Will Moore
MT 11

CIRCUIT ANALYSIS I

(DC Circuits)

Electrical and electronic devices are a feature of almost every aspect of our daily lives. Indeed most people carry around electronic circuits of one form or another – maybe in a watch, calculator, mobile phone, laptop *etc.* – all day long. It seems reasonable, therefore, since we are all so dependent on circuits that we spend a little time learning how they work and how to design them to do even more useful things.

At its simplest, an electrical circuit is merely a collection of components connected together in a particular way to produce a desired effect. Since this is the first course on electrical circuits we will concentrate on developing methods of circuit *analysis* that will enable us to calculate the voltages and currents in given circuits. This approach will provide us with a firm understanding of how the various circuit elements – resistors, capacitors and inductors – behave under a variety of conditions. Once we have developed confidence in analysing *given* circuits and understanding how they work we can proceed to the fun stage – designing our own circuits. Whether you are one of those people who already enjoys that kind of thing or whether you are one of those who cannot tell one end of a soldering iron from the other, you will all design and build a working transistor radio by the end of your first year.

Although the intention is that these notes should be reasonably self contained it would still be sensible to consult some of the vast number of books on this subject. A few possible titles might include:

Hughes E. *Electrical and Electronic Technology*, Pearson

A comprehensive text that covers practically the whole of the P2 course.

Smith R.J. & Dorf R.C. *Circuits, Devices and Systems*, Wiley

An alternative text that covers practically the whole of the P2 course.

Floyd T.L. & Buchla D. *Electronics Fundamentals: Circuits, Devices and Applications* , Pearson

Lots of illustrations, worked examples an*d* practice questions.

Nahvi M. & Edminster J. *Electric Circuits*, McGraw-Hill

Simple overview with lots of practice questions.

Howatson A.M. *Electric Circuits and Systems*, OUP

Written by a previous member of the Department and in a similar style to the way we still teach the subject, now out of print but available in most college libraries.

Bobrow L.S. *Elementary Linear Circuit Analysis*, OUP

A standard text, out of print but available in most college libraries.

... and last but not least:

HLT for data and also to see what information will be available to you in the examination!

This list is far from exhaustive and it may be that none of the above texts suit you – if so, please read around the subject and find the explanation/description that is the best for you. Go to the Library!

Syllabus

Charge conservation. Kirchhoff's laws, and mesh/nodal analysis. Concepts of ideal voltage and current sources, and impedances. Thévenin and Norton theorems with emphasis on concepts of input and output impedances.

Learning Outcomes

At the end of this course students should:

1. Appreciate the origins of current and conductivity
2. Be familiar with Ohm's law and its wider significance
3. Become familiar with linear components and power dissipation
4. Develop basic skills in circuit analysis and its relationship with Ohm's law
5. Appreciate the significance and utility of Kirchhoff's laws
6. Become confident in applying them to simple circuit analysis
7. Acquire higher-level skills in circuit analysis
8. Appreciate the importance of input and output impedance

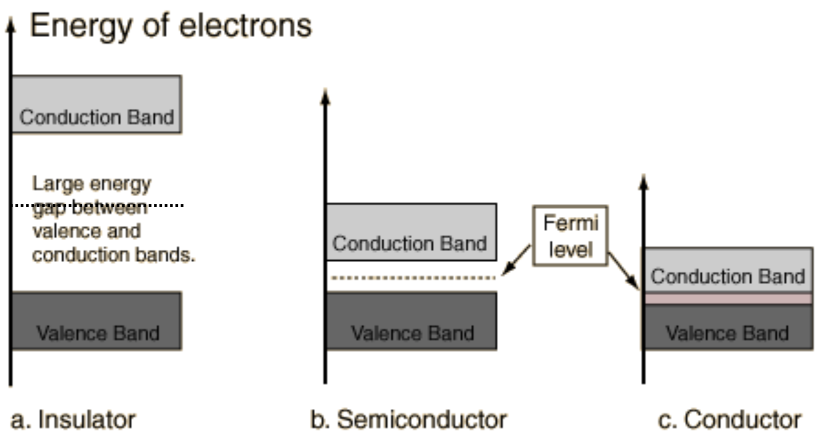
DC Circuits

1. Basic ideas

Circuit analysis is all about analysing the **currents** and the **voltages** in an electrical circuit. In this Circuit Analysis I course we will limit ourselves to **DC Circuits**. DC short for "Direct Current" but this is jargon for saying that all the currents and voltages are constant. For the purposes of this course (and indeed Circuit Analysis II), an electric circuit consists of **components** connected by **wires**. We will look in particular at three components, **resistors**, **voltage sources** and **current sources**. Each component can be characterised in terms of the relationship between the **current through it** and the **voltage across it**. We will assume the wires pass current but do not drop any voltage.

2. Conductors and Insulators

Let's start at the beginning. This is NOT a course about Physics or Chemistry and we will not dwell on them, but a little knowledge about such things can sometimes make sense of the things we see. The matter around us consists of atoms and the simplest model of atoms is to suppose that they each have a nucleus of protons and neutrons surrounded by a swirling cloud of electrons. Our model is further refined by supposing that the protons are positively charged and the electrons are negatively charged and that the there is a force of attraction between the two types of charge, which keeps the atom together. The light electrons whizz around the heavy protons like satellites around the earth under the effect of gravity. That would be fine, but physicists have also dreamt up an idea call **quantum theory** to explain that electrons can only whizz around the nucleus in particular orbits. The further the orbit from the nucleus, the higher is the kinetic energy of the electron. The behaviour of the atom is largely dictated by the electrons occupying the outermost orbits – those in the **valence band** are responsible for binding the atoms together and those in the **conduction band** are relatively free to hop from one atom to another. Electrical engineers divide the world into three types of material according to the three situations that can arise.



The Fermi level is the top energy level that would be occupied at absolute zero. At higher temperatures, the electrons are excited to higher energy levels.

In a **conductor**, the two bands merge into each other and at room temperature lots of electrons move up into the conduction band. That means that electrons can move around the material rather easily, we can for example use them to make the wires that we need to connect our circuits together.

In an **insulator**, there is a large energy gap between the valence band and the conduction band so that at normal temperatures, the valence band is full and the conduction band is empty. That means there is no possibility to move electrons around the material. That makes them rather useful for insulating our wires to stop them from connecting inadvertently. They can also have interesting dielectric properties that modify the forces acting when charges on either side.

In a **semiconductor** there is a small gap between the two bands and at room temperature only a modest number of electrons are excited into the conduction band. The spaces created in the conduction band mean that these electrons are also freed up and some really interesting behaviours arise which you will learn about later in the year.

3. Charge and Current

In circuit analysis we are very interested in the **charge** of the electrons and protons, **particularly when the charge moves**. As the protons are inextricably bound to the atoms, we usually only need consider the movement of electrons. This movement of charge as called **the electric current**. The current in a circuit is a measure of the rate at which charge, q, passes through the circuit. The instantaneous value of current, i, is given by

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

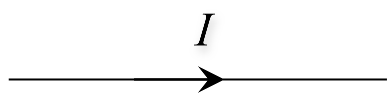
For the special case when the current is **constant**, $i = q/t$. We call this a "direct current" or **DC.**

We use the unit of charge called the **coulomb** with the symbol **C**. A current of one **coulomb per second** is called an **amp** with the symbol **A**.

It also follows that

$$q = \int i \,.\, \mathrm{d}t$$

Current is most commonly caused by the flow of negative electrons in a conductor, although other examples exist, *e.g.* positive ions in an electrolyte, or negative ions in a plasma. However, by an unfortunate accident of history, the convention of the direction of current is in the opposite sense to the flow of electrons.



Here is the depiction of a wire carrying a current *I* from left to right (By implication, the electrons will be flowing from right to left, but for the rest of this course we will not need to know this.)

We also note the convention that lower case letters, *e.g.* i, are used to denote an **instantaneous** value that varies with time. Sometimes we make this explicit by writing $i(t)$. In contrast, capital letters, *e.g.* I_0, are used for **steady state** (time independent) quantities. The other convention you should note is that **variables** are written in italics. That said, I am sure you will be able to catch me out sometimes in these notes!

4. Voltage

We have to put some energy into the system in order to make a current flow and this leads to the concept of electrical potential energy. When a current flows, it will generally flow from the higher electrical potential to the lower electrical potential. It's just like water in a pipe flowing from the higher gravitational potential to the lower. I say "generally" because of course we can supply energy to pump the water up again and in a battery we use chemical energy to take the current back up to the higher potential again.

The **potential difference ("pd")** between two points is measured in **volts** (**V**) and is usually called the **voltage**. The **voltage** 'across' or 'between' a pair of terminals is a measure of the work required to move a charge of one coulomb from one terminal to the other

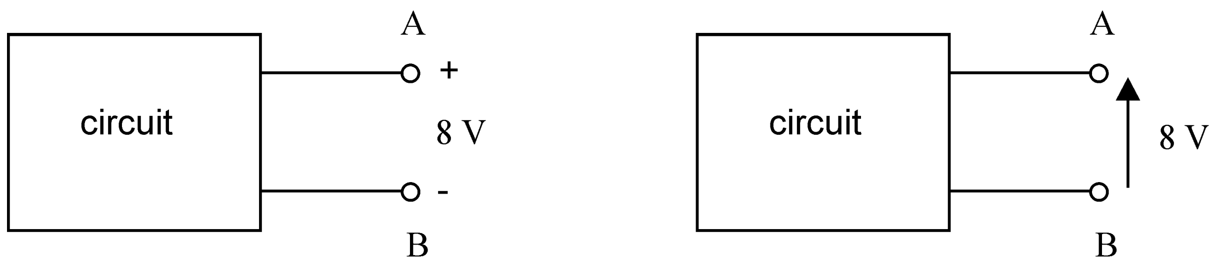
volts = **joules** per **coulomb**

Again instantaneous values are denoted as v whereas a capital V represents a steady state value.

Since voltage represents the potential needed to move charge between terminals it is clear that a voltage can exist between two points even if no current flows.

The energy converted per unit charge in an electrical **source** is also sometimes called the **electromotive force** (**e.m.f.**) of the source.

Voltage may be represented on a circuit diagram by a '+' and '-' pair of symbols or by an arrow.



In both cases

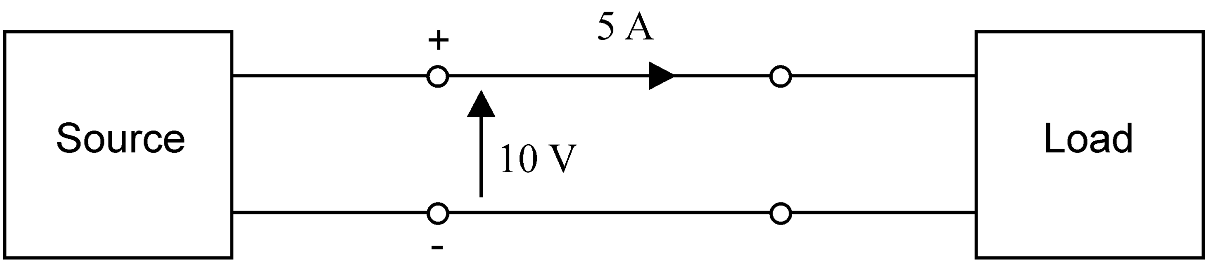
$$V_{\mathrm{A}} - V_{\mathrm{B}} = V_{\mathrm{AB}} = 8\mathrm{V}$$

i.e. terminal A is 8 V positive with respect to terminal B. Note that we have also introduced the notation, $V_{\mathrm{AB}} = V_{\mathrm{A}} - V_{\mathrm{B}}$. [This is the same notation that we use for "vectors" in our mathematics as you will find if (when) you have studied them.]

Power is the rate of transfer of energy, measured in **Watts**, it is given by

$$P = V.I$$

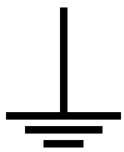
In general current flows out of the positive terminal of a source into the positive terminal of the load.



Here a power 5 A x 10 V = 50 W is transferred from the source to the load.

5. Earth (zero voltage reference)

Most real circuits also have a connection to “earth” (or, equivalently “ground”), which we may denote by the symbol below. By earthing our circuit to the earth pin of our mains plug (and thence to a metal stake in the ground somewhere nearby) we can reduce the risk of developing a dangerously high potential difference between the circuit and ourselves!



In this course we will only be analysing the potentials around the circuit and we will not be concerned about this connection to the outside world. Nevertheless, putting an earth symbol on our diagrams is equivalent to defining a “zero” reference voltage for our calculations, which may be quite sensible.

6. Linear passive circuit elements and Ohm's Law

The major part of many electrical circuits consists of **passive** elements, which can either dissipate energy (r**esistors**) or store energy (**capacitors** and **inductors**). A **linear** element is one in which the voltage across the element varies linearly with the current flowing through. We will deal solely with such elements in these lectures, although it is worth remembering that **practical** circuit elements will exhibit some (small) degree of non-linearity.

It is well known that as electrons move through a material they collide with the atoms and lose some of their energy. There is some 'resistance' to current flow and the loss of energy is usually converted to heat. Georg Simon Ohm studied the effect and found that the voltage drop across a piece of conductor was directly proportional to the current flowing through it. This is known, of course, as **Ohm's Law** and the constant of proportionality, R, is called the resistance. Thus

$$V = I.\mathrm{R}$$

If V is measured in volts and I in amps, the unit of R is the **Ohm** (Ω). The value of the resistance depends, of course, on the material used via its resistivity, ρ, its length, l, and cross-sectional area, A,

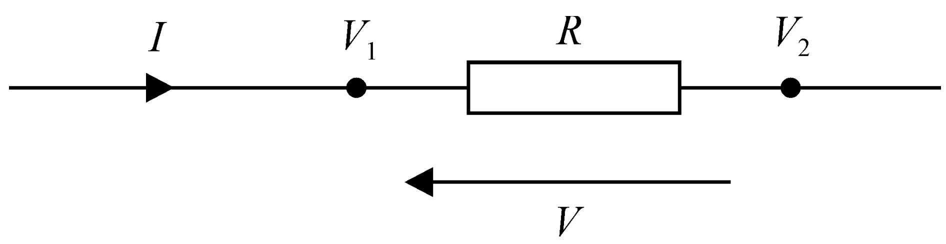
$$\mathrm{R} = \rho l / A.$$

HLT, for example, lists the resistivities of a number of materials. It is, of course, perfectly possible to write the proportionality between current and voltage in a form analogous to that above, *i.e.*

$$i = \mathrm{G}v$$

The constant of proportionality evidently has units of amps/volts or 1/ohms which are given the symbol S (Siemen) and $\mathrm{G} = 1/\mathrm{R}$ is called the **conductance** of the element.

Since Ohm's law is crucial in circuit analysis it is very important to take care to apply it correctly. Suppose a current *I* flows through a resistor of value R.

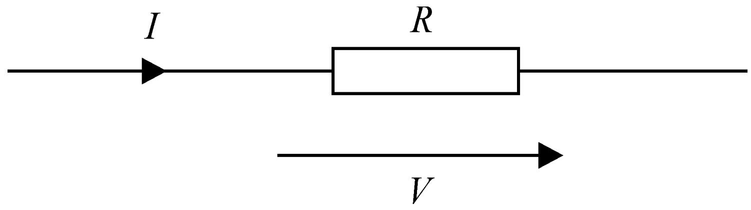


Ohm's law tells us that

$$V_1 - V_2 = V = I\,R.$$

The direction of the voltage arrow tells us we are **measuring** the potential on the left relative to the potential on the right and the current arrow tells us we are **measuring** the current flowing from left to right. If the potential on the left is **actually** higher than the potential on the right, *V* and *I* will both be positive. (Current going from a higher potential to a lower potential means we are dissipating energy in the resistor.) Of course if the left side is at a lower potential, both V and I will be negative and their product (the power dissipated in the resistor) will still be positive.

If we happen to measure the right side relative to the left, we would draw the voltage arrow the other way around.



Naturally power is still dissipated in the resistor so either V or I will have to be negative and we must write:

$$V = -IR$$

Thus, although Ohm's law is very simple we do need to be careful and "keep an eye on the signs".

Finally note that, since power dissipation is given by $P = IV$, we may write, for a resistor,

$$P = IV = I^2\,\mathrm{R} = V^2/\mathrm{R} \text{ Watts}$$

or, alternatively,

$$P = IV = I^2/\mathrm{G} = V^2\,\mathrm{G} \text{ Watts}$$

7. Practical Values

In engineering we have to deal with a wide range of variables and we make good use of "engineering notation" where values are given in the form n.nn × $10^{\mathbf{3n}}$, *e.g.* 1.23 × 10^6 and we have names for the powers:

tera	T	one trillion	10^{12}
giga	G	one billion	10^9
mega	M	one million	10^6
kilo	k	one thousand	10^3
milli	m	one-thousandth	10^{-3}
micro	µ	one-millionth	10^{-6}
nano	n	one-billionth	10^{-9}
pico	p	one-trillionth	10^{-12}

.... and more!

Thus 1.23 × 10^6 Ω ≡ 1.23 MΩ ≡ 1.23 mega-ohm. On a circuit diagram you may also see it written as 1M23.

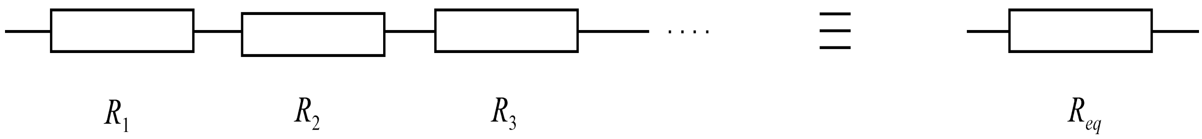
In this example, we have used "three significant digits" – 3 s.d. (or "three significant figures" – 3 s.f.). In engineering it is important to use an appropriate precision in our measurements and calculations. With a pocket calculator it is easy to write down a lot more digits than is sensible and you will probably be reprimanded by your tutor for doing so.

Let's look at rounding to 3 s.d. and suppose (say) you finish up writing down things between 1.00 and 9.99. To do that, you may be introduced a rounding error up to half the smallest digit, *i.e.* 0.005. The biggest percentage error you can introduce is for the number 1.00 when the error could be up to ±0.005/1.00. *i.e.* ±0.5%. On the other hand, for the number 9.99, your rounding error can't be any more than ±0.05%. Therefore 3 s.d. is appropriate when you are expecting your measurements or your answers to be accurate to about 1%. We usually want to avoid calculation errors, so we often use more significant digits in our intermediate calculations but just pause and think when you write down your final answer.

You will meet some real resistors when you get into the lab, but note that they come in quite a variety of shapes and sizes. They are commonly available in values from ohms (Ω) to mega-ohms ($M\Omega$) and in standard ranges and tolerances. For example, the E24 range (see HLT) has 24 equal ratios from 1 Ω to 10 Ω (and indeed in every other decade too) and provides the nominal values of 1.0, 1.1, 1.2, 1.3, 1.5, ... 8.2, 9.1, 10. This range is typically used for resistors with a tolerance of ±5% - that's handy because it means that any resistor manufactured can be labelled with one of the nominal values. Resistors also come in different power ratings from fractions of a watt upwards reflecting their differing abilities to dissipate the heat.

8. Resistors in Series and Parallel

We complete this 'basics' section by noting that elements that are connected together 'one after the other' such that the **same current** flows through each element are said to be connected in **series.**

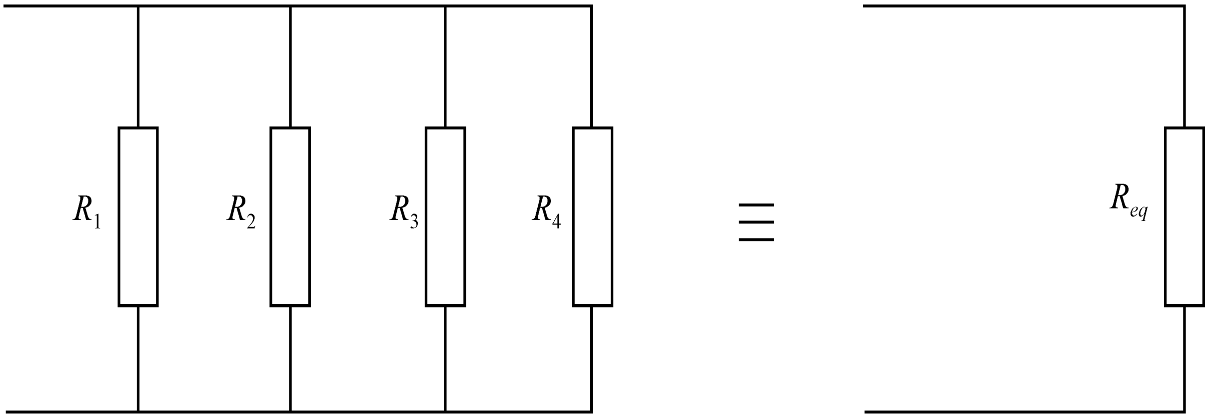


where $$R_{eq} = R_1 + R_2 + R_3$$

which is, of course, easily generalised to any number of elements, *N*, each of value R_n

$$R_{eq} = \sum_{i=1}^{N} R_n$$

Elements that are connected together such that the same voltage appears across each are said to be connected in **parallel**.



where
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

Since the inverse of resistances is involved in this case it is sometimes more convenient to work in terms of conductances, $G(= 1/R)$. In this case if there are N such elements of conductance G_n

$$G_{eq} = \sum_{i=1}^{N} G_n$$

We note that for the case of two resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

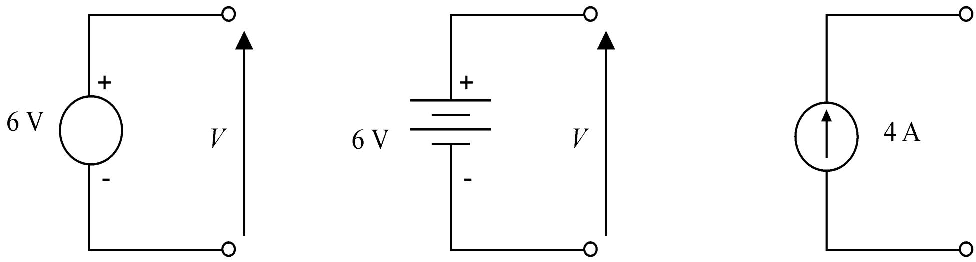
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} = \frac{\text{"product"}}{\text{"sum"}}$$

We will have occasion to make much use of this relationship in the future.

9. Independent and dependent sources

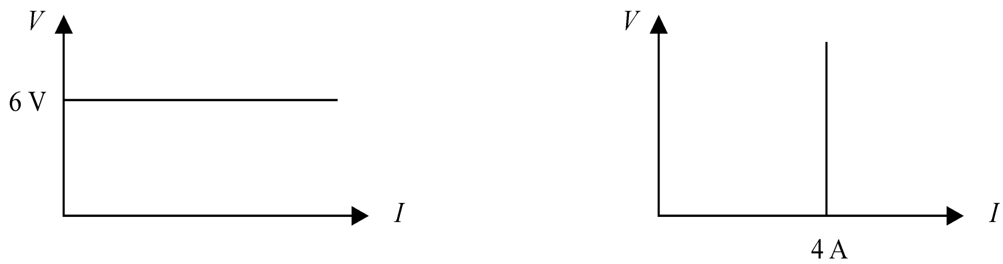
A **source** is an **active** element in the sense that it can deliver energy to an external device. Examples of source include batteries, alternators, oscillators *etc*.

There are two fundamental types of source. The first, with which we are all familiar is the **voltage source**. An example here is a battery. However, although perhaps less familiar at the moment, one can equally well conceive of a **current source**.



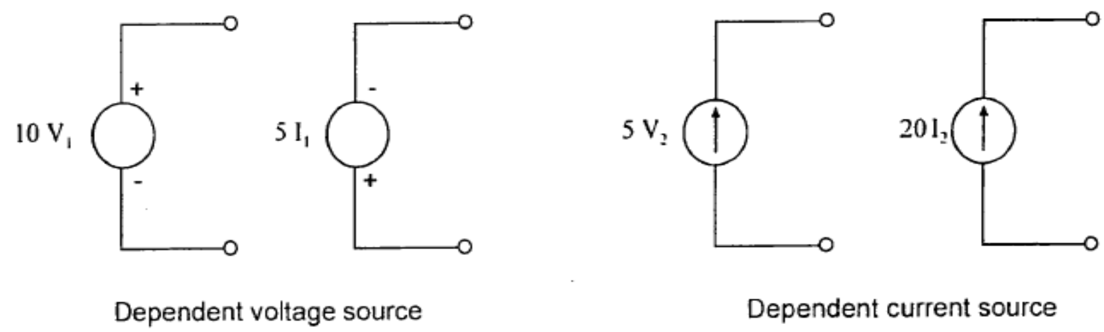
Ideal voltage source

Ideal current source



These sources are said to be i**ndependent** since their value is fixed independently of anything else in the circuit.

A **dependent** source on the other hand is one whose value depends on the current, or voltage, at some other point in the circuit to which it is connected. Such sources are drawn as

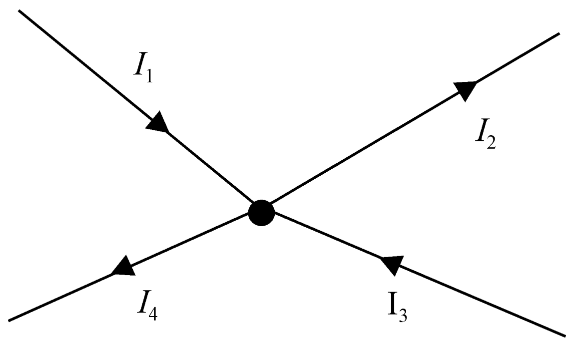


where V_1, I_1, V_2 & I_2, are voltages and currents somewhere else in the circuit. Although the dependent source concept may seem a little farfetched at the moment, we will have cause to return to it in connection with transformers and transistors.

10. Kirchhoff's laws

I suggested at the outset that we would look at circuits consisting of **components** and **wires**. Further to that we will also now assume that the wires are good conductors with negligible resistance such that they pass currents with negligible voltage drop. (This means that someone has chosen the wires to be thick enough that their resistance is very much less than that of the surrounding components.) Wires therefore have the same voltage at all points.

We will now state the two laws which we will permit us to analyse all electrical circuits. The **first** is **Kirchhoff's current law (KCL)** which tells us that the rate of flow of charge (current) into any point, or **node**, in a circuit is equal to the rate of flow of charge (current) out of it. In effect this says that charge cannot accumulate at any point in a circuit. Mathematically if currents flow into a node as shown



then the KCL tells us

$$I_1 - I_2 + I_3 - I_4 = 0$$

if there are N wires meeting at a point, each carrying a current I_n, then

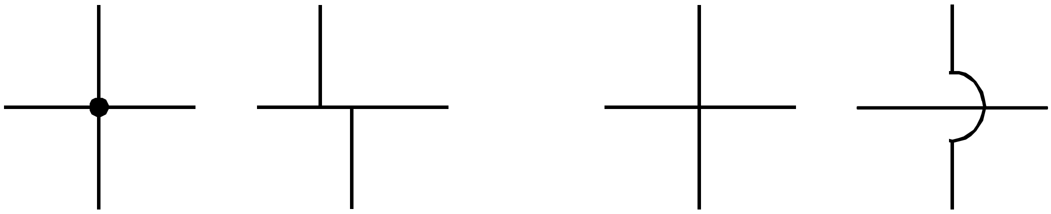
$$\sum_{n=1}^{N} I_n = 0$$

where due attention is paid to the signs so as to differentiate between current flowing into and away from the node.

Alternatively, if you prefer, you can sum the currents going into the node and equate them with the sum of currents leaving the node.

It is worth emphasising that, because we are assuming the wires have no resistance, **all** the wiring up until the next component (*e.g.* all that section of wiring shown in the diagram above) is at the same voltage. In principle this whole section of the wiring is one “node”. However, for convenience, we often put a blob at one particular point and refer to that as the node.

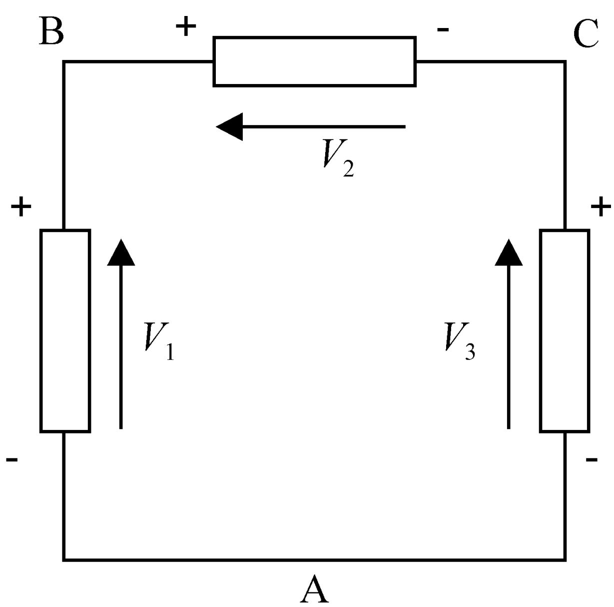
We **also** use blobs to clarify whether wires are joined together or not. The two drawings on the left show all the wires connected together, the two drawings on the right shows two separate wires crossing each other:



Kirchhoff further observed that, if we follow a path around a circuit and return to the starting point we'll get back to the same potential that we started at. This is the basis of the **second** law, **Kirchhoff's voltage law (KVL)** which tells us that the sum of the voltages around a closed path, taking due account of sign, must be zero. Thus if there are *N* elements with a voltage drop, V_n, across each individual one, then

$$\sum_{i=1}^{N} V_n = 0$$

Consider the circuit



KVL tells us that

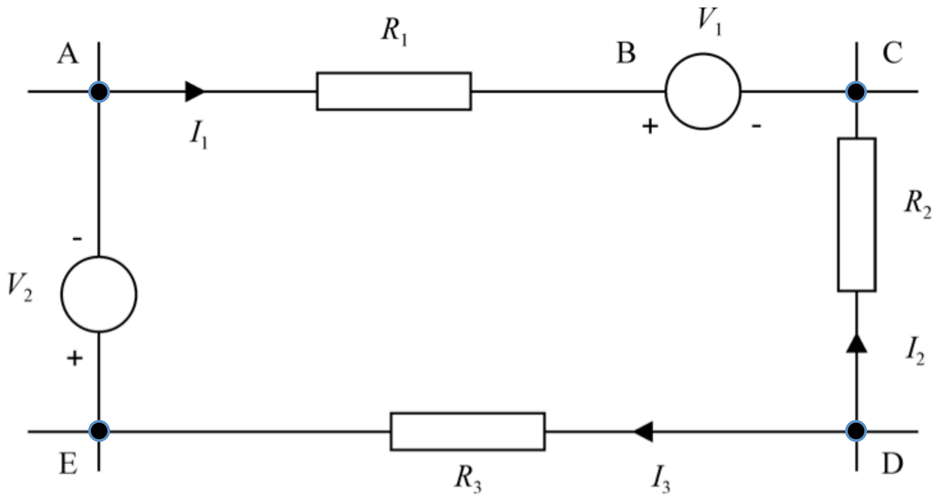
$$V_{BC} + V_{CA} + V_{AB} = 0$$

Now

$$V_{BC} = V_B - V_C = V_2$$
$$V_{CA} = V_C - V_A = V_3$$
$$V_{AB} = V_A - V_B = -V_1$$

Thus $$V_2 + V_3 - V_1 = 0$$

Let's look at another case where a loop is part of a larger circuit.



Again

$$V_{AB} + V_{BC} + V_{CD} + V_{DE} + V_{EA} = 0$$

and hence

$$I_1 R_1 + V_1 - I_2 R_2 + I_3 R_3 + V_2 = 0$$

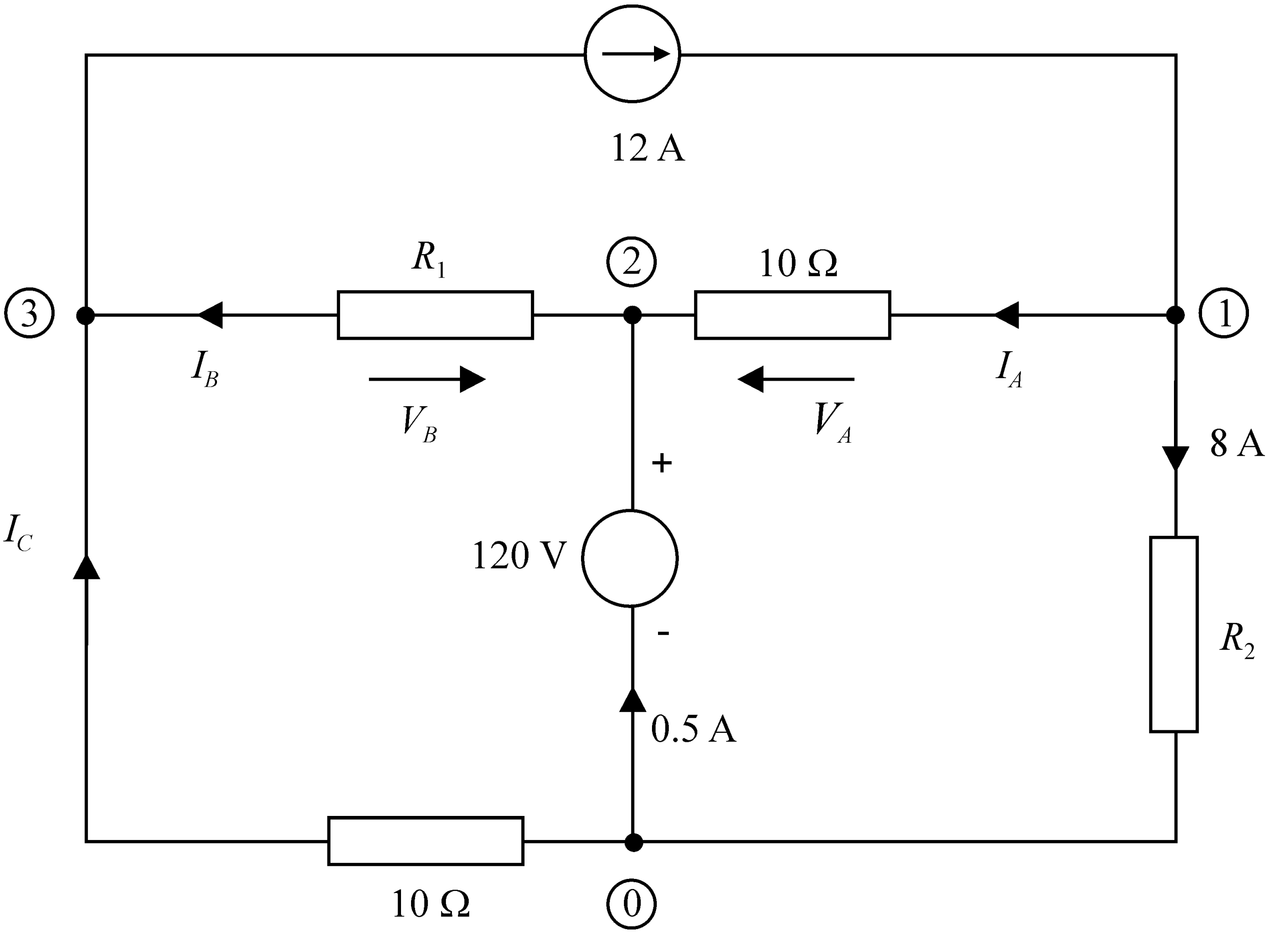
An alternative formulation of KVL is to say that the sum of the emfs applied must equal the sum of the pd's across the elements and we can see this by rearranging the equation as

$$V_1 + V_2 = I_2 R_2 - I_1 R_1 - I_3 R_3$$

It is a matter of choice which approach one takes.

Example

Find the unknown currents, voltages and resistor values in the following circuit:



At node (1), KCL gives

$$12 - I_A - 8 = 0$$
$$I_A = 4\text{A}$$

Ohm's law applied to the 10Ω resistor with I_A=4A flowing through it, gives, *taking account of the arrow* on V_A

$$V_A = -\,40\text{V}$$

At node (2) KCL gives

$$I_A + 0.5 - I_B = 0$$
$$I_B = 4.5\text{A}$$

Similarly at node (3) KCL gives

$$I_C + I_B - 12 = 0$$
$$I_C = 7.5\text{A}$$

If we now apply the Kirchhoff voltage law (KVL) around the left hand bottom loop we have, say,

$$V_{20} + V_{03} + V_{32} = 0$$
$$120 + 10I_C - V_B = 0$$
$$V_B = 195\text{V}$$

This also permits us to calculate $R_1 = \frac{V_B}{I_B} = \frac{195}{4.5} = 43.33\Omega$

Similarly applying KVL around the right hand loop gives, say

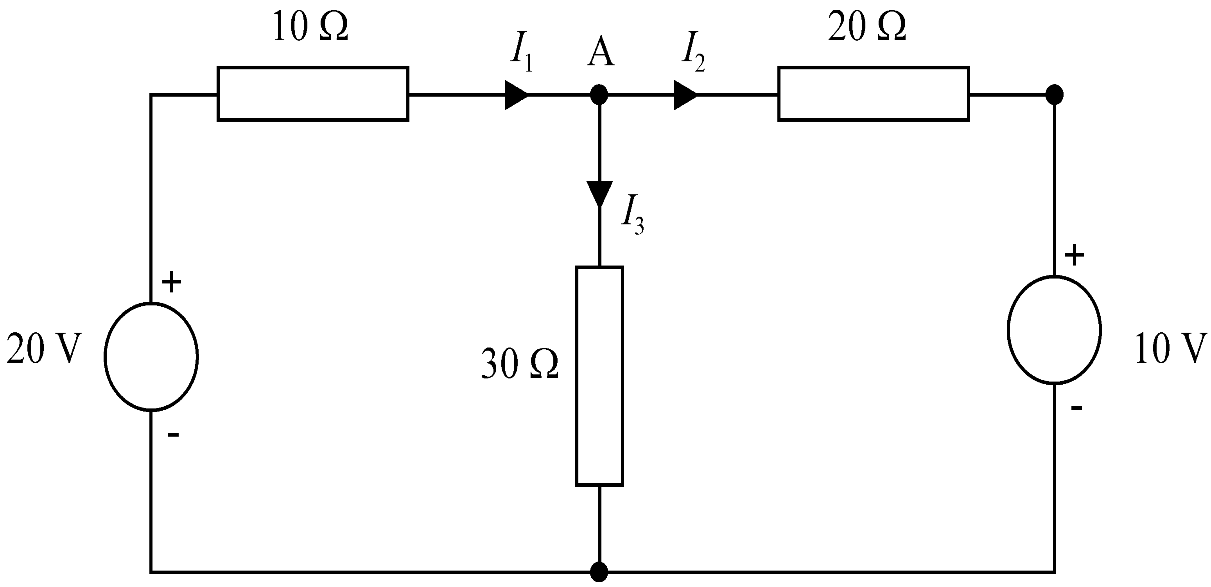
$$V_{10} + V_{02} + V_{21} = 0$$
$$8R_2 - 120 + V_A = 0$$
$$R_2 = \frac{40 + 120}{8} = 20\Omega$$

11. Loop analysis

Having introduced Kirchhoff's two laws we now proceed to describe two important tools, **loop analysis** and **nodal analysis**, which will provide systematic methods for us us to calculate analysing circuits. The two methods are complementary and the choice as to which method to use in practice is often determined by the specific problem or by personal preference. Let's start with:

Loop (or mesh) analysis

Consider the following circuit in which it is required to find the currents flowing through each of the resistors.



Our initial reaction might be to introduce the unknown currents, I_1, I_2 and I_3 and solve the problem by writing Kirchhoff's voltage law for the left hand loop as

$$20 = 10I_1 + 30I_3$$

and for the right hand loop

$$10 = -20I_2 + 30I_3$$

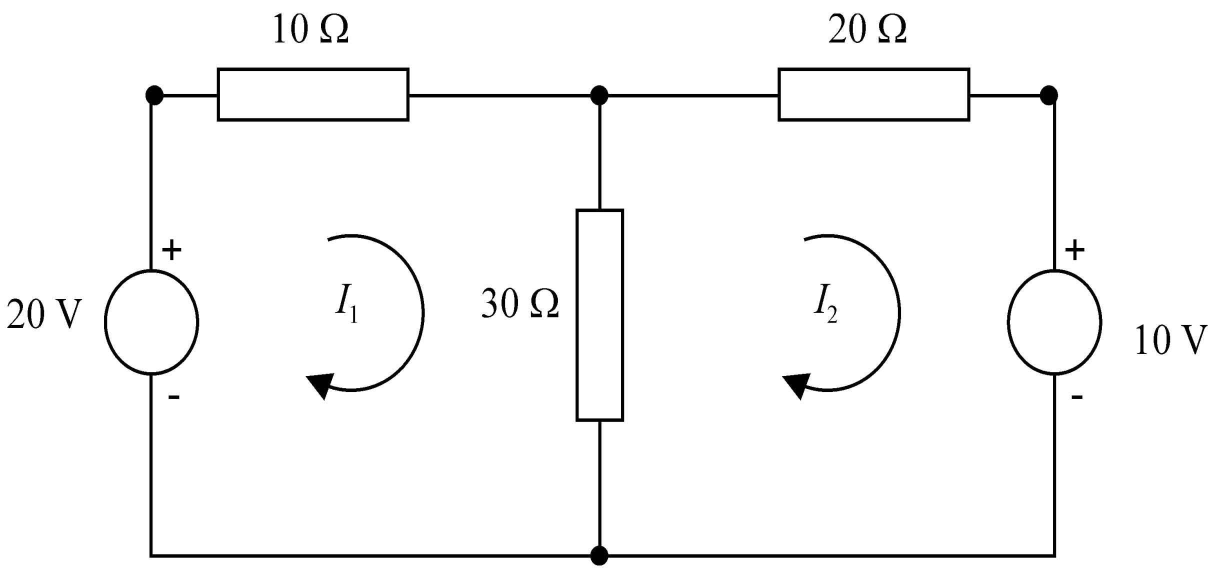
Finally Kirchhoff's current law for the point (node) *A* gives

$$I_1 = I_2 + I_3$$

We now have three equations and three unknowns which we can solve, eventually, to give:

$$I_1 = 0.636A, \qquad I_2 = 0.182A \qquad \text{and} \quad I_3 = 0.454A.$$

Although there is nothing wrong with this approach it's quite tedious to solve the three simultaneous equations and so we might wonder if there is an easier way to obtain the same result. The answer, as you will have guessed, is "yes" and this is the method of loop (or mesh) analysis. In this approach we assign currents to a specific loop rather than a specific piece of wire. In the previous example we would merely assign two loop currents I_1 and I_2 as follows



We use this notation to indicate that a current I_1 flows through the 10Ω resistor and a current I_2 through the 20Ω resistor. The current through the 30Ω resistor, on the other hand, is given by I_1-I_2 "downwards" or I_2-I_1 "upwards".

If we now write the Kirchhoff voltage law for the left hand loop we have

$$20 = 10I_1 + 30(I_1 - I_2)$$

and, for the right hand loop,

$$10 = -20I_2 - 30(I_2 - I_1)$$

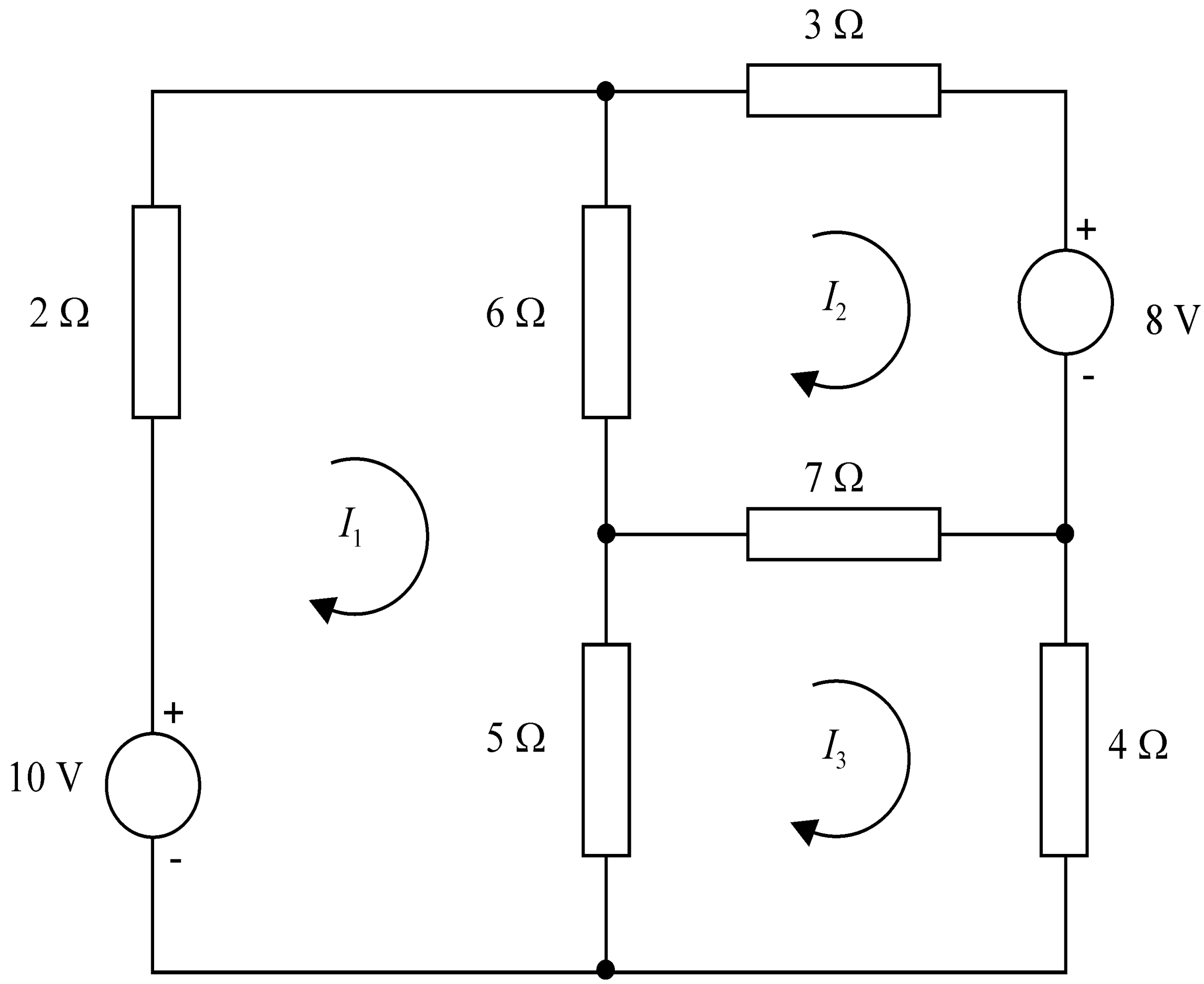
It is now straightforward to solve these *two* simultaneous equations

$$\left.\begin{aligned} 2 &= 4I_1 - 3I_2 \\ 1 &= 3I_1 - 5I_2 \end{aligned}\right\} \Rightarrow \begin{aligned} I_1 &= 7/11 = 0.636\text{A} \\ I_2 &= 2/11 = 0.182\text{A} \end{aligned}$$

Again the current in the 30Ω resistor is given as $I_1 - I_2 = 0.454$ A as before.

We note that the beauty of the approach is that we have reduced the number of equations to solve from three to.

We'll now do a few more examples to illustrate the method



We draw the three loops as indicated but we pay no particular attention to the directions of the currents I_1, I_2 and I_3. The mathematics will tell us the correct sign at the end of the calculation. It need not concern us when setting up the solution.

For the left hand loop we have

$$10 = 2I_1 + 6(I_1 - I_2) + 5(I_1 - I_3)$$

whereas the top right hand loop gives

$$8 = -3I_2 - 6(I_2 - I_1) - 7(I_2 - I_3)$$

and finally for the bottom left hand loop we have

$$0 = 4I_3 + 5(I_3 - I_1) + 7(I_3 - I_2)$$

From which – please check my arithmetic - -

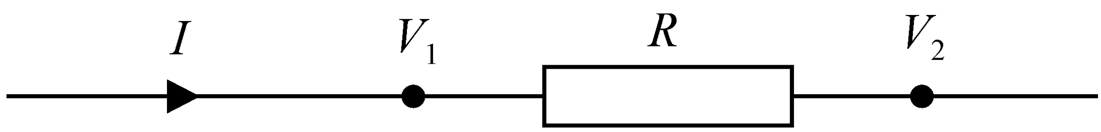
$$I_1 = 0.789A,\ \ I_2 = -0.119A \ \text{ and } \ I_3 = 0.195A.$$

Note that this calculation for the three loop currents permits us to calculate the currents through each of the *six* resistors. The more traditional approach would have required us to solve *six* simultaneous equations!!

At the beginning I called this loop **(or mesh)** analysis. **Mesh analysis** means that we treat the circuit like a wire mesh fence and associate a loop current for each and every "hole" in the mesh. Most times this gives us just the right number of equations but sometimes it doesn’t work out, as in the above example and *e.g.* if we have a circuit diagram with wires crossing each other. In **loop analysis** we are free to choose any loop which takes a closed path around the circuit, but a bit more thought is then needed to make sure that we have enough loops and that they are independent of each other, *i.e.* that our resulting simultaneous equations are sufficient and independent. (If they aren’t, we can’t solve them!)

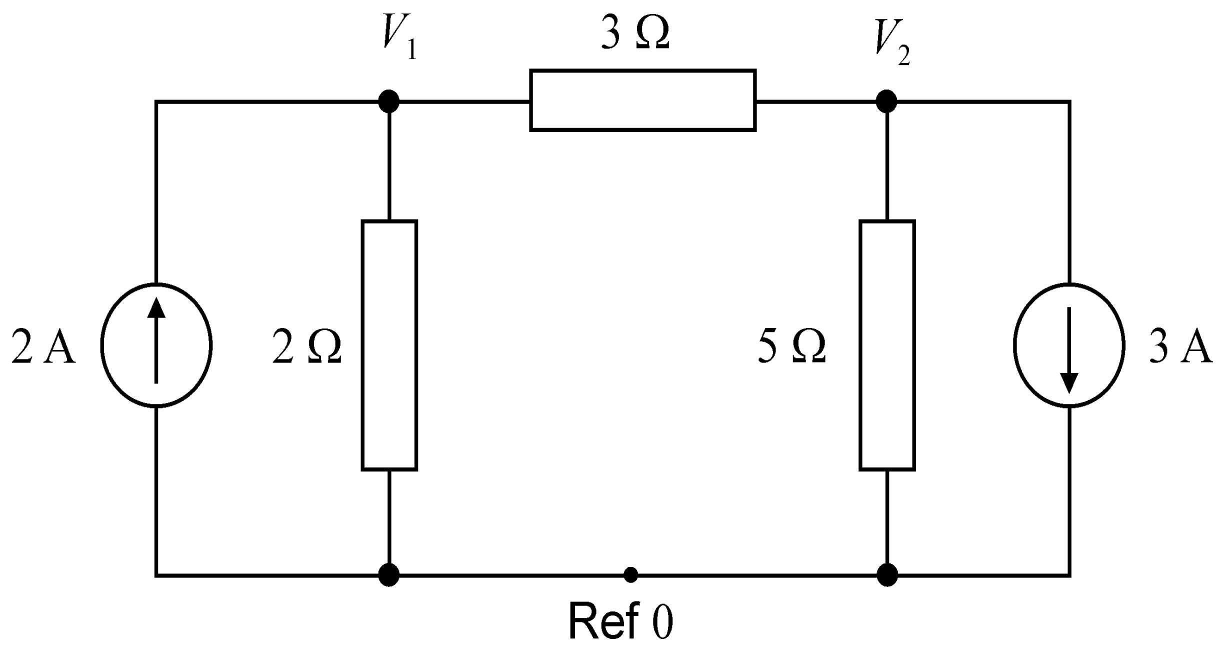
12. Nodal Analysis

In our previous analysis we have regarded the mesh currents as the unknowns from which voltages at various points around the circuit could be calculated. It is equally appropriate to regard the voltages at particular nodes (relative, of course, to some reference) as the unknowns. This is the basis of *nodal analysis* where we use Kirchhoff's current law at each node, other than the reference, to give a set of simultaneous equations which permits the 'node voltages' and hence, if required, branch currents to be found. Again we illustrate the method by way of an example. However, before doing so, it is sensible to remind ourselves of Ohm's law – re-stated here as



$$I = \frac{V_1 - V_2}{R}$$

Consider the following circuit and suppose we eventually want to know the voltage drop across the 3Ω resistor



We begin by introducing node voltage V_1 and V_2 with respect to the reference node 0. In general if a circuit has *n* principal nodes we need (*n*-1) simultaneous equations to solve the circuit.

Referring to node 1 we may write the Kirchhoff current condition at this node by summing the currents flowing into the node to zero as

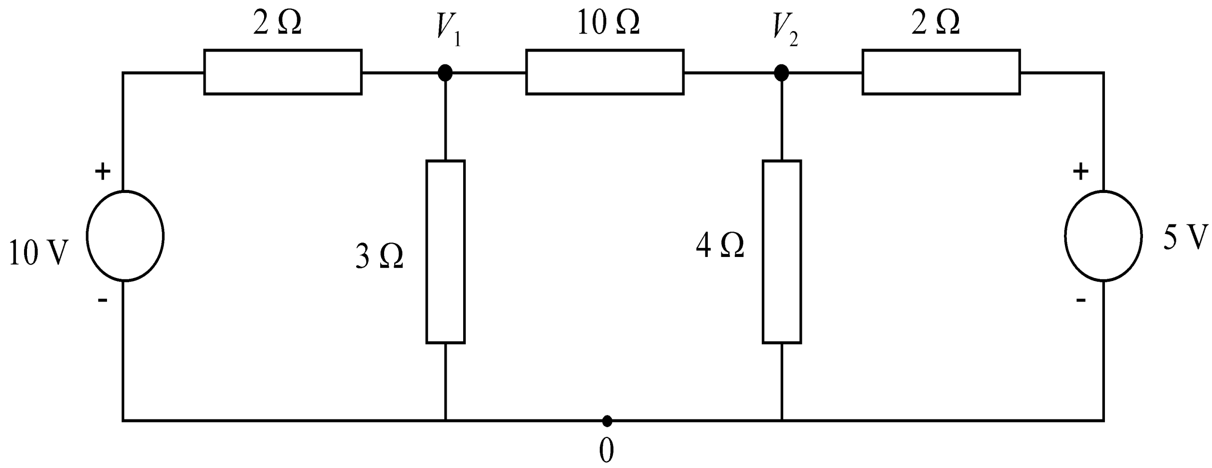
$$2+\frac{(0-V_1)}{2}+\frac{(V_2-V_1)}{3}=0$$

and for node 2 if we sum the current flowing out of the node to be zero we obtain

$$3+\frac{(V_2-0)}{5}+\frac{(V_2-V_1)}{3}=0$$

These equations may be solved to give $V_1 = 6.2V$ and $V_2 = 9.5V$. Hence the current flowing from node 2 towards node 1 is given by $(9.5 - 6.2)/3 = 1.1A$.

We now consider a circuit containing only voltage sources where we are required to find the node voltages V_1 and V_2 with respect to the reference 0,



If we decide to sum all the currents flowing into the nodes we may write for node 1

$$\frac{10 - V_1}{2} + \frac{V_2 - V_1}{10} + \frac{0 - V_1}{3} = 0$$

and for node 2

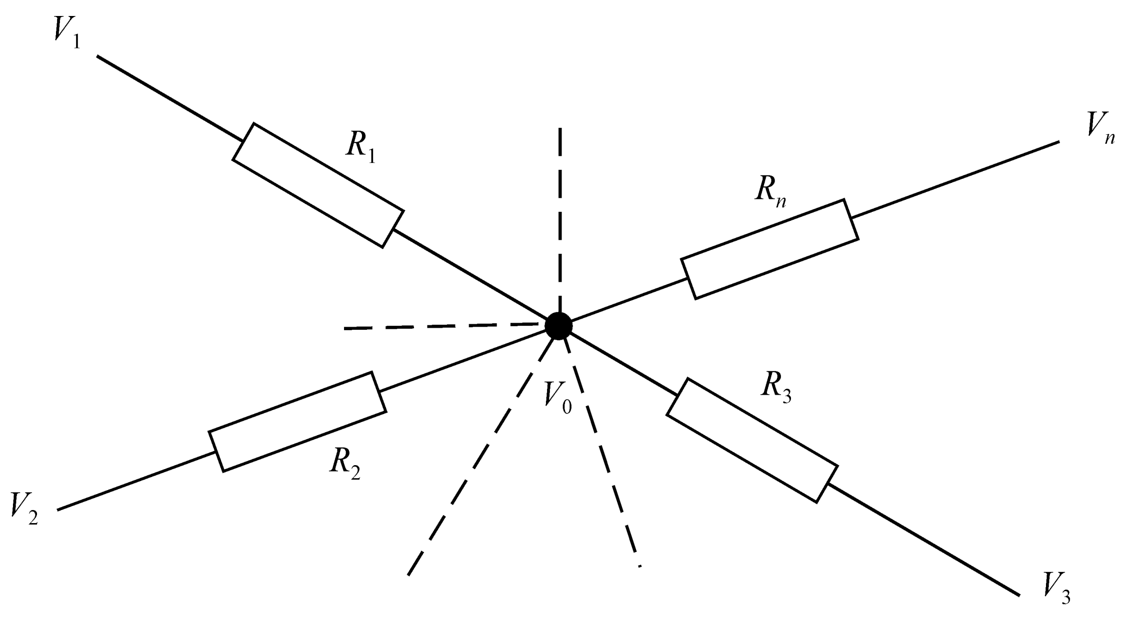
$$\frac{5 - V_2}{10} + \frac{V_1 - V_2}{10} + \frac{0 - V_2}{4} = 0$$

From which V_1 and V_2 may be found.

We could, of course, have decided to sum all the currents flowing out of the nodes to be zero. This would have given

$$\frac{V_1 - 10}{2} + \frac{V_1 - V_2}{10} + \frac{V_1 - 0}{3} = 0 \quad and \quad \frac{V_2 - 5}{10} + \frac{V_2 - V_1}{10} + \frac{V_2 - 0}{4} = 0$$

and, naturally would have made no difference to the final result. Indeed in solving problems like this I strongly suggest that you don't think too hard about what you are doing! I mean by this somewhat dramatic statement that you are merely **consistent** in the way that you write the equations. Thus for a particular node whose voltage is V_0, say, where n arms meet, each connected by a resistor, R_n with an "outer" potential V_n



Then either write

$$\sum_{n=1}^{N} \frac{(V_n - V_0)}{R_n} = 0$$

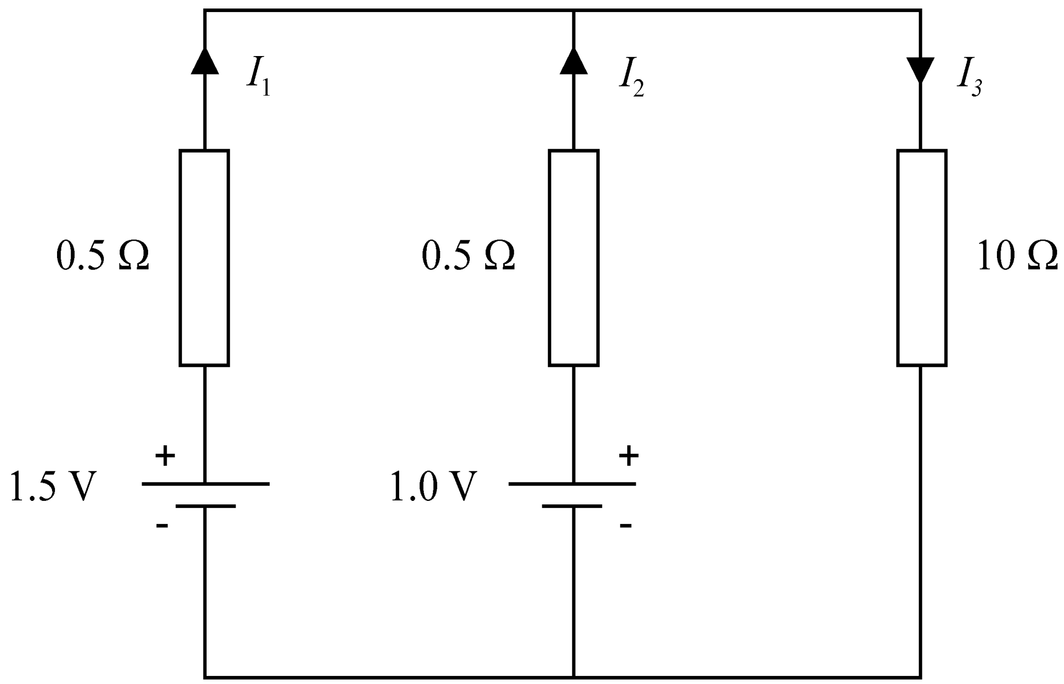
or

$$\sum_{n=1}^{N} \frac{(V_0 - V_n)}{R_n} = 0$$

The expressions are clearly equivalent. A good check that you haven't made a mistake is to check, in each term making up the current summation equation, that the *sign* of the *node voltage*, V_0, is the *same*.

This is probably the only thing you have to think about in the vast majority of cases when using Node voltage analysis.

Let’s look at a final example in which we are required to find the current flowing through each resistor.



(i) Brute force – not recommended!

Noting that the elements in parallel must have the same voltage across them gives us

$$1.5 - 0.5 I_1 = 1.0 - 0.5 I_2 = 10 I_3$$

Also

$$I_1 + I_2 = I_3$$

We now have three equations to solve for the three unknowns. We obtain $I_1 = 23/41A$; $I_2 = -18/41A$ and $I_3 = 5/41A$.

(ii) Use loop currents

Assume clockwise current loops I_4 and I_5 in the left and right hand loops respectively. The loop equations (KVL) give

$$1.5 - 0.5I_4 - 0.5(I_4 - I_5) - 1.0 = 0$$

and

$$1.0 - 0.5(I_5 - I_4) - 10I_5 = 0$$

This gives *two* equations to solve rather than three.

(iii) Use node voltages

Let the unknown node voltage at the "top" of the resistors be V. Then

$$\frac{V-1.5}{0.5} + \frac{V-1.0}{0.5} + \frac{V-0}{10} = 0$$

In this case we have only *one* equation to solve for V. Once we know V it is trivial to find the currents flowing through each resistor.

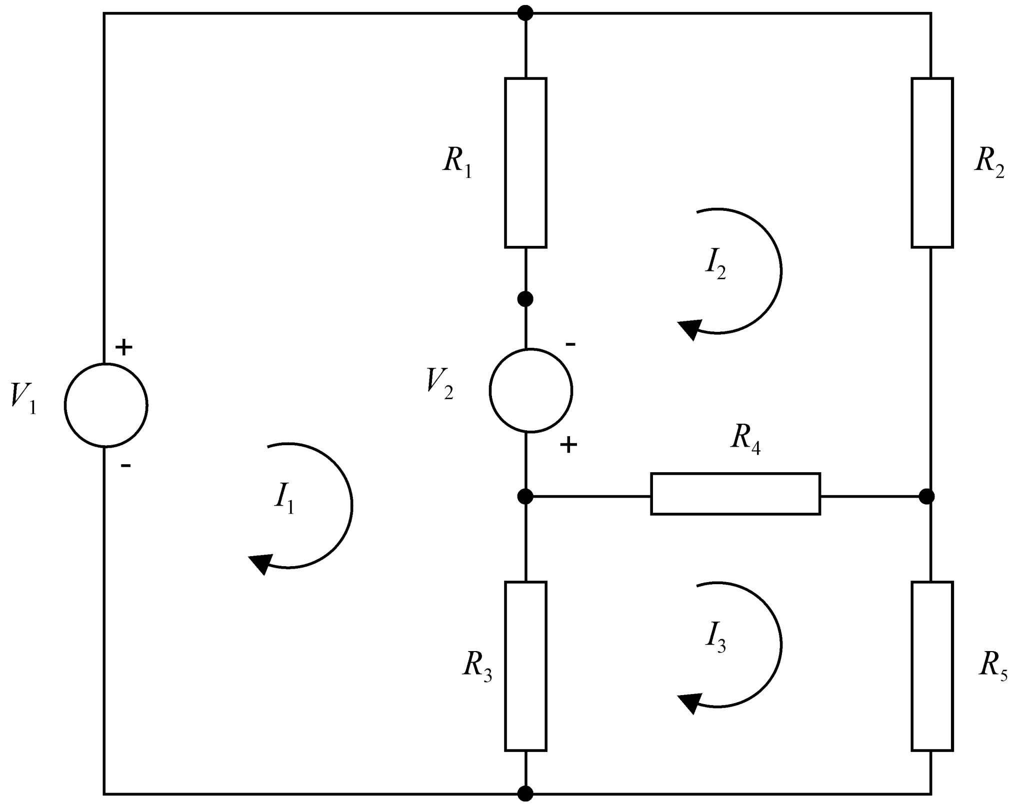
We emphasise that we have just used three methods to solve the same problem. They all, as they must, give the same answers. Some are easier to use than others. Practice will help you pick the easiest method. Indeed you might like to use node voltage analysis to check that we got the correct answers for the currents I_1, I_2 and I_3 in the circuit on the first page of the loop analysis section.

You may have noticed that Loop analysis seems more natural if you have voltage sources whereas Nodal analysis fits better with current sources. Later in these notes you will learn how a current sources can be translated into an equivalent voltage source, and *vice versa*, which is often a useful thing to do.

13. Matrix Notation

(This section is here for interest only and is NOT on the syllabus. If (when) you have studied matrices you may come to realise the power of matrix methods along with standard computer algorithms to solve very complicated circuit analysis problems way beyond anything you would want to solve "by hand".)

Since both loop/mesh analysis and nodal analysis result in a number of simultaneous equations there is, in a formal sense, advantage in writing the equations in matrix form. It means that we can describe large circuits in a succinct manner and that we can use standard computer algorithms to solve them. Let's illustrate this by following problem:



The three loop equations are given by

$$\begin{aligned} V_1 + V_2 &= (I_1 - I_2)R_1 + (I_1 - I_3)R_3 \\ -V_2 &= (I_2 - I_1)R_1 + I_2 R_2 + (I_2 - I_3)R_4 \\ 0 &= (I_3 - I_1)R_3 + (I_3 - I_2)R_4 + I_3 R_5 \end{aligned}$$

or, in matrix notation

$$\begin{pmatrix} V_1 + V_2 \\ -V_2 \\ 0 \end{pmatrix} = \begin{pmatrix} R_1 + R_3 & -R_1 & -R_3 \\ -R_1 & R_1 + R_2 + R_4 & -R_4 \\ -R_3 & -R_4 & R_3 + R_4 + R_5 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

which we can write as $\boldsymbol{v} = \boldsymbol{R}.\boldsymbol{i}$

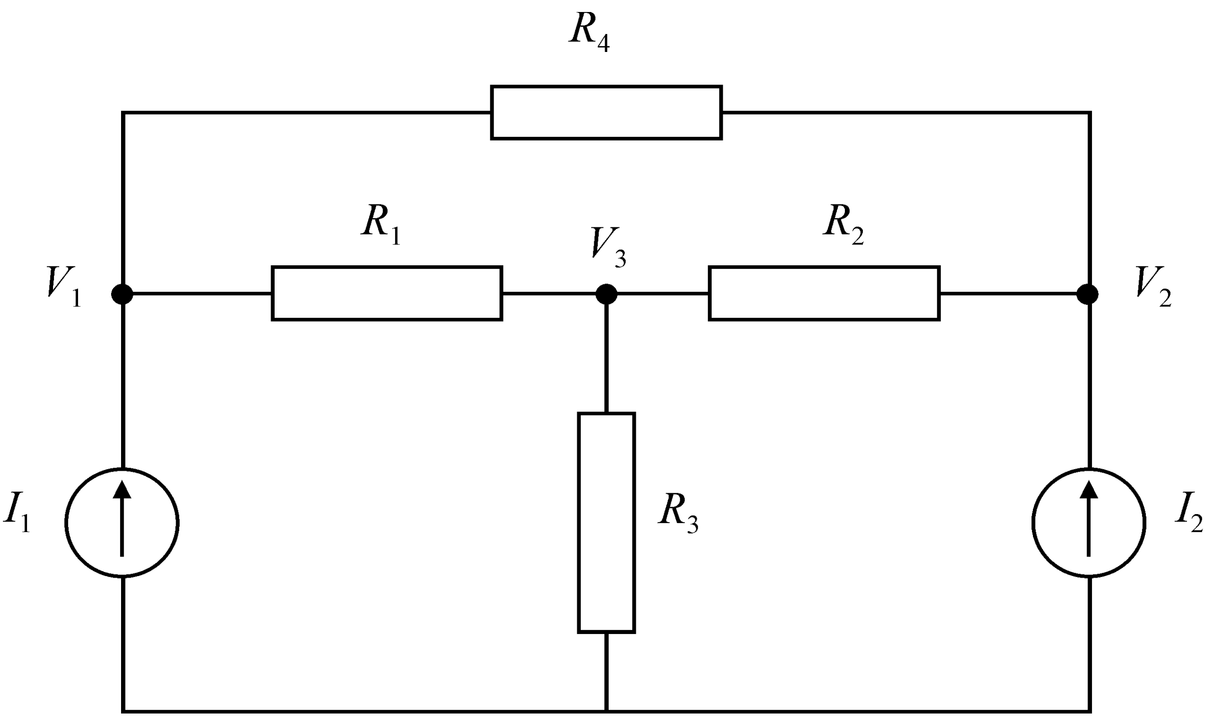
We note that the resistance matrix is square symmetric and in general it will take the form

$$\boldsymbol{R} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ : & : & : & : \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}$$

We observe that the diagonal elements, R_{ii}, represent the sum of the resistances in the mesh around which the current I_i flows. In our example, therefore, R_{22} is given by the sum of the resistors in the loop around which I_2 flows, $R_1 + R_2 + R_4$, and so on.

The off diagonal elements have the property that $R_{ij} = R_{ji}$ *and are given, if we are consistent with the directions of the currents, by the negative of the common or mutual resistance shared by the* i^{th} *and* j^{th} *loops. Thus* $R_{23} = R_{32} = -R_4$ *since the resistor* R_4 *is "common" to both the* I_2 *and* I_3 *loops.*

It is, of course, possible to make similar general remarks in the case of node-voltage analysis. In order to illustrate this let's consider the circuit below



where we have introduced node voltages, V_1, V_2 and V_3. *The node-voltage equations may be written as*

$$\frac{V_1 - V_2}{R_4} + \frac{V_1 - V_3}{R_1} - I_1 = 0$$

$$\frac{V_2 - V_3}{R_2} + \frac{V_2 - V_1}{R_4} - I_2 = 0$$

and

$$\frac{V_3 - V_1}{R_1} + \frac{V_3 - 0}{R_3} + \frac{V_3 - V_2}{R_2} = 0$$

which we can write neatly in matrix form in terms of conductance as

$$\begin{pmatrix} I_1 \\ I_2 \\ 0 \end{pmatrix} = \begin{pmatrix} G_1 + G_4 & -G_4 & -G_1 \\ -G_4 & G_2 + G_4 & -G_2 \\ -G_1 & -G_2 & G_1 + G_2 + G_3 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

or $\boldsymbol{i} = \boldsymbol{G}.\boldsymbol{v}$

We notice the conductance matrix is square symmetric and again general remarks can be made about its form.

$$\boldsymbol{G} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ : & : & : & : \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix}$$

Warning!

You might be tempted to solve circuit problems by writing these matrices directly from an examination of the circuit. However, this is a very risky strategy in which it is easy to make mistakes.

Nevertheless, when you write down the simultaneous equations of your mesh or nodal analysis, do look out for these symmetries as a check on your working. Although very useful commercially, matrix methods are unlikely to be the best way to solve the simple problems we will encounter on this course.

14. The principle of superposition

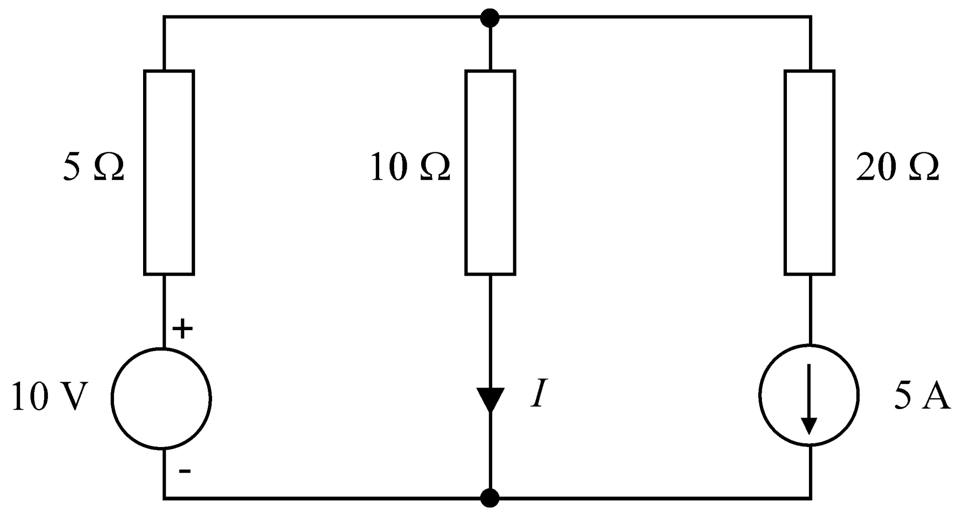
This is a very general principle which is useful in many branches of science where **linear** systems are involved. In our terms it tells us that if we have a circuit containing any number of independent sources that the currents and voltages in that circuit are given by the algebraic sum of the currents and voltages due to each of the sources acting independently with the others removed (set to zero).

Linearity implies that any particular voltage (or current), say V_x, is a linear function of all the sources, say E_y and I_z, as in

$$V_x = k_1 . E_y + k_2 . I_z.$$

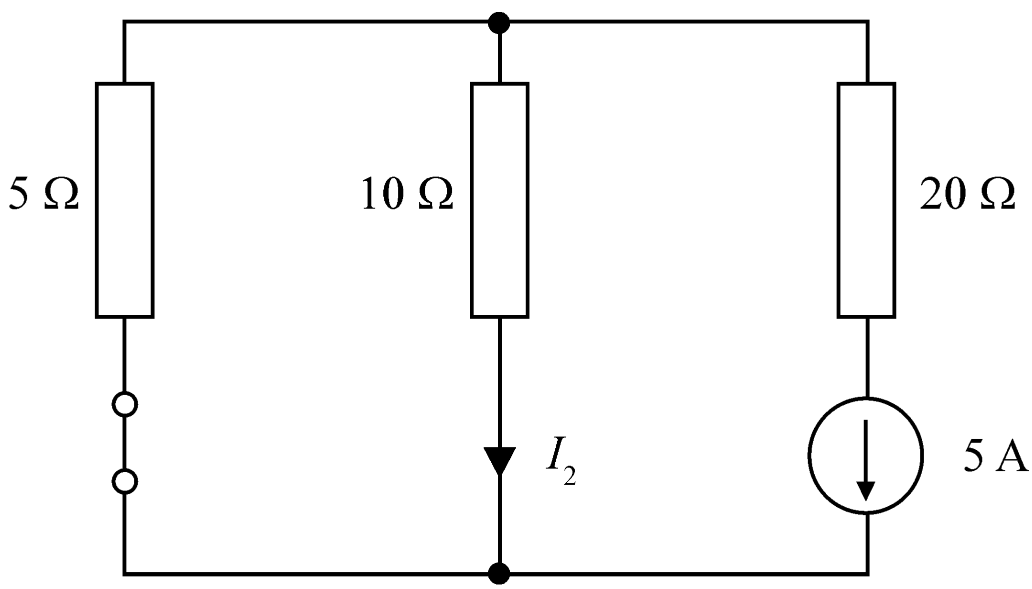
Linearity further implies that when I_z is zero, $V_x = k_1.E_y$ (= V_x', say) and when E_y is zero, $V_x = k_2.I_z$ (= V_x'', say) and superposition tells us that in general, V_x is the linear sum of the these components, *i.e.* $V_x = V_x' + V_x''$.

We'll illustrate the idea with the following example where we are asked to find the current, *I*, flowing through the 10Ω resistor:



We could of course solve the problem directly by introducing a (clockwise) loop current I_1 into the left hand loop. The loop equation $10 = 5I_1 + 10(I_1 - 5)$ yields $I = I_1 - 5 = -1\,\mathrm{Amp}$. We now confirm this result using the principle of superposition.

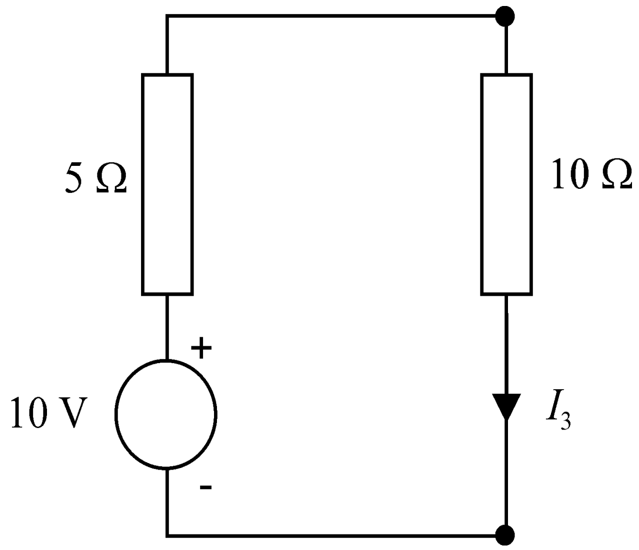
(i) We solve the problem when the 10V voltage source is removed – *i.e.* set to zero. We note that when a voltage source is zero there is no voltage drop across it and so, in circuit terms it is replaced by a *short circuit*. The circuit now becomes



The 5A current is now split between 5Ω and 10Ω resistors in parallel. The same voltage is developed across both resistors thus

$$(5+I_2)5 = -10I_2 \Rightarrow I_2 = -\frac{5}{3}\,\text{A}$$

(ii) When the 5A current source is set to zero no current flows through the 20Ω resistor and so the circuit reduces to



in which $$I_3 = \frac{10}{5+10}\,\text{A} = 2/3\,\text{A}.$$

The total current, I, which flows when both sources are present is merely the sum of these two currents. Thus

$$I = I_2 + I_3 = -\frac{5}{3} + \frac{2}{3} = -1\text{A}$$

which, of course, agrees with the value obtained by direct calculation.

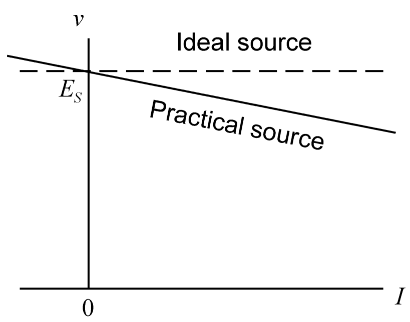
Although the illustrative example here was easy to solve directly we note that this is a powerful principle which is often very helpful when dealing with more complicated situations.

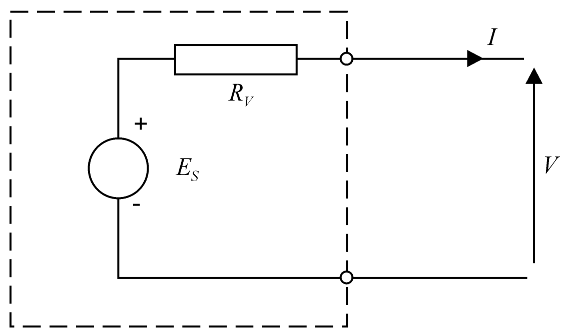
Caution!

Earlier, we introduced the idea of a **dependent source**. In this case we cannot arbitrarily set it to zero because its value depends on something else in the circuit. Best to avoid superposition in this case.

15. Practical (non-ideal) sources

As we have seen, an ideal voltage source can, in principle, supply any current to any load as evidenced by the 'flat' *V-I* characteristic of a few pages ago. In practice the voltage supplied falls as the current increases. We model this behaviour by placing a resistor in series with our ideal voltage source.





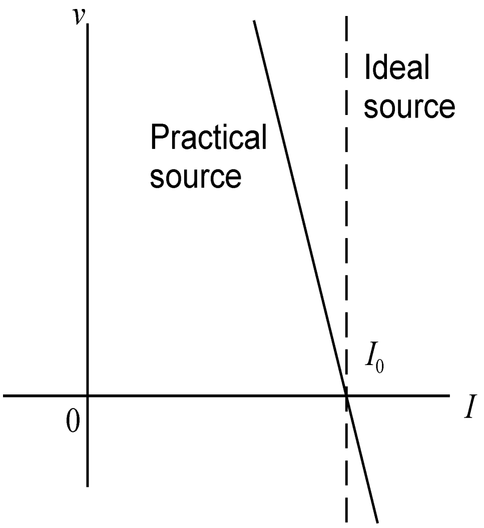
In this case the actual voltage supplied is given by

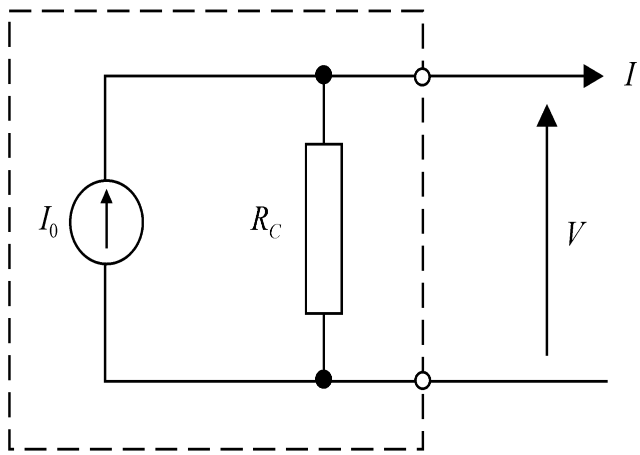
$$V = E_s - I\,R_V$$

which is, of course, only equal to E_s when $I = 0$. It is usually a design objective to keep R_V as small as possible so as to be able to provide a constant voltage over a range of current. We note that the resistor R_V is variously, and equivalently, called the **output** resistance of the circuit, the **internal** resistance of the source or just the **source** resistance.

The importance of what we have just done is that we have created a very simple circuit – a voltage source in series with a resistor – whose behaviour is **equivalent**, as far as the outside world is concerned, to that of the, perhaps complicated, device that is the actual source. This is an example of an **"equivalent circuit"**. It is a concept we will use many times in the future since it permits us to analyse the effects of a device without getting bogged down in the minutiae of its internal details. This particular instance is so common that we give it a name, the **Thévenin equivalent circuit**.

Since an ideal voltage course cannot maintain a constant voltage for all currents it will come as no surprise that a practical current source cannot provide a constant current for all voltages. An appropriate *equivalent* circuit in this case would be





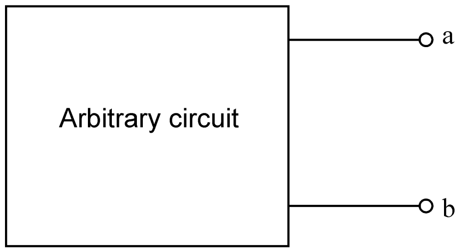
Since the current through the resistor R_c, is $I_0 - I$ ‘downwards’, the terminal voltage is given, by Ohm's law, as $V = (I_0 - I)\mathrm{R_c}$.
Therefore we may write

$$I = I_0 - V/\mathrm{R_c}$$

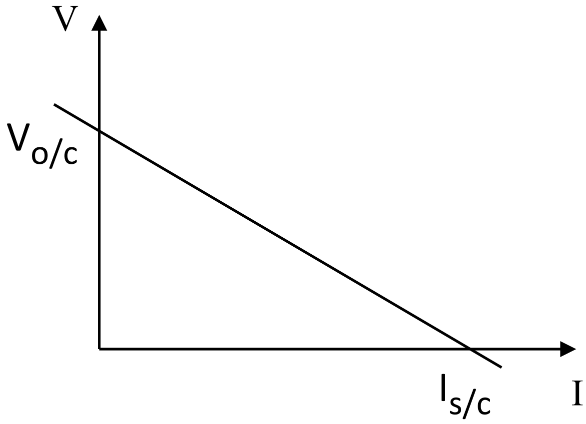
In this case it is usually desirable that R_c be large such that $I \approx I_0$ over as large a range of V as possible.

This particular circuit is called the **Norton equivalent circuit**.

16. Thévenin and Norton Equivalent Circuits



Consider an arbitrarily complicated **linear** circuit in which we only have access to two nodes, a and b. We can measure the voltage V_{ab} and extract a current $I_a = -I_b$. If the circuit is linear, there must be a **linear** relation between this voltage and current and a graph of voltage *vs.* current will be a straight line:

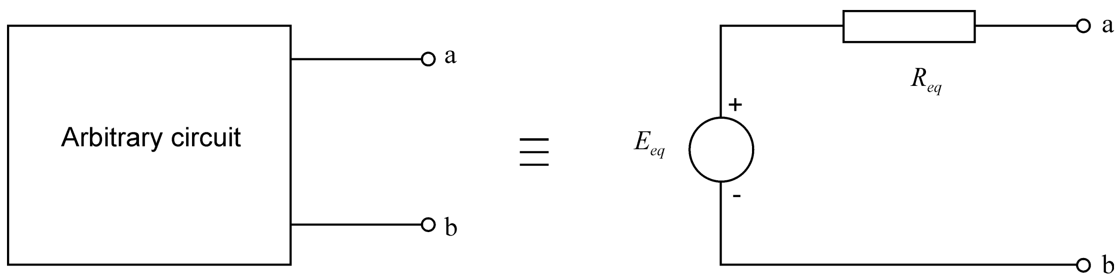


The intercept with the voltage axis occurs when I is zero and can be measured as the **open-circuit voltage, $V_{o/c}$.** The intercept with the current axis occurs when V is zero and can be measured as the **short-circuit current, $I_{s/c}$.** We can therefore represent it by the equation:

$V = V_{o/c} - I \cdot R_{eq}$,

where $R_{eq} = V_{o/c} / I_{s/c}$, the negative slope of the line.

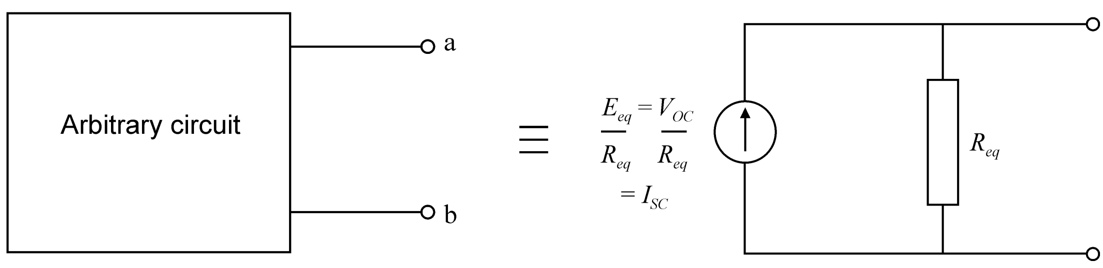
However complicated the actual circuit is, we can completely describe its behaviour by this simple graph. In electronics we also like to represent things by circuits - if the line is relatively flat, we may choose to represent it by a **Thévenin equivalent circuit**.



Here $V_{ab} = E_{eq} - R_{eq}.I$,

where $E_{eq} = V_{o/c}$ and $R_{eq} = V_{o/c} / I_{s/c}$.

If the line is relatively steep, we may instead choose to represent it by a **Norton equivalent circuit**.



Here $I_a = I_{eq} - R_{eq}.\ V_{ab}$,

where $I_{eq} = I_{s/c}$ and $R_{eq} = V_{o/c} / I_{s/c}$.

[Our ability to represent the external behaviour of any linear circuit by either of these two equivalent circuits is sometimes stated as **Thévenin's Theorem** and **Norton's Theorem**.]

Our straight line is of course defined by any two points. The **open-circuit voltage** and the **short-circuit current** are often convenient to analyse and/or to measure in the practice but any other two points will do (*e.g.* in the lab where we don't want to short out our circuit!).

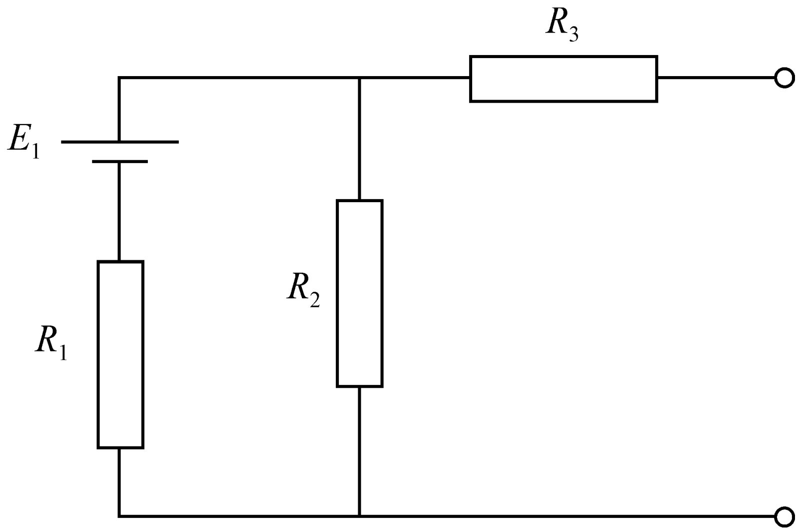
Alternatively the line can be defined by one point and the slope. The slope is $\Delta V/\Delta I = -R_{eq}$, and is the **resistance of the circuit when all sources are set to zero** (as with superposition, voltage sources set to zero = short circuit; current sources set to zero = open circuit; and we cannot arbitrarily set dependant sources to zero). For some circuits this is an easy thing to work out.

Therefore to determine the Thévenin or the Norton equivalent circuit we usually work calculate two out of the three parameters:

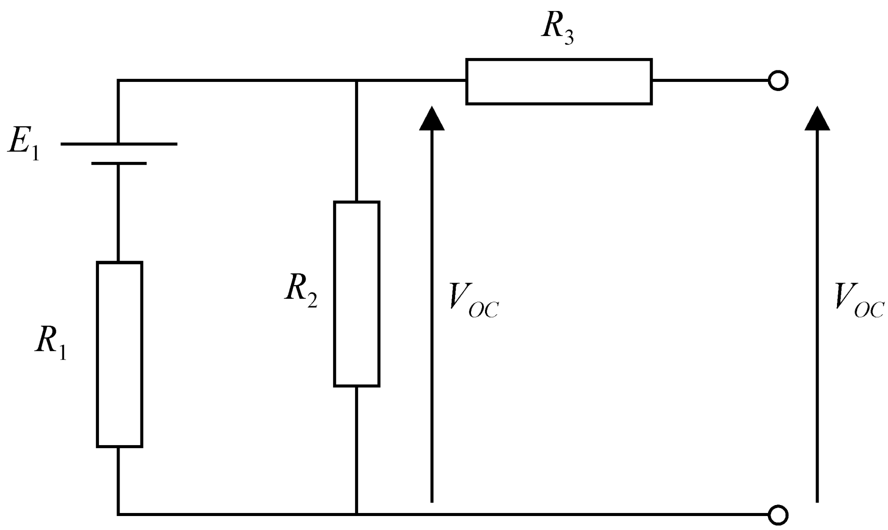
(i) Open-circuit voltage
(ii) Short-circuit current
(iii) Passive resistance.

Example

Find the Thévenin equivalent of the following circuit



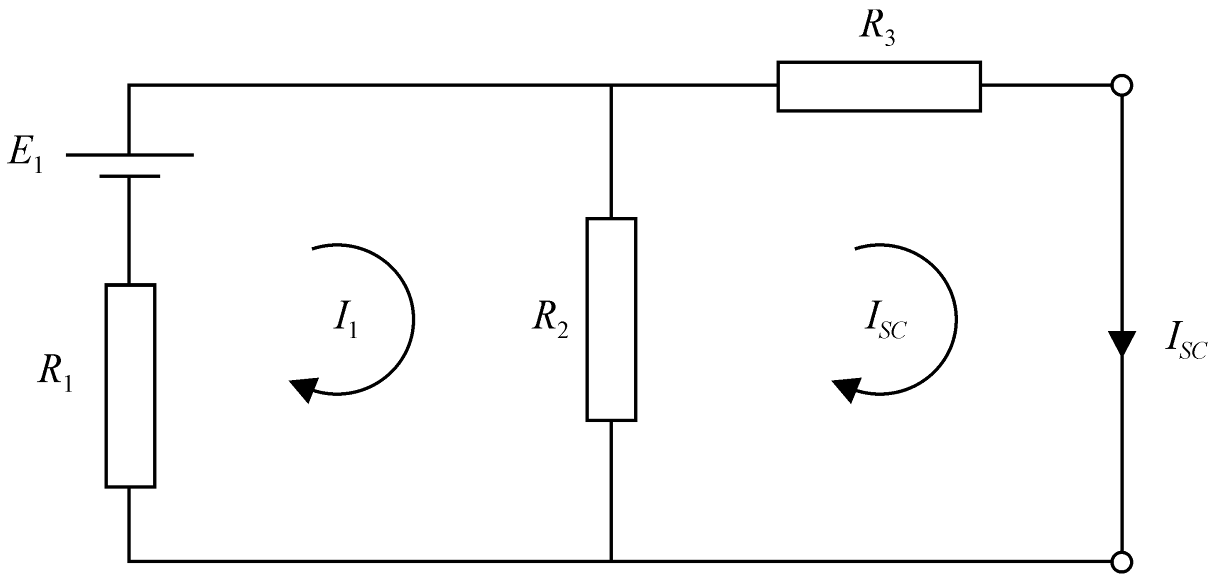
We first calculate the open circuit voltage, V_{oc}. Since no current flows through the resistor R_3, V_{oc} will appear across R_2.



Since current flows only in the left hand loop $\left(=E_1/\left[R_1+R_2\right]\right)$ it is simple to write

$$E_{eq} = V_{oc} = \frac{R_2}{R_1 + R_2} E_1$$

We must now find the short circuit current which flows when the terminals a and b are connected together



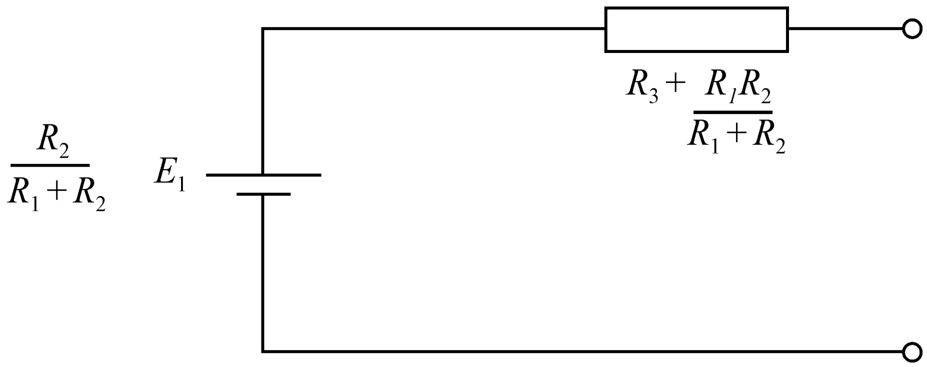
The two KVL loop equations may be written as

$$E_1 = (I_1 - I_{sc}) R_2 + I_1 R_1$$
$$0 = I_{sc} R_3 + (I_{sc} - I_1) R_2$$

which gives $I_{sc} = E_1 R_2 / [(R_1 + R_2)(R_2 + R_3) - R_2^2]$ from which we may calculate R_{eq}, after a little algebra, as

$$R_{eq} = \frac{V_{oc}}{I_{sc}} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

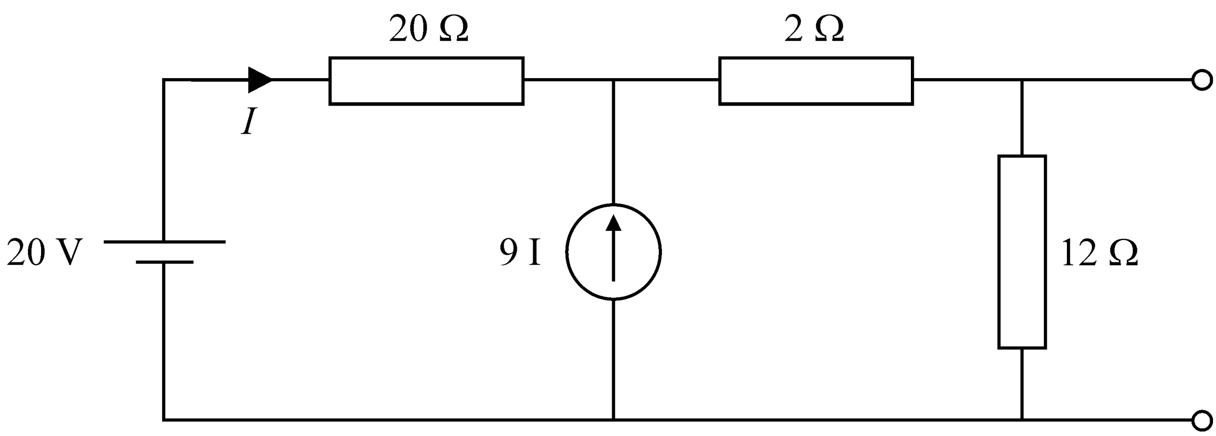
Thus as far as the outside world is concerned this circuit behaves as if it were



At this point another possibility may occur to you. Suppose we set the sources to zero in our arbitrary circuit (as we did with superposition earlier). In this case it is a simple matter of setting E_1 to zero. The circuit is now entirely resistive and the resistance between the terminals is just R_3 in series with a parallel combination of R_1 and R_2 giving us the above Thévenin resistance more directly.

Now try another example:

Find the Thévenin equivalent of the following circuit which contains a **dependent** current source whose value is given by 9I where I is the current flowing through the 20Ω resistor

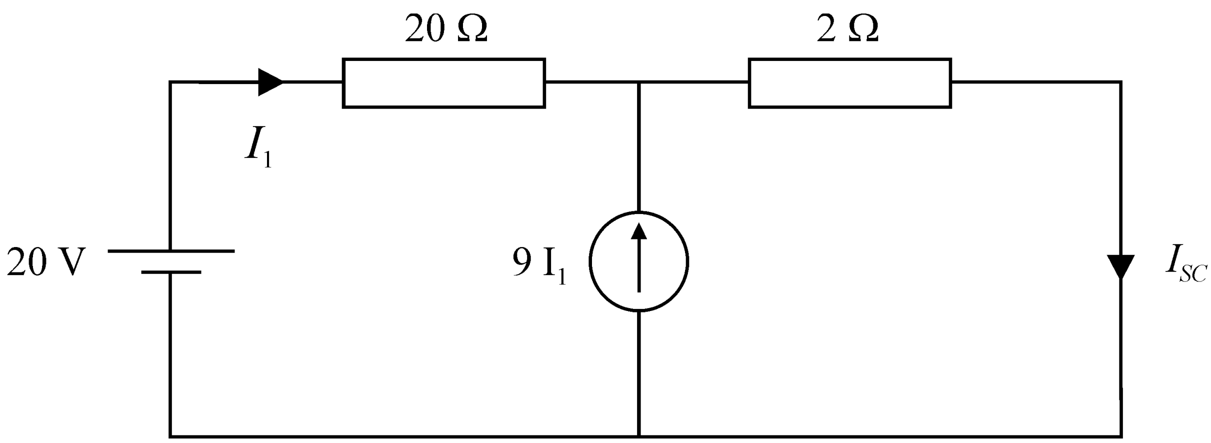


We begin by calculating the open circuit voltage and note that, since a current of 9I + I = 10I flows through both the 2Ω and 12Ω resistor under these conditions that $V_{oc} = 12 \cdot 10I = 120I$. We may find I by writing a KVL equation around the outer perimeter as

$$20 = 20I + 2\ \ 10I + 12\ \ 10I$$

hence $$V_{oc} = 120I = 120.20/60 = 15\text{V}.$$

We now need to find the short circuit current. When a short is connected between a and b all the current flows through this and not the 12Ω resistor. Thus the circuit to analyse becomes

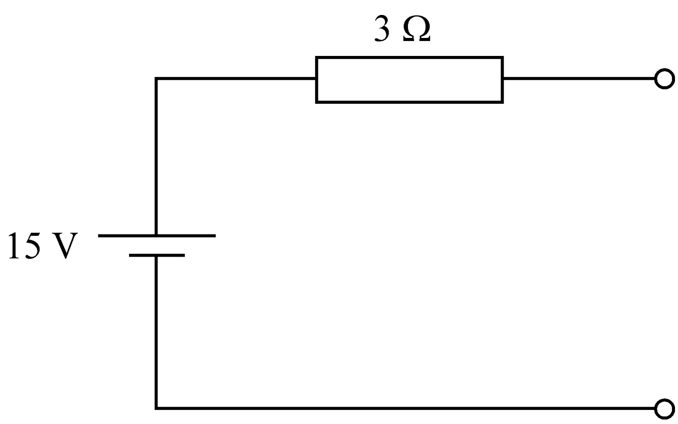


where we have re-labelled the current I as I_1, to emphasise that it is now a different value since we are considering a different circuit.

Again a current $10I_1$ flows through the 2Ω resistor and the short circuit. Therefore $I_{sc} = 10I_1$. We find I_1 by again writing a KVL equation around the outer loop as

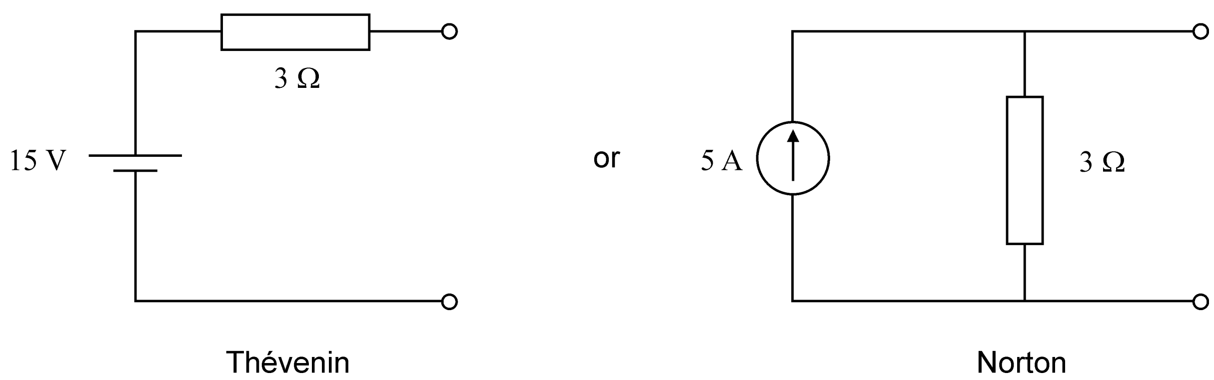
$$20 = 20I_1 + 2\quad 10I_1$$

Whence $I_1 = 0.5A$ and so $I_{sc} = 5A$; $R_{eq} = V_{oc}/I_{sc} = 15/5 = 3\Omega$. Thus the Thévenin equivalent is



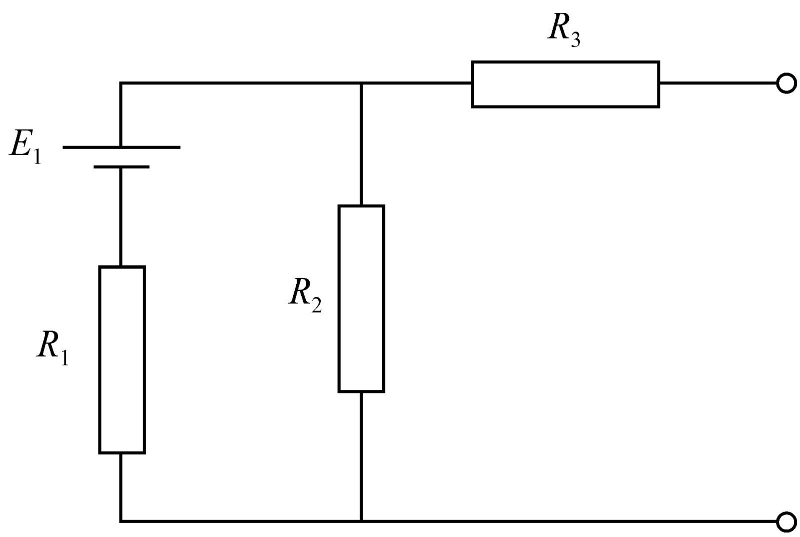
17. Transformation between Thévenin and Norton

Since we can use either a Thévenin circuit or a Norton circuit, we can replace one by the other whenever we feel like it:



Sometimes this is rather helpful. We already noted that this may be useful in connection with Loop and Nodal analysis, and it is a useful trick in lots of situations.

Let's look at the recent example again:



E_1 in series with R_1 comprise a Thévenin circuit and can be replaced by the Norton circuit of $I_N = E_1/R_1$ in parallel with R_1.

This Norton R_1 is now in parallel to R_2 and can be combined become $R_x = R_1R_2/(R_1+R_2)$.

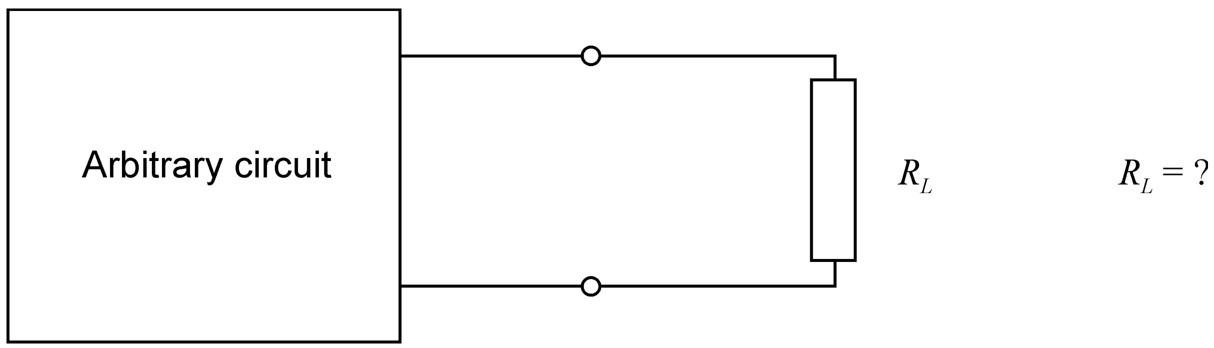
Now the new Norton circuit of I_N in parallel with R_x can be transformed to a Thévenin circuit $E_T = I_N/R_x$ in series with R_x.

Finally we combine the series resistors R_x and R_3 and obtain the same answer as before.

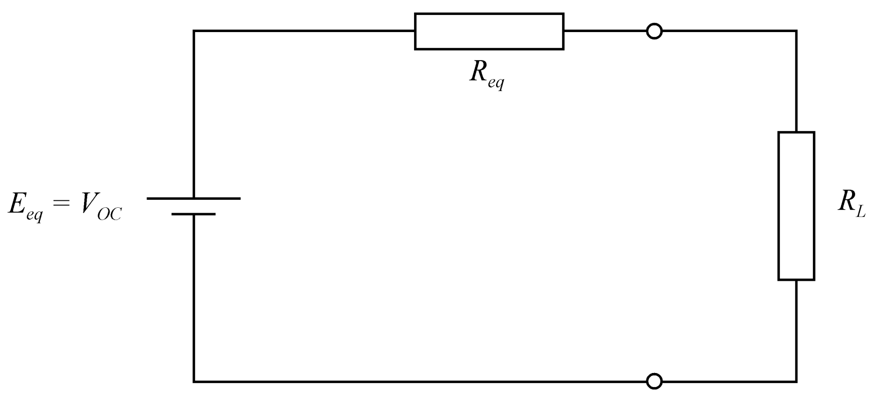
Draw the circuits corresponding to this development and verify for yourself that the answer is the same as before.

18. Maximum power transfer

Suppose we have an arbitrary circuit containing many sources and resistors connected together in as complicated a fashion as we like or, perhaps, dislike. Suppose further that we connect this circuit to a resistor, R_L, (the load resistor) and we want to know, *e.g.*, what value of R_L to choose so that the resistor will absorb the maximum amount of power.



If we represent the circuit by its Thévenin equivalent the problem becomes trivial.



The current flowing in the circuit $I = V_{oc}/(R_{eq} + R_L)$ and the power dissipated in the load, P, is given by

$$P = I^2 R_L = V_{oc}^2 \frac{R_L}{\left(R_{eq} + R_L\right)^2}$$

In order to maximise this power as a function of R_L we need to solve

$$\frac{dP}{dR_L} = 0$$

The differentiation[1] is routine and yields

$$R_L = R_{eq}$$

which you can check gives the maximum value. Thus the maximum power is delivered when the load resistance is *equal* to the Thévenin (or Norton) resistance.

[1] I hope that you will find this differentiation easy, but in case your maths is rusty from the summer break, you may find it slightly easier to minimize 1/P.

19. Input and Output Impedance & Voltage Divider

This brings us to the realization that connecting one circuit to another places demands on the output of the driving circuit (the one that's providing the voltage and current) and the load circuit (the one that's receiving the voltage and current). In general the term we use to describe the ability of a circuit to deliver a given current at a given voltage is the ***output impedance*** ("impedance" is a generalisation of the concept of resistance to include also capacitors and inductors – see Circuit Analysis II). Similarly, for a receiving circuit, the sinking of a certain current at a given input voltage is termed the ***input impedance.***

The Input Impedance is merely the input voltage divided by the input current. So we can write (using the symbol *Z* for generalized impedance, for DC conditions it is obviously *R*):

$$Z_{in} = \frac{V_{in}}{I_{in}}$$

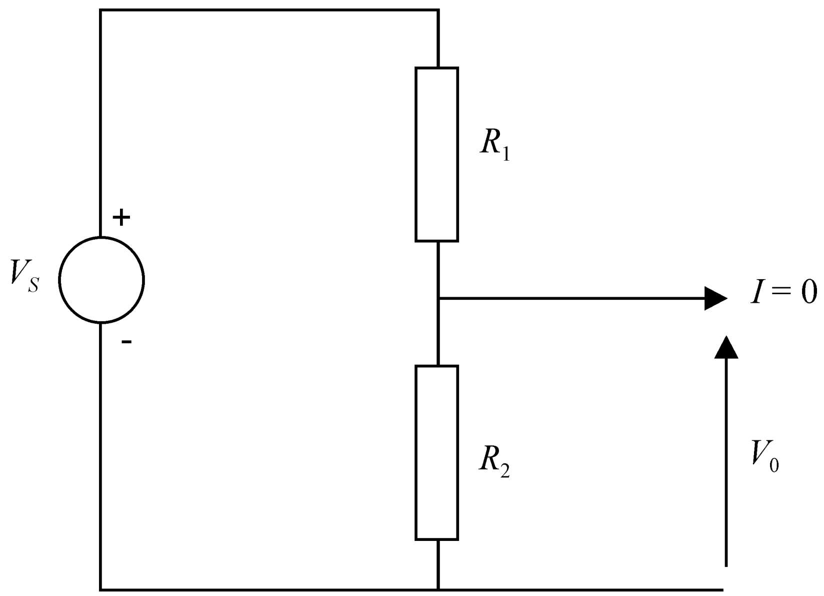
The Output Impedance is simply the resistance of the equivalent source:

$$Z_{out} = \frac{V_{oc}}{I_{sc}} = \frac{E_{eq}}{E_{eq}/R_{eq}} = R_{eq}$$

Therefore a re-statement of the power transfer theorem is that output impedance of the source must equal input impedance of the load for maximum power transfer. This principal is extremely important in a wide range of applications.

On the other hand, in many designs, we would like the output voltage of a circuit to be specified irrespective of what load we may apply to it. In this case, we arrange that $Z_{out} \ll Z_{in}$ in which case $V_{out} \approx E_{eq}$.

A common example of this is the **voltage divider** (or **"potential divider")**, which will see very frequently in the future.



The current flowing through the two resistors is $V_s/(R_1 + R_2)$ and hence the voltage appearing across the resistor R_2 is given by

$$V_0 = \frac{R_2}{R_1 + R_2} V_s$$

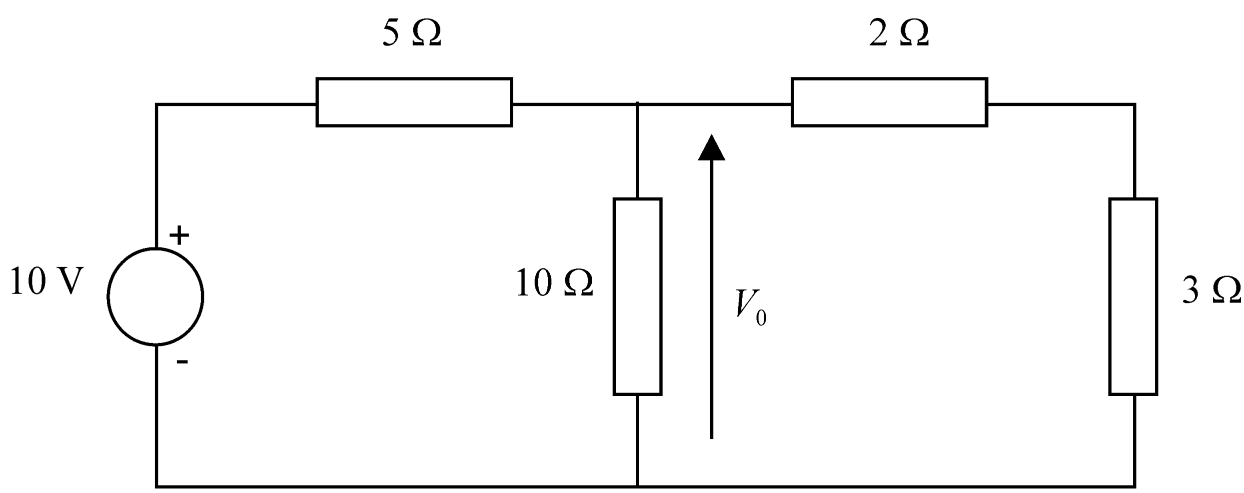
Of course, this is only true if $I \approx 0$, or, equivalently if R_1, R_2 << any load resistance that is added.

20. Redrawing your circuit

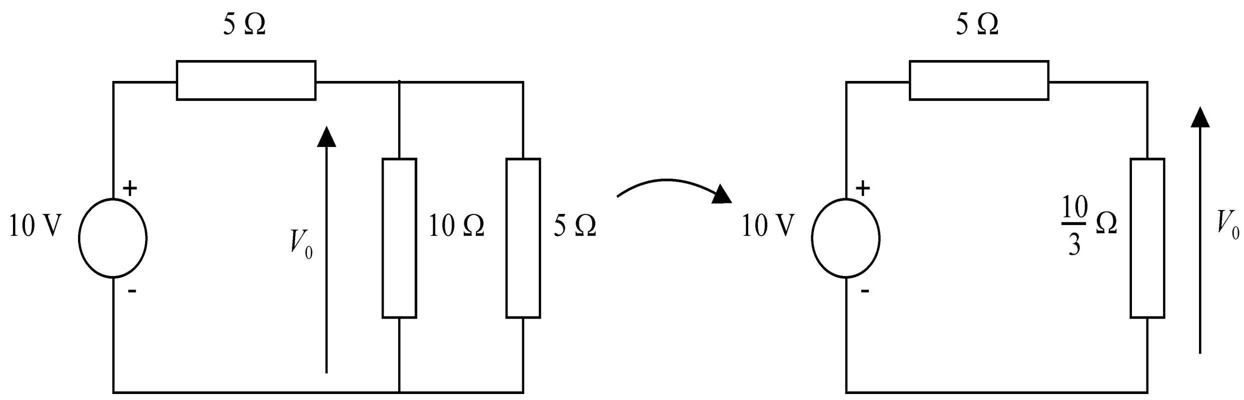
Sometimes circuits are drawn in a haphazard way that makes them very difficult to understand. It is often a good idea to **re-draw** the circuit in a way that makes it clear ***to you*** what is going on.

By convention, we tend to draw circuit diagrams with sources on the left supplying loads to the right, and we tend to draw our zero reference (or ground) as the bottom line in our circuit.

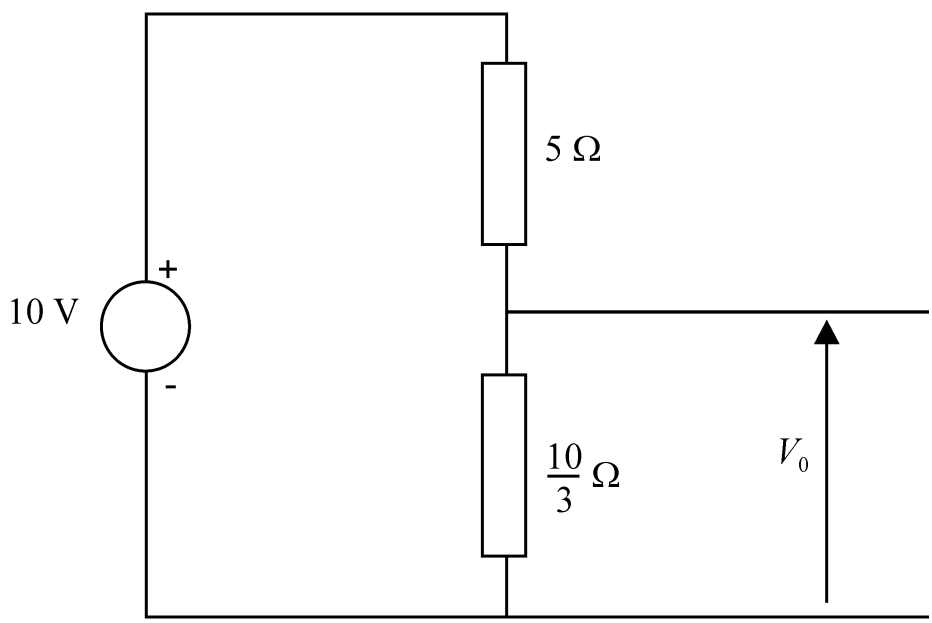
Sometimes we do a bit of simplification when we are redrawing the circuit. For example, suppose we are required to find the voltage V_0 across the 10Ω resistor.



We first note that the 2Ω and 3Ω resistors are connected in series and so may be replaced by a single 5Ω resistor. Thus the circuit can be redrawn as on the left below:



However the 10Ω and 5Ω resistors are now seen to be connected in parallel with the desired voltage V_0 appearing across both of them. This combination may be replaced by a single resistor of value $\frac{10\ .\ 5}{10+5} = 10/3\Omega$ as shown in the right hand diagram. This may, if necessary, be further re-drawn in the standard voltage divider configuration as



from which $V_0 = \frac{10/3}{5+10/3} \cdot 10 = 4.0V$

21. Summary

By now you should have an understanding of the basic concepts of electrical circuits and have developed skills in simple circuit analysis. These tools and concepts will be developed and applied to more complicated circuits and devices in the sequel “Circuit Analysis II”. Before we can do that, you need to learn the mathematics of complex numbers, differential equations and frequency analysis!