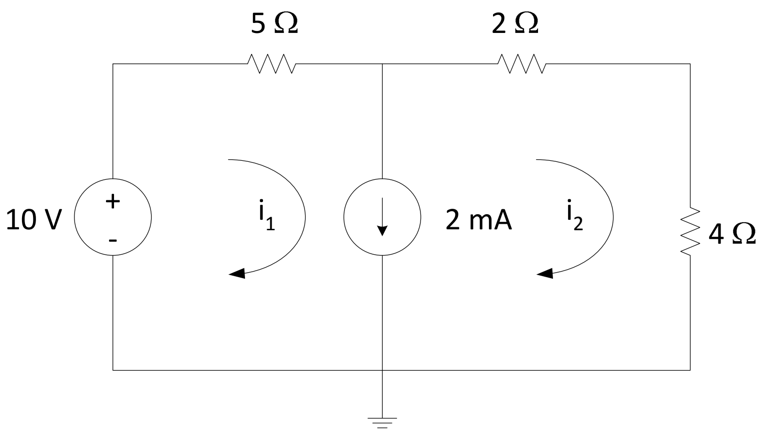
By current division:

$$I_1''(t) = \frac{3k}{3k+3k} I_2''(t) = \frac{-2V_2(t)}{15k}\frac{3}{6} = \frac{-V_2(t)}{15k}$$

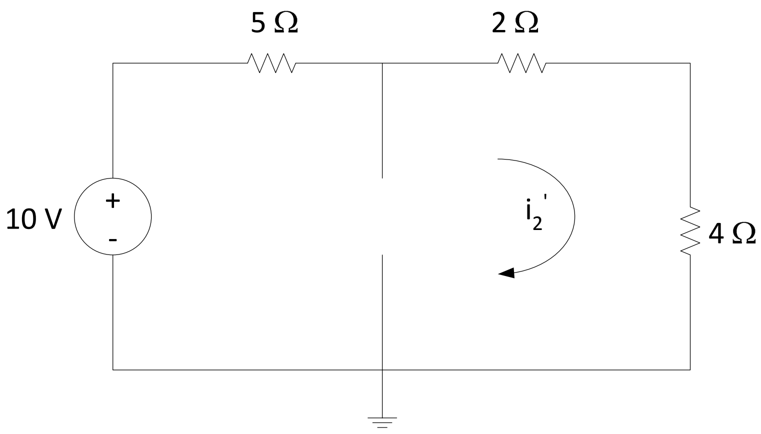
Now

$$I_1(t) = I_1' + I_1'' = \frac{V_1}{5k} - \frac{V_2}{15k}$$

Example: Determine the current i_2 in the circuit below using the principle of superposition.



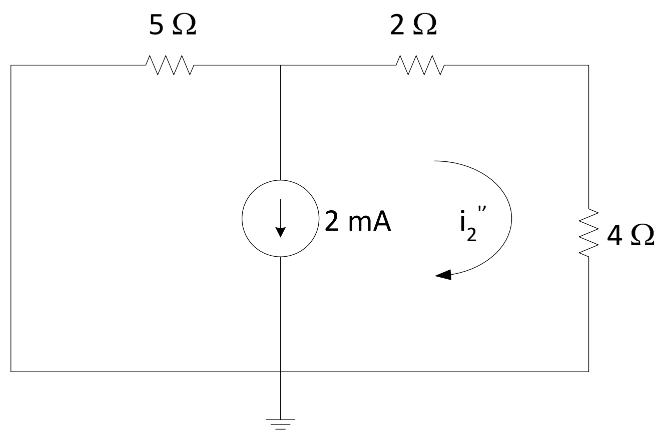
Step 1: Zero the current source by replacement by an open circuit.



Then

$$i_2' = \frac{10}{5+2+4} = 0.909\ A$$

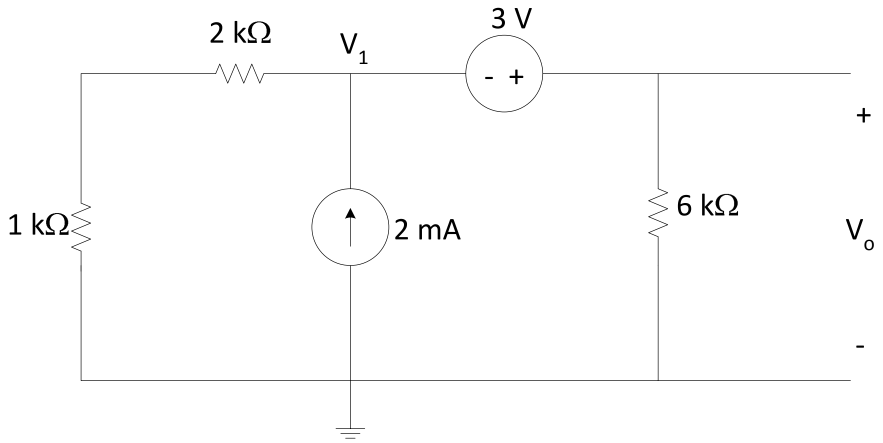
Step 2: Zero the voltage source by replacing it by a short circuit.



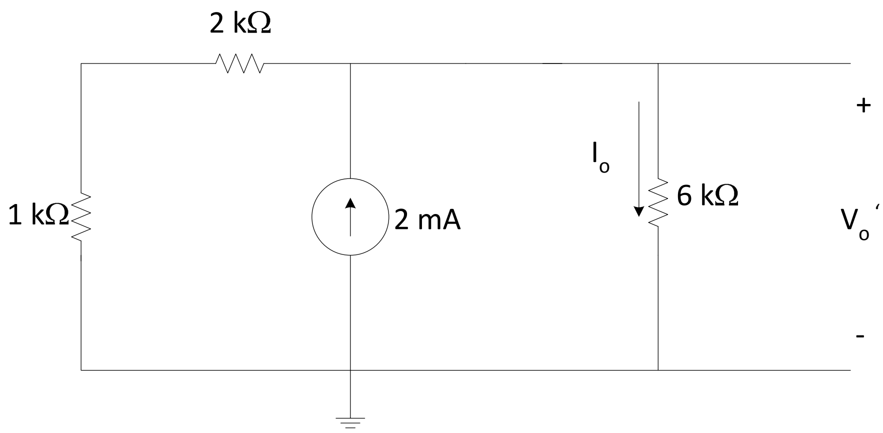
$$i_2'' = -\frac{5}{5+6}2 = -0.909\ A$$

Therefore $i_2 = i_2' + i_2'' = 0$

Example: Use superposition to find V_o.



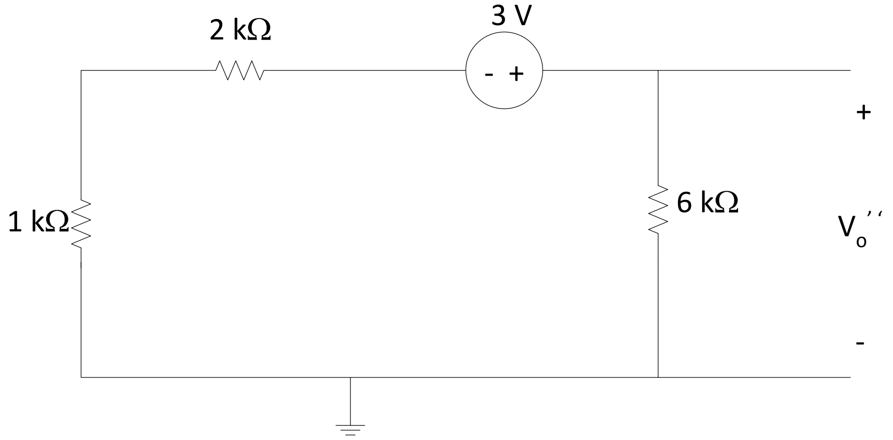
Step 1: Remove (short circuit) the 3 V source



$$I_o = \frac{3}{3+6} \times 2 = \frac{2}{3}\ mA$$

and $V_o' = I_o \times 6k = 4\ V$

Step 2: remove (open circuit) the 2 mA current source



$$V_o'' = \frac{6}{1+2+6} \times 3 = 2\ V$$

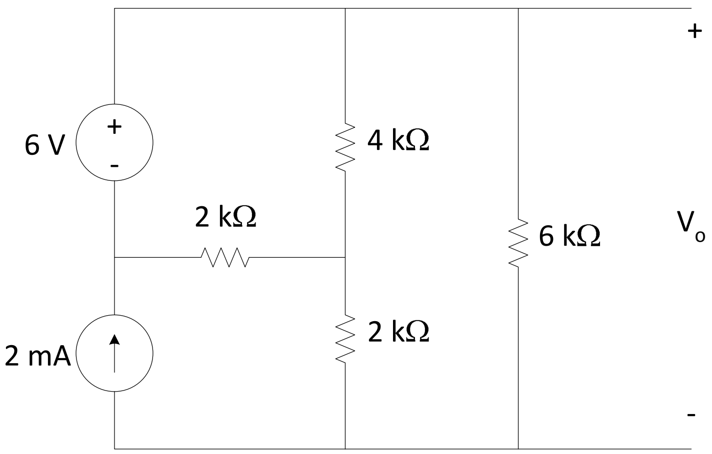
Hence $V_o = V_o' + V_o'' = 4 + 2 = 6\ V$

As a check: using KCL and recognizing the supernode, we have:

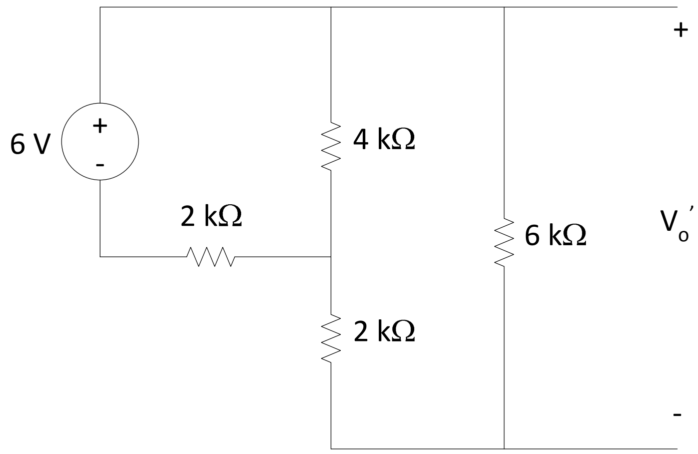
$$\frac{V_1}{3k} - 2 \times 10^{-3} + \frac{V_0}{6k} = 0$$

But $V_o - V_1 = 3$ V which implies that $V_o = 6$ V

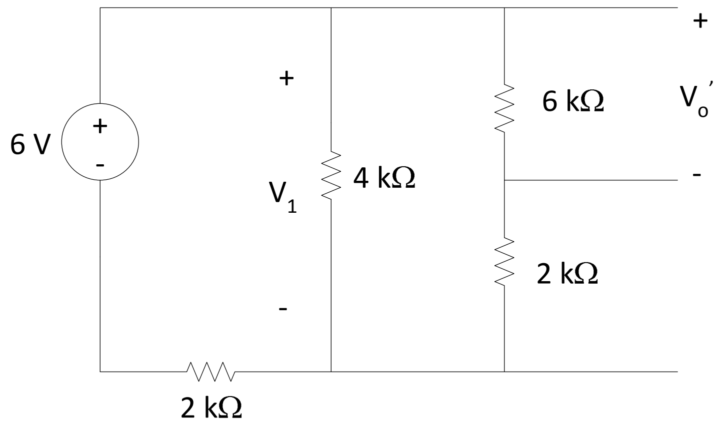
Example: Find V_o using the Superposition Principle.



Step 1: Remove (open-circuit) the current source.



Or



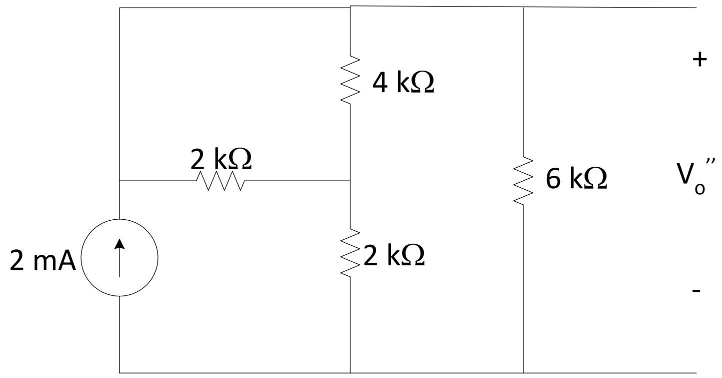
$$R_{eq} = (6+2)//4 = \frac{8}{3} k\Omega$$

$$V_1 = \frac{(8/3)}{(8/3)+2} \times 6 = \frac{24}{7} V$$

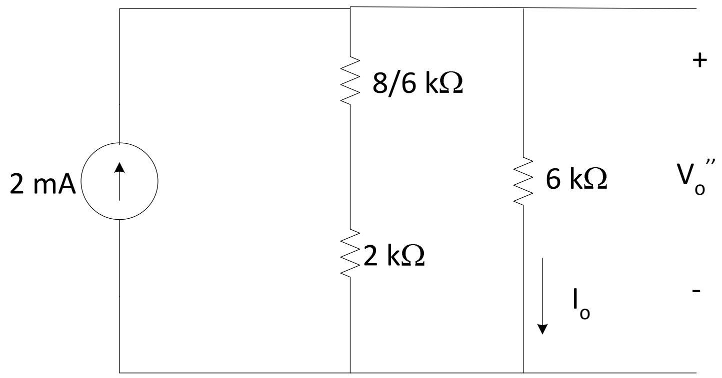
and

$$V_o' = \frac{6}{6+2} \times V_1 = \frac{18}{7}\ V$$

Step 2: Remove (short-circuit) voltage source.



Or



$$I_o = \frac{(10/3)}{(10/3)+6} \times 2 = \frac{5}{7}\ mA$$

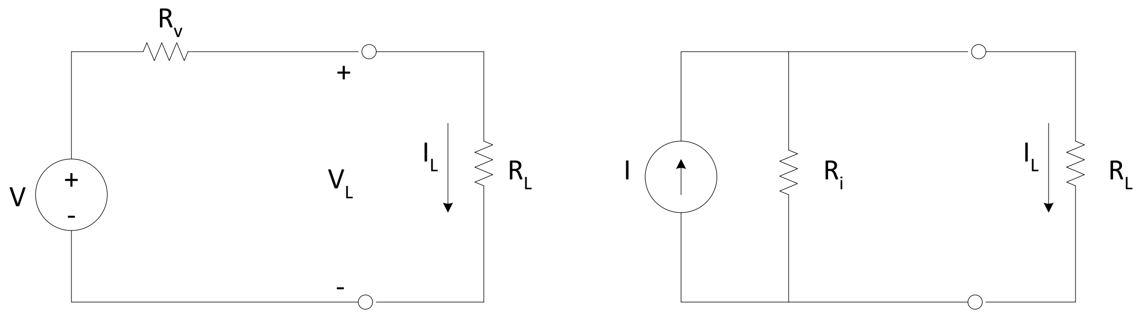
and

$$V_o'' = 6k \times I_o = \frac{30}{7}V$$

$$V_O = V_O' + V_O'' = \frac{48}{7}\ V$$

Source Transformation

Real sources differ from ideal models. In general a practical voltage source does not produce a constant voltage regardless of the load resistance or the current it delivers, nor does a practical current source deliver a constant current regardless of the load resistance or the voltage across its terminals. Practical sources contain internal resistance.



(a) (b)

$$P_L = I_L^2 \times R_L = \left(\frac{V}{R_V + R_L}\right)^2 R_L = \frac{V^2}{R_L}\left(\frac{1}{1 + {}^{R_V}/_{R_L}}\right)^2$$

If $R_L \gg R_V$ then

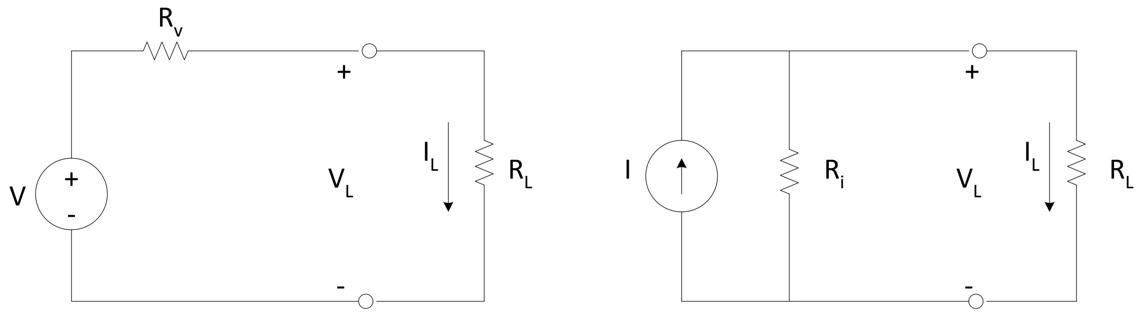
$P_L = \frac{V^2}{R_L}$ which is the power delivered by an ideal source.

Similarly,

$$P_L = I_L^2 \times R_L = \left(\frac{IR_i}{R_L + R_I}\right)^2 R_L = I^2 R_L\left(\frac{1}{1 + \frac{R_L}{R_i}}\right)^2$$

If $R_i \gg R_L$ then $P_L = I^2 R_L$ which is the power delivered by an ideal current source.

Equivalent Sources



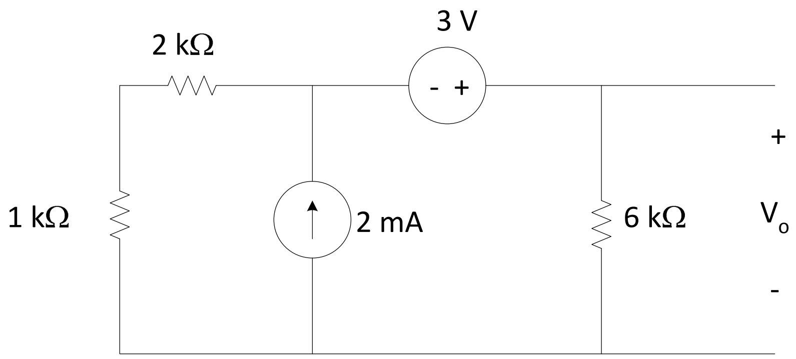
(a) (b)

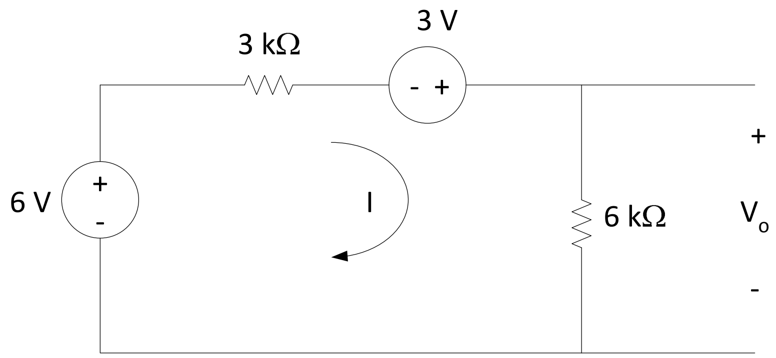
For circuit (a): $V = I_L R_V + V_L$

For circuit (b):$I = I_L + \frac{V_L}{R_I}$ $or\ IR_I = R_I I_L + V_L$

For the networks in (a) and (b) to be equivalent, their terminal characteristics must be identical, i.e. $V = IR_i$ $and\ R_i = R_V$.

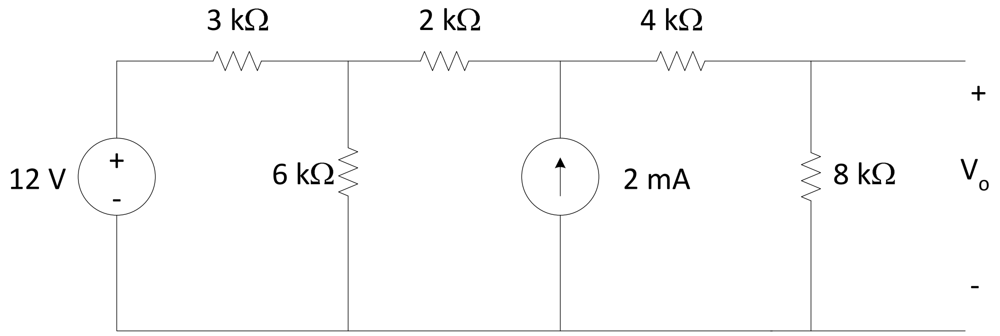
Example: Determine V_o using the Source Transformation technique.

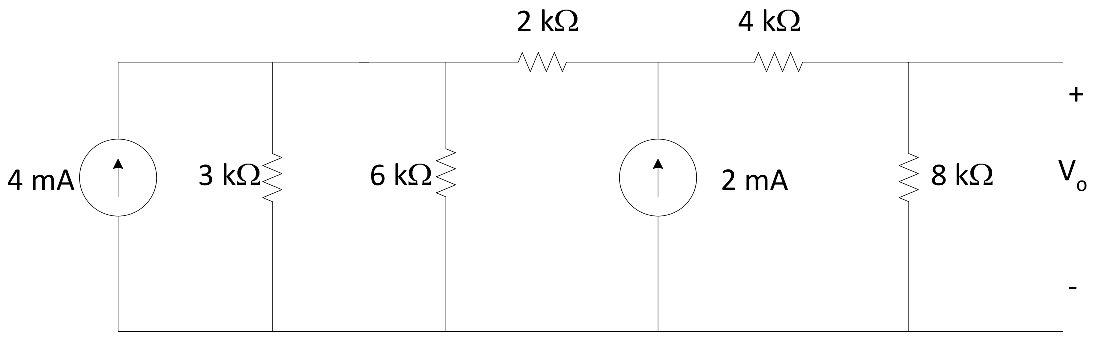


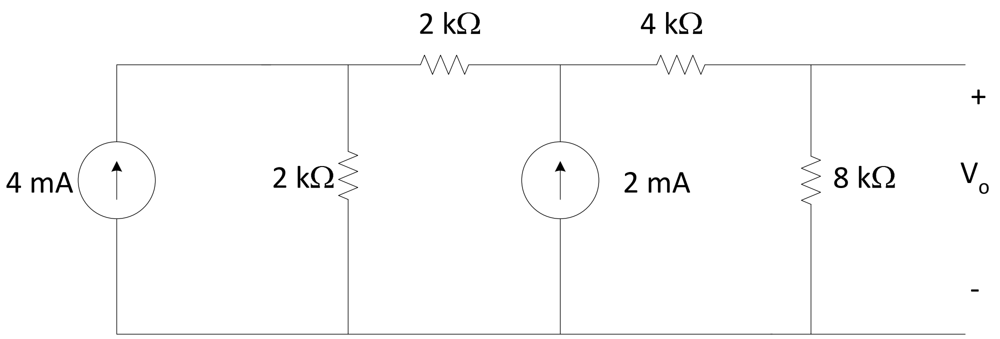


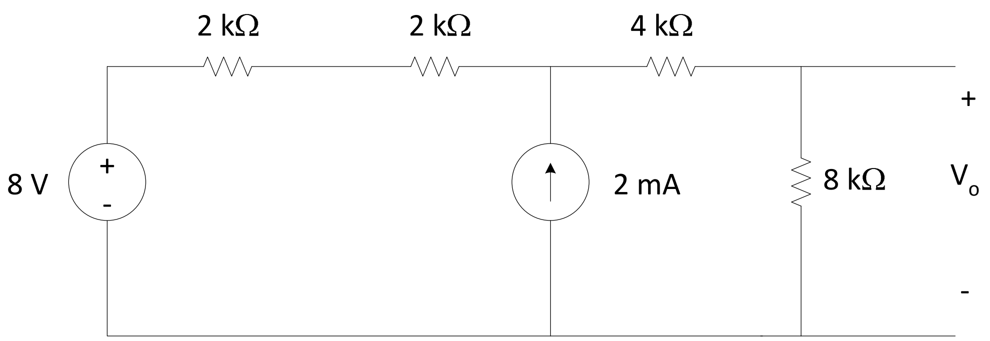
Mesh 1: - 6 + 3kI – 3 + 6kI = 0 which yields I = 1 mA and V_o = 6kI =6 V

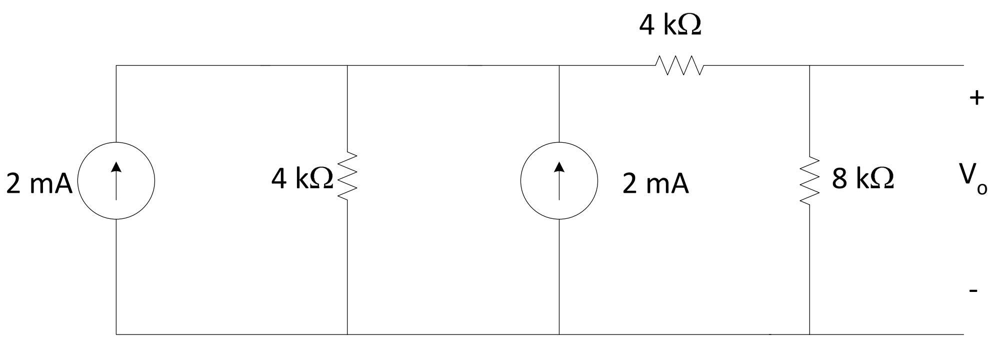
Example: Use Source transformation to find V_o.

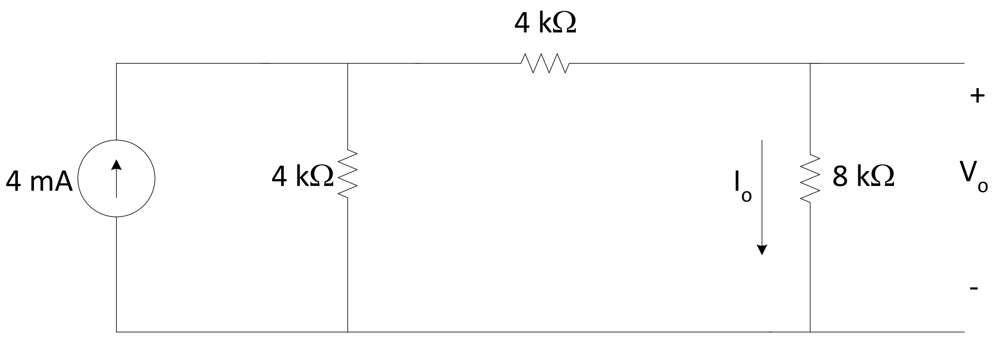












$$I_o = (4 \times 10^{-3})\left(\frac{4}{4+4+8}\right) = 1\,mA$$

and $$V_o = I_o \times 8k = 8\,V$$

Thévenin' and Norton's Theorems

In this section we will discuss one of the most important topics in the analysis of electric circuits; the concept of an **equivalent circuit**. Very complicated circuits can be viewed in terms of much simpler equivalent source and load circuits.

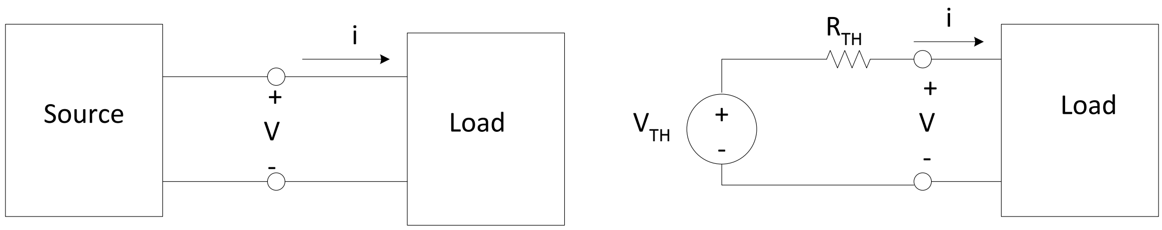
Suppose we are given a circuit and we wish to determine the current, voltage or power that is delivered to some resistor of that network. Thevenin's theorem tells us that we can replace the entire network, excluding the load, by an equivalent circuit, that contains only an independent voltage source in series with a resistor in such a way that the I – V relationship at the load is unchanged.

The Thévenin Theorem

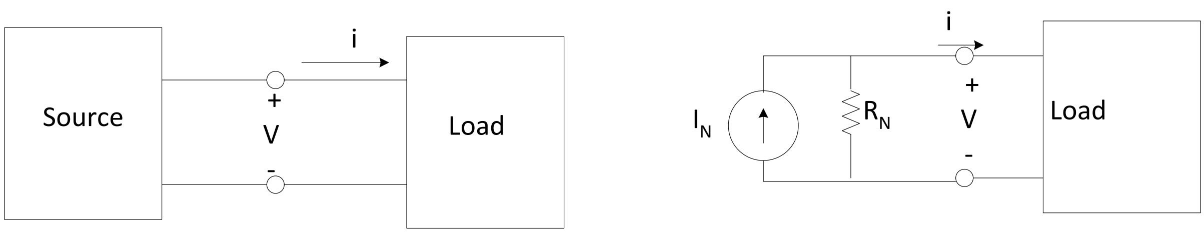
When viewed from the load, any network composed of ideal voltage and current sources, and of linear resistors, may be represented by an equivalent circuit consisting of an ideal voltage source V_{TH} in series with an equivalent resistance R_{TH}.

The Norton Theorem

When viewed from the load, any network composed of ideal voltage and current sources, and of linear resistors, may be represented by an equivalent circuit consisting of an ideal current source I_N in series with an equivalent resistance R_N.



(a) Illustration of Thevenin's Theorem



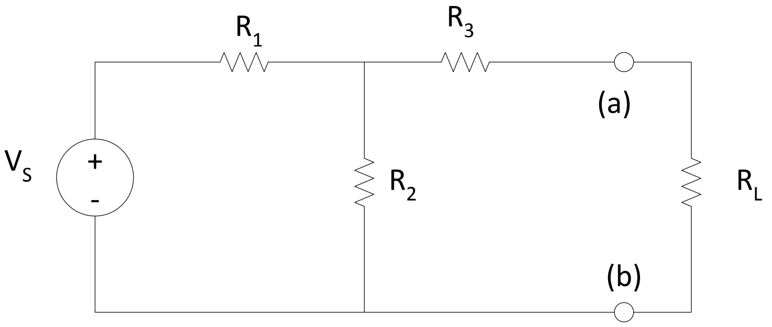
(b) Illustration of Norton's Theorem

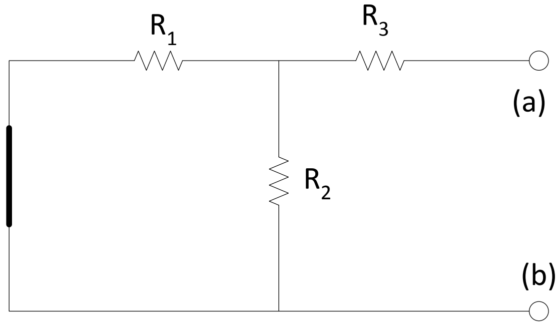
Determination of Norton or Thévenin Equivalent Resistance

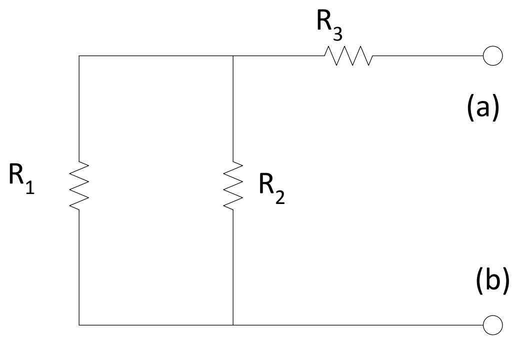
Method

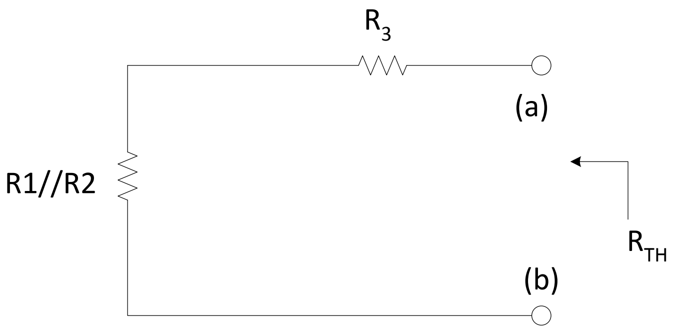
Find the equivalent resistance presented by the circuit at its terminals by setting all sources in the circuit equal to zero and computing the effective resistance between the terminals. Voltage sources are replaced by short-circuits and current sources are replaced by open-circuits.

Example:



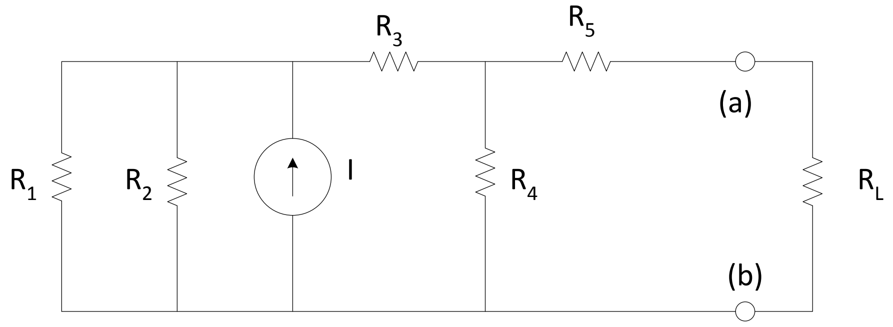


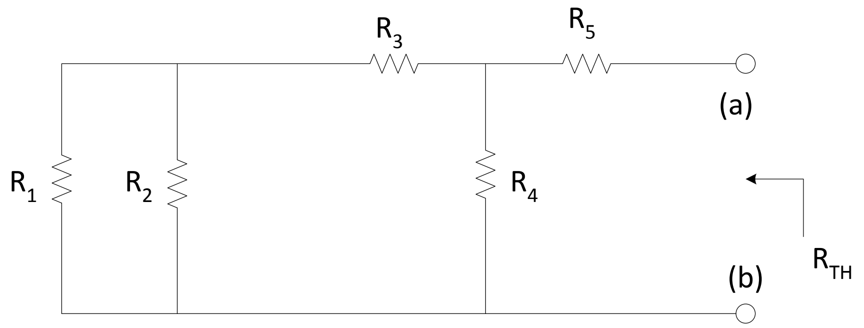




$R_{TH} = R_3 + R_1//R_2$

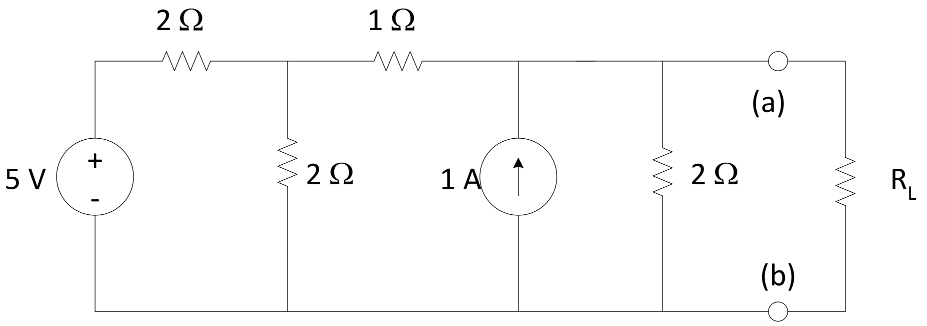
Example: Find the Thévenin Equivalent Resistance seen by the load R_L in the circuit below given that: $R_1 = 20\ \Omega$; $R_2 = 20\ \Omega$; $I = 5$ A; $R_3 = 10\ \Omega$; $R_4 = 20\ \Omega$; $R_5 = 10\ \Omega$

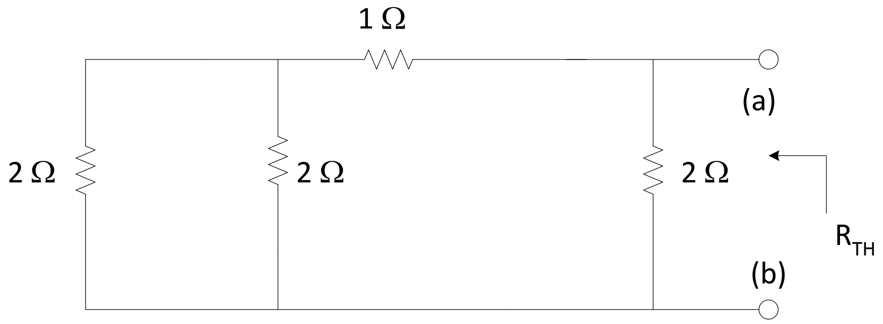




$R_{TH} = [((R_1//R_2) + R_3)//R_4] + R_5 = 20\ \Omega$.

Example: Find the Thévenin Equivalent Resistance of the circuit below as seen by the load resistance R_L.





R_{TH} = ((2//2) + 1)//2 = 1 Ω

Note: The Thévenin and the Norton equivalent resistances are one and the same quantity.

Computing the Thévenin Voltage.

The equivalent (Thévenin) source voltage is equal to the open-circuit voltage present at the load terminals.

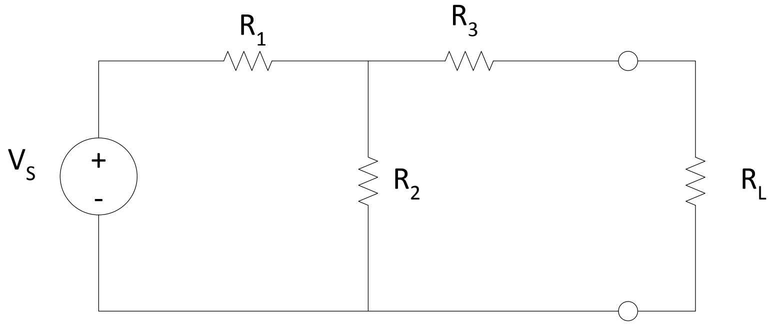
Step 1: Remove the load, leaving the load terminals open-circuited.

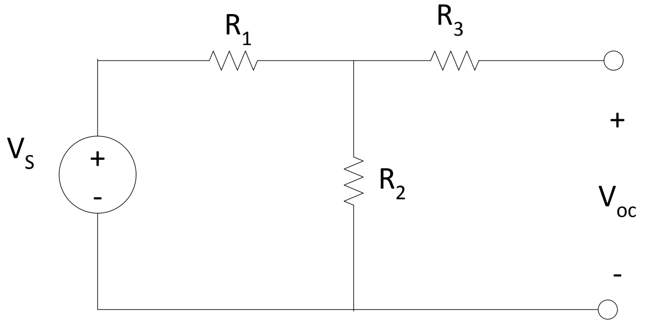
Step 2: Define the open-circuit voltage V_{oc} across the open load terminals.

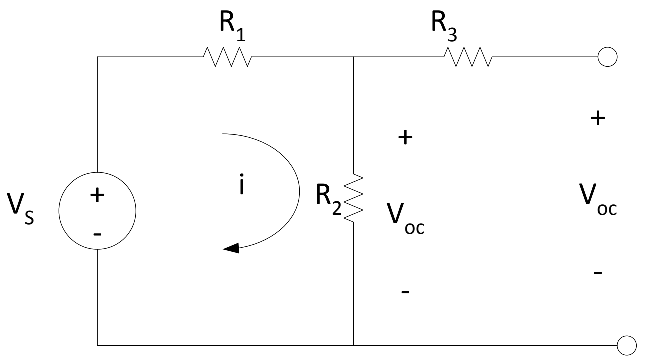
Step 3: Apply any preferred method (e.g. node analysis, mesh analysis) to solve for V_{oc}.

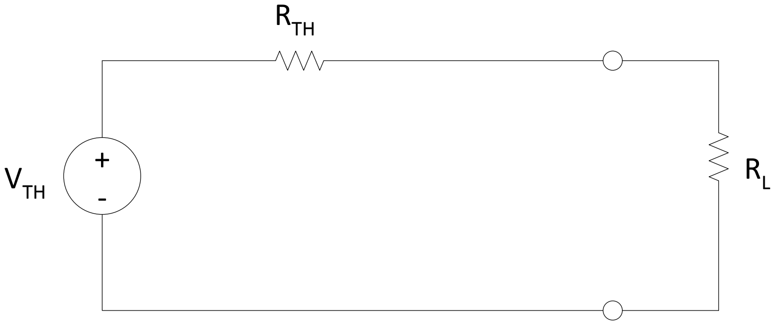
Step 4: The Thévenin voltage is $V_{TH} = V_{oc}$.

Example: Computing the Thévenin voltage.





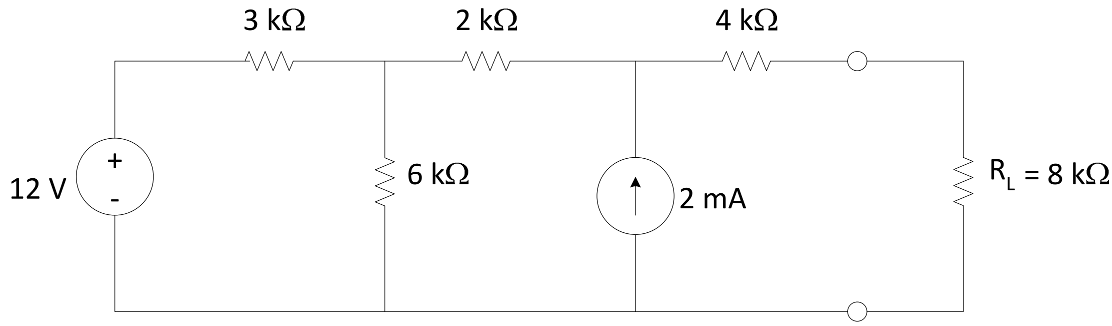


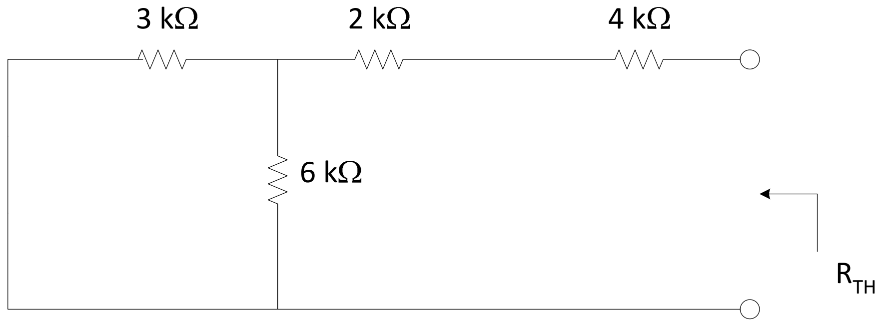


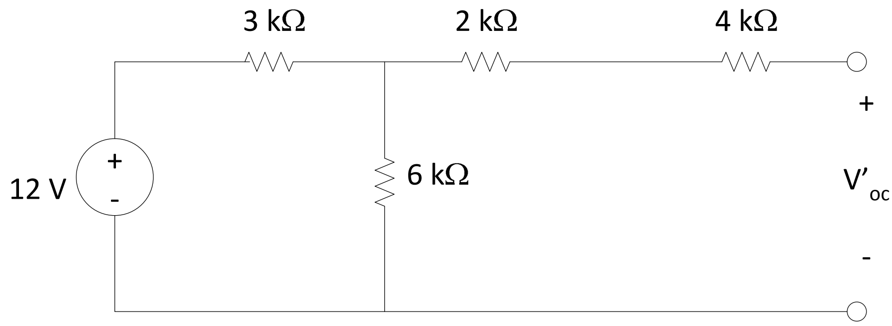
Using voltage division;

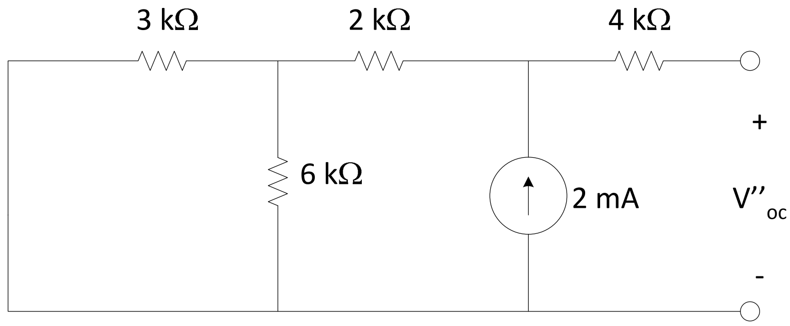
$$V_{TH} = V_{OC} = \frac{R_2}{R_1 + R_2} V_S$$

Example: Determine the Thévenin equivalent circuit for the network shown below.







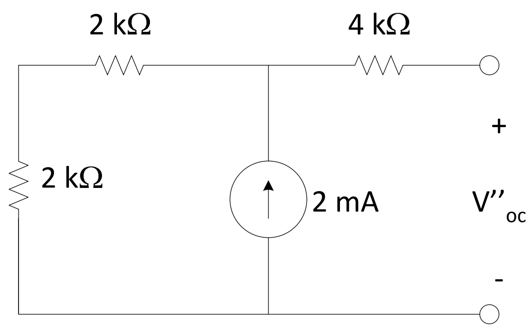


Step 1: Remove load and independent sources to find R_{TH}.

$R_{TH} = (3//6) + 2 + 4 = 8\ k\Omega$

Step 2: Use the superposition technique to find V_{oc}.

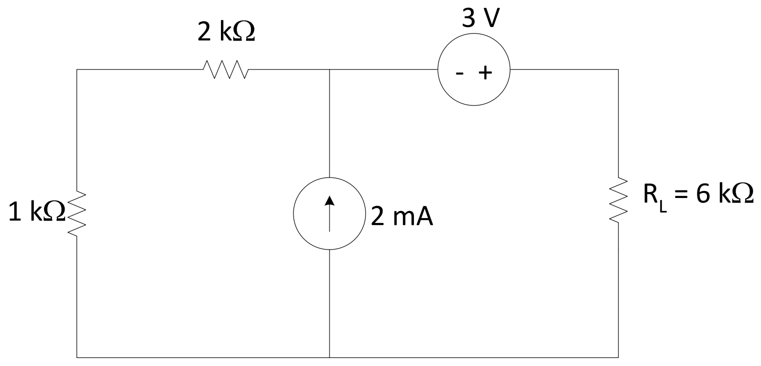
$$V'_{oc} = \frac{6}{3+6} \times 12 = 8\,V$$

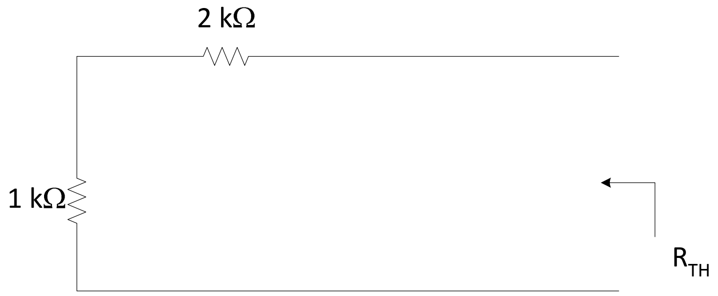


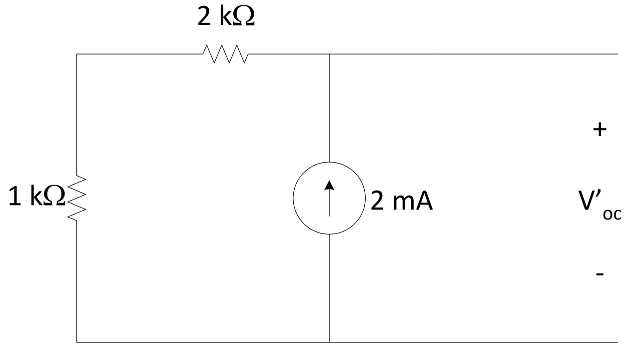
$V''_{oc} = 2\,mA \times 4\,k\Omega = 8$ V

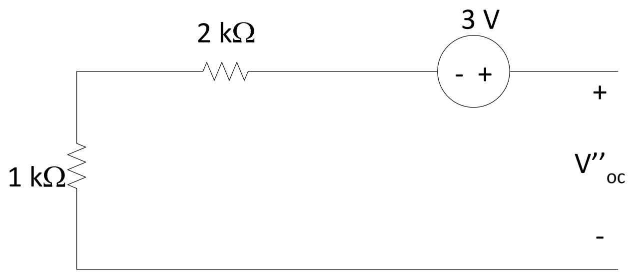
$$V_{OC} = V'_{OC} + V''_{OC} = 16\,V$$

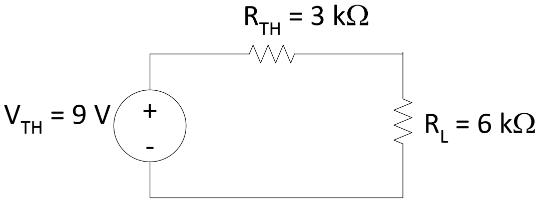
Example: Fnd the Thévenin and Norton equivalent circuits for the network shown below.

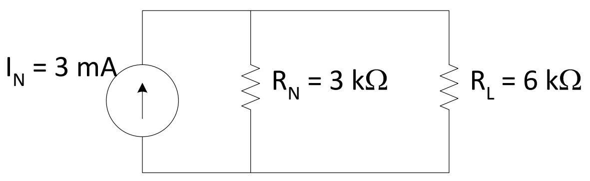






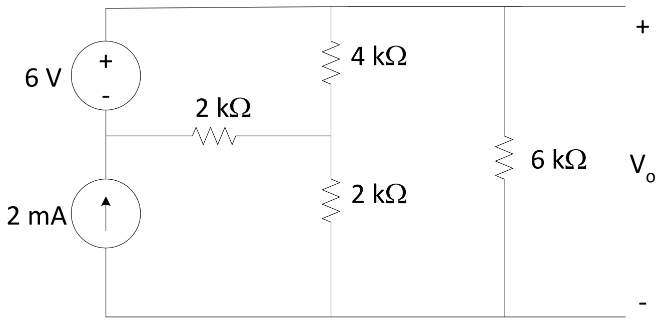


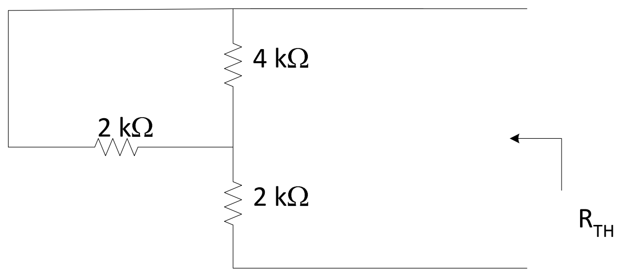


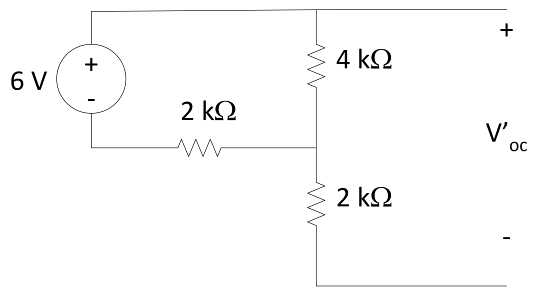


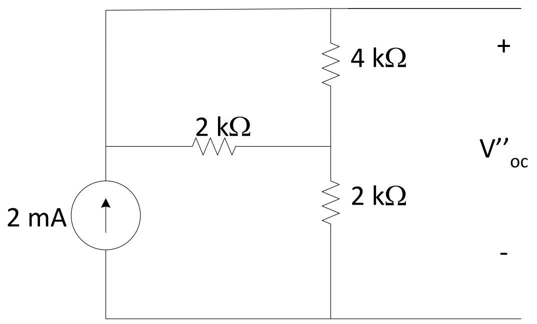
R_{TH} = 3 kΩ; $V'_{oc} = 2\ mA\ \times 3\ k\Omega = 6\ V$;By KVL $-3 +\ V''_{oc} = 0$ or $V''_{oc} = 3\ V$
Hence $V_{oc} = V'_{oc} + V''_{oc} = 9\ V$

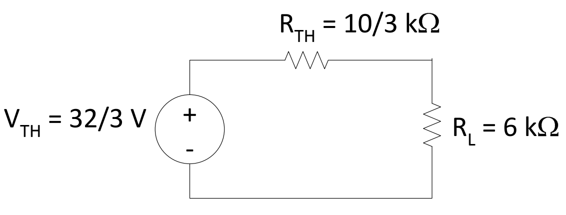
Example: Use Thévenin's theorem to determine V_o.

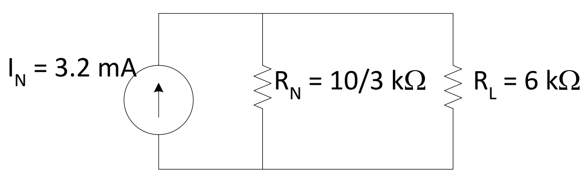












$$R_{TH} = \frac{10}{3}\Omega$$

$$V'_{OC} = \frac{4}{2+4} \times 6 = 4\,V$$

$$V''_{oc} = [(2//4) + 2] \times 2\,mA = \frac{20}{3}\,V$$

$$V_{oc} = V'_{oc} + V''_{oc} = \frac{32}{3}\,V$$

Circuits containing only Dependent Sources

If dependent sources are present, the Thévenin equivalent circuit will be determined by calculating V_{oc} and I_{SC}. i.e. $R_{TH} = V_{oc}/ I_{SC}$

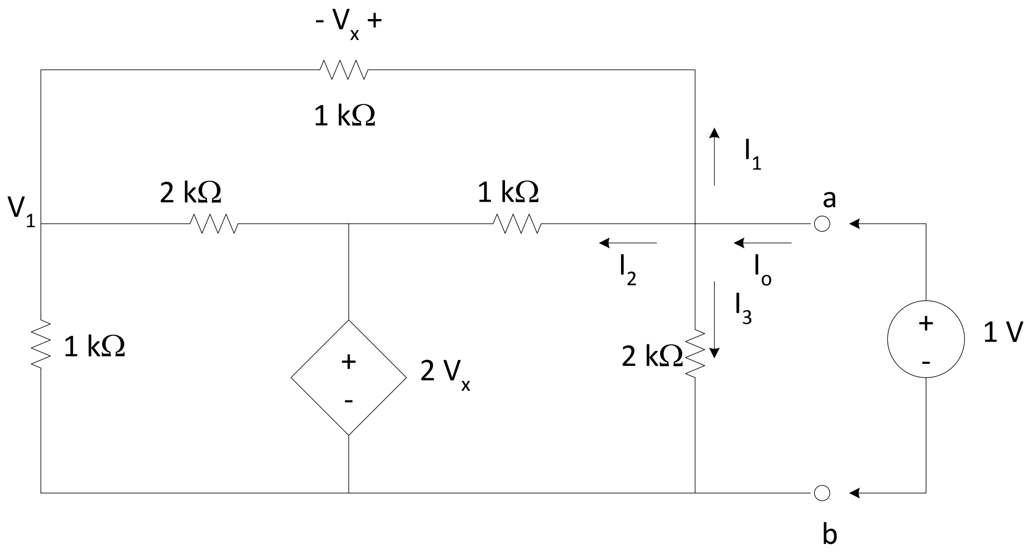
If there are no independent sources then both V_{oc} and I_{SC} will be necessarily zero and R_{TH} therefore cannot be determined by V_{oc}/ I_{SC}.

If $V_{oc} = 0$ then the equivalent circuit is merely the unknown resistance R_{TH}.

If we apply an external source to the network (a test source) V_T and determine the current I which flows into the network from V_T, then $R_{TH} = V_T/I_T$

V_T can be set to 1-V so that $R_{TH} = 1/I_T$.

Example: Determine the Thévenin equivalent circuit as seen from *a-b* for the network below.



Solution: Apply a test source of 1-V at terminals a-b. Compute current I_o and RTH = $1/I_o$.

Applying KVL around the outer loop: $-1 + V_x + V_1 = 0$ from which we obtain $V_x = -V_1$

At node 1 using KCL:

$$\frac{V_1}{1k} + \frac{V_1 - 2V_x}{2k} + \frac{V_1 - 1}{1k} = 0$$

Therefore

$$V_x = \frac{3}{7}\,V$$

and

$$I_1 = \frac{V_x}{1k}$$

$$I_2 = \frac{1 - 2V_x}{1k} = \frac{1}{7}\,mA$$

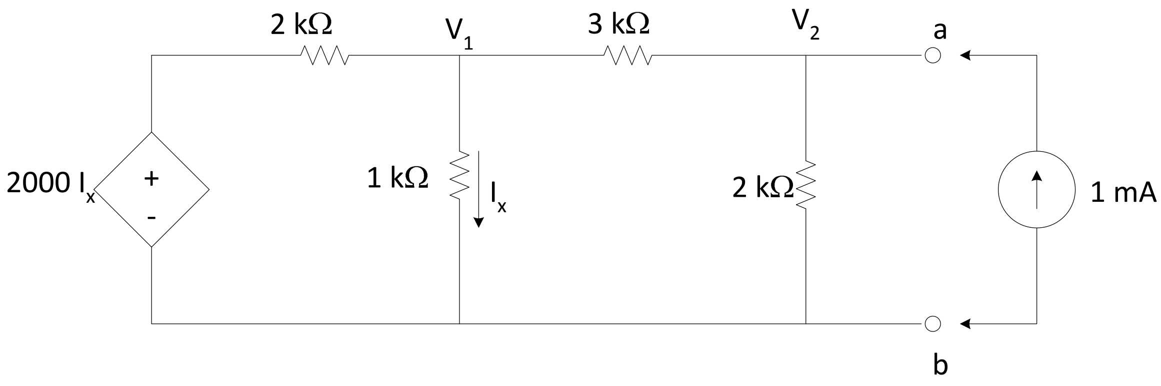
$$I_3 = \frac{1}{2k} = 2\,mA$$

Since $I_o = I_1 + I_2 + I_3$, then

$$I_o = \frac{15}{14}\,mA$$

$$R_{TH} = \frac{14}{15}k\Omega$$

Example: Determine the Thévenin equivalent circuit as seen from *a*-*b* for the network below.



We observe

$$R_{TH} = \frac{V_2}{1\,mA}$$

At node 1:

$$\frac{V_1 - 2000I_x}{2k} + \frac{V_1}{1k} + \frac{V_1 - V_2}{3k} = 0$$

At node 2:

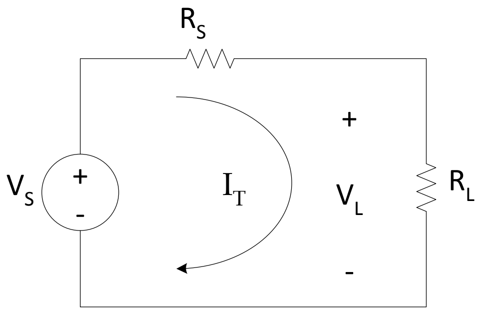
$$\frac{V_2 - V_1}{3k} + \frac{V_2}{2k} - 1mA = 0$$

$$R_{TH} = \frac{V_2}{1\,mA} = \frac{10}{7}\,k\Omega$$

Maximum Power Transfer Theorem

The reduction of any linear resistive network to its Thévenin or Norton equivalent circuits is a very convenient conceptualization, as far as this allows relatively easy computation of load related quantities. The power absorbed by a load is one such computation.

The Thévenin or Norton model implies that some of the generated power is absorbed by the internal circuits and resistance within the source. Since this power loss is unavoidable the question to be answered is how power can be transferred to the load from the source under the most ideal circumstances? We wish to know the value of load resistance that will absorb maximum power from the source.



Consider the network above.

The power absorbed by the load is:

$$P_L = I_T^2 R_L$$

and the load current is given by:

$$I_T = \frac{V_S}{R_S + R_L}$$

Therefore

$$P_L = \left(\frac{V_S}{R_S + R_L}\right)^2 R_L$$

To find the value of R_L that maximizes the expression for P_L (assuming V_T and R_T are constant), we differentiate with respect to R_L and set equal to zero.

$$\frac{dP_L}{dR_L} = \frac{(R_S + R_L)^2 V_S^2 - V_S^2 R_L 2(R_S + R_L)}{(R_S + R_L)^4} = 0$$

Which leads to:$(R_S + R_L)^2 - 2R_L(R_S + R_L) = 0$ and $R_L = \ R_S$

The *Maximum Power Transfer Theorem*

An independent voltage source in series with a resistance R_S, or an independent current source in parallel with a resistance R_S, delivers maximum power to the load resistance R_L for the condition $R_L = R_S$.

To transfer maximum power to a load, the equivalent source and load resistances must be **matched** i.e. equal to each other.

First-Order Transient Circuits

Introducing the study of circuits characterized by a single storage element. Although the circuits have an elementary appearance, they have significant practical applications. They find use as coupling networks in electronic amplifiers; as compensating networks in automatic control systems; as equalizing networks in communication channels.

The study of these circuits will enable us to predict the accuracy with which the output of an amplifier can follow an input which is changing rapidly with time or to predict how quickly the speed of a motor will change in response to a change in its field current. The knowledge of the performance of the simple RL and RC circuits will enable us to suggest modifications to the amplifier or motor in order to obtain a more desirable response.

The analysis of such circuits is dependent upon the formulation and solution of integro-differential equations which characterize the circuits. The special type of equation we obtain is a homogeneous linear differential equation which is simply a differential equation in which every term is of the first degree in the dependent variable or one of its derivatives.

A solution is obtained when an expression is found for the dependent variable as a function of time, which satisfies the differential equation and also satisfies the prescribed energy distribution in the inductor or capacitor ata prescribed instant of time, usually t = 0.

The solution of the differential equation represents a response of the circuit and it is known by many names. Since this response depends upon the general " nature" of the circuit (the types of elements, their sizes, the interconnection of the elements), it is often called the natural response. It is also obvious that any real circuit cannot store energy forever as the resistances necessarily associated with the inductors and capacitors will all convert all stored energy to heat. The response must eventually die out and is therefore referred to as the transient response. (Mathematicians call the solution of a homogeneous linear differential equation, a *complementary function*. When we consider independent sources acting on a circuit, part of the response will partake of the nature of the particular source used.

In summary, The analysis of First-Order circuits involves an examination and description of the behaviour of a circuit as a function of time after a sudden change in the network occurs due to switches opening or closing. When only a single storage element is present in the network, the network can be described by a first-order differential equation.

Because a storage element is present, the circuit response to a sudden change will go through a transition period prior to settling down to a steady-state value.

General Form of the Response Equations

In the study of first-order transient circuits it will be shown that the solution of these circuits (i.e. finding a voltage or current) requires the solution of a firs-order differential equation of the form:

$$\frac{dx(t)}{dt} + ax(t) = f(t)$$

A fundamental theorem of differential equations states that if $x(t) = x_p(t)$ is any solution to the differential equation and $x(t) = x_c(t)$ is any solution to the homogeneous equation,

$$\frac{dx(t)}{dt} + ax(t) = 0$$

then $x(t) = x_p(t) + x_c(t)$ is a solution of the original differential equation. The term $x_p(t)$ is called the *particular integral solution* or *forced response* and $x_c(t)$ is called the *complementary* or *natural response*.

The general solution of the differential equation then consists of two parts that are obtained by solving the two equations:

i. $\frac{dx(t)}{dt} + ax(t) = A$

where f(t) = A (a constant)

ii. $\frac{dx(t)}{dt} + ax(t) = 0$

Since the right-hand side of equation (i) is constant, it is reasonable to assume that the solution $x_p(t)$ must also be a constant. If we assume $x_p(t) = K_1$ and substitute in equation (i), we obtain

$$K_1 = \frac{A}{a}$$

Rewriting equation (ii),

$$\frac{dx_c(t)/_{dt}}{x_c(t)} = -a \xrightarrow{yields} \frac{d[lnx_c(t)]}{dt} = -a$$

$$lnx_c(t) = -at + c$$

And

$$x_c(t) = K_2e^{-at}$$

The complete solution is; $x(t) = x_p(t) + x_c(t) = \frac{A}{a} + K_2e^{-at}$

Generally

$$x(t) = K_1 + K_2e^{-\frac{t}{\tau}}$$

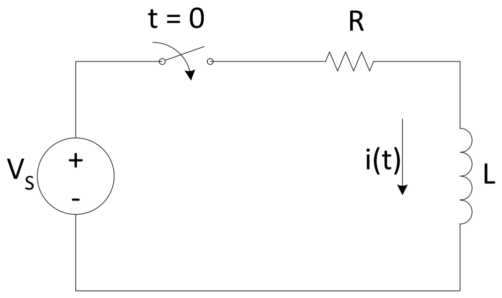
Where $K_1 \triangleq$ steady state solution which is the value of $x(t)\ as\ t \to \infty$

And $\tau \triangleq$ time constant of the circuit.

The Differential Equation Approach

State-Variable approach – write the equation for the voltage across the capacitor and/or the equation for the current through the inductor. These quantities (voltage across the capacitor; current through the inductor) do no change instantaneously.

The Simple RL Circuit



Using KVL:

$$L\frac{di(t)}{dt} + Ri(t) = V_s$$

The solution to the above differential equation is of the form: $i(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$

If we substitute in the differential equation, we get:

$$-L\frac{K_2}{\tau}e^{\frac{-t}{\tau}} + RK_1 + RK_2 e^{\frac{-t}{\tau}} = V_s$$

Equating the constant and exponential terms, we get:

$$RK_1 = V_s \ or \ K_1 = \frac{V_s}{R}$$

And

$$-L\frac{K_2}{\tau} = RK_2 \ \ or \ \ \tau = \frac{L}{R}$$

Therefore

$$i(t) = \frac{V_s}{R} + K_2 e^{\frac{-t}{\tau}}$$

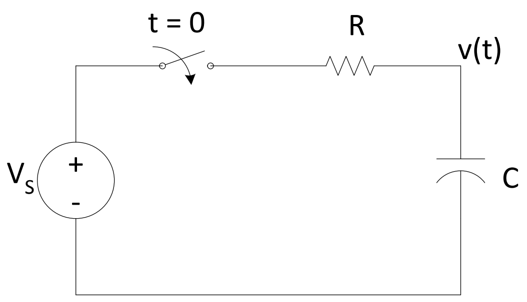
If I(0) = 0 then

$$\frac{V_s}{R} + K_2 = 0 \ \ or \ K_2 = -\frac{V_s}{R}$$

Hence

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{\frac{-t}{\tau}}$$

The Simple RC Circuit



Using KCL:

$$C\frac{dv(t)}{dt} + \frac{v(t) - V_S}{R} = 0$$

which can be rewritten as:

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_S}{R}$$

We know the solution is of the form: $v(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$

If we substitute in the differential equation, we get:

$$-\frac{K_2}{\tau}e^{\frac{-t}{\tau}} + \frac{K_1}{RC} + \frac{K_2}{RC}e^{\frac{-t}{\tau}} = \frac{V_s}{RC}$$

Equating like terms, we get:

$$\frac{K_1}{RC} = \frac{V_s}{RC} \ or \ K_1 = V_s$$

$$\tau = \frac{1}{RC}$$

Hence

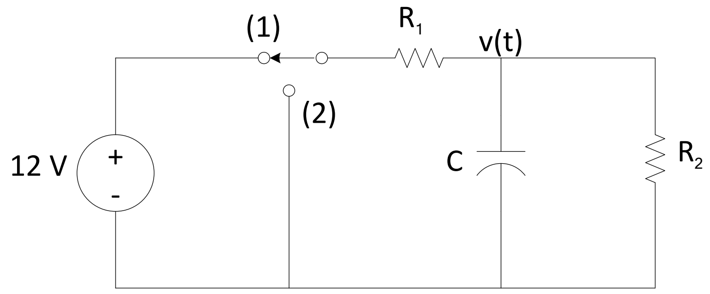
$$v(t) = V_s + K_2 e^{\frac{-t}{\tau}}$$

If the capacitor is initially uncharged then $\mathrm{v}(0) = 0$,

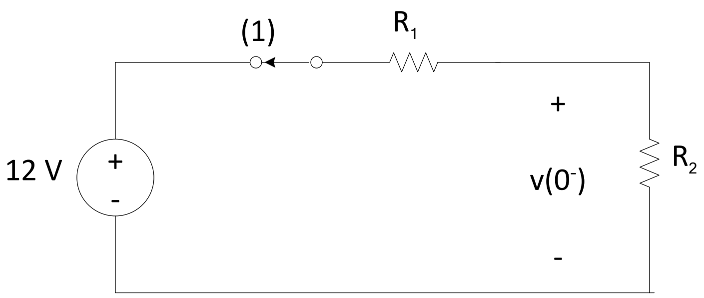
Therefore $0 = V_s + K_2 \ or \ K_2 = -V_s$

$$v(t) = V_s - V_s e^{\frac{-t}{\tau}} = \ V_s\left(1 - e^{-t\frac{1}{RC}}\right)$$

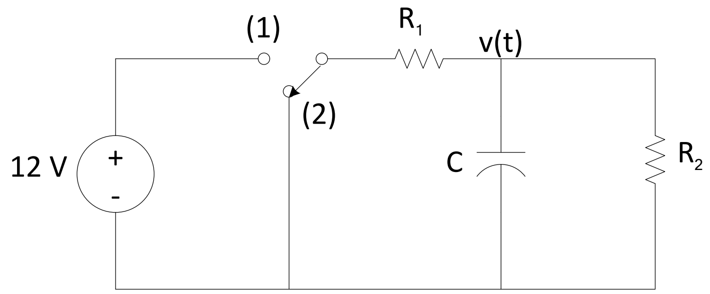
Example: The switch has been in position (1) for a long time. At t = 0 the switch is moved to position (2). Calculate i(t) for t > 0 given that R_1 = 6 kΩ; R_2 = 3 kΩ; C = 100 μF.



For t = 0^-



For t = 0^+



For t = 0^-,

The capacitor is fully charged - v(t) is not changing – and conducts no current.

So by voltage division

$$\frac{v(0-)}{3} = \frac{12}{6+3} \; or \; v(0\,-) = 4\;V$$

For t = 0^+

At node with v(t) by KCL,

$$\frac{v(t)}{R_1} + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2} = 0$$

Substituting we get ,

$$\frac{dv(t)}{dt} + 5v(t) = 0$$

whose solution is of the form $v(t) = K_2 e^{\frac{-t}{\tau}}$

Substituting in the differential equation, we get:

$$\frac{K_2}{R_1} e^{\frac{-t}{\tau}} - \frac{CK_2}{\tau} e^{\frac{-t}{\tau}} + \frac{K_2}{R_2} e^{\frac{-t}{\tau}} = 0$$

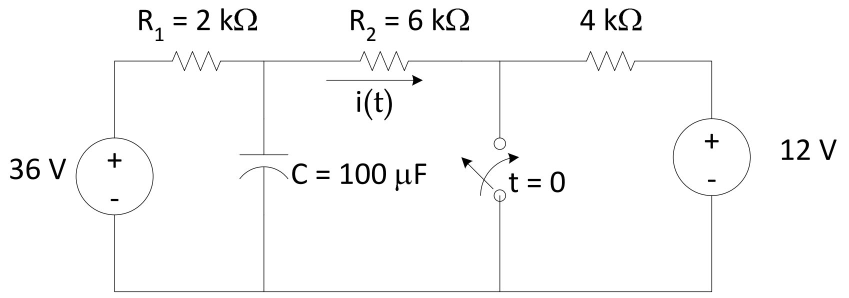
Therefore

$$\frac{1}{R_1} - \frac{C}{\tau} + \frac{1}{R_2} = 0 \;\; or \;\; \tau = \frac{R_1 R_2}{R_1 + R_2} C = 0.2\ s$$

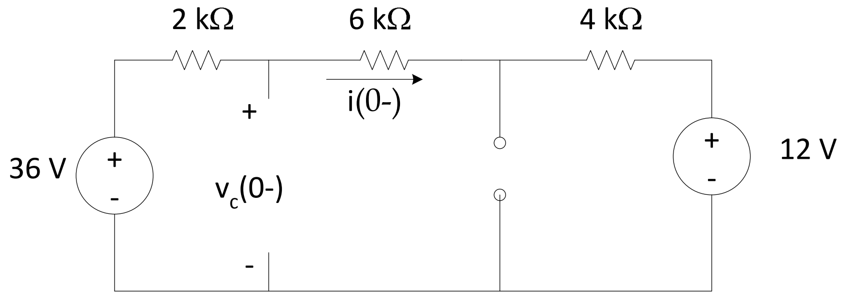
$v(t) = K_2 e^{-5t}$ but v(0-) = v(0+) = 4 = K_2

$$v(t) = 4e^{-5t} and\ i(t) = \frac{v(t)}{R_2} = \frac{4}{3} e^{-5t}\ mA$$

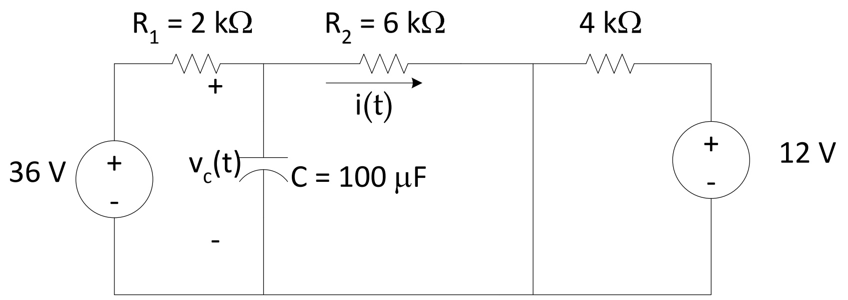
Example: Find i(t) for t > 0

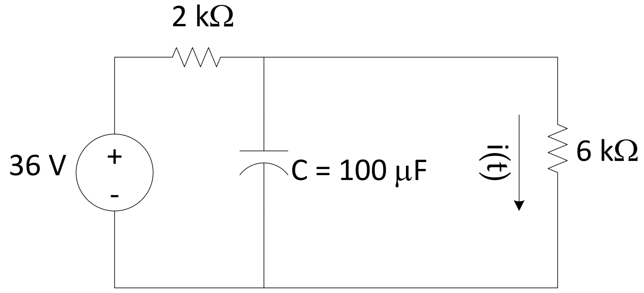


Step 1
t = 0-



Step 2
t = 0+





$v_c(t)$ is of the form $v_c(t) = K_1 + K_2 e^{-t/\tau}$

Step 1: Find $v_c(0\text{-})$

By KVL; $-36 + (2 + 4 + 6)\text{ki}(0\text{-}) + 12 = 0$ which gives $\text{i}(0\text{-}) = 2\ \text{mA}$

$\text{v}_c(0\text{-}) - 36 = i(0\text{-}) \times 2\ \text{mA} = 32\ \text{V}$

Step 2: Write differential equation with $v_c(t)$

$$\frac{v_c - V_s}{R_1} + C\frac{dv_c(t)}{dt} + \frac{v_c}{R_2} = 0$$

which can be rewritten as:

$$\frac{dv_c(t)}{dt} + \frac{v_c}{R_1C} - \frac{V_s}{R_1C} + \frac{v_c}{R_2C} = 0$$

or

$$\frac{dv_c(t)}{dt} + \frac{v_c}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_s}{R_1C}$$

Substituting $\text{v}_c(\text{t}) = \text{K}_1 + \text{K}_2\text{e}^{-t/\tau}$ in the above differential equation we get;

$$-\frac{K_2}{\tau}e^{\frac{-t}{\tau}} - \frac{R_1 + R_2}{R_1R_2C}K_1 + \frac{R_1 + R_2}{R_1R_2C}K_2e^{\frac{-t}{\tau}} = \frac{V_s}{R_1C}$$

Therefore

$$\frac{R_1 + R_2}{R_1R_2C}K_1 = \frac{V_s}{R_1C}\ or\ K_1 = \frac{216}{8}$$

$$\tau = \frac{R_1R_2C}{R_1 + R_2} = 0.15\ s$$

Since $\text{v}_c(0\text{-}) = 32\ \text{V}$, we have

$$32 = \frac{216}{8} + K_2$$

Or

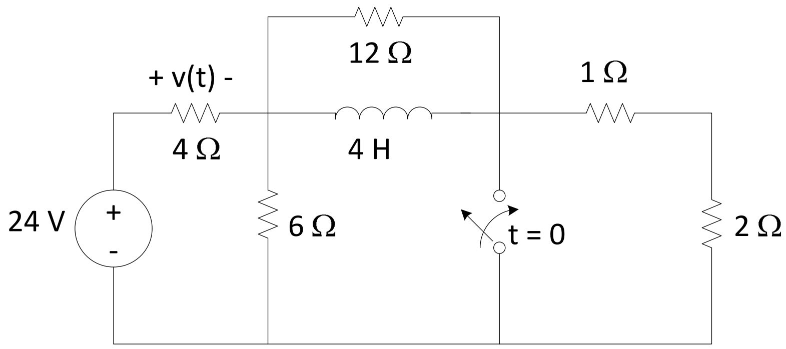
$$K_2 = 5$$

$$v_c(t) = \frac{216}{8} + 5e^{\frac{-t}{0.15}}\,V$$

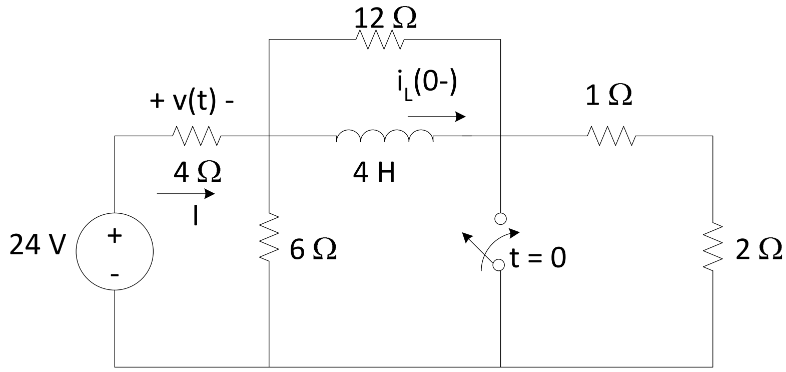
And

$$i(t) = \frac{v_c(t)}{6} = \frac{36}{8} + \frac{5}{6}e^{\frac{-t}{0.15}}\,mA$$

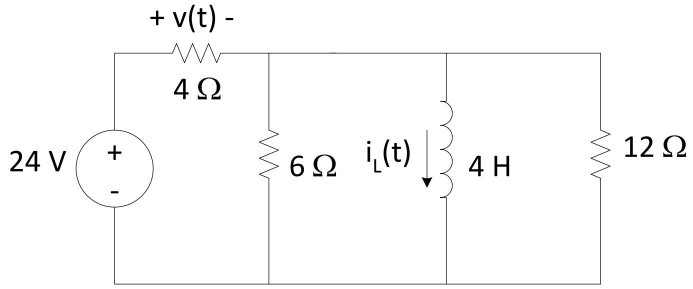
Example: Find *v(t)*.

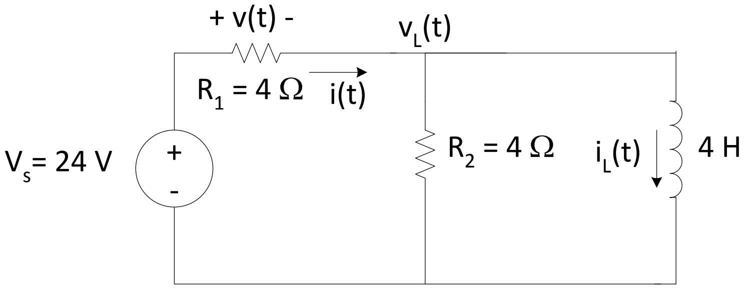


Step 1
t = 0-



Step 2
t = 0+





v(t) is of the form; $i(t) = K_1 + K_2 e^{-t/\tau}$

Step1: $R_{Total} = 4 + (3//6) = 6\ \Omega$

$$I = \frac{24}{R_{Total}} = 4\ A$$

$$i_L(0-) = \frac{6}{6+3} \times 4 = \frac{8}{3}\ A$$

Step 2: t > 0

$V_s - v_L(t) = R_1 i(t)$ and

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$i(t) = \frac{v_L}{R_2} + i_L$$

Now

$$L\frac{di_L(t)}{dt} + R_1 i(t) = V_s$$

or

$$L\frac{di_L(t)}{dt} + R_1\frac{v_L}{R_2} + R_1 i_L = V_s$$

Substituting for $v_L(t)$, R_1, R_2 and L and rearranging, we get:

$$\frac{di_L(t)}{dt} + \frac{1}{2}i(t) = 3$$

Substituting for i(t),

$$-\frac{K_2}{\tau}e^{\frac{-t}{\tau}} - \frac{K_1}{2}e^{\frac{-t}{\tau}} + \frac{K_2}{2}e^{\frac{-t}{\tau}} = 3$$

Equating like terms, we get:

$K_1 = 6$ and $\tau = 2$ which gives

$$i_L(t) = 6 + K_2 e^{\frac{-t}{2}}$$

Recall

$$i_L(0-) = i_L(0+) = \frac{8}{3}A$$

Therefore

$$\frac{8}{3}A = 6 + K_2$$

or

$$K_2 = -\frac{10}{3}$$

$$v_L(t) = L\frac{di_L(t)}{dt} = \frac{20}{3}e^{\frac{-t}{2}}$$

Hence

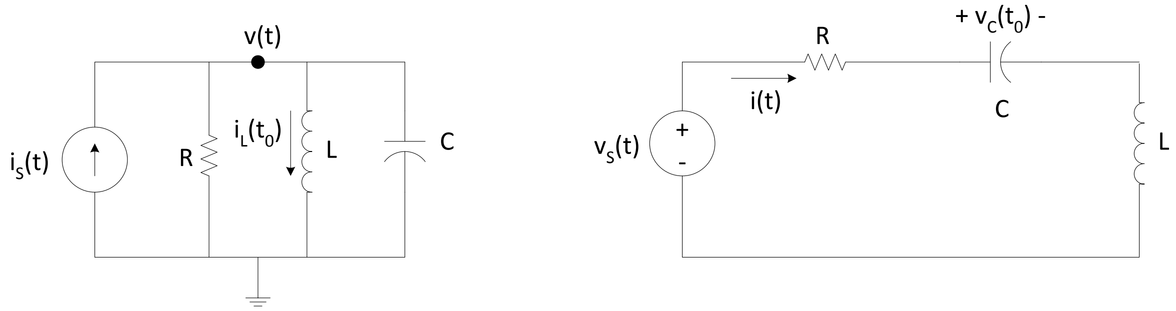
$$v(t) = V_s - v_L = 24 - \frac{20}{3}e^{\frac{-t}{2}}$$

Second-Order Circuits

Second order systems occur very frequently in nature. They are characterized by the ability of a system to store energy in one of two forms – potential or kinetic – and to dissipate this energy. Second-order systems always contain two energy storage elements.

Second-order circuits are characterized linear second-order differential equations.

Consider the RLC circuits shown below. Assume that energy may have been initially stored in both the inductor and capacitor.



For the parallel RLC circuit, we have by KCL:

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v(x)dx + i_L(t_0) + C\frac{dv}{dt} = i_s(t)$$

For the series RLC circuit, we have by KVL:

$$Ri + \frac{1}{C}\int_{t_0}^{t} i(x)dx + v_C(t_0) + L\frac{di}{dt} = v_s(t)$$

If the two equations are differentiated with respect to time, we obtain:

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt}$$

And

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv_S}{dt}$$

We know that if $x(t) = x_p(t)$ is a solution to the second-order differential equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = f(t)$$

And if $x(t) = x_c(t)$ is a solution to the homogeneous equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$$

Then

$$x(t) = x_p(t) + x_c(t)$$

is a solution to the original equation.

If *f(t)* is a constant, i.e. a constant forcing function f(t) = A, then

$$x(t) = \frac{A}{a_2} + x_c(t)$$

For the homogeneous equation:

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$$

where a_1 and a_2 are constants, we can rewrite the equation in the form:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0x(t) = 0 \;\; where\; a_1 = \; 2\zeta\omega_0 \; and \;\; a_2 = \omega_0^2$$

The solution of the above differential equation is of the form:

$$x(t) = Ke^{st}$$

Substituting this expression into the differential equation, we get:

$$s^2Ke^{st} + 2\zeta\omega_0 sKe^{st} + \omega_0^2Ke^{st} = 0$$

Dividing by Ke^{st} yields:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

The above equation is called the characteristic equation; ζ is called the exponential damping ratio, and ω_0 is referred to as the undamped natural frequency.

Now

$$\mathrm{s} = \frac{-\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2}$$

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

giving two values of *s* as:

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

In general the complementary solution is of the form:

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

where K_1 and K_2 are constants that can be evaluated via the initial conditions

$$x(0)\ and\ \frac{dx(0)}{dt}$$

The form of the solution of the homogeneous equation is dependent on the value of ζ.

If $\zeta > 1$, the roots of the characteristic equation, s_1 and s_2, also called the natural frequencies because they determine the natural (unforced) response of the network, are real and unequal; if $\zeta < 1$, the roots are complex numbers; if $\zeta = 1$, the roots are real and equal.

Case 1, ζ **> 1**

The circuit is *overdamped*. The natural frequencies s_1 and s_2, are real and unequal and

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

where K_1 and K_2 are found from the initial conditions. The natural response is the sum of two decaying exponentials.

Case 2, ζ **< 1**

This is the *underdamped* case. The roots of the characteristic equation can be written as:

$$s_1 = -\zeta\omega_0 + \mathrm{j}\omega_0\sqrt{1-\zeta^2} = -\sigma + \mathrm{j}\omega_\mathrm{d}$$

$$s_2 = -\zeta\omega_0 - \mathrm{j}\omega_0\sqrt{1-\zeta^2} = -\sigma - \mathrm{j}\omega_\mathrm{d}$$

where

$$j = \sqrt{-1}, \qquad \sigma = \zeta\omega_0\ \ and\ \ \omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$x_c(t) = e^{-\zeta\omega_o t}\left(A_1 \cos\omega_0 \sqrt{1-\zeta^2}\mathrm{t} + \mathrm{A}_2 \sin\omega_0\sqrt{1-\zeta^2}\mathrm{t}\right)$$

where $A_1 and\ \mathrm{A}_2$ are constants

Case 3, ζ **= 1**

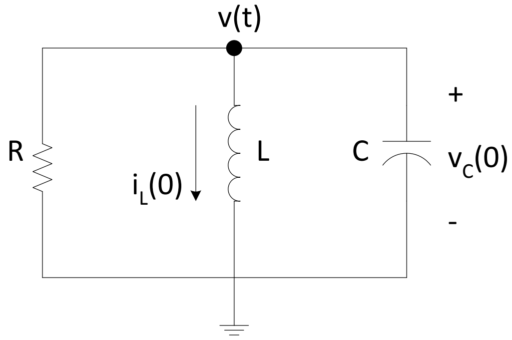
This is the *critically* damped case where $s_1 = s_2 == -\zeta\omega_0$

For a characteristic equation with repeated roots, the general solution is of the form:

$x_c(t) = B_1 e^{s_1 t} + B_2 e^{s_2 t}$ where

$B_1\ and\ B_2\ are\ constants\ to\ be\ determined\ from\ the\ initial\ conditions.$

Example: Consider the parallel circuit shown below where R = 2 Ω; C = 1/5 F; L = 5 H:



The second-order differential equation that describes the voltage $v(t)$is:

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

or

$$\frac{d^2v}{dt^2}+2.5\frac{dv}{dt}+v=0$$

Assume that the initial conditions on the storage elements are $i_L(0)=$ - 1 A and $v_C(0)=4$ V, find the node voltage $v(t)$ and the inductor current.

The damping term is:

$$\alpha=\frac{1}{2RC}$$

And the undamped natural frequency is:

$$\omega_0=\frac{1}{\sqrt{LC}}$$

The characteristic equation for the network is:

$$s^2+2.5s+1=0$$

and the roots are $s_1 =$ - 2 and $s_2 =$ -0.5.

Since the roots are real and unequal, the circuit is *overdamped* and

$$v(t)=K_1e^{-2t}+K_2e^{-0.5t}$$

$$v_C(0)=v(t)=K_1+K_2=4$$

Also

$$\frac{dv(t)}{dt}=-2K_1e^{-2t}-0.5K_2e^{-0.5t}$$

By KCL, we can write;

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+i_L(t)=0$$

or

$$\frac{dv(t)}{dt} = -\frac{1}{RC}v(t) - \frac{i_L(t)}{C}$$

At t = 0

$$\frac{dv(0)}{dt} = -\frac{1}{RC}v(0) - \frac{i_L(0)}{C} = -2.5 \times 4 - 5(-1) = -5$$

Since

$$\frac{dv(0)}{dt} = -\frac{1}{RC}v(0) - \frac{i_L(0)}{C}$$

then

$$-2K_1 - 0.5K_2 = -5$$

Solving for K_1 and K_2 yields $K_1 = 2$ and $K_2 = 2$, and therefore

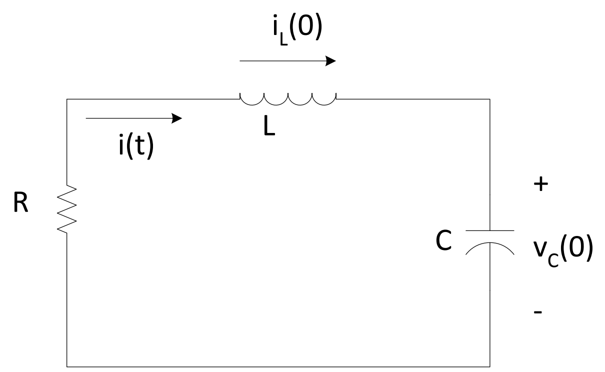
$$v(t) = 2e^{-2t} + 2e^{-0.5t}\ V$$

The inductor current is related to $v(t)$by

$$i_L(t) = \frac{1}{L}\int v(t)\,dt = \frac{1}{5}\int [2e^{-2t} + 2e^{-0.5t}]dt$$

$$i_L(t) = -\frac{1}{5}e^{-2t} - \frac{4}{5}e^{-0.5t}\ A$$

Example: The series RLC circuit shown has the following parameters: $C = 0.04$ F; $L = 1$ H; $R = 6\ \Omega$; $i_L(0) = 4$ A and $v_C(0) = -4$ V. Find expressions for the current and capacitor voltage.



The equation for the current in the circuit is given by:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

The damping term is:

$$\alpha = \frac{R}{2L}$$

and the undamped natural frequency is:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Substituting the values of the circuit elements, we get;

$$\frac{d^2 i}{dt^2} + 6\frac{di}{dt} + 25i = 0$$

The characteristic equation is:

$$s^2 + 6s + 25 = 0$$

and it has roots

$$s_1 = -3 + j4$$

$$s_2 = -3 - j4$$

Since the roots are complex, the circuit is *underdamped,* and the expression for *i(t)* is:

$$i(t) = K_1 \, \mathrm{e}^{-3\mathrm{t}} \cos 4t + K_2 e^{-3t} \sin 4t$$

Using the initial condition

$$i(0) = 4 = K_1$$

and

$$\frac{di}{dt} = -4K_1 \, \mathrm{e}^{-3\mathrm{t}} \sin 4t - 3K_1 \, \mathrm{e}^{-3\mathrm{t}} \cos 4t + 4K_2 e^{-3t} \cos 4t - 3K_2 e^{-3t} \sin 4t$$

$$\frac{di(0)}{dt} = -3K_1 + 4K_2$$

We can find $\frac{di(0)}{dt}$ using KVL.

$$Ri(0) + L\frac{di(0)}{dt} + v_c(0) = 0$$

or

$$\frac{di(0)}{dt} = \frac{R}{L} i(0) - \frac{v_c(0)}{L} = -\frac{6}{1}(4) + \frac{4}{1} = -20$$

hence

$$-3K_1 + 4K_2 = -20$$

giving

$$K_1 = 4, \qquad K_2 = -2$$

Therefore

$$i(t) = 4\,\mathrm{e}^{-3\mathrm{t}}\cos 4t - 2e^{-3t}\sin 4t \ A$$

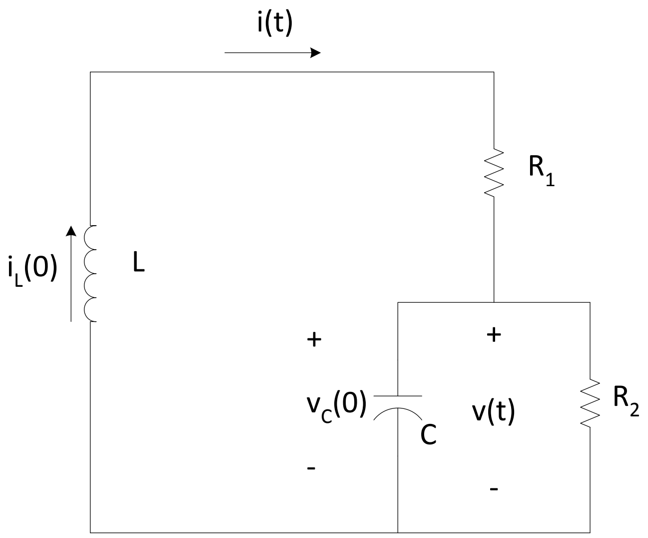
and from above

$$v_c(t) = -Ri(t) - L\frac{di(t)}{dt}$$

$$v_c(t) = -4\,\mathrm{e}^{-3\mathrm{t}}\cos 4t + 22e^{-3t}\sin 4t \ V$$

Example: Consider the circuit shown below, determine *i(t)* and *v*(t) given that

$R_1 = 10\ \Omega$; $R_2 = 8\ \Omega$; C = 1/8 F; L = 2 H; $i_L(0) = ½$ A



The two equations that describe the network are:

$$L\frac{di}{dt} + R_1 i(t) + v(t) = 0$$

and

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R_2}v(t)$$

Substituting the second equation into the first yields;

$$\frac{d^2v}{dt^2} + \left(\frac{1}{R_2C} + \frac{R_1}{L}\right)\frac{dv}{dt} + \frac{R_1 + R_2}{R_2LC}v = 0$$

Substituting the values of the elements we get;

$$\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 9v = 0$$

The characteristic equation is:

$$s^2 + 6s + 9 = 0$$

And the roots are: $s_1 = -3$

$s_2 = -3$

Since the roots are real and repeated, the circuit is *critically damped*.

$$v(t) = K_1e^{-3t} + K_2te^{-3t}$$

Since

$v(t) = v_C(t)$ then $v(0) = v_C(0) = 1 = K_1$

Also,

$$\frac{dv(t)}{dt} = -3K_1e^{-3t} + K_2e^{-3t} - 3K_2te^{-3t}$$

Recall from above:

$$\frac{dv(t)}{dt} = \frac{i(t)}{C} - \frac{1}{R_2C}v(t)$$

Equating the two expressions and evaluating at t = 0, we get:

$$\frac{i(t)}{C} - \frac{1}{R_2C}v(t) = -3K_1e^{-3t} + K_2te^{-3t} - 3K_2te^{-3t}$$

$$\frac{i(0)}{C} - \frac{1}{R_2C}v(0) = -3K_1 + K_2$$

or

$$\frac{1/2}{1/8} - \frac{1}{8 \times 1/8}1 = -3K_1 + K_2$$

$$3 = -3K_1 + K_2$$

Giving $K_1 = 1 \ and \ K_2 = 6$

Hence

$$v(t) = e^{-3t} + 6te^{-3t}\,V$$

By KCL:

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R_2}v(t)$$

Substituting for $v(t)$ we get:

$$i(t) = \frac{1}{8}(-3e^{-3t} + 6e^{-3t} - \ 18te^{-3t}\ ) + \frac{1}{8}(e^{-3t} + 6te^{-3t})\ A$$

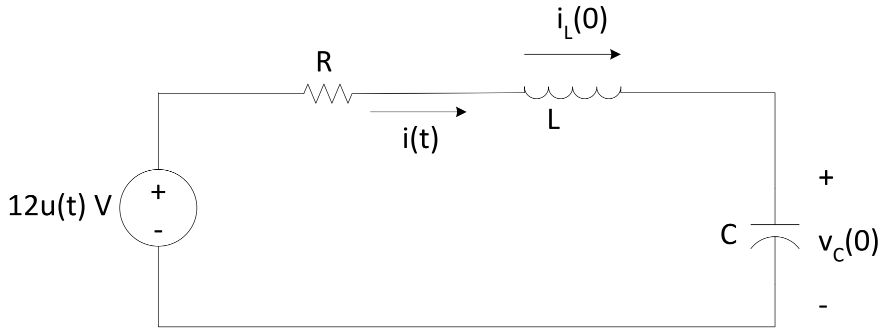
or

$$i(t) = \frac{1}{2}e^{-3t} - \frac{3}{2}\ te^{-3t}\ A$$

Example of a series RLC circuit with a step function

Consider the circuit shown – similar to one previously analyzed – except with a constant forcing function present. The following are the circuit parameters:

C = 0.04 F; L = 1 H; R = 6 Ω; $i_L(0)$ = 4 A and $v_C(0)$ = - 4 V. Find $v_C(t)$ for t > 0.



We will recall that general solution will consist of a particular solution plus a complementary solution.

We have already determined that the complementary solution is of the form:

$$K_3\ \mathrm{e}^{-3t}\cos 4t + K_4e^{-3t}\sin 4t$$

The particular solution is a constant since the input is a constant. The general solution is:

$$v_C(t) = \ K_3\ \mathrm{e}^{-3t}\cos 4t + K_4e^{-3t}\sin 4t + K_5$$

When the circuit has reached the steady-state condition, the inductor is a short-circuit, the capacitor is an open-circuit and hence the final value of $v_C(t)$ will be 12 V .

$$v_C(\infty) = 12\ V$$

Substituting in the equation above, we get:

$$v_C(\infty) = 12\ V = K_5$$

Therefore

$$v_C(t) = K_3\,\mathrm{e}^{-3t}\cos 4t + K_4 e^{-3t}\sin 4t + 12$$

Using the initial conditions, we can calculate K_3 and K_4.

$$v_C(0) = K_3 + 12 = -4 \; or \;\; K_3 = -16$$

Since $i_C = C\frac{dv_C}{dt}$ or

$$\frac{dv_C(0)}{dt} = \frac{i(0)}{C} = \frac{4}{0.04} = 100$$

Therefore

$$\frac{dv_C(t)}{dt} = -3K_3\,\mathrm{e}^{-3t}\cos 4t - 4K_3\,\mathrm{e}^{-3t}\sin 4t - 3K_4 e^{-3t}\sin 4t + 4\,K_4 e^{-3t} cos4t$$

and

$$\frac{dv_C(0)}{dt} = -3K_3 + 4K_4 = 100$$

hence $K_4 = 13$. The general solution for $v_C(t)$is:

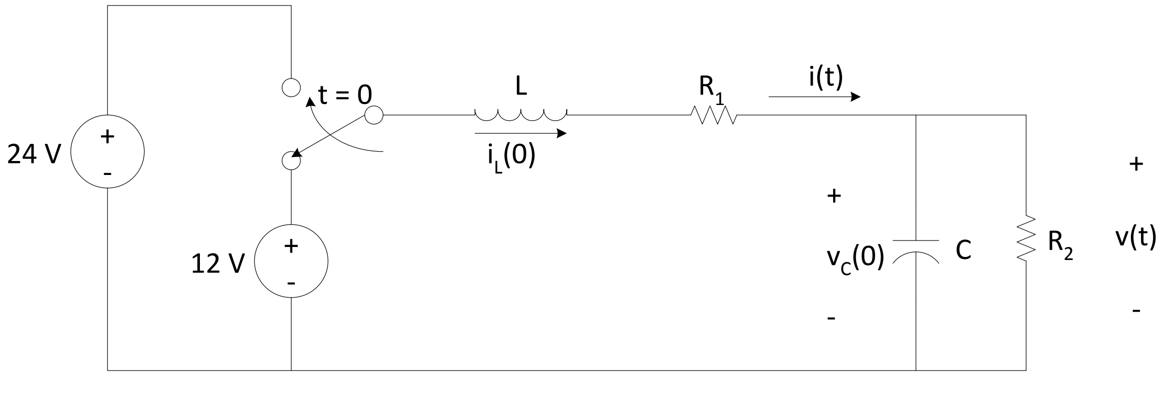
$$v_C(t) = 12 - 16\,\mathrm{e}^{-3t}\cos 4t + 13e^{-3t}\sin 4t\;V$$

Example of a series RLC circuit with a step function

Consider the circuit shown below. Given the following:

$R_1 = 10\ \Omega$; $R_2 = 2\ \Omega$; $C = 1/4$ F; $L = 2$ H; $i_L(0) = ½$ A

Determine the output voltage v(t).



We assume the switch has been connected to the 12 V supply for a long time so that the circuit is in steady state at t = 0-

For t > 0, the equations that describe the circuit are:

$$L\frac{di}{dt} + R_1 i(t) + v(t) = \frac{24}{LC}$$

and

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R_2}v(t)$$

Combining these equations, we get:

$$\frac{d^2v(t)}{dt^2} + \left(\frac{1}{R_2C} + \frac{R_1}{L}\right)\frac{dv(t)}{dt} + \frac{R_1 + R_2}{R_2LC}v(t) \ = \ \frac{24}{LC}$$

Substituting the values of the circuit elements, we get:

$$\frac{d^2v(t)}{dt^2} + 7\frac{dv(t)}{dt} + 12v(t) = 48$$

The characteristic equation is:

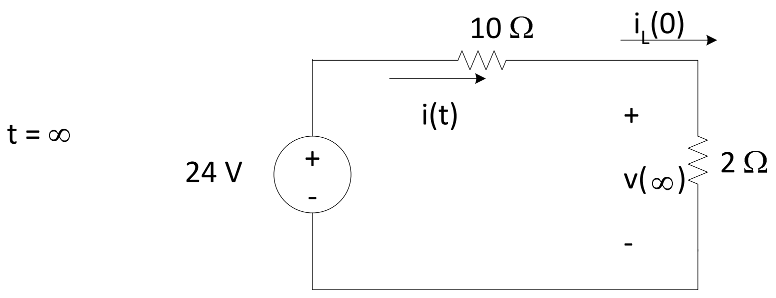
$$s^2 + 7s + 12 = 0$$

And the roots are: s_1 = - 3

s_2 = - 4

The circuit is *overdamped* and therefore the general solution is:

$$v(t) = K_1e^{-3t} + K_2e^{-4t} + K_3$$

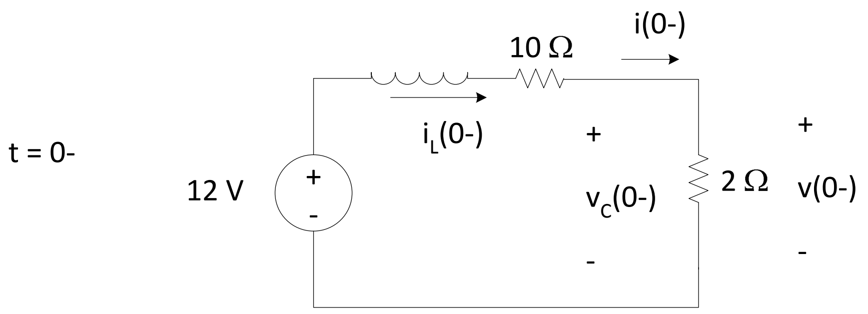


Now

$$v(\infty) = K_3 = 4$$

Hence

$$v(t) = K_1e^{-3t} + K_2e^{-4t} + 4$$



Now

$$i_l(0-) = i(0-) = \frac{12}{10+2} = 1\ A = i_L(0+)$$

and $\quad v_C(0-) = v(0-) = 2 \times i(0-) = 2\ V$

hence

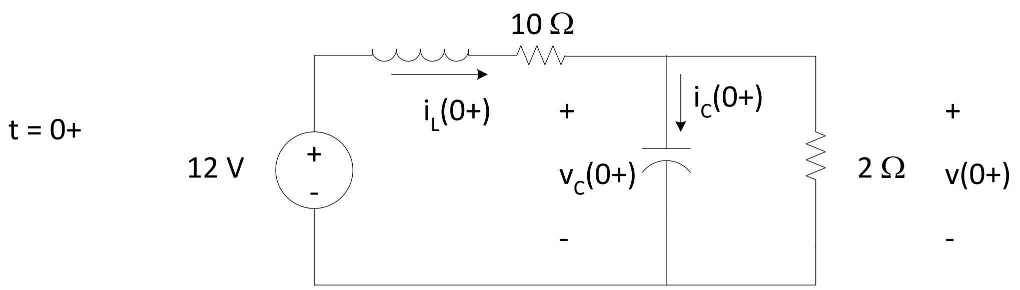
$$v(0+) = K_1 + K_2 + 4 = 2$$

or

$$K_1 + K_2 = -2$$

Now

$$\frac{dv(0)}{dt} = -3K_1 - 4K_2$$



$$i_L(0+) = i_C(0+) + \frac{v(0+)}{R_2} = C\frac{dv(0)}{dt} + \frac{v(0+)}{R_2}$$

$$\frac{dv(0)}{dt} = \frac{i_L(0+)}{C} - \frac{v(0+)}{R_2 C}$$

$$\frac{dv(0)}{dt} = \frac{1}{1/4} - \frac{2}{2 \times 1/4} = 0$$

$$-3K_1 - 4K_2 = 0$$

Solving for K_1 and K_2 we get $K_1 = -8$ and $K_2 = 6$.

The general solution for the voltage response is:

$$v(t) = 4 - 8e^{-3t} + 6e^{-4t}$$

Sinusoidal forcing Functions

Consider the sinusoidal wave $x(t) = X_m sin(\omega t)$ where X_m is the amplitude of the sine Type equation here.wave, ω is the radian or angular frequency and ωt is the argument of the sine function.

The function repeats itself every 2π radians which is described mathematically as:

$$x(\omega t + 2\pi) = x(\omega t)$$

Or

$$x[\omega(t + T)] = x(\omega t)$$

Consider the general expression for a sinusoidal function:

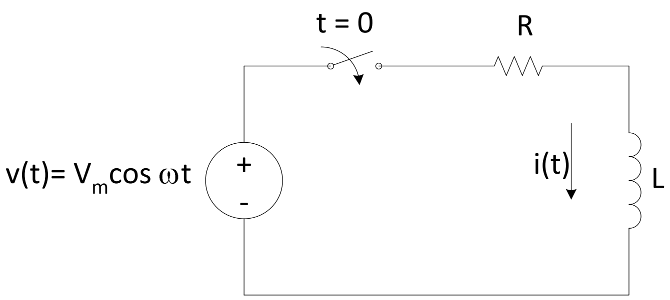
$$x_1(t) = X_{m_1} sin(\omega t + \theta)$$

and

$$x_2(t) = X_{m_2} sin(\omega t + \emptyset)$$

If $\theta \neq \emptyset$ the functions are said to be out of phase.

A Simple RL Circuit with a sinusoidal forcing function



By KVL:

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos \omega t$$

Since the forcing function is $V_m \cos \omega t$, we assume that the forced response component of the current i(t) is of the form $i(t) = A\, cos(\omega t + \emptyset)$ which can be rewritten as:

$$i(t) = A\, cos\, \emptyset \cos \omega t - A \sin \emptyset \sin \omega t$$

$$i(t) = A_1 \cos \omega t + A_2 \sin \omega t$$

Substituting in the differential equation, we get:

$$L\frac{d}{dx}(A_1 \cos \omega t + A_2 \sin \omega t) + R(A_1 \cos \omega t + A_2 \sin \omega t) = V_m \cos \omega t$$

$$-A_1 \omega L\, sin\, \omega t + A_2\, \omega L\, cos\;\; \omega t + R\, A_1 \cos \omega t + RA_2 \sin \omega t = V_m \cos \omega t$$

Equating coefficients of sine and cosine, we get:

$$-A_1\omega L + A_2 R = 0$$

$$A_1 R + A_2\omega L = V_m$$

Solving for A_1 andA_2 gives:

$$A_1 = \frac{RV_m}{R^2 + \omega^2 L^2}$$

$$A_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

Hence

$$i(t) = \frac{RV_m}{R^2 + \omega^2 L^2}\cos\omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin\omega t$$

Now

$$i(t) = A\,cos(\omega t + \emptyset)$$

where

$$A\cos\emptyset = \frac{RV_m}{R^2 + \omega^2 L^2}$$

$$A\sin\emptyset = -\frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$\tan\emptyset = -\frac{\omega L}{R}$$

Since

$$(A\cos\emptyset)^2 + (A\sin\emptyset)^2 = A^2 = \frac{R^2 {V_m}^2}{(R^2 + \omega^2 L^2)^2} + \frac{(\omega L)^2 {V_m}^2}{(R^2 + \omega^2 L^2)^2}$$

$$A^2 = \frac{{V_m}^2}{(R^2 + \omega^2 L^2)^2}$$

And

$$A = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

Therefore

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\,cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

If L = 0, then $\emptyset = 0\ and\ i(t) is\ in\ phase\ with\ v(t)$.

If R = 0, then $\emptyset = -90^{\circ}\ and\ the\ current\ lags\ the\ voltage\ by\ 90^{\circ}$.

If L and R are both present, the current lags the voltage by some angle between $0°\ and\ 90°$.

It should become clear that solving a simple one-loop circuit containing one resistor and one inductor is complicated when compared to a single-loop circuit containing only two resistors.

Recall Euler's equation: $e^{j\omega t} = \cos \omega t + j \sin \omega t$

Hence

$$Re\left(e^{j\omega t}\right) = \cos \omega t$$

and

$$Im\left(e^{j\omega t}\right) = \sin \omega t$$

Suppose a forcing function is: $v(t) = V_m e^{j\omega t}$, we can rewrite

$$v(t) = V_m \cos \omega t + jV_m \sin \omega t$$

The complex forcing function can be viewed as two forcing functions, a real one and an imaginary one. Because of *linearity*, the superposition principle can be applied and hence the current response can be written as:

$$i(t) = I_m \cos(\omega t + \emptyset) + jI_m \sin(\omega t + \emptyset)$$

where

$$I_m \cos(\omega t + \emptyset)\ is\ the\ response\ due\ to\ V_m \cos \omega t$$

And

$$jI_m \sin(\omega t + \emptyset)\ \ is\ the\ response\ due\ to\ jV_m\ \sin \omega t$$

The expression for the current containing both a real and an imaginary term can be written by Euler's equation as:

$$i(t) = I_m e^{j(\omega t + \emptyset)}$$

We can apply $V_m e^{j\omega t}$ and calculate the response $I_m e^{j(\omega t+\emptyset)}$.

Redo example with simple RL circuit.

The forcing function is now:

$$V_m e^{j\omega t}$$

The forced response will be of the form:

$$i(t) = I_m e^{j(\omega t + \emptyset)}$$

Substituting in the differential equation:

$$L\frac{di(t)}{dt} + Ri(t) = V_m e^{j\omega t}$$

We get:

$$L\frac{d\left[I_m e^{j(\omega t+\emptyset)}\right]}{dt} + RI_m e^{j(\omega t+\emptyset)} = V_m e^{j\omega t}$$

Or

$$RI_m e^{j(\omega t+\emptyset)} + j\omega L I_m e^{j(\omega t+\emptyset)} = V_m e^{j\omega t}$$

Dividing by $e^{j\omega t}$ we get:

$$RI_m e^{j\emptyset} + j\omega L I_m e^{j\emptyset} = V_m$$

Rewrite as:

$$I_m e^{j\emptyset} = \frac{V_m}{R + j\omega L}$$

In polar form:

$$I_m e^{j\emptyset} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} e^{j\left[-\tan^{-1}\left(\frac{\omega L}{R}\right)\right]}$$

Hence

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}$$

And

$$\emptyset = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Since the actual forcing function was $V_m \cos \omega t$ rather than $V_m e^{j\omega t}$, our actual response is the real part of the complex response .

$$i(t) = Re\left[I_m e^{j(\omega t+\emptyset)}\right]$$

$$i(t) = I_m \cos(\omega t + \emptyset) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

PHASORS

Assume that the forcing function for a linear network is of the form:

$$v(t) = V_m e^{j\omega t}$$

Then all steady-state voltages or currents in the network will have the same form and same frequency. As we note the frequency, ω, the $e^{j\omega t}$ can be suppressed as it is common to every term in the equations that describe the network. All voltages and currents can be fully described by a magnitude and phase. That is a voltage $v(t) = V_m \cos(\omega t + \theta)$ can be written in exponential form as: $v(t) = V_m \cos(\omega t + \theta) = Re\left[V_m e^{j(\omega t+\theta)}\right]$ or as a complex number: $v(t) = Re\left(V_m \angle\theta \; e^{j\omega t}\right)$. As we are only interested in the real part as this is the actual forcing function and we can suppress $e^{j\omega t}$, we can work with the complex number $V_m \angle \theta$. This complex representation is referred as a *phasor*. So,

$$v(t) = V_m \cos(\omega t + \emptyset) = Re\left[V_m e^{j(\omega t + \theta)}\right] \; is\ written\ as\ \boldsymbol{V} = V_m \angle \theta$$

And $i(t) = I_m \cos(\omega t + \emptyset) = Re\left[I_m e^{j(\omega t + \emptyset)}\right] \; is\ written\ as\ \mathbf{I} = I_m \angle \emptyset$ in phasor notation.

Redo RL example:

The differential equation that describes the RL series circuit is:

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos \omega t$$

The forcing function can be replaced by a complex forcing function written as $\boldsymbol{V}e^{j\omega t}$ with phasor $\boldsymbol{V} = V_m \angle 0°$. Similarly the forced response component of the current can be written as $\boldsymbol{I}e^{j\omega t}$ with phasor $\mathbf{I} = I_m \angle \emptyset$. We will recall that the solution of the differential equation is the real part of this current.

The differential equation becomes:

$$L\frac{d}{dt}\left(\boldsymbol{I}e^{j\omega t}\right) + R\boldsymbol{I}e^{j\omega t} = V_m e^{j\omega t}$$

Dividing by $e^{j\omega t}$, we get:$j\omega L\boldsymbol{I} + R\boldsymbol{I} = \boldsymbol{V}$ or

$$\mathbf{I} = \frac{\boldsymbol{V}}{\mathbf{R} + \mathrm{j}\omega\mathrm{L}} = I_m \angle \emptyset = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \angle - \tan^{-1}\frac{\omega L}{R}$$

And

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

Summary

v(t) represents a voltage in the time domain, the phasor $\boldsymbol{V}$ represents the voltage in the frequency domain.

Example: Convert the following voltages to phasors.

(i) $v_1(t) = 12\ \cos(377t - 425°)\ V$

(ii) $v_2(t) = 18\ \sin(2513t + 4.2°)\ V$

Solution (i)

Recall $v(t) = V_m \cos(\omega t + \theta) = Re\left(\ V_m \angle \theta\ e^{j\omega t}\right)$

So $\omega = 377$; $\theta = -425°$; $V_m = 12$ hence in phasor notation $\boldsymbol{V_1} = 12 \angle - 425°\ V$

Solution (ii)

Recall

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

Therefore $v_2(t) = 18\ \cos(2513t - 85.8°)$

So $\omega = 2513;\ \ \theta = -85.8°;\ \ V_m = 18$ hence in phasor notation $\boldsymbol{V_2} = 18\angle - 85.8°\ V$

Example: Convert the following phasors to the time domain given the frequency is 400 Hz.

(i) $\boldsymbol{V_1} = 10\angle 20°\ V$

(ii) $\boldsymbol{V_2} = 12\angle - 60°\ V$

Solution (i) Recall $\boldsymbol{V} = V_m\angle\theta°$ so that $V_{m_1} = 10;\quad \theta_1 = 20°$

Since $v(t) = V_m\cos(\omega t + \theta) = \boldsymbol{V}e^{j\omega t}$, we have $\omega = 2\pi f = 800\pi$

So

$$v_1(t) = 10\cos(800\pi t + 20°)\ V$$

Similarly in (ii)

$$v_2(t) = 12\cos(800\pi t - 60°)\ V$$

Deriving the current-voltage relationship for a resistor using phasors.

For a resistor $v(t) = Ri(t)$.

Applying the complex voltage $V_m e^{j(\omega t+\theta_v)}$ results in a complex current $I_m e^{j(\omega t+\theta_i)}$.

$$V_m e^{j(\omega t+\theta_v)} = I_m e^{j(\omega t+\theta_i)}$$

which can be rewritten as:

$$V_m e^{j\theta_v} = I_m e^{j\theta_i}$$

In phasor form $\boldsymbol{V} = \boldsymbol{RI}$ where $\boldsymbol{V} = V_m\angle\ \theta_v$ and $\boldsymbol{I} = I_m\angle\theta_i$

We observe $\theta_v = \theta_i$ which means the current and voltage are *in phase*.

For an inductor where $v(t) = L\frac{di(t)}{dt}$ if we substitute the complex voltage and current we get:

$$V_m e^{j(\omega t+\theta_v)} = L\frac{d}{dt}\left(I_m e^{j(\omega t+\theta_i)}\right)$$

which can be reduced to:

$$V_m e^{j\theta_v} = j\omega L I_m e^{j\theta_i}$$

In phasor notation;

$$\boldsymbol{V} = j\omega L\boldsymbol{I}$$

Now

$$V_m e^{j\theta_v} = \omega L I_m e^{j(\theta_i+90°)}$$

The current lags the voltage by 90°

Example: The voltage $v(t) = 12\cos(377t + 20°)\ V$ is applied to a 20-mH inductor. Find the resulting current.

The phasor current is:

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{j\omega L} = \frac{12\angle 20°}{\omega L\angle 90°} = \frac{12\angle 20°}{(377)(20\times 10^{-3})\angle 90°}$$

$$\boldsymbol{I} = 1.59\angle 70°$$

Therefore

$$i(t) = 1.59\cos(377t - 70°)\ A$$

For a capacitor where $i(t) = C\frac{dv(t)}{dt}$ if we substitute the complex voltage and current we get:

$$I_m e^{j(\omega t+\theta_i)} = C\frac{d}{dt}\left[V_m e^{j(\omega t+\theta_v)}\right]$$

Which reduces to:

$$I_m e^{j\theta_i} = j\omega C V_m e^{j\theta_v} = \omega C V_m e^{j(\theta_v+90°)}$$

In phasor notation

$$\boldsymbol{I} = j\omega C\boldsymbol{V}$$

The current leads the voltage by 90°

Impedance

Impedance **Z** is defined as the ratio of the phasor voltage $\boldsymbol{V}$ to the phasor current $\boldsymbol{I}$.

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V_m e^{j\theta_v}}{I_m e^{j\theta_i}} = \frac{V_m}{I_m}\angle(\theta_v - \theta_i) = Z\angle\theta_z$$

In rectangular form

$$\boldsymbol{Z}(\omega) = R(\omega) + jX(\omega)$$

where $R(\omega)$ is the real or resistive component and $X(\omega)$ is the imaginary or reactive component.

Z is a complex number but NOT a phasor since phasors denote sinusoidal functions.

$$Z\angle\theta_z = R + jX$$

Therefore $Z = \sqrt{R^2 + X^2}$ and $\theta_z = \tan^{-1}\frac{X}{R}$

$$R = Z\cos\theta_z$$

$$X = Z\sin\theta_z$$

The Laplace Transform

The Laplace transform of a function *f(t)* is defined by the equation:

$$L[f(t)] = \boldsymbol{F}(s) = \int_0^\infty f(t)e^{-st}dt$$

Where *s* is the complex frequency

$$s = \sigma + j\omega$$

We assume *f(t)* = 0 for t < 0. For *f(t)* to have a Laplace transform, it must satisfy the condition:

$$\int_0^\infty e^{-\sigma t}|f(t)|dt < \infty$$

for some real value of σ.

The inverse Laplace transform is defined by:

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_2+j\infty}^{\sigma_1+j\infty} \boldsymbol{F}(s)e^{st}ds$$

where σ_1 is real.

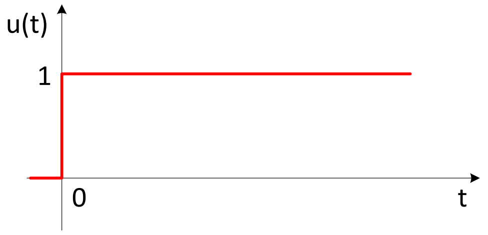
The Laplace transform has a uniqueness property i.e. for a given *f(t)* there is a unique F(s)

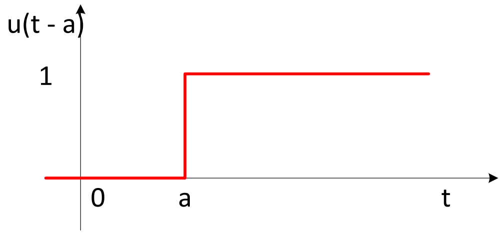
Singularity Functions

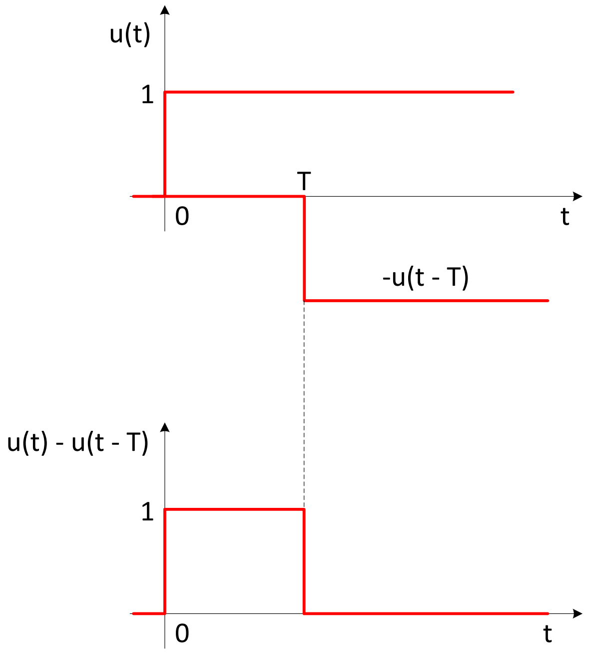
The unit step function u(t) and the unit impulse or delta function δ(t). They are called *singularity functions* because they are either not finite or they do not possess finite derivatives everywhere.

The Unit Step Function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$







The Laplace Transform of the Unit Step function

$$\boldsymbol{F}(s) = \int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} 1e^{-st}dt$$

$$\boldsymbol{F}(s) = -\frac{1}{s}e^{-st}\bigg|_0^{\infty} = \frac{1}{s} \qquad \sigma > 0$$

Therefore

$$\mathcal{L}[u(t)] = \boldsymbol{F}(s) = \frac{1}{s}$$

For the time-shifted unit step function u(t – a),

$$\boldsymbol{F}(s) = \int_0^{\infty} u(t-a)e^{-st}dt$$

We note that

$$u(t-a) = \begin{cases} 1, & a < t < \infty \\ 0, & t < 0 \end{cases}$$

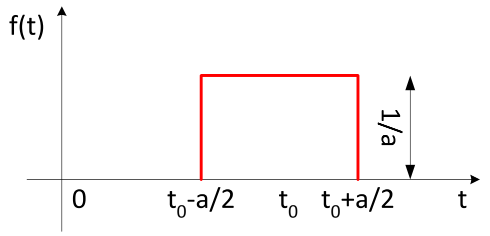
Hence

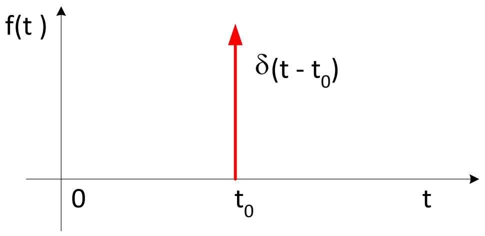
$$\boldsymbol{F}(s) = \int_a^{\infty} 1e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_a^{\infty} = \frac{e^{-as}}{s} \qquad \sigma > 0$$

And for the pulse $u(t) - u(t-T)$ the Laplace transform is:

$$\boldsymbol{F}(s) = \int_0^{\infty} [u(t) - u(t-T)]e^{-st}dt = \frac{1 - e^{-Ts}}{s} \quad \sigma > 0$$

The Unit Impulse Function





The unit impulse function can be represented in the limit by the rectangular pulse shown as

$$a \to 0$$

The function is defined as:

$$\delta(t - t_0) = 0 \qquad\qquad t \neq t_0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)dt = 1 \qquad\qquad \epsilon > 0$$

The unit impulse is zero except at t = t_0, where it is undefined. It has unit area.

An important property of the unit impulse function is its ability to sample or its sampling property.

$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & t_0 < t_1, \quad t_0 > t_2 \end{cases}$$

The above is valid for a finite t_0 and any $f(t)$ *continuous at* t_0.

The unit impulse function samples the value of $f(t)$ *at* $t = t_0$

The Laplace transform of an impulse function

$$\boldsymbol{F}(s) = \int_0^\infty \delta(t - t_0)e^{-st}dt$$

Using the sampling property of the delta function, we get:

$$\mathcal{L}[\delta(t - t_0)] = e^{-t_0 s}$$

In the limit as

$$t_0 \to 0, \ \ e^{-t_0 s} \to 1$$

and therefore

$$\mathcal{L}[\delta(t)] = \boldsymbol{F}(s) = 1$$

Example: Find the Laplace transform of *f(t) = t*.

$$\boldsymbol{F}(s) = \int_0^\infty te^{-st}dt$$

Integrating by parts, setting

$u = t$ and $dv = e^{-st}\,dt$

$du = dt$ and $v = \int e^{-st}\,dt = -\frac{1}{s}e^{-st}$

Therefore

$$\boldsymbol{F}(s) = \frac{-t}{s}e^{-st}\bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}dt = \frac{1}{s^2} \quad \sigma > 0$$

Example: Find the Laplace transform of *f(t) = cos ωt*.

$$\boldsymbol{F}(s) = \int_0^\infty \cos\omega t\, e^{-st}dt$$

$$= \int_0^\infty \frac{e^{+j\omega t} + e^{-j\omega t}}{2}e^{-st}dt$$

$$= \int_0^\infty \frac{e^{-(s-j\omega)t} + e^{-(s+j\omega)t}}{2}e^{-st}dt$$

$$= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \qquad \sigma > 0$$

$$\boldsymbol{F}(s) = \frac{s}{s^2+\omega^2}$$

If *f(t)* = sin (ωt) then

$$\boldsymbol{F}(s) = \frac{\omega}{s^2+\omega^2}$$

Some Laplace Transform Pairs

f(t)	**F(s)**
δ(t)	1
u(t)	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$

Some Useful Properties of the Laplace Transform

Property	**f(t)**	**F(s)**
1. Magnitude scaling	**A**f(t)	A**F**(s)
2. Addition /subtraction	$f_1(t) \pm f_2(t)$	$\boldsymbol{F_1}(s) \pm \boldsymbol{F_2}(s)$
3. Time scaling	f(at)	$\frac{1}{a}\boldsymbol{F}\left(\frac{s}{a}\right), a > 0$
4. Time shifting	$f(t-t_0)u(t-t_0),\ t_0 \geq 0$ $f(t)u(t-t_0)$	$e^{-t_0 s}\boldsymbol{F}(s)$ $e^{-t_0 s}F\mathcal{L}[f(t+t_0)]$
5. Frequency shifting	$e^{-at}f(t)$	**F**(s+a)
6. Differentiation	$\frac{d^n f(t)}{dt^n}$	$s^n\boldsymbol{F}(s) - s^{n-1}f(0) - s^{n-2}f^1(0)$ $\cdots s^0 f^{n-1}(0)$
7. Multiplication by *t*	$tf(t)$ $t^n f(t)$	$-\frac{d\boldsymbol{F}(s)}{ds}$ $(-1)^n\frac{d^n\boldsymbol{F}(s)}{ds^n}$
8. Integration	$\int_0^t f(\lambda)d\lambda$	$\frac{1}{s}\boldsymbol{F}(s)$

Properties of the Laplace Transform

$$\mathcal{L}[f(at)] = \frac{1}{a}\boldsymbol{F}\left(\frac{s}{a}\right) \quad a > 0$$

Time-shifting theorem

$$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}\boldsymbol{F}(s) \qquad t_0 \geq 0$$

Frequency-shifting or modulation theorem

$$\mathcal{L}[e^{-at}f(t)] = \boldsymbol{F}(s+a)$$

Example: Find the Laplace transform of $e^{-at}\cos\omega t$

Since $\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$, then

$$\mathcal{L}[e^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2}$$

Inverse Laplace Transform

The algebraic solution of the circuit equations in the complex frequency domain results in a rational function of *s* of the form:

$$\boldsymbol{F}(s) = \frac{\boldsymbol{P}(s)}{\boldsymbol{Q}(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

The roots of the polynomial $\boldsymbol{P}(s)$ (i.e., $-z_1, -z_2, \cdots -z_m$) are called the *zeros* of the function $\boldsymbol{F}(s)$ because at these values of *s*, $\boldsymbol{F}(s) = 0$.

Similarly the roots of the polynomial $\boldsymbol{Q}(s)$ (i.e., $-p_1, -p_2, \cdots -p_n$) are called poles of $\boldsymbol{F}(s)$, since at these values $\boldsymbol{F}(s)$ becomes infinite.

1. If the roots are simple, then $\boldsymbol{P_1}(s)/\boldsymbol{Q}(s)$ can be expressed in partial fraction form

$$\frac{\boldsymbol{P_1}(s)}{\boldsymbol{Q}(s)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n}$$

2. If $\boldsymbol{Q}(s)$ has simple complex roots, they will appear in complex-conjugate pairs, and the partial fraction expansion of $\boldsymbol{P_1}(s)/\boldsymbol{Q}(s)$ for each pair of complex-conjugate roots will be of the form:

$$\frac{\boldsymbol{P_1}(s)}{\boldsymbol{Q_1}(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{K_1}{(s+\alpha-j\beta)} + \frac{K_1^*}{(s+\alpha+j\beta)} + \cdots$$

Where $\boldsymbol{Q}(s) = \boldsymbol{Q_1}(s)(s+\alpha-j\beta)(s+\alpha+j\beta) and\ K_1^*$ is the complex conjugate of K_1.

3. If $\boldsymbol{Q}(s)$has a root of multiplicity *r*, the partial fraction expansion for each such root will be of the form

$$\frac{\boldsymbol{P_1}(s)}{\boldsymbol{Q_1}(s)(s+p_1)^r} = \frac{K_{11}}{(s+p_1)} + \frac{K_{12}}{(s+p_1)^2} + \cdots + \frac{K_{1r}}{(s+p_1)^r} + \cdots$$

Example: Given

$$\boldsymbol{F}(s) = \frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)}$$

Find $f(t) = \mathcal{L}^{-1}[\boldsymbol{F}(s)]$

Let us express $\boldsymbol{F}(s)$ in a partial fraction expansion.

$$\frac{12(s+1)(s+3)}{s(s+2)(s+4)(s+5)} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

To determine K_0 multiply both sides of the equation by *s* and evaluate at *s = 0*.

$$\frac{12(s+1)(s+3)}{(s+2)(s+4)(s+5)} = K_0 + \frac{K_1 s}{s+2} + \frac{K_2 s}{s+4} + \frac{K_3 s}{s+5}$$

At s= 0 we have

$$\frac{12(1)(3)}{(2)(4)(5)} = K_0 + 0 + 0 + 0$$

$$K_0 = \frac{36}{40}$$

Similarly multiplying both sides of the equation by (s + 2) and evaluating at s = -2

$$(s+2)\boldsymbol{F}(s)|_{s=-2} = \left.\frac{12(s+1)(s+3)}{(s+4)(s+5)}\right|_{s=-2} = K_1$$

Gives $K_1 = 1$

Similarly

$$(s+4)\boldsymbol{F}(s)|_{s=-4} = \left.\frac{12(s+1)(s+3)}{(s+2)(s+5)}\right|_{s=-4} = K_2$$

Gives

$$K_2 = \frac{36}{8}$$

$$K_3 = -\frac{32}{5}$$

Hence

$$\boldsymbol{F}(s) = \frac{{}^{36}\!/_{40}}{s} + \frac{1}{s+2} + \frac{{}^{36}\!/_{8}}{s+4} - \frac{{}^{32}\!/_{5}}{s+5}$$

And

$$f(t) = \left(\frac{36}{40} + 1e^{-2t} + \frac{36}{8}e^{-4t} - \frac{32}{5}e^{-5t}\right)u(t)$$

Complex-Conjugate Poles

Consider the case where $\boldsymbol{F}(s)$ has one pair of complex-conjugate poles.

$$\boldsymbol{F}(s) = \frac{\boldsymbol{P_1}(s)}{\boldsymbol{Q_1}(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{K_1}{(s+\alpha-j\beta)} + \frac{K_1^*}{(s+\alpha+j\beta)} + \cdots$$

Then

$$K_1 = (s+\alpha-j\beta)\boldsymbol{F}(s)|_{s=-\alpha+j\beta}$$

K_1 is generally a complex number that can be expressed as $|K_1|\angle\theta$ and $K_1^* = |K_1|\angle-\theta$

Hence

$$\boldsymbol{F}(s) = \frac{|K_1|\angle\theta}{(s+\alpha-j\beta)} + \frac{|K_1|\angle-\theta}{(s+\alpha+j\beta)} + \cdots$$

The corresponding time function is then

$$f(t) = \mathcal{L}^{-1}[\boldsymbol{F}(s)] = |K_1|e^{j\theta}e^{-(\alpha-j\beta)t} + |K_1|e^{-j\theta}e^{-(\alpha+j\beta)t}$$

$$f(t) = |K_1|e^{-\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)} + \cdots\right]$$

$$f(t) = |K_1|e^{-\alpha t}\cos(\beta t+\theta) + \cdots$$

Example: Determine the time function y(t) for the function

$$\boldsymbol{Y}(s) = \frac{10(s+2)}{s(s^2+4s+5)}$$

Expressing in partial fraction, we obtain

$$\frac{10(s+2)}{s(s+2-j1)(s+2+j1)} = \frac{K_0}{s} + \frac{K_1}{s+2-j1} + \frac{K_1^*}{s+2+j1}$$

$$K_0 = \left.\frac{10(s+2)}{(s+2-j1)(s+2+j1)}\right|_{s=0} = \left.\frac{10(s+2)}{(s^2+4s+5)}\right|_{s=0} = 4$$

$$K_1 = \frac{10(s+2)}{s(s+2+j1)}\bigg|_{s=-2+j1} = 2.236\angle -153.43°$$

$$\boldsymbol{Y}(s) = \frac{4}{s} + \frac{2.236\angle -153.43°}{s+2-j1} + \frac{2.236\angle 153.43°}{s+2+j1}$$

Hence

$$y(t) = [4 + 4.472e^{-2t}\cos(t - 153.43°)]u(t)$$

Multiple Poles

Consider the case where has a pole of multiplicity *r*.

$$\boldsymbol{F}(s) = \frac{\boldsymbol{P_1}(s)}{\boldsymbol{Q_1}(s)(s+p_1)^r} = \frac{K_{11}}{(s+p_1)} + \frac{K_{12}}{(s+p_1)^2} + \cdots + \frac{K_{1r}}{(s+p_1)^r} + \cdots$$

$$K_{1r} = (s+p_1)^r\, \boldsymbol{F}(s)|_{s=-p_1}$$

$$K_{1r-1} = \frac{d}{ds}[(s+p_1)^r \boldsymbol{F}(s)]$$

$$(2!)K_{1r-2} = \frac{d^2}{ds^2}(s+p_1)^r \boldsymbol{F}(s)|_{s=-p_1}$$

Hence

$$K_{1j} = \frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}[(s+p_1)^r \boldsymbol{F}(s)]\bigg|_{s=-p_1}$$

Example: Given that

$$\boldsymbol{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)}$$

Find *f(t)*.

$$\boldsymbol{F}(s) = \frac{10(s+3)}{(s+1)^3(s+2)} = \frac{K_{11}}{(s+1)} + \frac{K_{12}}{(s+1)^2} + \frac{K_{13}}{(s+1)^3} + \frac{K_2}{s+2}$$

Then

$$K_{13} = (s+1)^3\, \boldsymbol{F}(s)|_{s=-1} = 20$$

$$K_{12} = \frac{d}{ds}[(s+1)^3 \boldsymbol{F}(s)]\bigg|_{s=-1} = -\frac{10}{(s+2)^2}\bigg|_{s=-1} = -10$$

$$2K_{11} = \frac{d^2}{ds^2}(s+1)^3 \boldsymbol{F}(s)|_{s=-p_1} = \frac{20}{(s+2)^3}\bigg|_{s=-1} = 20$$

$$K_{11} = 10$$

$$K_2 = (s+2)\,\boldsymbol{F}(s)|_{s=-2} = K_2$$

$$K_2 = -10$$

Then

$$\boldsymbol{F}(s) = \frac{10}{(s+1)} - \frac{10}{(s+1)^2} + \frac{20}{(s+1)^3} - \frac{10}{s+2}$$

Since

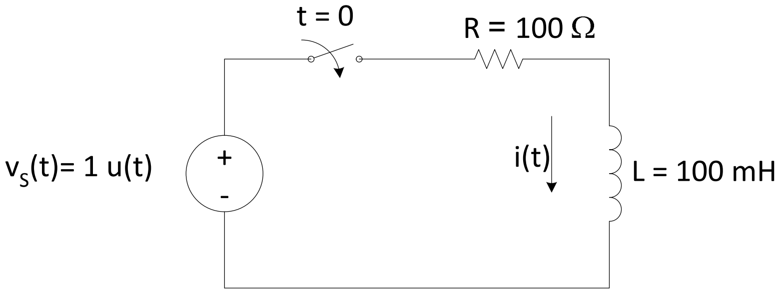
$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{t^n}{n!}e^{-at}$$

Then

$$f(t) = (10e^{-t} - 10te^{-t} + 10t^2e^{-t} - 10e^{-2t})u(t)$$

Laplace Transform in Circuit Analysis

Consider the RL series circuit shown.



$$L\frac{di(t)}{dt} + Ri(t) = v_s(t)$$

The complementary differential equation is

$$L\frac{di(t)}{dt} + Ri(t) = 0$$

which has a solution of the form

$$i_c(t) = K_c e^{-\alpha t}$$

After substitution we get

$$R - \alpha L = 0$$

Or

$$\alpha = \frac{R}{L} = 1000$$

The particular solution is of the same form as the forcing function

$$i_p(t) = K_p$$

After substitution we get

$$1 = RK_p$$

Or

$$K_p = \frac{1}{R} = \frac{1}{100}$$

The complete solution is then

$$i(t) = K_p + K_c e^{-1000t} = \frac{1}{100} + K_c e^{-1000t}$$

Since

$$i(0-) = i(0+) = 0$$

$$0 = \frac{1}{100} + K_c$$

$$K_c = -\frac{1}{100}$$

$$i(t) = \frac{1}{100} - \frac{1}{100} e^{-1000t}\, A = 10(1 - e^{-1000t})u(t) mA$$

Solution with the Laplace Transform

$$L\frac{di(t)}{dt} + Ri(t) = v_s(t)$$

Taking the Laplace of the above equation we get

$$\boldsymbol{V}(s) = L[s\boldsymbol{I}(s) - \ i(0)] + R\boldsymbol{I}(s)$$

Since

$$i(0) = 0$$

$$\boldsymbol{I}(s) = \frac{\boldsymbol{V_s}(s)}{sL + R}$$

But

$$\boldsymbol{V_s}(s) = \mathcal{L}[v_s(t)] = \mathcal{L}[1\, u(t)] = \frac{1}{s}$$

Hence

$$\boldsymbol{I}(s) = \frac{1}{s(sL + R)}$$

To find the *i(t*), we use the inverse Laplace Transform

$$\boldsymbol{I}(s) = \frac{1/\mathrm{L}}{s\left(s + \frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{\left(s + \frac{R}{L}\right)}$$

$$\boldsymbol{I}(s) = \frac{1}{sR} - \frac{1}{R\left(s + \frac{R}{L}\right)}$$

Hence

$$i(t) = \frac{1}{R}\left(1 - e^{\frac{-R}{L}t}\right) = 10(1 - e^{-1000t})u(t)\ mA$$

Notice that the complete solution is derived in one step as opposed to the solution in the time domain.

For a resistor of value R, the current-voltage relationship in the time domain is

$$v(t) = Ri(t)$$

The relationship in the frequency domain, *s*, is

$$\boldsymbol{V}(s) = R\boldsymbol{I}(s)$$

For a capacitor of value C, the current-voltage relationship in the time domain is

$$v(t) = \frac{1}{C}\int_0^t i(x)dx + v(0)$$

$$i(t) = C\frac{dv(t)}{dt}$$

The relationship in the frequency domain, *s*, is

$$\boldsymbol{V}(s) = \frac{\boldsymbol{I}(s)}{sC} + \frac{v(0)}{s}$$

$$\boldsymbol{I}(s) = sC\boldsymbol{V}(s) - C\ v(0)$$

For a inductor of value L, the current-voltage relationship in the time domain is

$$v(t) = L\frac{di(t)}{dt}$$

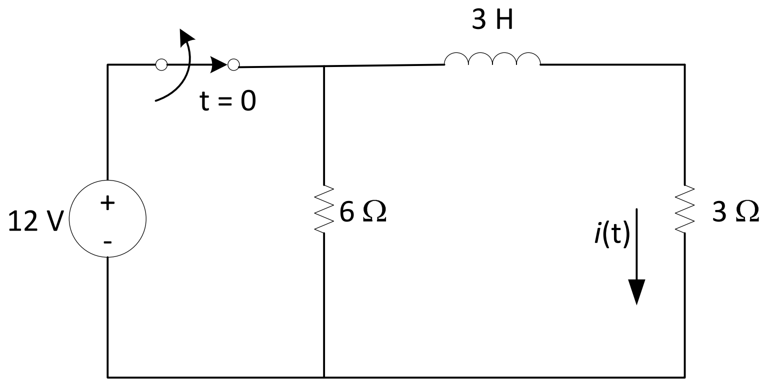
$$i(t) = \frac{1}{L}\int_0^t v(x)dx + i(0)$$

The relationship in the frequency domain, *s*, is

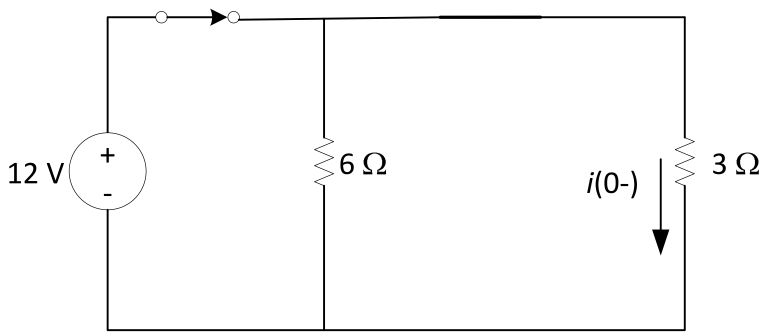
$$V(s) = sLI(s) - Li(0)$$

$$I(s) = \frac{V(s)}{sL} + \frac{i(0)}{s}$$

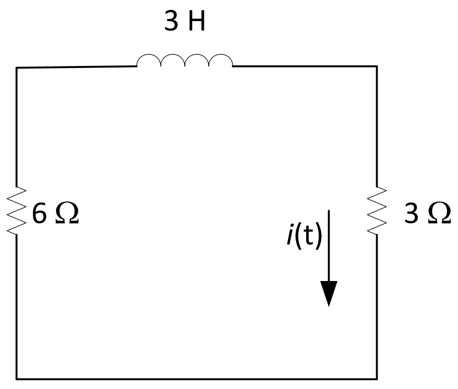
Example: For the network shown below, the switch opens at t = 0. Use Laplace transforms to find *i(t)*, for t > 0.



t < 0



t > 0



For t < 0, we have;

$$i(0^-) = \frac{12}{3} = 4\ A$$

For t > 0, we have;

$$3\frac{di(t)}{dt} + 3i(t) + 6i(t) = 0$$

Taking the Laplace transform of the differential equation, we get:

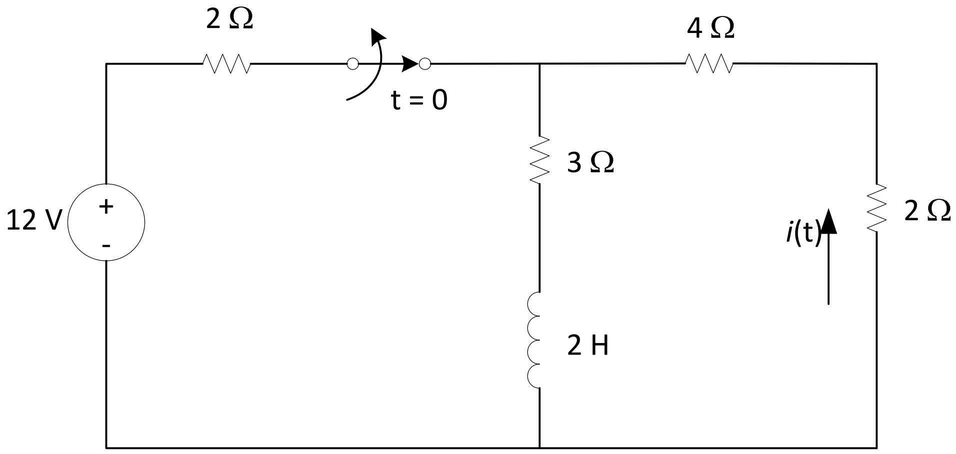
$$3s\boldsymbol{I}(s) - 3i(0^-) + 3\boldsymbol{I}(s) + 6\boldsymbol{I}(s) = 0$$

$$(3s + 9)\boldsymbol{I}(s) = 12$$

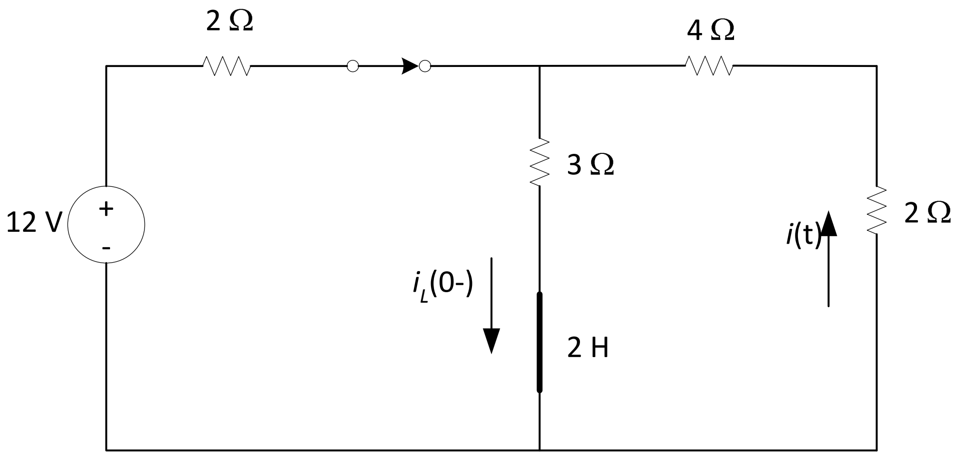
$$\boldsymbol{I}(s) = \frac{12}{3(s+3)} = \frac{4}{s+3}$$

$$i(t) = 4e^{-3t}u(t)\ A\ for\ t > 0$$

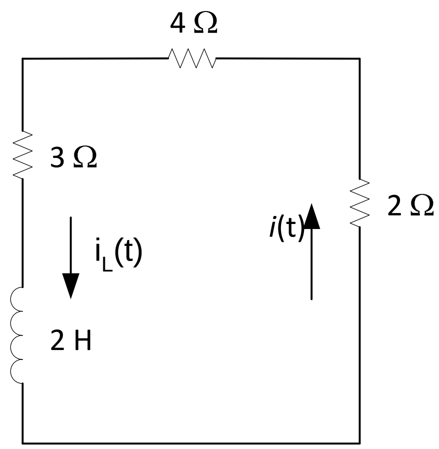
Example: The switch in the circuit below opens at t = 0. Find *i(t)* for t > 0 using Laplace transforms.



t < 0



t > 0



For t < 0

The equivalent resistance R_{eq} is: ((4+2)//3)+2 = 4 Ω

The total current in the circuit is I_{Tot} = 12/4 = 3 A

$$i_L(0^-) = \frac{6}{6+3}3 = 2\,A$$

For t > 0

$$2\frac{di(t)}{dt} + 2i(t) + 4i(t) + 3i(t) = 0$$

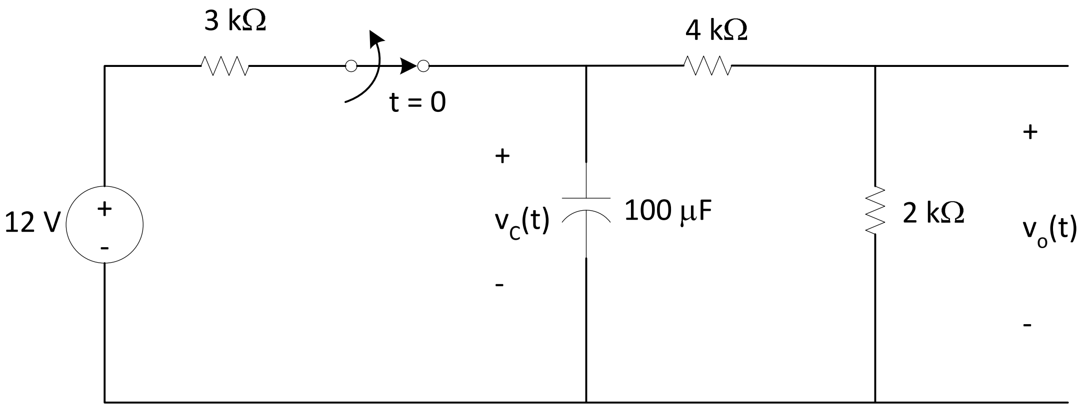
$$2s\boldsymbol{I}(s) - 2i(0^-) + 2\boldsymbol{I}(s) + 4\boldsymbol{I}(s) + 3\boldsymbol{I}(s) = 0$$

$$(2s+9)\boldsymbol{I}(s) = 4$$

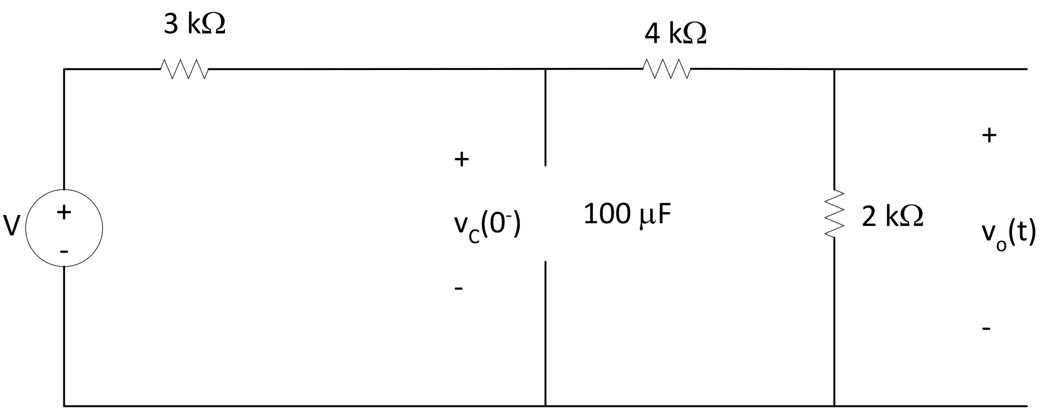
$$\boldsymbol{I}(s) = \frac{4}{2s+9} = \frac{2}{s + {}^{9}/_{2}}$$

$$i(t) = 2e^{\frac{-9}{2}t}u(t)\;A\;for\;t > 0$$

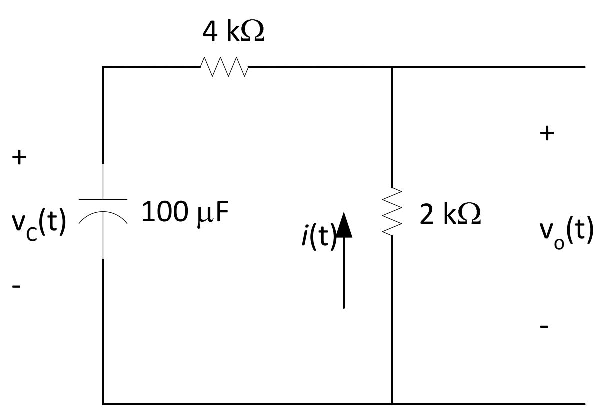
Example: In the network shown, the switch opens at $t = 0$. Use Laplace transforms to find $v_O(t)$ for $t > 0$.



t < 0



t > 0



For t < 0

$$v_C(0^-) = \frac{4+2}{3+4+2}12 = 8\,V$$

For t > 0:

By KVL we have:

$$v_C(t) + 2ki(t) + 4ki(t) = 0$$
$$i(t) = C\frac{dv_C(t)}{dt}$$

Hence

$$v_C(t) + (2k)(100\mu)\frac{dv_C(t)}{dt} + 4k(100\mu)\frac{dv_C(t)}{dt} = 0$$

$$0.6\frac{dv_C(t)}{dt} + v_C(t) = 0$$

Taking the Laplace transform of the above equation

$$0.6s\boldsymbol{V_c}(s) - 0.6v_C(0^-) + \boldsymbol{V_c}(s) = 0$$
$$0.6s\boldsymbol{V_c}(s) + \boldsymbol{V_c}(s) = 4.8$$
$$\boldsymbol{V_c}(s)(0.6s + 1) = 4.8$$
$$\boldsymbol{V_c}(s) = \frac{4.8}{0.6s+1} = \frac{8}{s+1.67}$$

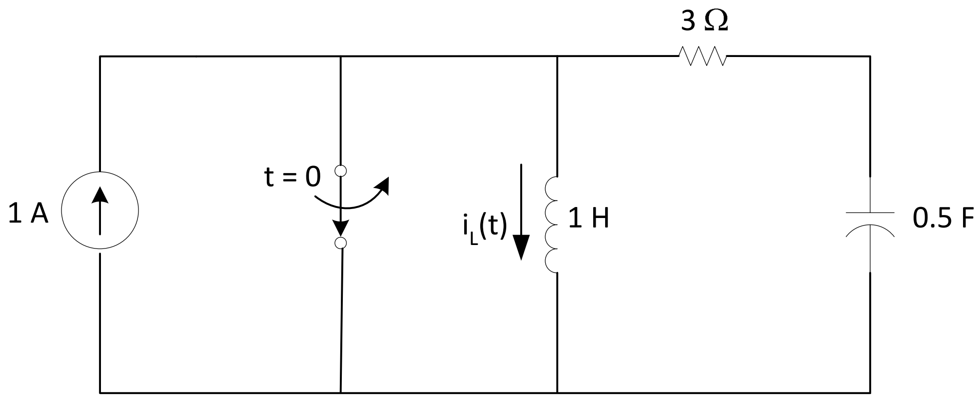
therefore

$$v_C(t) = 8e^{-1.67t}u(t)\,V, \qquad t > 0$$

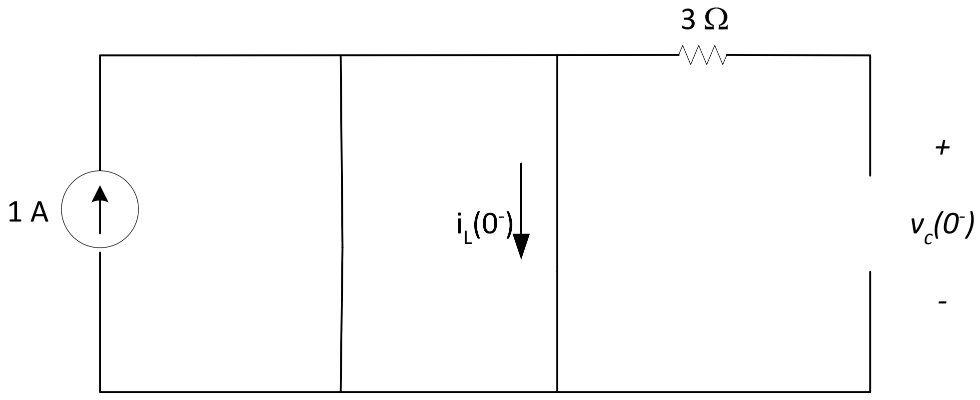
And

$$v_0(t) = \frac{2}{2+4}8e^{-1.67t} = 2.67e^{-1.67t}u(t)\,V\ for\ t > 0$$

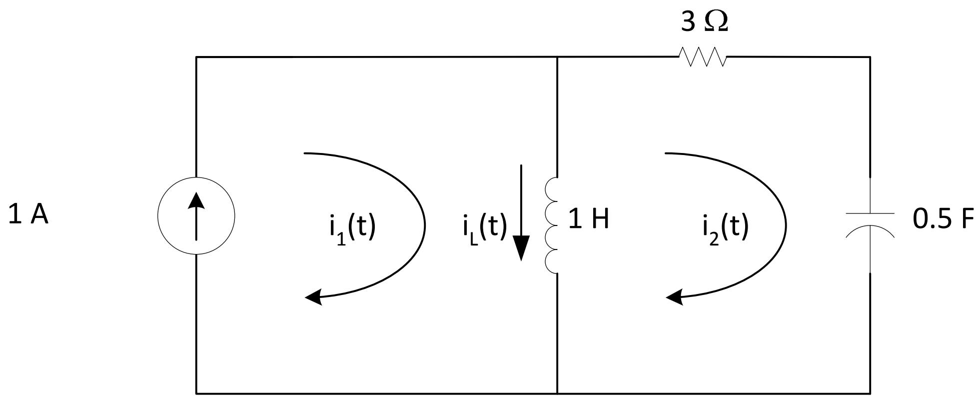
Example: In the network shown, the switch opens at $t = 0$. Use Laplace transforms to find $i_L(t)$ for $t > 0$.



t < 0



t > 0



LM

For t < 0

$$i_L(0^-) = 0; \quad v_C(0^-) = 0$$

For t > 0

$$i_1(t) = 1A$$

$$i_1(t) = i_L(t) + i_2(t)$$

By KVL:

$$3i_2(t) + 2\int i_2\,(t)dt - \frac{di_L(t)}{dt} = 0$$

$$3[1 - i_L(t)] + 2\int [1 - i_L(t)]dt - \frac{di_L(t)}{dt} = 0$$

$$3 - 3i_L(t) + 2\int dt - 2\int i_L(t)dt - \frac{di_L(t)}{dt} = 0$$

Taking the Laplace transform of the above equation;

$$-3I_L(s) + \frac{2}{s} - \frac{2}{s}I_L(s) - sI_L(s) = -3$$

$$I_L(s)\left[-s - \frac{2}{s} - 3\right] = -3 - \frac{2}{s}$$

$$I_L(s)\left[\frac{s^2 + 3s + 2}{s}\right] = \frac{3s + 2}{s}$$

$$I_L(s) = \frac{3s + 2}{s^2 + 3s + 2} = \frac{3s + 2}{(s + 2)(s + 1)}$$

$$I_L(s) = \frac{A}{(s + 2)} + \frac{B}{(s + 1)}$$

Hence

$$\frac{3s + 2}{(s + 2)(s + 1)} = \frac{A}{(s + 2)} + \frac{B}{(s + 1)}$$

$$3s + 2 = A(s + 1) + B(s + 2)$$

Let s = -1

$$3(-1) + 2 = B(-1 + 2)$$

$$B = -1$$

Let s = -2

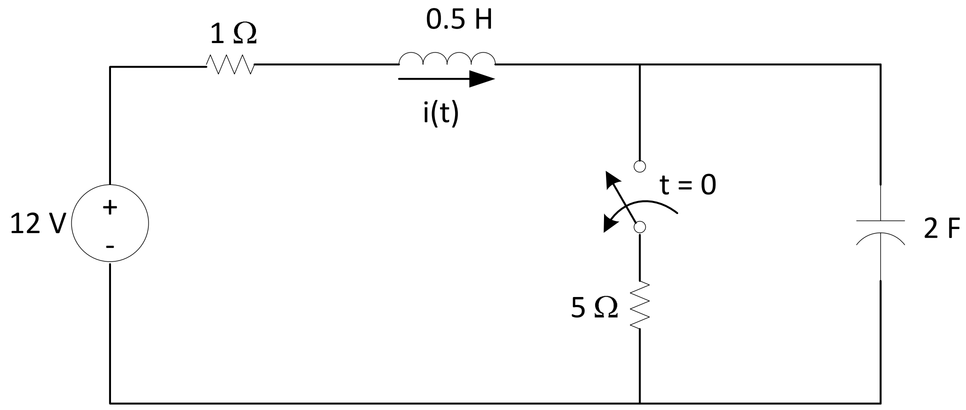
$$3(-2) + 2 = A(-2 + 1)$$

$$A = 4$$

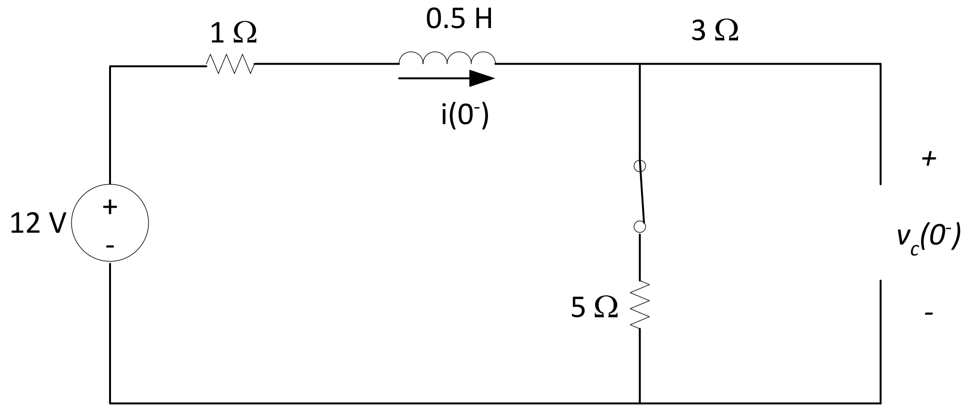
$$I_L(s) = \frac{4}{(s+2)} + \frac{-1}{(s+1)}$$

$$i_L(t) = (4e^{-2t} - e^{-t})u(t)\ A$$

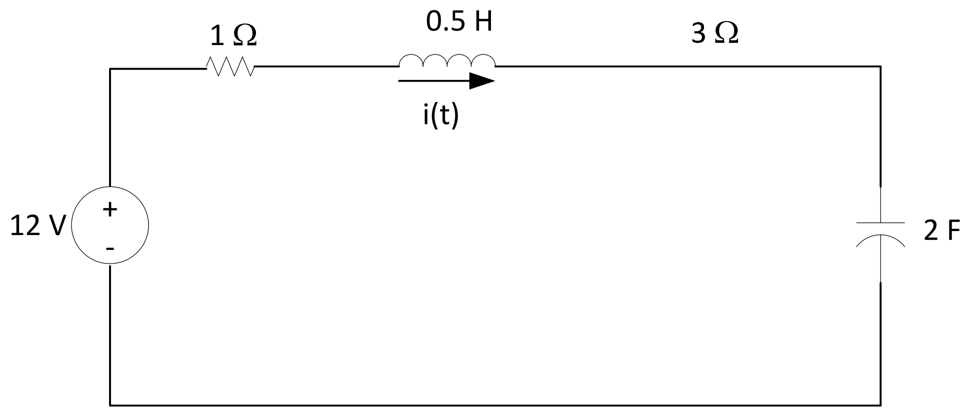
Example: In the circuit shown below, the switch has been closed for a long time and is opened at $t = 0$. Find $i(t)$ for t > 0, using Laplace transforms.



t < 0



t > 0



For t < 0;

$$i(0^-) = \frac{12}{1+5} = 2\,A$$

$$v_C(0^-) = 5i(0^-) = 10\,V$$

For t > 0:

By KVL:

$$0.5\frac{di(t)}{dt} + i(t) + \frac{1}{2}\int i(t) = 12$$

Taking the Laplace transform, we get

$$0.5I(s) - 0.5i(0^-) + I(s) + \frac{1}{2s}I(s) = \frac{12}{s}$$

$$0.5I(s) + I(s) + \frac{1}{2s}I(s) = \frac{12}{s} + 1$$

$$0.5I(s) + I(s) + \frac{1}{2s}I(s) = \frac{12}{s} + 1$$

$$I(s)\left[0.5s + \frac{1}{2s} + 1\right] = \frac{12+s}{s}$$

$$I(s)\left[\frac{s^2+2s+1}{2s}\right] = \frac{12+s}{s}$$

$$I(s) = \frac{\frac{12+s}{s}}{\frac{s^2+2s+1}{2s}} = \frac{2(s+12)}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$2(s+12) = A(s+1) + B$$

Let s = -1

$$2(-1+12) = B$$

$$B = 22$$

Let s = 0

$$24 = A + B$$

$$A = 2$$

$$I(s) = \frac{2}{s+1} + \frac{22}{(s+1)^2}$$

$$i(t) = (2e^{-t} + 22te^{-t})u(t)\,A$$

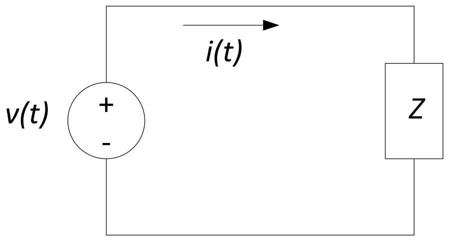
AC Power

Instantaneous Power

When a linear electric circuit is excited by a sinusoidal source, all voltages and currents in the circuit are also sinusoids of the same frequency as that of the excitation sources.

The instantaneous power supplied or absorbed by any device is the product of the instantaneous voltage and current.

Consider the ac network shown



Let $\quad v(t) = V_m \cos(\omega t + \theta_v)$

And $\quad i(t) = I_m \cos(\omega t + \theta_i)$

The instantaneous power is then $p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$

Using

$$\cos \emptyset_1 \cos \emptyset_2 = \frac{1}{2}[\cos(\emptyset_1 - \emptyset_2) + \cos(\emptyset_1 + \emptyset_2)]$$

We can write

$$p(t) = \frac{V_m I_m}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]$$

The instantaneous power consists of two terms, the first being a constant, the second is a cosine wave of twice the excitation frequency.

Average Power

The average power is computed by integrating the instantaneous power over a complete period and dividing this result by the period.

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)\, dt$$

Where t_0 is arbitrary, $T = {}^{2\pi}/_{\omega}$ is the period of the voltage or current and P has unit of watts.

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_m I_m}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]dt$$

The first term of the integrand is a constant (independent of *t*). Integrating the constant over a period and dividing by the period results in the original constant.

The second term is a cosine wave which when integrated over one complete period is zero.

Hence

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

For a purely resistive circuit where $\theta_v = \theta_i$

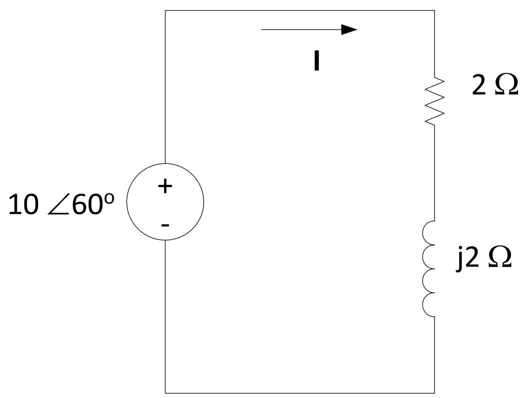
$$P = \frac{1}{2} V_m I_m$$

For a purely reactive circuit where $\theta_v - \theta_i = \pm 90°$

$$P = \frac{1}{2} V_m I_m \cos(90°) = 0$$

Purely reactive impedances absorb no average power. They are referred to as lossless elements. A purely reactive network stores energy over one part of the period and releases it over another.

Example: Determine the average power absorbed by the impedance in the circuit below.



$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}} = \frac{V_m \angle \theta_v}{2 + j2} = \frac{10\angle 60°}{2.83\angle 45°} = 3.53\angle 15°\ A$$

hence

$$I_m = 3.53\ A \ \ and\ \theta_v = 15°$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$P = \frac{1}{2}(10)(3.53)\cos(60° - 15°)$$

$$= 12.5\ W$$

We will recall that the inductor absorbs no power. We can then calculate the power absorbed by the resistor.

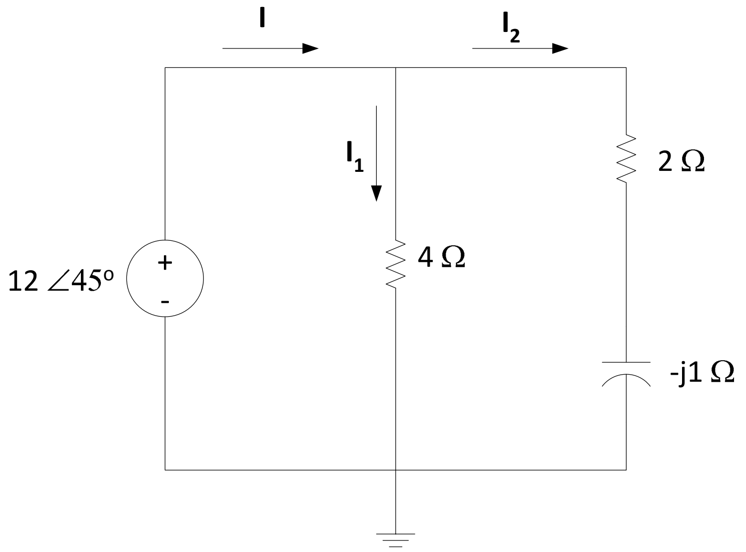
The voltage across the resistor is

$$\boldsymbol{V}_R = \frac{2}{2+j2} 10\angle 60° = 7.07\angle 15°$$

Power absorbed by the resistor is:

$$P = \frac{1}{2} V_m I_m = \frac{1}{2}(7.07)(3.53) = 12.5\ W$$

Example: Determine the total average power absorbed and the total power delivered in the circuit shown below.



$$\boldsymbol{I_1} = \frac{12\angle 45°}{4} = 3\angle 45°$$

$$\boldsymbol{I_2} = \frac{12\angle 45°}{2-j1} = \frac{12\angle 45°}{2.24\angle -26.57°} = 5.36\angle 71.57°$$

$$\boldsymbol{I} = \boldsymbol{I_1} + \boldsymbol{I_2}$$

$$\boldsymbol{I} = 3\angle 45° + 5.36\angle 71.57° = 8.15\angle 62.1°$$

The average power absorbed in the 4 Ω resistor is:

$$P_4 = \frac{1}{2} V_m I_m = \frac{1}{2}(12)(3) = 18\ W$$

The average power absorbed in the 2 Ω resistor is:

$$P_2 = \frac{1}{2} \boldsymbol{I}_m^2 R = \frac{1}{2}(5.36)^2(2) = 28.7\ W$$

The total power absorbed is:

$$P_A = 18 + 28.7 = 46.7\ W$$

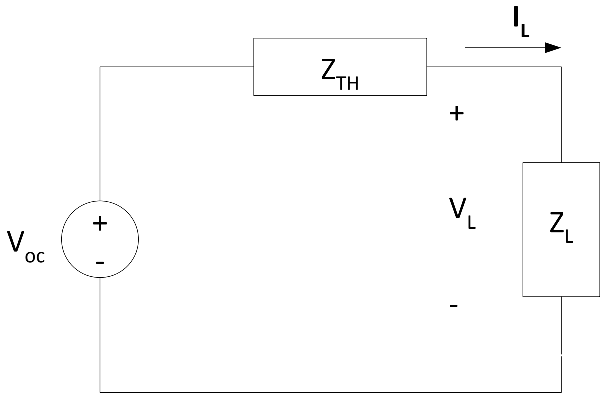
The total average power supplied by the source is:

$$P_S = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$P_S = \frac{1}{2}(12)(8.15)\cos(45° - 62.1°) = 46.7\ W$$

Maximum Average Power Transfer

Consider the circuit shown.



Average power at the load is:

$$P_L = \frac{1}{2} V_m I_m \cos\left(\theta_{v_L} - \theta_{i_L}\right)$$

Phasor current and voltage at the load is:

$$\boldsymbol{I}_L = \frac{\boldsymbol{V}_{oc}}{Z_{Th} + Z_L}$$

where

$$Z_{Th} = R_{TH} + jX_{TH}$$

and

$$Z_L = R_L + jX_L$$

Now

$$I_L = \frac{V_{OC}}{[(R_{TH} + R_L)^2 + (X_{TH} + X_L)^2]^{\frac{1}{2}}}$$

$$V_L = \frac{V_{OC}(R_L^2 + X_L^2)^{\frac{1}{2}}}{[(R_{TH} + R_L)^2 + (X_{TH} + X_L)^2]^{\frac{1}{2}}}$$

$$\theta_{v_L} - \theta_{i_L} = \theta_{z_L}$$

$$\cos\theta_{z_L} = \frac{R_L}{(R_L^2 + X_L^2)^{\frac{1}{2}}}$$

$$P_L = \frac{1}{2}\frac{V_{OC}^2 R_L}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}$$

By examination the following is observed:

Voc is a constant.

The quantity $(X_{TH} + X_L)$ absorbs no power. Any non-zero value of this quantity will reduce P_L . To minimize this quantity $X_{TH} = -X_L$.

We are left with:

$$P_L = \frac{1}{2}\frac{V_{OC}^2 R_L}{(R_{TH}+R_L)^2}$$

Earlier analysis of a similar expression showed that the quantity is maximised when:

$R_{TH} = R_L$

Therefore for maximum average power transfer to the load,

$$Z_L = R_L + jX_L = R_{TH} - jX_{TH} = Z_{TH}^*$$

If the load impedance is purely resistive, that is $X_L = 0$, then maximum average power transfer occurs when $\frac{dP_L}{dR_L} = 0$.

The value of R_L that maximises P_L when $X_L = 0$ is

$$R_L = \sqrt{R_{TH}^2 + X_{TH}^2}$$

Effective or rms values

The average power absorbed by a resistive load is a function of the type of source that is delivering power to the load. If the source is dc the average power absorbed is I^2R. If the source is sinusoidal, the average power absorbed is $\frac{1}{2}I_m R$. These are by no means the only waveforms available.

The effectiveness of a source, of whatever periodic waveform, in delivering power to a resistive load is what we seek to establish. The concept of the effective value of a periodic waveform is defined as that constant (dc) value of the periodic waveform that would deliver the same average power.

If that constant current is I_{eff}, then the average power delivered to a resistor is:

$$P = I_{eff}^2 R$$

The average power delivered to a resistor by a periodic current $i(t)$ is:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)Rdt$$

Equating both expressions, we obtain:

$$I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt}$$

This is called the root mean square value, I_{rms}.

Compute the rms value of $i(t) = I_m \cos(\omega t - \theta)$ with a period of$T = {}^{2\pi}/_{\omega}$.

$$I_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)Rdt} = \left[\frac{1}{T}\int_{t_0}^{t_0+T} I_m^2 \cos^2(\omega t - \theta)\, dt\right]^{\frac{1}{2}}$$

Using $\cos^2 \emptyset = \frac{1}{2} + \frac{1}{2}\cos 2\emptyset$

$$I_{rms} = I_m \left\{\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega t - 2\theta)\right] dt\right\}^{\frac{1}{2}}$$

The average or mean value of a cosine wave is zero, therefore

$$I_{rms} = I_m \left(\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2} dt\right)^{\frac{1}{2}} = I_m \left[\frac{\omega}{2\pi}\left(\frac{t}{2}\right)\Big|_0^{2\pi/\omega}\right]^{\frac{1}{2}}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

A sinusoidal current with a maximum value I_m delivers the same average power to a resistor R as a dc current with a value of $\frac{I_m}{\sqrt{2}}$.

Recall average power

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$

Or

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

Power absorbed by a resistor R is

$$P = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

Power Factor

The phase angle of the load impedance plays a very important role in the absorption of power by a load impedance.

In steady-state average power delivered to a load is

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

The average power is dependent on the cosine term.

The product $V_{rms} I_{rms}$ is referred to as the ***apparent power*** with units volt-amperes (VA).

The power factor (pf) is defined as:

$$pf = \frac{P}{V_{rms} I_{rms}} = \cos(\theta_v - \theta_i) = \cos\theta_{Z_L}\ (power\ factor\ angle)$$

For a purely resistive load where $\theta_{Z_L} = 0$, the *pf = 1*.

For a purely reactive load where $\theta_{Z_L} = \pm 90°$, the *pf = 0*

If the current leads the voltage as it does in an RC circuit load, the *pf* is said to be *leading*.

If the current lags the voltage as it does in an RL circuit load, the *pf* is said to be *lagging*.

Example:

For $Z_L = 1 - j1\ \Omega,\ \theta_{Z_L} = \tan^{-1}\left(\frac{-1}{1}\right) = -45°,$ the $pf = \cos\theta_{Z_L} = \cos(-45)° = 0.707$ which is a *leading pf*.

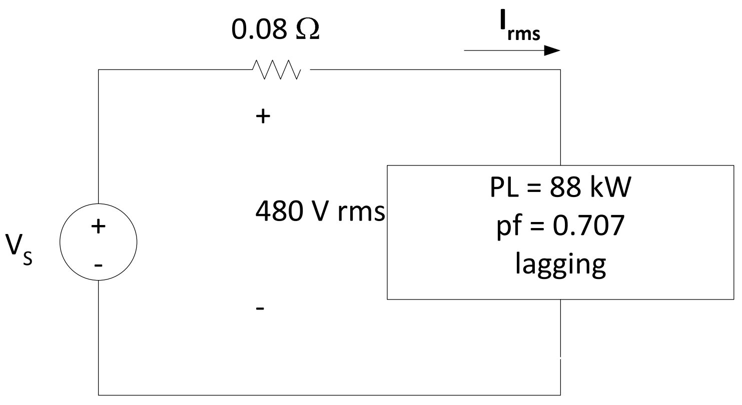
For $Z_L = 2 + j1\ \Omega,\ \theta_{Z_L} = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°,$ the $pf = \cos\theta_{Z_L} = \cos(26.57)° = 0.894$ which is a *lagging pf*.

Example: An industrial load consumes 88 kW at a *pf* of 0.707 lagging from a 480-V rms line. The transmission line resistance from the power company's transformer to the plant is 0.08 Ω. Determine the power that must be supplied by the power company

a) Under present conditions, and
b) If the pf is changed to 0.90 lagging

Solution

a) Consider the circuit shown



$$pf = \frac{P}{V_{rms} I_{rms}} = \cos(\theta_v - \theta_i) = \cos\theta_{Z_L}$$

$$I_{rms} = \frac{P_L}{(pf)(V_{rms})} = \frac{88 \times 10^3}{(0.707)(480)} = 259.3\ A$$

The power company must supply

$$P_S = P_L + (0.08)I_{rms}^2$$

$$P_S = 88{,}000 + (0.08)(259.3)^2 = 93.38\ kW$$

b) For a pf = 0.90 lagging,

$$I_{rms} = \frac{P_L}{(pf)(V_{rms})} = \frac{88 \times 10^3}{(0.9)(480)} = 203.7\ A\ rms$$

The power company must now supply

$$P_S = P_L + (0.08)I_{rms}^2 = 93.38\ kW$$

$$P_S = 88{,}000 + (0.08)(203.7)^2 = 91.32\ kW$$

Note the difference in power that must be supplied

For a pf = 0.707, line loss is 5.38 kW

For a pf = 0.90, line loss is 3.32 kW

Complex Power

Complex power, **S**, is defined as:

$$\boldsymbol{S} = \boldsymbol{V_{rms}}\boldsymbol{I_{rms}^*} = V_{rms}\angle\theta_v I_{rms}\angle{-\theta_i} = V_{rms}I_{rms}\angle\theta_v - \theta_i$$

$$\boldsymbol{S} = V_{rms}I_{rms}\cos(\theta_v - \theta_i) + jV_{rms}I_{rms}\sin(\theta_v - \theta_i)$$

The real part of the complex power is the real or average power.

The imaginary part of the complex power is called the reactive or quadrature power.

$$\boldsymbol{S} = P + jQ$$

Where $P = Re(\boldsymbol{S}) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$

And $Q = Im(\boldsymbol{S}) = V_{rms}I_{rms}\sin(\theta_v - \theta_i)$

The magnitude of **S** is called the *apparent power*.

Complex power is measured in volt-amperes.

For a resistor $\theta_v - \theta_i = 0°$, $\cos(\theta_v - \theta_i) = 1$; $\sin(\theta_v - \theta_i) = 0$. The resistor absorbs real power but does not absorb any reactive power.

For an inductor, $\theta_v - \theta_i = 90°$,

$$P = Re(\boldsymbol{S}) = V_{rms}I_{rms}\cos(90°) = 0$$

$$Q = Im(\boldsymbol{S}) = V_{rms}I_{rms}\sin(90°) > 0$$

An inductor absorbs reactive power but does not absorb real power.

For a capacitor, $\theta_v - \theta_i = -90°$,

$$P = Re(\boldsymbol{S}) = V_{rms}I_{rms}\cos(-90°) = 0$$

$$Q = Im(\boldsymbol{S}) = V_{rms}I_{rms}\sin(-90°) < 0$$

A capacitor does not absorb any real power, but absorbs (negative) reactive power. Simply means the capacitor is supplying reactive power. Capacitors for this reason are used in pf correction.

Recall

$$\boldsymbol{S} = \boldsymbol{V_{rms}}\boldsymbol{I_{rms}^*}$$

Now

$$\boldsymbol{V_{rms}} = \boldsymbol{I_{rms}}\boldsymbol{Z}$$

Hence

$$\boldsymbol{S} = \boldsymbol{I_{rms}}\boldsymbol{Z}\boldsymbol{I_{rms}^*} = \boldsymbol{I_{rms}}\boldsymbol{I_{rms}^*}\boldsymbol{Z}$$

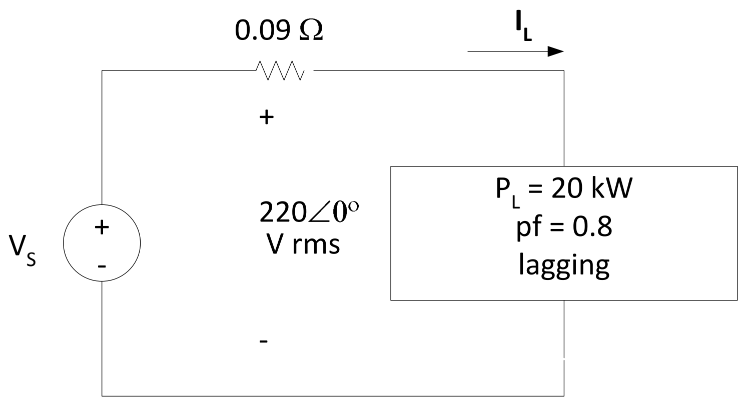
Since

$$\boldsymbol{I_{rms}}\boldsymbol{I^*_{rms}} = I_{rms}\angle\theta_i I_{rms}\angle{-\theta_i} = I^2_{rms}$$

Then

$$\boldsymbol{S} = \boldsymbol{I_{rms}}\boldsymbol{I^*_{rms}}\boldsymbol{Z} = I^2_{rms}\boldsymbol{Z} = I^2_{rms}(R + jX) = P + jQ$$

Example: A load operates at 20 kW, 0,8 pf lagging. The load voltage is $220\angle 0°\ V\ rms\ at\ 60\ Hz$. the impedance of the line is 0.09 + j0.3 Ω. Determine the voltage and power factor at the input to the line.



We know

$$P = S\cos\theta\ or\ S = \frac{P}{\cos\theta}$$

$$S = \frac{P}{\cos\theta} = \frac{P}{pf} = \frac{20{,}000}{0.8} = 25{,}000\ VA$$

$$\boldsymbol{S_L} = 25{,}000\angle\theta = 25{,}000\angle 36.87° = 20{,}000 + j15{,}000\ VA$$

At the load

$$\boldsymbol{S_L} = \boldsymbol{V_L}\boldsymbol{I^*_L}$$

Therefore

$$\boldsymbol{I^*_L} = \frac{25{,}000\angle 36.87°}{220\angle 0°} = 113.64\angle 36.87°\ A\ rms$$

And

$$\boldsymbol{I_L} = 113.64\angle - 36.87°\ A\ rms$$

The complex losses in the line are:

$$\boldsymbol{S}_{line} = I^2_L\boldsymbol{Z}_{line}$$

$$\boldsymbol{S}_{line} = (113.64)^2(0.09 + j0.3) = 1162 + j3874.21\ VA$$

The total power absorbed must be equal to the power supplied.

Hence

$$\boldsymbol{S}_S = \boldsymbol{S}_L + \boldsymbol{S}_{line}$$

$$\boldsymbol{S}_S = (20{,}000 + j15{,}000) + (1162.26 + j3874.21) = 21{,}162.26 + j18{,}874.21$$

$$\boldsymbol{S}_S = 28{,}356.25\angle 41.73°\ VA$$

The generator voltage will then be

$$V_S = \frac{|\boldsymbol{S}_s|}{I_L} = \frac{28{,}356.25}{113.64} = 249.53\ V\ rms$$

The generator pf is:

$$\cos(41.73°) = 0.75\ lagging$$

Another way to solve by KVL

Having calculated

$$\boldsymbol{I}_L = 113.64\angle -36.87°\ A\ rms$$

The voltage drop on the transmission line is

$$\boldsymbol{V}_{line} = \boldsymbol{I}_L\boldsymbol{Z}_{line} = (113.64\angle -36.87°\ )(0.09 + j0.3)$$

$$\boldsymbol{V}_S = 35.59\angle 36.43°\ V\ rms$$

The generator voltage is then

$$\boldsymbol{V}_S = 220\angle 0° + 35.59\angle 36.43° = 249.53\angle 4.86°\ V\ rms$$

Now

$$\theta_v - \theta_i = 4.86° - (-36.87°) = 41.73°$$

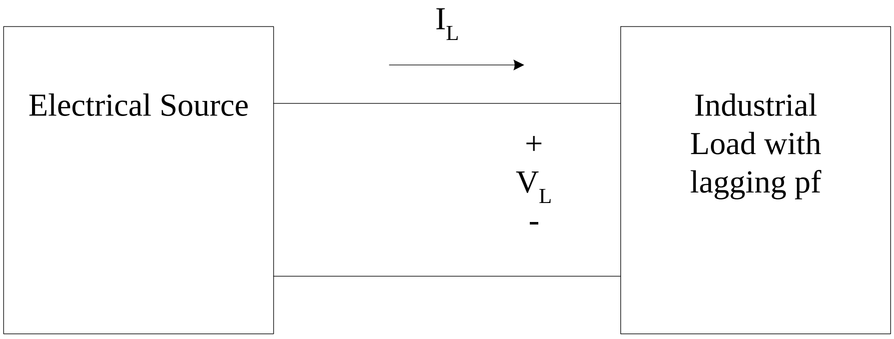
Hence

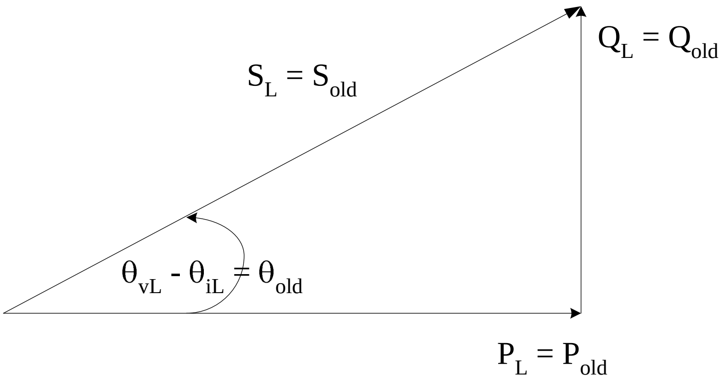
$$pf = \cos(41.73°) = 0.75\ lagging$$

Power Factor Correction

The need to have a high pf is now known. Less losses, smaller conductors etc. the nature of most loads is pf lagging.

We need to find a way to increase the pf of a load economically. How to decrease the pf angle is our objective (increase the pf).



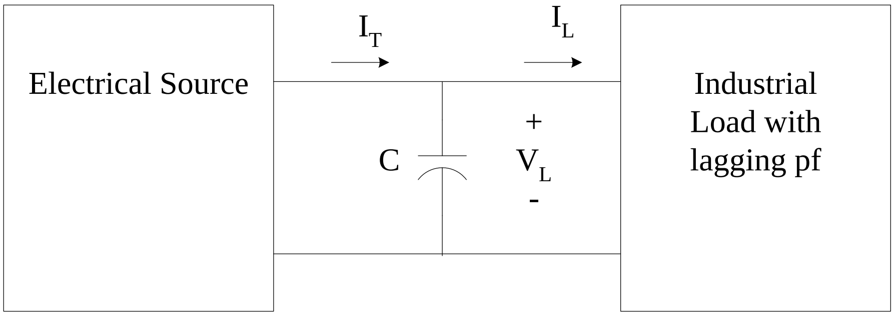


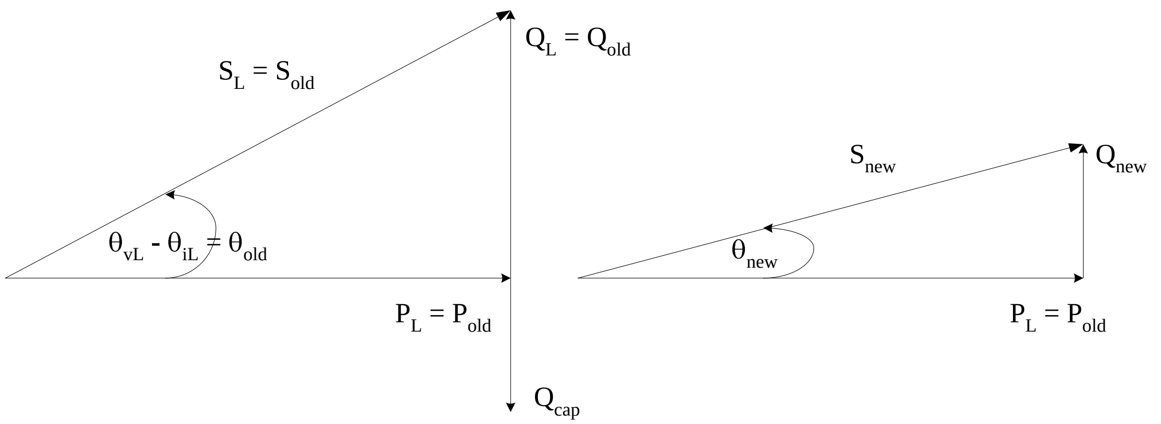
Since

$$\tan\theta = \frac{Q}{P}$$

To decrease θ, P could be increased. This is not practical or economically feasible as power consumption would increase and the cost of electricity would increase.

Another option is to decrease Q by connecting a capacitor across the load





$$\boldsymbol{S}_{old} = P_{old} + jQ_{old} = |\boldsymbol{S_{old}}|\angle\theta_{old}$$

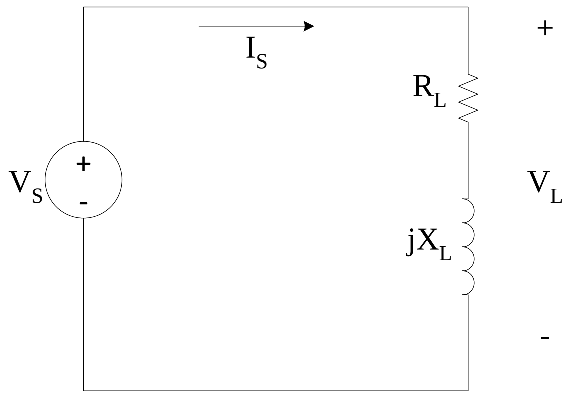
And

$$\boldsymbol{S}_{new} = P_{old} + jQ_{new} = |\boldsymbol{S}_{new}|\angle\theta_{new}$$

$$\boldsymbol{S}_{new} - \boldsymbol{S}_{old} = \boldsymbol{S}_{cap} = (P_{old} + jQ_{new}) - (P_{old} + jQ_{old}) = j(Q_{new} - Q_{old}) = jQ_{cap}$$

Example:

Calculate the complex power for the circuit shown and correct the pf to unity by connecting a parallel reactance given the following: $V_S = 117\angle 0$; $R_L = 50\ \Omega$; $\mathrm{jX_L} = 86.7\ \Omega$; $\omega = 377\ \mathrm{rad/s}$ (all rms values)



We know $\boldsymbol{S} = P + jQ$ for the complex load

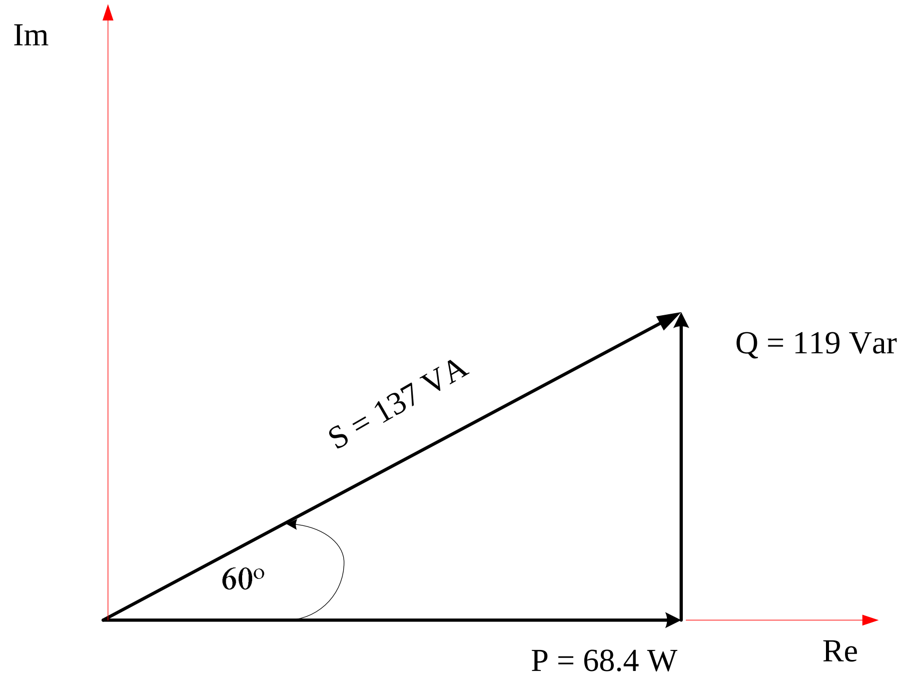
$$Z_L = R + jX_L = 50 + J86.7 = 100\angle(1.047)\ \Omega$$

$$\boldsymbol{I_L} = \frac{V_L}{Z_L} = \frac{117\angle 0}{100\angle(1.047)} = 1.17\angle - 1.047\ \mathrm{A}$$

$$\boldsymbol{S} = \boldsymbol{V}_L \boldsymbol{I}_L^* = 117\angle 0 \times 1.17\angle 1.047 = 137\angle 1.047 = 68.4 + J118.6\, VA$$

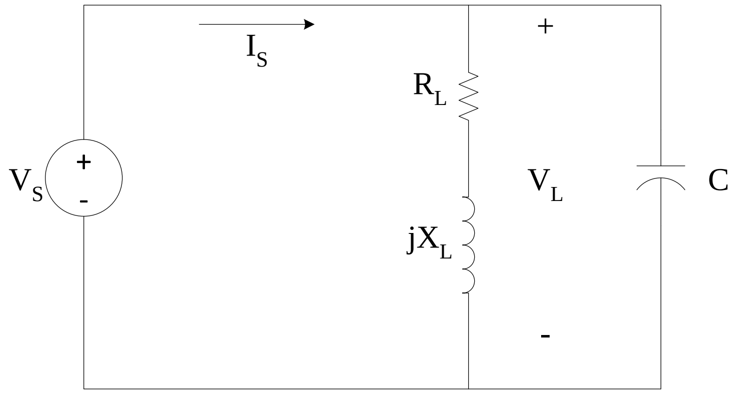
Hence

$$P = 68.4\, W \quad and \quad Q = 118.5\, Vars$$



To eliminate the reactive power due to the inductance, add an equal and opposite reactive component – Q_L.

Therefore choose C that Q_C = -118.5 Vars.



The reactance

$$X_C = \frac{|\boldsymbol{V}_L|^2}{Q_C} = \frac{(117)^2}{-118.5} = 115\ \Omega$$

Since

$$C = -\frac{1}{\omega X_C} = -\ \frac{1}{377(-115)} = 23.1\ \mu F$$