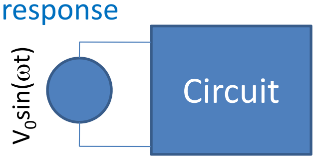
AC CIRCUIT ANALYSIS

Our analysis so far has been for direct current (DC) circuits, where voltages and currents are assumed to be constant with time

To look at the effect of capacitors and inductors, we will have to perform alternating current (AC) analysis

With AC circuit analysis, we assume that a sinusoidal signal with a radian frequency of ω is applied, and we solve for the circuit's response



Notes: 2π radians = $360°$

ω has units of radians/sec

So, ωt has units of radians

Also note: $\omega = 2\pi f$ where f is frequency in Hz

Hertz (Hz) is in cycles per second, and each cycle is $360°$ or 2π radians

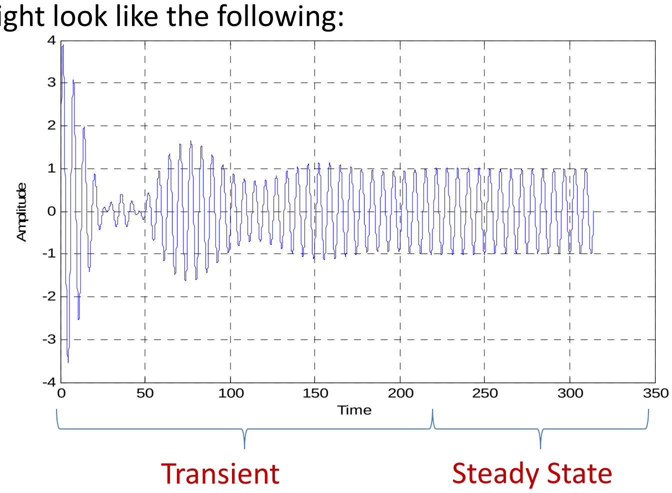
We can perform our calculations using radians or degrees as long as we are careful

So, $\sin(\omega t + 45°)$ and $\sin\left(\omega t + \pi/4\right)$ are equivalent

SINUSOIDAL STEADY STATE

When a signal or power are applied to a circuit, oftentimes transients occur. These transients typically die out quickly, and after they have died out, it is said that the circuit is in steady state.

For example, if we applied $V_0 \sin(\omega t)$ to a circuit at $t=0$, the current might look like the following:



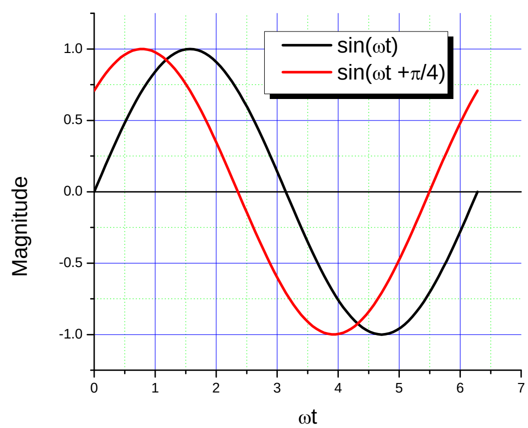
SINUSOIDAL STEADY STATE ANALYSIS

- When we analyze a circuit, we assume that it is in steady state
- If we know how a circuit responds as a function of frequency, we can determine its response to any waveform using Fourier Analysis
- If a sinusoidal signal is fed to a linear circuit, all voltages and currents in that circuit will be sinusoids that are scaled in magnitude and shifted in phase

PHASE

If a sinusoid is shifted in time, there will be a corresponding shift in phase

Example: what would be the phase shift if a 10 MHz sinusoid were delayed by 12.5 nsec (nanoseconds or 10^{-9} sec)?



Original sinusoid, $\sin(\omega t) = \sin\left(2\pi 10^7 t\right)$

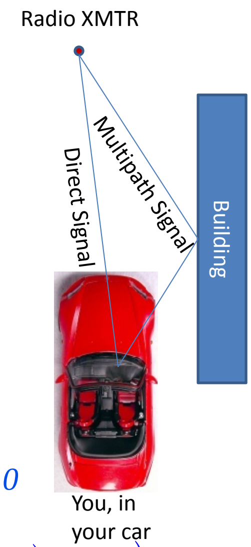
Delayed sinusoid, $\sin\left(\omega\left(t + t_d\right)\right)$

$$= \sin\left(2\pi 10^7 \left(t + 12.5\times 10^{-9}\right)\right) = \sin\left(2\pi 10^7 t + 0.25\pi\right) = \sin\left(2\pi 10^7 t + \pi/4\right)$$

Phase measurements are often used as a means to measure distance

EXAMPLE: EFFECT OF PHASE SHIFT

MULTIPATH INTERFERENCE



The signal delivered by your antenna to your radio is the sum of the direct signal and any multipath signals

The multipath signals can often be as strong as the direct signal

Numerical Example: If you are listening to WHEB (100.3 MHz) and there is a strong multipath signal (as strong as the direct signal), how will that multipath affect reception if the multipath signal travels a path that is *1.5* meters longer than the direct signal?

The multipath signal will arrive *1.5 m/3x10*^{8} m/sec = *5.0* nsec after the direct signal.

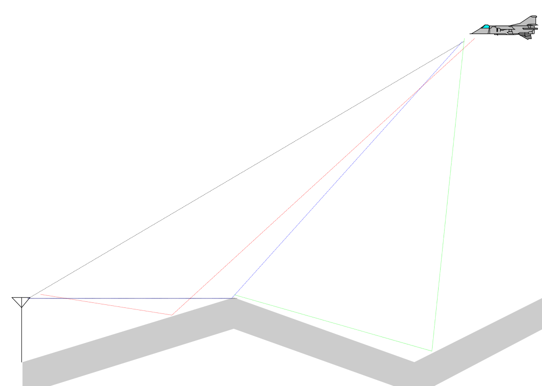
$$V_{antenna} = V_0 \sin\left(2\pi\left(100.3\times10^6\right)t\right) + V_0 \sin\left(2\pi\left(100.3\times10^6\right)\left(t+t_d\right)\right)$$

The phase shift caused by the delay is: $\Delta\theta = 2\pi\left(100.3\times10^6\right)\left(t_d\right) \approx \pi$

For any : $\sin\left(\omega t\right) + \sin\left(\omega t + \pi\right) = 0 \Rightarrow$ No signal received

EXAMPLE: EFFECT OF TERRAIN MULTIPATH ON LANDING SYSTEM PERFORMANCE

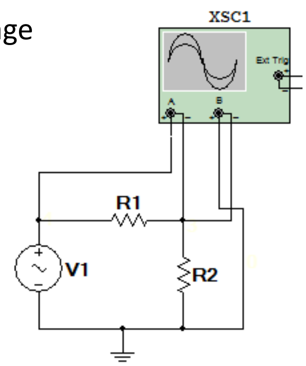
The received signal will be the sum of 4 sine waves, each with a different amplitude and phase



PHASE DIFFERENCE BETWEEN VOLTAGE & CURRENT

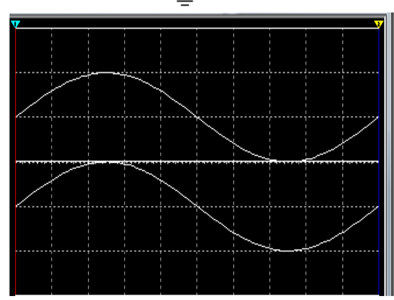
Measuring the phase difference between voltage and current

Example: Setup to measure the phase of the voltage across and the current through a resistor



Note: because of its high input impedance, the oscilloscope will not affect the operation of the circuit

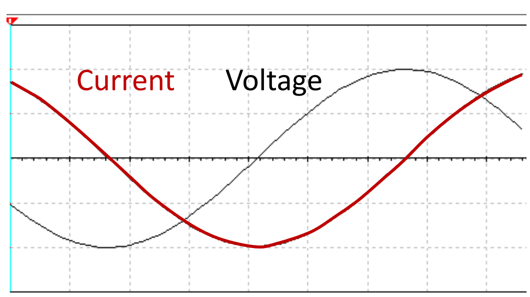
The same current, i, will flow through both resistors

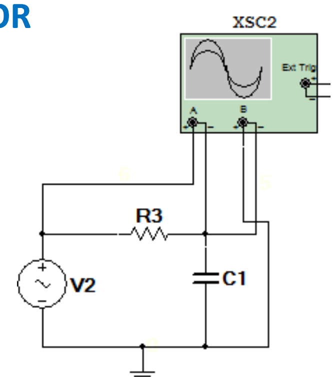


The voltage across R1 (Channel A) is proportional to i. The voltage across R2 (Channel B) is the voltage across R2

For resistors, there is no phase shift between voltage and current

PHASE DIFFERENCE BETWEEN VOLTAGE & CURRENT ON A CAPACITOR





Note: the measurement setup causes the current waveform to be inverted

In this case, the current leads the voltage by $\pi/2$ or $90°$

In words: when the voltage is changing fastest, the current is at its maximum- when the voltage is not changing, the current is at zero

In math:

If $V(t) = \cos(\omega t)$, then $i = C\dfrac{dV}{dt} = C\dfrac{d\cos(\omega t)}{dt} = -\omega C\sin(\omega t)$

IMPEDANCE

Resistance is the ratio of voltage to current: $R=v/i$

If a circuit is said to be purely resistive, it is assumed that voltage and current are in phase

Impedance, Z, is also the ratio of voltage to current, but it accounts for the phase relationship between voltage and current

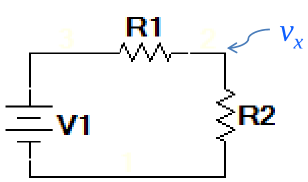
Example: calculate the impedance of a capacitor. Start by assuming a sinusoidal voltage across the capacitor => $v_c=V_0 cos(\omega t)$

For this applied voltage $i = C\dfrac{dv_c}{dt} = C\dfrac{dV_0\cos(\omega t)}{dt} = -\omega C V_0 \sin(\omega t)$

So, $Z = \dfrac{v}{i} = \dfrac{V_0\cos(\omega t)}{-\omega C V_0 \sin(\omega t)} = \dfrac{\cos(\omega t)}{-\omega C\cos(\omega t + 90°)} = \dfrac{1}{\omega C}\angle -90°$

The $1/(\omega C)$ gives the ratio of the magnitudes of v to i, and the $-90°$ gives the phase relationship between v and i

ACCOUNTING FOR PHASE SHIFT USING CIRCUIT EQUATIONS



Before going to a circuit equation that involves phase, let's go over a DC circuit analysis example we worked before that does not involve phase

Apply KCL at the node containing v_x:

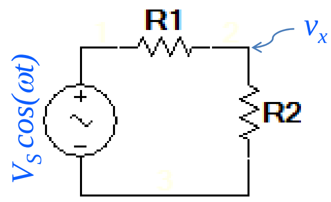
$$\sum_{j=1}^{N} i_j = 0 \Rightarrow \frac{v_x - V_1}{R_1} + \frac{v_x}{R_2} = 0 \Rightarrow \frac{R_2\left(v_x - V_1\right) + R_1 v_x}{R_1 R_2} = 0$$

Multiply both sides by $R_1 R_2$: $R_2\left(v_x - V_1\right) + R_1 v_x = 0$

Rearranging: $v_x\left(R_2 + R_1\right) = V_1 R_2$

If we call the gain of this circuit $\left(\dfrac{v_x}{V_1}\right)$, then Gain = $\dfrac{R_2}{R_1 + R_2}$

ACCOUNTING FOR PHASE SHIFT USING CIRCUIT EQUATIONS (2)



Now, look at a circuit with an AC source:

Recall that for a linear circuit, if the input is a sinusoid, all currents and voltages in the circuit will be sinusoidal at the same frequency

Again, apply KCL at the node containing v_x:

$$\frac{v_x \cos(\omega t+\theta)-V_S \cos(\omega t)}{R_1}+\frac{v_x \cos(\omega t+\theta)}{R_2}=0$$

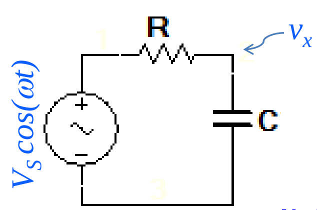
The magnitude of the unknown voltage is v_x, and its phase is θ

Rearranging: $(R_2+R_1)v_x \cos(\omega t+\theta)=R_2 V_S \cos(\omega t)$

The only way for this equality to hold is if $\theta=0$. This is the expected result, since there is nothing in this circuit to shift the phase of the current with respect to the voltage

$$v_x \cos(\omega t)=\frac{R_2 V_S \cos(\omega t)}{(R_2+R_1)} \Rightarrow v_x=\frac{R_2 V_S}{(R_2+R_1)}$$

ACCOUNTING FOR PHASE SHIFT USING CIRCUIT EQUATIONS (3)



The capacitor in this circuit will introduce phase shift

Again, apply KCL at the node containing v_x:

$$\frac{v_x \cos(\omega t+\theta)-V_S \cos(\omega t)}{R}+\frac{v_x \cos(\omega t+\theta-90°)}{\frac{1}{\omega C}}=0$$

We are summing currents here, and the $-90°$ accounts for the phase shift between voltage and current in the capacitor

Rearranging our equation above:

$$\frac{1}{\omega C}\left(v_x \cos(\omega t+\theta)-V_S \cos(\omega t)\right)+R v_x \cos(\omega t+\theta-90°)=0$$

We can solve problems like this one using trig identities, and we will use a shorthand notation to make this easier

ADDING SINUSOIDS OF DIFFERENT MAGNITUDES & PHASES (EE'S DO THIS A LOT)

Start by applying a trig identity to $cos(\omega t+\theta)$:

$$\cos(\omega t+\theta)=\cos(\omega t)\cos(\theta)-\sin(\omega t)\sin(\theta)$$

This means that we can express $cos(\omega t+\theta)$ in terms of $sin(\omega t)$ and $cos(\omega t)$ only: $cos(\theta)$ and $sin(\theta)$ are constants

Example

$$\cos(\omega t+60°)=\cos(\omega t)\cos(60°)-\sin(\omega t)\sin(60°)$$

$$=0.5\cos(\omega t)-0.866\sin(\omega t)$$

Significance: sin & cos are orthogonal functions (separated by $90°$ in phase) => the coefficients of the cos terms and sin terms must be the same on either side of an equality

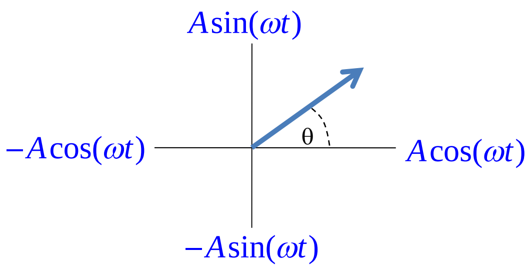
Example:

$$A\cos(\omega t)+B\sin(\omega t)=15\cos(\omega t)-8\sin(\omega t)+2\cos(\omega t)$$

$\Rightarrow A=17$ and $B=-8$

ADDING SINUSOIDS OF DIFFERENT MAGNITUDES & PHASES (2)

A graphical way to think about $A\cos(\omega t+\theta)$



We can use this to find the magnitude and phase of a variable that is expressed in terms of $sin(\omega t)$ and $cos(\omega t)$

Example: find A and θ given that $A\ cos(\omega t+\theta)= 4sin(\omega t)+3cos(\omega t)$

Solution: $A=\sqrt{4^2+3^2}=5 \quad \theta=\tan^{-1}\left(4/3\right)=53.1°$

SHORTHAND NOTATION FOR ADDING SINUSOIDS OF DIFFERENT MAGNITUDES & PHASES

Since sinusoids have a magnitude and phase, we can add them like vectors, which have a magnitude and direction

As with the plot on the previous slide, we will put the $cos(\omega t)$ component on the x-axis and the $sin(\omega t)$ component on the y-axis

Notation: in steady-state analysis, we assume that all currents and voltages are varying sinusoidally, so we do not have to keep writing the $sin(\omega t)$ and $cos(\omega t)$ terms (also, they will cancel algebraically since they will appear on both sides of circuit equations)

To make our notation more compact, we will place use a "$-j$" to represent $sin(\omega t)$ and a "1" to represent the $cos(\omega t)$ terms

Thus, the previous example can be written as:

$$\cos(\omega t + 60°) = \cos(\omega t)\cos(60°) - \sin(\omega t)\sin(60°)$$
$$= 0.5 + j0.866$$

SUMMING SINUSIODS EXAMPLES

If $A = 2\cos(\omega t + 30°)$, $B = -3\cos(\omega t - 50°)$ and $C = \cos(\omega t)$, calculate $A + B + C$

Rewriting in shorthand notation:

$$A = 2\cos(\omega t + 30°) = 2\cos(30°) + j2\sin(30°) \quad = 1.732 \;+ j1$$
$$B = -3\cos(\omega t - 50°) = -3\cos(-50°) - j3\sin(-50°) = -1.928 + j2.298$$
$$C = \cos(\omega t) = \cos(0°) + j\sin(0°) = 1 + j0$$

Summing the components separately, as in vector addition:

$$A + B + C = (1.732 - 1.928 + 1) + j(1 + 2.298 + 0) = 0.804 + j\,3.298$$

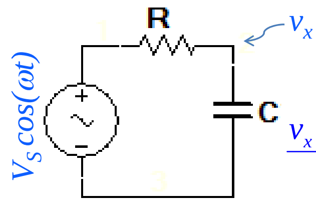
We can convert this into a magnitude and phase:

Magnitude $= \sqrt{0.804^2 + 3.298^2} = 3.4$ Phase $= \tan^{-1}(3.298 / 0.804) = 76.3°$

We can write this as: $A + B + C = 3.4\cos(\omega t + 76°)$

The above notation is called Complex Number Notation

RETURNING TO OUR CAPACITOR CIRCUIT



After applying KCL, we came up with:

$$\frac{v_x\cos(\omega t+\theta)-V_S\cos(\omega t)}{R}+\frac{v_x\cos(\omega t+\theta-90°)}{\frac{1}{\omega C}}=0$$

Using our new *notation*, we can write this as:

$$\frac{v_x-V_S}{R}+j\frac{v_x}{\frac{1}{\omega C}}=0 \Rightarrow v_x\left(1+j\omega RC\right)=V_S$$

If we define gain as *being* v_x/V_s

$$\text{Gain}=\frac{1}{1+j\omega RC}$$

Gives the expected gain of 1 at $\omega=0$

Note that this gain introduces a phase shift that is a function of frequency. For many circuits, gain will have a magnitude and phase.

NUMERICAL EXAMPLE

If $R=10K\Omega$ and $C=50\ pF$ ($pf = 10^{-12}$ Farads), what is the gain of our low-pass filter at $318\ KHz$?

$$\text{Gain}=\frac{1}{1+j\omega RC}=\frac{1}{1+j\left(2\pi\left(3.18\times10^5\right)\right)10^4\left(5\times10^{-11}\right)}=\frac{1}{1+j}$$

$$\frac{1}{1+j}=\frac{1}{\sqrt{1^2+1^2}\angle\tan^{-1}\left(1/1\right)}=\frac{1}{\sqrt{2}\angle45°}=\frac{1\angle-45°}{\sqrt{2}}$$

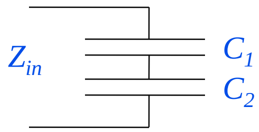
$$=0.707\angle-45°\ \text{(Polar Form)}$$

$$\text{Note}=\frac{1}{A\angle\theta}=\frac{1\angle-\theta}{A}\ \text{ just as }\ \frac{1}{10^A}=10^{-A}$$

The gain of our low-pass filter is 0.707 at $318\ KHz$, and it introduces a phase shift of $-45°$ between the input and output at that frequency.

IMPEDANCES IN SERIES

Just as resistances add when in series, impedances add when in series



$$Z_{in} = Z_{C_1} + Z_{C_2} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Example: Find the net capacitance for 2 capacitors in series as shown above.

$$Z_{in} = \frac{1}{j\omega C_{in}} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Multiply both sides by $j\omega$

$$\frac{1}{C_{in}} = \frac{1}{C_1} + \frac{1}{C_2}$$

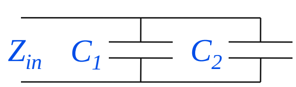
Rearrange to find C_{in}

$$C_{in} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Unlike resistors, capacitance does not add when in series although the impedance of capacitors does add

IMPEDANCES IN PARALLEL

Impedances act in parallel in the same way as resistance acts in parallel.



$$Z_{in} = \frac{V_{in}}{i_{in}} = \frac{V_{in}}{i_{C_1} + i_{C_2}} = \frac{V_{in}}{\frac{V_{in}}{Z_{C_1}} + \frac{V_{in}}{Z_{C_2}}}$$

$$\frac{1}{\frac{1}{Z_{C_1}} + \frac{1}{Z_{C_2}}} = \frac{1}{\frac{1}{\left(\frac{1}{j\omega C_1}\right)} + \frac{1}{\left(\frac{1}{j\omega C_2}\right)}} = \frac{1}{j\omega\left(C_1 + C_2\right)}$$

Capacitance adds in parallel: $C_{in} = C_1 + C_2$

Note: $\frac{1}{\left(\frac{1}{A}\right)} = A$