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OPTIMAL CONTROL OF SINGULARLY PERTURBED LINEAR SYSTEMS AND APPLICATIONS

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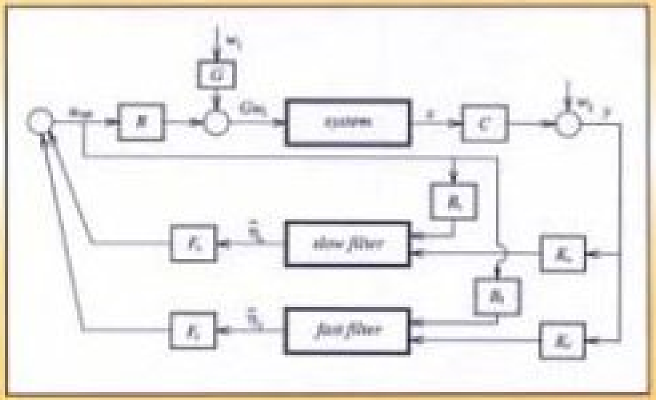

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HIGH-ACCURACY TECHNIQUES

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Series Introduction

Many textbooks have been written on control engineering, describing new techniques for controlling systems, or new and better ways of mathematically formulating existing methods to solve the ever-increasing complex problems faced by practicing engineers. However, few of these books fully address the applications aspects of control engineering. It is the intention of this new series to redress this situation.

The series will stress applications issues, and not just the mathematics of control engineering. It will provide texts that present not only both new and well-established techniques, but also detailed examples of the application of these methods to the solution of real-world problems. The authors will be drawn from both the academic world and the relevant applications sectors.

There are already many exciting examples of the application of control techniques in the established fields of electrical, mechanical (including aerospace), and chemical engineering. We have only to look around in today's highly automated society to see the use of advanced robotics techniques in the manufacturing industries; the use of automated control and navigation systems in air and surface transport systems; the increasing use of intelligent control systems in the many artifacts available to the domestic consumer market; and the reliable supply of water, gas, and electrical power to the domestic consumer and to industry. However, there are currently many challenging problems that could benefit from wider exposure to the applicability of control methodologies, and the systematic systems-oriented basis inherent in the application of control techniques.

This new series will present books that draw on expertise from both the academic world and the applications domains, and will be useful not only as academically recommended course texts but also as handbooks for practitioners in many applications domains.

Neil Munro

Preface

This book is intended for engineers, mathematicians, physicists, and computer scientists interested in control theory and its applications. The book studies a special class of linear control systems known as singularly perturbed systems. These systems, characterized by the presence of slow and fast variables, describe dynamics of many real physical systems such as aircraft, power systems, nuclear reactors, chemical reactors, electrical circuits, dc and induction motors, robots, large space flexible structures, synchronous machines, cars, and so on. In general, all systems that have components of different physical nature (for example, electrical vs mechanical components) display slow-fast phenomena. Mathematically, the slow and fast phenomena are characterized by small and large time constants, or by system eigenvalues that are clustered into two disjoint sets. The slow system variables correspond to the set of the eigenvalues closer to the imaginary axis, and the fast system variables are represented by the set of the eigenvalues that are far from the imaginary axis.

Mathematical theory of singularly perturbed systems, also known as theory of differential equations with small parameters multiplying certain derivatives, originated in the papers of A. Tikhonov, J. Levin, and N. Levinson at the beginning of the 1950s and gained its maturity during the 1960s and 1970s in the works of A. Vasileva, V. Butuzov, W. Wasow, F. Hoppensteadt, R. O'Malley, K. Chang, and their coworkers. One of the most important results in mathematical theory of linear singularly perturbed systems is the development of the Chang transformation, which facilitates exact decomposition of singularly perturbed *linear systems* into pure-slow and pure-fast subsystems.

Singularly perturbed *control* systems became an extensive subject of research by the end of the 1960s and during the 1970s in the papers published by P. Kokotovic and his graduate students, among whom P. Sannuti, J. Chow, H. Khalil, and D. Young were the most productive. A large number of journal papers on singularly perturbed control systems were published during the 1970s, 1980s, and 1990s in both mathematics and engineering. The approaches taken in engineering during the 1970s and 1980s were based on the expansion methods (power series, asymptotic expansions, Taylor series)—the methods developed by previously mentioned mathematicians. The approaches were in most cases accurate only with an $O(\epsilon)$ accuracy, where ϵ is a small positive singular perturbation parameter. Generating higher order expansions for those methods has been analytically cumbersome and numerically inefficient, especially for higher dimensional control systems. Even more, it has been demonstrated in the control literature that for some applications the $O(\epsilon)$ accuracy either is not sufficient or in some cases has not solved the considered singularly perturbed control problems.

The development of high accuracy efficient techniques for singularly perturbed control systems started in the middle of the 1980s along the lines of slow-fast integral manifold theory of E. Fridman, V. Sobolev, and V. Strygin, and the recursive approach based on fixed-point iterations of Z. Gajic. At the beginning of the 1990s, the fixed-point recursive approach culminated in the so-called Hamiltonian approach for the *exact* slow-fast decomposition of singularly perturbed, linear-quadratic, deterministic and stochastic, optimal control and filtering problems.

This book represents a comprehensive overview of the current state of knowledge of the Hamiltonian approach to singularly perturbed linear optimal control systems. The book devises a unique powerful method whose core result seems to be repeated and slightly modified over and over again, while the method solves more and more challenging problems of linear singularly perturbed optimal continuous- and discrete-time systems, including nonstandard singularly perturbed linear systems, high gain feedback and cheap control problems, small measurement noise problem, sampled data systems, and H_∞ optimization and filtering problems. It should be pointed out that some related problems still remain unsolved, especially corresponding problems in the discrete-time domain, and the optimization problems over a finite horizon. These problems are identified in the book as open problems for future research.

The presentation is based on the recent research work of the authors and their coworkers. The book presents a unified theme about the exact pure-slow pure-fast decoupling of the corresponding optimal control problems owing to the existence of a transformation that exactly decouples the nonlinear algebraic Riccati equation into the pure-slow and pure-fast, reduced-order, independent, algebraic Riccati equations. In that direction, we show how to study independently in slow and fast time scales with very high accuracy (theoretically with perfect accuracy) deterministic and stochastic, continuous- and discrete-time, linear-quadratic optimal control and filtering problems. Some of the results presented appear for the first time in this book.

Each chapter is organized to represent an independent entity so that the readers interested in a particular class of linear singularly perturbed control systems can find complete information within the particular chapter. The book demonstrates theoretical results on many practical applications using examples from aerospace, chemical, electrical, and automotive industries. In that direction, we apply theoretical results obtained to optimal control and filtering problems represented by real mathematical models of aircraft, cars, power systems, chemical reactors, and so on.

The authors are thankful for support and contributions from their colleagues, Professors S. Bingulac, E. Fridman, V. Kecman, M. Qureshi, X. Shen, and W. Su, Drs. Z. Aganovic and H. Hsieh, and doctoral students C. Coumarbatch, D. Popescu, and V. Radisavljevic.

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1

Introduction

This book represents a continuation of the work on parallel algorithms for optimal control of singularly perturbed linear deterministic and stochastic control problems of (Gajic and Shen, 1993). The book presents the most efficient methods for solving exactly (or with very high accuracy) optimal control and filtering problems of singularly perturbed linear systems by removing numerical ill-conditioning and obtaining well-conditioned, reduced-order, *exact* (or highly accurate) pure-slow and pure-fast subproblems. The class of problems solvable by the newly presented techniques are steady state linear-quadratic optimal control and filtering problems whose Hamiltonian matrices under appropriate scaling and permutation preserve singularly perturbed forms such that they can be block diagonalized into pure-slow and pure-fast Hamiltonian matrices. We call this method the *Hamiltonian approach to singularly perturbed linear control systems*. The problems presently solvable by the Hamiltonian method are: linear-quadratic optimal regulator and Kalman filter in continuous- and discrete-time domains, optimal open-loop control of continuous- and discrete-time linear systems, multimodeling estimation and control, H_∞ optimal control and filtering of linear systems, linear-quadratic zero-sum differential games, linear-quadratic high gain, cheap control, small measurement noise problems, sampled data control systems, nonstandard linear singularly perturbed systems, and limited classes of finite horizon optimal linear control and filtering problems. Some other classes of linear-quadratic type optimal control problems that can be solved by the methodology considered in this book may emerge in the near future.

The algorithms of (Gajic and Shen, 1993), termed *parallel recursive fixed-point approach to singular perturbations*, remain powerful tools for all other classes of singularly perturbed linear and bilinear optimal control systems for which high order of accuracy is required, especially for finite horizon linear-quadratic optimization problems, output feedback, and steady state Nash and Stackelberg differential games, and steady state jump parameter linear stochastic systems.

It is well documented in the literature that theory of singular perturbations has been a very fruitful control engineering research area in the last thirty five years, (Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986; Gajic and Shen, 1993). Singularly perturbed control systems have been studied using Taylor series, asymptotic expansions, and power-series methods—techniques traditionally used in mathematics for studying singularly perturbed systems of differential equations (O'Malley, 1974, 1991). Being nonrecursive in nature, these expansion methods become very cumbersome and computationally very expensive (the size of computations required can be considerable) when a higher order of accuracy, $O\left(\epsilon^k\right)^*$, $k \geq 2$, where ϵ represents a small positive singular perturbation parameter, is required. In such cases, the advantage of using the expansion methods (important theoretical tools to remove ill-conditioning of the original problems and produce well-conditioned, approximate, reduced-order subproblems) is questionable from the numerical point of view, and sometimes these methods are almost not applicable in practice (Grodt and Gajic, 1988; Gajic *et al.*, 1989; Skataric and Gajic, 1992; Mizukami and Suzumura, 1993). It can be said, in general, that until the middle of the 1980s, the singular perturbation methods used in control engineering were efficient for solving control problems for which only the accuracy of $O(\epsilon)$ was sufficient. In the era of an increased application of modern control theory results in real physical systems, this is a serious problem. Even more, the standard statement of singular perturbation theory that the approximate results obtained are valid under the assumption that "it exists ϵ small enough" limits the practical implementation of $O(\epsilon)$-theory of singular perturbations to real physical systems. In order to broaden the class of real physical systems for which theory of singular perturba-

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* $O\left(\epsilon^k\right)$ is defined by $O\left(\epsilon^k\right) < c\epsilon^k$, where c is a bounded constant and k is a real number.

tions can be successfully applied, the development of $O(\epsilon^k)$ theory is a necessary requirement.

The high accuracy approach to singularly perturbed control systems started in the middle of the 1980s in the works of (Gajic, 1984; Fridman and Strygin, 1984; Sobolev 1984; Srtygin *et al.*, 1985; Fridman, 1986) and culminated in a series of papers by Gajic, Fridman and their coworkers that cover a broad range of linear and nonlinear optimal control problems.

The work of Gajic, based on the fixed point iterations, originally known under the name of the recursive approach to singular perturbations (Gajic, 1986; Gajic *et al.*, 1990) culminated in the so-called Hamiltonian approach for the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal regulator (Su, 1990; Su *et al.*, 1992a) and Kalman filter (Gajic and Lim, 1994). In (Su *et al.*, 1992a), the algebraic Riccati equation of the singularly perturbed control problem has been completely and exactly decomposed into the reduced-order pure-slow and pure-fast algebraic Riccati equations. The closed-loop decomposition results of (Su *et al.*, 1992a) are valid only at the steady state. However, for the finite time optimization the corresponding open-loop exact decomposition result is obtained in (Su *et al.*, 1992b). These results allow us to perform exact optimal regulation and filtering completely from subsystem levels, (Gajic and Lim 1994). The extension to the linear-quadratic optimal Gaussian control in continuous-time has been done in (Lim, 1994, 1999). The corresponding discrete-time results of (Su *et al.*, 1992a; Gajic and Lim, 1994) are obtained in (Lim, 1994; Lim *et al.*, 1995; Gajic *et al.*, 1995). Most recently, in (Kecman *et al.*, 1999), the eigenvector approach is proposed for simultaneous block diagonalization of the Hamiltonian matrix of singularly perturbed systems and solution of the associated algebraic equations.

The work of (Fridman and Strygin, 1984; Sobolev 1984; Strygin *et al.*, 1985; Fridman, 1990a,b) based on slow-fast integral manifold theory resulted also in the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal control problems as demonstrated in (Fridman, 1990a, 1995, 1996a). It remains an open question whether or not the integral manifold approach to decomposition of singularly linear-quadratic control problems (Fridman, 1995, 1996a) can be directly related to the results obtained in (Su *et al.*, 1992a). It should be pointed out that the results of (Fridman, 1995, 1996a) hold for both finite-time (horizon) and steady state optimization problems. The slow-fast integral manifold theory is extended to linear singularly perturbed systems with delays (Fridman, 1990a, 1996b) and to some classes of nonlinear optimal control problems in (Sobolev, 1984; Fridman, 1999, 2000; Fridman and Shaked, 2000). The essence of slow-fast integral manifold theory for optimal singularly perturbed linear systems will be given in Chapter 8.

Since the recursive approach is an integral part of the Hamiltonian approach, in the following, we review the main results obtained using the recursive approach to singularly perturbed linear and bilinear control systems (Gajic and Shen, 1993). In addition, in this chapter we indicate the main features of the Hamiltonian approach and give the book's overview.

Many examples of real physical control systems are included throughout the book. All examples are solved using MATLAB and SIMULINK.*

1.1 The Recursive Approach

Singularly perturbed systems display multiple time scale phenomena, hence they are parallel in nature and very well suited for parallel computations and parallel processing of information. The recursive methods for singularly perturbed control systems are presented in (Gajic and Shen, 1993) in the spirit of parallel and distributed computations (Bertsekas and Tsitsiklis, 1991) and parallel processing of information in terms of reduced-order, independent, approximate slow and fast filters. The recursive techniques are applicable to almost all important areas of optimal linear control theory, in the context of continuous and discrete, deterministic and stochastic, singularly perturbed systems (Gajic and Shen, 1993). A generalization of the recursive methods to the optimal control of singularly perturbed bilinear systems has been done in (Aganovic, 1993; Aganovic and Gajic, 1995).

The development of the recursive techniques based on the fixed-point reduced-order parallel algorithms that produce $O\left(\epsilon^k\right), k = 1, 2, 3, ...,$ accuracy for singularly perturbed linear-quadratic optimal *steady state* control problems has been done in (Gajic, 1984; Gajic, 1986; Grodt and Gajic, 1988; Gajic *et al.*, 1990; Shen, 1990; Gajic and Shen 1991a,b; Qureshi, 1992; Qureshi *et al.*, 1992; Skataric and Gajic, 1992). The

* MATLAB and SIMULINK are registered trademarks of The Math Works Inc.

corresponding methods for *finite horizon* optimal linear-quadratic singularly perturbed control systems have been developed in (Grodt and Gajic, 1988; Su *et al.*, 1992b; Shen, 1990, 1992). A special class of singularly perturbed systems known as quasi singularly perturbed systems has been considered in (Skataric and Gajic, 1992). The recursive approach is extended to the linear-quadratic Stackelberg games in (Mizukami and Suzumura, 1993), to a special class of linear-quadratic Nash games in (Skataric and Petrovic, 1998), and to optimal control of linear jump parameter stochastic systems in (Borno and Gajic, 1995). The application of the above recursive approach to the linear-quadratic regulator for loop shaping for high frequency compensation is considered (Geray and Looze, 1996). The recursive approach to singularly perturbed H_∞ control problems is presented in (Mukaidani *et al.*, 1999).

The main algebraic equations of linear steady state optimal control theory for singularly perturbed systems, namely, the algebraic Lyapunov and Riccati equations have been studied in (Gajic, 1986; Shen *et al.*, 1991; Gajic and Shen, 1991b, 1993; Gajic and Qureshi, 1995). The recursive algorithms for the solution of these equations have been obtained in the most general case when the problem matrices are functions of a small perturbation parameter. The numerical decomposition has been achieved so that only low-order systems are involved in algebraic computations. The introduced recursive methods are of the fixed point type and can be implemented as parallel synchronous algorithms. Both continuous- and discrete-time versions of the algebraic Lyapunov and Riccati equations are studied in the above references. It should be pointed out that the partitioned form of the singularly perturbed algebraic Riccati equation is very complicated in the discrete-time domain. That problem can be overcome by using a bilinear transformation of (Kondo and Furuta, 1986), that is applicable under a mild assumption, so that the solution of the discrete algebraic Riccati equation of singularly perturbed systems can be obtained by using results derived for the corresponding continuous-time algebraic Riccati equations (Gajic and Shen 1991b). It is shown that the singular perturbation recursive methods converge with the rate of convergence of $O(\epsilon)$. Hence, each fixed-point iteration improves (theoretically) the order of accuracy by $O(\epsilon)$. Having obtained approximate solutions of the algebraic Lyapunov and Riccati equations, corresponding approximate linear-quadratic optimal control problems can be solved in terms of these solutions. Below, we outline the basic steps in recursive procedures of solving the algebraic Lyapunov and Riccati equations of singularly perturbed linear systems.

Lyapunov Equations

The algebraic Lyapunov equation of singularly perturbed systems has the form

$$KA^T + AK + GG^T = 0 \tag{1.1}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon}G_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & K_2 \\ K_2^T & \frac{1}{\epsilon}K_3 \end{bmatrix} \tag{1.2}$$

with the following dimensions $A_1^{n_1 \times n_1}$, $A_4^{n_2 \times n_2}, G_1^{n_1 \times m}$, $G_2^{n_2 \times m}, K_1^{n_1 \times n_1}, K_3^{n_2 \times n_2}$, $n = n_1 + n_2$, compatible to the system decomposition into n_1 slow and n_2 fast state space variables. In addition, the matrix A_4 is assumed to be nonsingular, which is the standard assumption in theory of singularly perturbed linear systems (Kokotovic *et al.*, 1986). The small positive singular perturbation parameter ϵ affects the coefficient matrices in (1.1) in such a way that it makes the problem of solving the algebraic Lyapunov equation (1.1) numerically ill-conditioned. Using the recursive fixed-point algorithm, the numerical ill-conditioning can be removed and the solution of (1.1) can be obtained in terms of reduced-order well conditioned algebraic Lyapunov equations corresponding, respectively, to the slow and fast system state variables as follows. The $O(\epsilon)$ accurate solutions are obtained from

$$\begin{gathered} K_1^{(0)}A_0^T + A_0K_1^{(0)} + G_0G_0^T = 0 \\ A_0 = A_1 - A_2A_4^{-1}A_3, \quad G_0 = G_1 - A_2A_4^{-1}G_2 \\ K_3^{(0)}A_4^T + A_4K_3^{(0)} + G_2G_2^T = 0 \\ K_2^{(0)} = -\left(A_2K_3^{(0)} + K_1^{(0)}A_3^T + G_1G_2^T\right)A_4^{-T} \end{gathered} \tag{1.3a}$$

The required solution is given by

$$K_j = K_j^{(0)} + \epsilon E_j, \quad j = 1, 2, 3 \tag{1.3b}$$

where

$$E_1^{(i+1)} A_0^T + A_0 E_1^{(i+1)} = \mathcal{L}_1\left(E_2^{(i)}\right)$$

$$E_3^{(i+1)} A_4^T + A_4 E_3^{(i+1)} = \mathcal{L}_3\left(E_2^{(i)}\right)$$

$$E_2^{(i+1)} = \mathcal{L}_3\left(E_1^{(i+1)}, E_2^{(i)}, E_3^{(i+1)}\right)$$

$$E_2^{(0)} = 0, \ i = 0, 1, 2, ...$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are linear matrix functions. Note that this decomposition requires stability of matrices A_0 and A_4, which guarantees the existence of the unique solutions for $K_1^{(0)}$ and $K_3^{(0)}$ (Gajic and Qureshi, 1995). The above fixed point algorithm has the rate of convergence of $O(\epsilon)$, which indicates that after k iterations the accuracy of $O\left(\epsilon^k\right)$ is achieved (Gajic *et al.*, 1990; Gajic and Shen, 1993).

It is interesting to point out that the fixed-point algorithm for solving the discrete-time domain algebraic Lyapunov equation defined by

$$A^T P A - P = -Q \tag{1.4}$$

with the problem matrices having the singularly perturbed structure established in (Litkouhi and Khalil, 1984, 1985) as

$$A = \begin{bmatrix} I + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, P = \begin{bmatrix} \frac{1}{\epsilon} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{1.5}$$

leads to the *continuous-time* reduced-order well-conditioned algebraic Lyapunov equations for $P_1^{(0)}$ and E_1 with $P_1 = P_1^{(0)} + \epsilon E_1$. The reduced-order fast subsystem algebraic Lyapunov equations (for $P_3^{(0)}$ and E_3) remain the discrete-time ones, and the equations for $P_2^{(0)}$ and E_2 are linear, easily solvable, reduced-order linear algebraic equations (see Gajic and Shen, 1993; Gajic and Qureshi, 1995).

Riccati Equations

The continuous-time, regulator type, algebraic Riccati equation, whose positive semidefinite stabilizing solution solves the linear-quadratic optimization problem of singularly perturbed systems, is defined by

$$PA + A^T P + Q - PSP = 0 \tag{1.6}$$

where

$$P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, S = \begin{bmatrix} S_1 & \frac{1}{\epsilon} S_2 \\ \frac{1}{\epsilon} S_2^T & \frac{1}{\epsilon^2} S_3 \end{bmatrix} \tag{1.7}$$

The $O(\epsilon)$–approximate slow and fast algebraic Riccati equations of (1.6)–(1.7) are derived in (Chow and Kokotovic, 1976)

$$\begin{aligned} P_1^{(0)} A_s + A_s^T P_1^{(0)} + Q_s - P_1^{(0)} S_s P_1^{(0)} &= 0 \\ P_3^{(0)} A_4 + A_4^T P_3^{(0)} + Q_3 - P_3^{(0)} S_3 P_3^{(0)} &= 0 \end{aligned} \tag{1.8}$$

In addition

$$P_2^{(0)} = \mathcal{F}_2\left(P_1^{(0)}, P_3^{(0)}\right) \tag{1.9}$$

where $\mathcal{F}_2$ is a linear matrix function. The slow subsystem matrices introduced in (1.8) can be obtained by using the corresponding formulas of (Chow and Kokotovic, 1976) or even in a simpler manner by using the results of (Wang and Frank, 1992). The required solutions of approximate slow and fast Riccati equations (1.8) exist under the standard stabilizability-detectability conditions imposed on the slow and fast subsystems. It has been known since the work of (Chow and Kokotovic, 1976; see also Gajic, 1986) that

$$P_j = P_j^{(0)} + \epsilon E_j, \quad j = 1, 2, 3 \tag{1.10}$$

The derivations of the error equations, E_j, and the development of the fixed-point algorithm for their efficient numerical solution were done in (Gajic, 1986). The corresponding algorithm has the form

$$\begin{aligned} E_1^{(i+1)} D_1 + D_1^T E_1^{(i+1)} &= H_1\left(E_1^{(i)}, E_2^{(i)}, E_3^{(i)}, \epsilon\right) \\ E_3^{(i+1)} D_3 + D_3^T E_3^{(i+1)} &= H_3\left(E_2^{(i)}, E_3^{(i)}, \epsilon\right) \\ E_2^{(i+1)} &= H_2\left(E_1^{(i+1)}, E_2^{(i)}, E_3^{(i+1)}, \epsilon\right) \\ E_1^{(0)} = 0, \ E_2^{(0)} = 0, \ E_3^{(0)} &= 0, \ i = 0, 1, 2, \ldots \end{aligned} \tag{1.11}$$

This algorithm requires the solution of the reduced-order algebraic Lyapunov equations, where $H_j, j = 1, 2, 3$, are quadratic matrix functions.

The newly defined matrices in (1.8)-(1.11) and the corresponding matrix functions can be found in (Gajic, 1986; Gajic and Shen, 1993). The algorithm of (1.11) converges to the exact solution for the error equations with the rate of convergence of $O(\epsilon)$.

Chang Transformation

The celebrated Chang transformation decouples *exactly linear* singularly perturbed systems into independent slow and fast subsystems, (Chang, 1972). This transformation plays the fundamental role in modern theory of singularly perturbed control systems, and it is an essential part of the high accuracy techniques based on the recursive fixed-point and Hamiltonian approaches. The required matrices for the Chang transformation can be obtained from two coupled matrix algebraic equations. Algorithms that efficiently generate solutions of these algebraic equations are derived in (Kokotovic *et al.*, 1980; Gajic, 1986; Grodt and Gajic, 1988). The highlights of the Chang transformation are presented below. A detailed coverage of the Chang transformation in both continuous- and discrete-time domains including the new versions of the Chang transformation can be found in (Gajic and Shen, 1993, Chapter 3).

The linear singularly perturbed deterministic system defined by

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) \\ \epsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) \end{aligned} \tag{1.12}$$

where ϵ is a small positive singular perturbation parameter is transformed via the Chang transformation into pure-slow and pure-fast subsystems

$$\begin{aligned} \dot{\eta}_1(t) &= (A_1 - A_2L)\eta_1(t) \\ \epsilon\dot{\eta}_2(t) &= (A_4 + \epsilon LA_2)\eta_2(t) \end{aligned} \tag{1.13}$$

with

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon HL & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.14}$$

where the matrices L and H satisfy

$$\begin{aligned} A_4L - A_3 - \epsilon L(A_1 - A_2L) &= 0 \\ HA_4 - A_2 + \epsilon(HLA_2 - A_1H + A_2LH) &= 0 \end{aligned} \tag{1.15}$$

The unique solution of equations (1.15) exist, by the implicit function theorem, for sufficiently small values of ϵ under the assumption that the matrix A_4 is nonsingular. The fixed point algorithm for solving (1.15) is derived in (Kokotovic *et al.*, 1980). It has the following form

$$A_4L^{(i+1)} = A_3 + \epsilon L^{(i)}\Big(A_1 - A_2L^{(i)}\Big)$$

$$H^{(i+1)}A_4 = A_2 - \epsilon\Big(H^{(i)}L^{(i)}A_2 - A_1H^{(i)} + A_2L^{(i)}H^{(i)}\Big) \tag{1.16}$$

$$L^{(0)} = A_4^{-1}A_3, \quad H^{(0)} = A_2A_4^{-1}, \quad i = 0, 1, 2, ...$$

This algorithm converges with the rate of convergence of $O(\epsilon)$. The Newton method for iterative solution of equations (1.15) is derived in (Grodt and Gajic, 1988) as follows

$$D_1^{(i)}L^{(i+1)} + L^{(i+1)}D_2^{(i)} = Q^{(i)}$$

$$D_1^{(i)} = A_4 + \epsilon L^{(i)}A_2, \quad D_2^{(i)} = -\epsilon\Big(A_1 - A_2L^{(i)}\Big) \tag{1.17}$$

$$Q^{(i)} = A_3 + \epsilon L^{(i)}A_2L^{(i)}, \quad L^{(0)} = A_4^{-1}A_3$$

This algorithm has quadratic convergence of $O\left(\epsilon^{2^i}\right)$. Having obtained the solution for L with the required accuracy, the H-equation can be solved directly as a linear Sylvester equation

$$H^{(i+1)}D_1^{(i+1)} + D_2^{(i+1)}H^{(i+1)} = A_2 \tag{1.18}$$

Note that the L- and H-equations have to be solved sequentially, first the L-equation and then the H-equation.

A new version of the Chang transformation is developed in (Qureshi and Gajic, 1992), in which equations for L and H are completely decoupled, hence it can be solved in parallel. The transformation of (Qureshi and Gajic, 1992) is given by

$$\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} = \begin{bmatrix}I_{n_1} & -\epsilon L_{new}\\ -H_{new} & I_{n_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{1.19}$$

where the matrices L_{new} and H_{new} satisfy

$$\begin{aligned} L_{new}A_4 - A_2 - \epsilon(A_1 - L_{new}A_3)L_{new} &= 0\\ H_{new}A_4 + A_3 - \epsilon H_{new}(A_1 + A_2H_{new}) &= 0 \end{aligned} \tag{1.20}$$

These equations can be also solved efficiently either by using the fixed point iterations or the Newton method as demonstrated above.

The discrete-time version of the original Chang's results are obtained in (Borno, 1994). The discrete-time versions of the Chang transformation can be found in several papers, see for example (Gajic and Shen, 1991a). The new version of the discrete-time Chang transformation is presented in (Gajic and Shen, 1993, Borno, 1994). Note that the Chang transformation also exactly decouples the singularly perturbed algebraic, differential, and difference Lyapunov equations into the corresponding reduced-order, independent, pure-slow and pure-fast, Lyapunov-type equations (Gajic and Shen, 1993).

In the following we survey linear optimal control and filtering problems of singularly perturbed systems based on the recursive approach.

Linear Optimal Control and Filtering Problems

Based on the previously established results on the Chang transformation and algebraic Lyapunov and Riccati equations, the linear-quadratic Gaussian control problem of singularly perturbed systems has been solved in (Khalil and Gajic, 1984; Gajic, 1986). The approach of (Khalil and Gajic, 1984) was based on the Taylor series expansions, and the approach of (Gajic, 1986) on the fixed point iterations for calculating the solutions of the Chang decoupling equations, solutions of the regulator and filter algebraic Riccati equations, and the coefficients of the optimal (and approximate) filter and controller. The main idea of the corresponding slow-fast decoupling and parallelism of the optimal control and filtering tasks are explained in the next paragraph.

The linear singularly perturbed stochastic control system with the corresponding measurements is defined by

$$
\begin{aligned}
\dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + B_1u(t) + G_1w(t) \\
\epsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) + B_2u(t) + G_2w(t) \\
y(t) &= C_1x_1(t) + C_2x_2(t) + v(t)
\end{aligned}
\tag{1.21}
$$

where $u(t)$ represents an m-dimensional control vector, $y(t)$ are r_2-dimensional system measurements, $w(t) \in \Re^{r_1}$ and $v(t) \in \Re^{r_2}$ are system and measurement disturbances assumed to be zero-mean, stationary, mutually uncorrelated, Gaussian white noise stochastic processes with intensities $W > 0$ and $V > 0$. The quadratic performance criterion to be minimized is

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{ \int_0^{t_f} \left[\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T Q \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + u^T(t) R u(t) \right] dt \right\} \tag{1.22}$$

It has been shown in (Khalil and Gajic, 1984; Gajic, 1986) that the optimal solution to the above linear-quadratic stochastic optimization problem can be obtained in terms of reduced-order slow and fast Kalman filters as follows

$$\begin{aligned} u_{opt}(t) &= -f_1 \hat{\eta}_1(t) - f_2 \hat{\eta}_2(t) \\ \dot{\hat{\eta}}_s(t) &= a_s \hat{\eta}_1(t) + g_s \nu(t) \\ \epsilon \dot{\hat{\eta}}_f(t) &= a_f \hat{\eta}_1(t) + g_f \nu(t) \\ \nu(t) &= y(t) - c_1 \hat{\eta}_s(t) - c_2 \hat{\eta}_f(t) \end{aligned} \tag{1.23}$$

Note that the slow and fast Kalman filters are driven by the innovation process $\nu(t)$, hence communications of optimal slow and fast estimates are needed in order to form the innovation process. The proposed method allows parallel processing of information and reduces considerably the size of required off-line and on-line computations, since it introduces full parallelism in the design procedure. The corresponding singularly perturbed discrete stochastic problem is considered in (Shen, 1990; Gajic and Shen, 1991a). It will be shown in the next section that a decomposition technique based on the Hamiltonian approach can produce independent slow and fast Kalman filters driven by the system measurements.

The recursive approach to deterministic output feedback control of singularly perturbed linear systems is considered in (Gajic *et al.*, 1989). The well-defined recursive numerical technique for the solution of nonlinear algebraic matrix equations, associated with the output feedback control problem of singularly perturbed systems has been developed. The numerical slow-fast decomposition is achieved so that only low-order systems are involved in algebraic computations. The paper (Gajic *et al.*, 1989) shows that each iteration step of the fixed-point algorithm improves the accuracy by an order of magnitude, that is, the accuracy of $O(\epsilon^k)$ can be obtained by performing only k iterations. This represents the significant improvement since all results on the output feedback control problems for singularly perturbed systems have been previously obtained with the accuracy of $O(\epsilon)$ only. As an example, an industrial important reactor—fluid catalytic cracker—demonstrates the efficiency of the proposed algorithm and the failure of $O(\epsilon)$ theory. The static output feedback control problem for discrete linear singularly perturbed stochastic systems is studied in (Qureshi *et al.*, 1992), where a recursive algorithm is presented to solve the corresponding nonlinear algebraic equations. The algorithm removes the ill-conditioning by decomposing the higher order equations into lower order equations corresponding to the fast and slow time scales.

In (Skataric and Gajic, 1992; Skataric, 1993) a special class of linear control systems represented by the standard singularly perturbed system matrix and with the control input matrix having three different nonstandard forms is studied. The obtained results are quite simplified (compared to the standard singularly perturbed control systems), and in one case the optimal solution of the algebraic Riccati equation is completely determined in terms of the reduced-order algebraic Lyapunov equations. The proposed method is successfully applied to the reduced-order design of optimal controllers for a hydro power plant (Skataric and Gajic, 1992). It is important to point out that the solutions to the real 11th- and 14th-order hydro power control systems are obtained by the presented reduced-order parallel algorithms, but the global method fails to produce the answers in both cases.

The problem of high gain feedback and cheap control is studied in (Huey *et al.*, 1993). The singular perturbation methodology is used to describe the problems under consideration (Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986). The reduced-order parallel algorithm producing any arbitrary order of accuracy is obtained under the control oriented assumptions. It is important to point out that in the presented methodology there is no need to study the high gain feedback and cheap control problems in the limit when a small parameter ϵ tends to zero. This avoids the impulsive behavior and the presence of singular controls. The efficiency of the algorithm obtained is demonstrated on an example of a flexible space structure.

The recursive approach to singularly perturbed linear control systems is extended in the work of (Aganovic, 1993; Aganovic and Gajic, 1995) to bilinear control systems. The composite near-optimal control of singularly perturbed bilinear systems is obtained in (Aganovic and Gajic, 1995) by combining the ideas from (Chow and Kokotovic, 1976) and (Cebuhar and Constanza, 1984). Obtained results are demonstrated on a fourth-order induction motor drives. The extension of the near-optimal composite control to the optimal reduced-order control is also considered. The reduced-order open-loop optimal control of singularly perturbed bilinear systems is presented in (Aganovic and Gajic, 1995).

More details about the recursive approach can be found in the book by Gajic and Shen, 1993. It should be emphasized that the recursive approach remains an important research area especially for more complex linear-quadratic optimal control problems such as Nash and Stackelberg games, H_∞-optimization, jump parameter stochastic systems, output feedback control (Mizukami and Suzumura, 1993; Borno and Gajic, 1995; Mukaidani *et al.*, 1999).

Recursive Approach of Derbel

In the recent results of (Derbel *et al.*, 1994a; Derbel and Kamoun, 1994, 1996), the coefficients for the Taylor series expansion of some singularly perturbed control problems have been obtained in a recursive manner. The approach has been successfully used for order reduction of linear singularly perturbed systems (Alimi and Derbel, 1995; Derbel and Kamoun, 1996). An extension of this approach is presented in (Toumi, 1998). The corresponding applications to synchronous machines have been considered in (Derbel *et al.*, 1994b; Djemel *et al.*, 1996). The results of Derbel and his coworkers might help that the classical approach to singularly perturbed linear control systems, based either on Taylor series or asymptotic expansions or power-series methods, becomes a high accurate technique. In that direction, an extension of the results obtained is needed to cover various types of optimal control and filtering problems of linear singularly perturbed systems.

1.2 The Essence of the Hamiltonian Approach

The Hamiltonian approach to singularly perturbed linear optimal control systems is based on block diagonalization of the Hamiltonian matrix compatible to its slow-fast structure. It represents the most efficient method for solving singularly perturbed linear optimal control and filtering problems, including their nonstandard (Kecman and Gajic, 1999) and H_∞ formulations (Fridman, 1996a, Hsieh and Gajic, 1998, Lim and Gajic, 2000, Fridman and Shaked, 2000). One of the main results of this method is the complete and exact decomposition of the corresponding algebraic Riccati equations into the reduced-order, completely independent, pure-slow and pure-fast, algebraic Riccati equations.

It is well known that the algebraic Riccati equations can be studied in terms of corresponding Hamiltonian matrices. The Hamiltonian matrix that corresponds to the algebraic Riccati equation (1.6) has the following form (Kwakernaak and Sivan, 1972)

$$\mathbf{H} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \tag{1.24}$$

It is easy to show that the eigenvalues of $\mathbf{H}$ are symmetrically distributed with respect to the imaginary axis. Namely, using the similarity transformation (it preserves the matrix eigenvalues) of the form

$$\mathbf{T} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{1.25}$$

it is easy to show that

$$\mathbf{T}^{-1}\mathbf{H}\mathbf{T} = -\begin{bmatrix} A^T & -Q \\ -S & -A \end{bmatrix} = -\mathbf{H}^T \tag{1.26}$$

Since the eigenvalues of the matrix transpose are equal to the original matrix eigenvalues, we conclude that $\mathbf{H}$ and $-\mathbf{H}$ must have the same eigenvalues, which can happen only of the eigenvalues of the matrix $\mathbf{H}$ are symmetrically distributed with respect to the imaginary axis.

The Hamiltonian matrices of singularly perturbed linear optimal control systems retain the singularly perturbed form by interchanging and appropriately scaling some of the state and costate variables, hence they can be block diagonalized via the decoupling transformation of (Chang, 1972). The block diagonalization procedure produces the pure-slow and pure-fast Hamiltonian matrices, each corresponding to the pure-slow and pure-fast nonsymmetric algebraic Riccati equations. The nonsymmetric algebraic Riccati equations obtained can be easily solved via the Newton method since their $O(\epsilon)$ perturbations are symmetric algebraic Riccati equations whose solutions represent excellent initial guesses for the Newton method.

The algebraic Riccati equation of singularly perturbed *continuous-time* control systems, defined by (1.6)-(1.7), can be written as

$$A^TP + PA + Q - PSP = 0, \quad \dim\{P\} = n_s + n_f = n_1 + n_2$$
$$A = O\left(\frac{1}{\epsilon}\right), \quad S = O\left(\frac{1}{\epsilon^2}\right), \quad Q = O(1) \tag{1.27}$$

This algebraic equation is numerically ill-conditioned due to the special structures of matrices A and S. By using the Hamiltonian approach, this equation is completely and exactly decomposed into two reduced-order algebraic Riccati equations corresponding to slow and fast time scales, (Su *et al.*, 1992a), as

$$\begin{aligned} &P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0, \quad \dim\{P_s\} = n_s = n_1 \\ &P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0, \quad \dim\{P_f\} = n_f = n_2 \end{aligned} \tag{1.28}$$

with $a_i, b_i = O(1), i = 1,2,3,4$. Equations (1.28) are well-conditioned, reduced-order, pure-slow and pure-fast, algebraic Riccati equations.

The pure-slow and pure-fast algebraic Riccati equations (1.28) are nonsymmetric, but their $O(\epsilon)$ perturbations are symmetric, that is

$$\begin{gathered} P_s^{(0)}a_1^{(0)} + a_1^{(0)^T}P_s^{(0)} + a_3^{(0)} - P_s^{(0)}a_2^{(0)}P_s^{(0)} = 0, \quad \dim\left\{P_s^{(0)}\right\} = n_s \\ a_i = a_i^{(0)} + O(\epsilon), \quad i = 1,2,3,4 \\ a_4^{(0)} = -a_1^{(0)T}, \quad a_3^{(0)} = a_3^{(0)^T}, \quad a_2^{(0)} = a_2^{(0)^T} \\ \\ P_f^{(0)}b_1^{(0)} + b_1^{(0)^T}P_f^{(0)} + b_3^{(0)} - P_f^{(0)}b_2^{(0)}P_f^{(0)} = 0, \quad \dim\left\{P_f^{(0)}\right\} = n_f \\ b_i = b_i^{(0)} + O(\epsilon), \quad i = 1,2,3,4 \\ b_4^{(0)} = -b_1^{(0)^T}, \quad b_3^{(0)} = b_3^{(0)^T}, \quad b_2^{(0)} = b_2^{(0)^T} \end{gathered} \tag{1.29}$$

The approximate slow and fast algebraic Riccati equations obtained in (1.29) are identical to the corresponding algebraic Riccati equations of (Chow and Kokotovic, 1976). The unique positive semidefinite stabilizing solutions of (1.29) exist under standard stabilizability-detectability assumptions imposed on slow and fast subsystems. These solutions can be easily obtained by using any standard method for solving the symmetric algebraic Riccati equation. It is shown in (Su *et al.*, 1992a) that the Newton method is very efficient for solving the pure-slow and pure-fast, nonsymmetric algebraic Riccati equations. The Newton method for solving (1.28) is given in terms of Lyapunov iterations (Su *et al.*, 1992a)

$$\begin{gathered}
P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)} \\
P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)} \\
i = 0, 1, 2, \ldots
\end{gathered} \tag{1.30}$$

It converges in four to five iterations.

Having found the solutions for P_s and P_f, the required solution of (1.27) is obtained as a simple matrix function of P_s and P_f, (Su *et al.*, 1992a), that is

$$P = \mathcal{F}_C(P_s, P_f) \tag{1.31}$$

The above results about the exact pure-slow and pure-fast decomposition of the algebraic Riccati equation applied to the linear-quadratic optimal control problem defined by

$$\begin{aligned}
\dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) \\
\epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t)
\end{aligned} \tag{1.32}$$

and

$$J = \min_u \int_0^\infty \left[\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T Q \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + u^T(t) R u(t)\right] dt \tag{1.33}$$

where

$$u(x(t)) = -F_1 x_1(t) - F_2 x_2(t) \tag{1.34}$$

lead to the following fundamental lemma, which can be deduced from the results of (Su *et al.*, 1992a).

Lemma 1.1 *Consider the optimal closed-loop linear system*

$$\begin{aligned}
\dot{x}_1(t) &= (A_1 - B_1 F_1) x_1(t) + (A_2 - B_1 F_2) x_2(t) \\
\epsilon \dot{x}_2(t) &= (A_3 - B_2 F_1) x_1(t) + (A_4 - B_2 F_2) x_2(t)
\end{aligned} \tag{1.35}$$

Under standard stabilizability-detectability conditions imposed on the slow and fast subsystems, there exists a nonsingular transformation $\mathbf{T}$

$$\begin{bmatrix} \xi_s(t) \\ \xi_f(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.36}$$

such that

$$\begin{aligned} \dot{\xi}_s(t) &= (a_1 + a_2 P_s)\xi_s(t) \\ \epsilon\dot{\xi}_f(t) &= (b_1 + b_2 P_f)\xi_f(t) \end{aligned} \tag{1.37}$$

where P_s *and* P_f *are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati equations (1.28). The nonsingular* transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\Pi_1 + \Pi_2 P) \tag{1.38}$$

Known matrices Π_1, Π_2 *are given in terms of solutions of the Chang decoupling equations. Even more, the global solution* P *can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations using formula (2.28).*

◇

The above decomposition of the algebraic Riccati equations and their variants produce new insights into the slow-fast time scale optimal filtering and control for several important problems of linear singularly perturbed systems, which will be demonstrated in the subsequent chapters of this book. The results to be presented are characterized by well-conditioning, complete and exact decoupling of the slow and fast time scale phenomena, reduction in off-line and on-line computational requirements, and parallel and distributed processing of information. It should be emphasized that the complete results have been obtained for the steady state optimal control and filtering, and that the pure-slow pure-fast decoupling of the corresponding differential/difference Riccati equations appears to be both analytically and computationally much more difficult.

The new pure-slow and pure-fast filter decomposition scheme is used in (Lim, 1994; Lim *et al.*, 1995) to solve the linear-quadratic optimal Gaussian control problem defined in (1.21)-(1.22). The optimal solution is obtained in the form of

$$u_{opt}(t) = -F_s\hat{\eta}_s(t) - F_f\hat{\eta}_f(t)$$

$$\dot{\hat{\eta}}_s(t) = (a_{1F} + a_{2F}P_{sF})^T\hat{\eta}_s(t) + B_su(t) + K_sy(t) \tag{1.39}$$

$$\epsilon\dot{\hat{\eta}}_f(t) = (b_{1F} + b_{2F}P_{fF})^T\hat{\eta}_1(t) + B_fu(t) + K_fy(t)$$

where $a_{1F}, a_{2F}, b_{1F}, b_{2F}, P_{sF}, P_{fF}$ come from the pure-slow and pure-fast filter algebraic equations dual to (1.28), that is (Gajic and Lim, 1994)

$$P_{sF}a_{1F} - a_{4F}P_{sF} - a_{3F} + P_{sF}a_{2F}P_{sF} = 0, \quad \dim\{P_{sF}\} = n_1$$

$$P_{fF}b_{1F} - b_{4F}P_{fF} - b_{3F} + P_{fF}b_{2F}P_{fF} = 0, \quad \dim\{P_{fF}\} = n_2 \tag{1.40}$$

In a similar manner, the numerically ill-conditioned algebraic Riccati equation of singularly perturbed *discrete-time* control systems, given by

$$P_d = A_d^TP_dA_d + Q_d - A_d^TP_dB_d\left(R_d + B_d^TP_dB_d\right)^{-1}B_d^TP_dA_d = 0$$

$$\dim\{P_d\} = n_s + n_f = n_1 + n_2 \tag{1.41}$$

is exactly solved in terms of two reduced-order algebraic *continuous-time* Riccati equations corresponding to slow and fast time scales having the form of (1.28). The sought solution of (1.41) is obtained, under the standard stabilizability-detectability assumptions imposed on the slow and fast subsystems, as a simple matrix function of the solutions of the pure-slow and pure-fast algebraic Riccati equations (Lim, 1994; Lim *et al.*, 1995; Gajic *et al.*, 1995)

$$P_d = f_d\left(P_s^d, P_f^d\right) \tag{1.42}$$

The decomposition of the discrete-time algebraic Riccati equation in terms of independent, reduced-order, continuous-time algebraic Riccati equations represents a pretty powerful result since the continuous-time algebraic Riccati equation is much better understood and easier for solving than the discrete-time algebraic Riccati equation.

The finite time optimal open-loop control problems (linear two-point boundary value problem) for singularly perturbed control systems can also be studied from the Hamiltonian approach point of view. The main idea is to exploit the reduced-order subsystems to find efficiently the optimal open-loop control in the new coordinates. The change of coordinates is particularly important for singularly perturbed systems, where the original, numerically ill-conditioned, two-point boundary value problem is transformed into the pure-slow and pure-fast reduced-order completely decoupled initial value problems. By doing this, the stiffness of the singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined system of linear algebraic equations (Su *et al.*, 1992b). The study of the open-loop control problem presented for singularly perturbed continuous-time systems is extended to the corresponding discrete-time domain in (Qureshi *et al.*, 1991; Qureshi, 1992). It is important to notice that the complete results for the finite time closed-loop slow-fast decoupling, based on the Hamiltonian approach and given in terms of the Riccati differential (difference) equations, have not been obtained yet. The partial results in that direction are available in the paper by (Grodt and Gajic, 1988). The study is underway to find the complete answer to this important slow-fast time scale decoupling problem.

Variable structure singularly perturbed systems, including a design technique for a sliding surface, have been considered in (Su, 1999).

Finally, we want to point out that in some instances, the presentation of the last two sections of this chapter, follows closely the recent overview paper of Gajic *et al.*, 1999.

1.3 Overview

This book is organized in nine chapters. Chapters 2 and 3 present the most fundamental results about the Hamiltonian approach for the standard linear-quadratic optimal control problems of singularly perturbed linear control systems, respectively, in continuous- and discrete-time domains. The results for the closed-loop optimal control are presented at the steady state, and the open-loop optimal control is studied for the finite time optimization period. The presentation is mostly based on the results of (Su *et al.*, 1992a,b; Qureshi *et al.*, 1991; Qureshi, 1992; Gajic and Lim, 1994; Lim, 1994; Lim *et al.*, 1995; Gajic *et al.*, 1995). These chapters include, either new results or new interpretations and improvements of previously published results.

Chapter 4 on the optimal control and filtering of the multimodeling structures of linear dynamic systems is based on the very recent research work of Professor Z. Gajic and his doctoral student C. Coumarbatch. Some results of Chapter 4 are presented in (Coumarbatch, 2000; Coumarbatch and Gajic, 2000a,b). The results obtained have been successfully applied to the Kalman filtering problem of a passenger car under road disturbances and to the optimal control problem of a power system. This chapter gives important fundamentals that can be extended to the development of the exact Pareto multimodeling strategies (Khalil and Kokotovic, 1978) and the exact decoupling for the quasi-decentralized multimodeling estimation (Gajic, 1988).

In Chapter 5, we present the results on the continuous-time H_∞ optimal filtering and control of linear singularly perturbed systems by following the results of (Hsieh and Gajic, 1998; Lim and Gajic, 2000). We indicate the difficulties encountered in the H_∞ optimization of singularly perturbed linear systems, and the necessity for an additional transformation to exactly decouple the slow and fast H_∞ filters—in contrast to the results of Chapter 2, where the same transformation decouples both the algebraic filter Riccati equation and the corresponding Kalman filter.

The open-loop cheap (and high gain) control problem in continuous-time and the problem of complete decomposition of the corresponding algebraic "cheap (high gain)" Riccati equation into the reduced-order pure-slow and pure-fast Riccati equations are studied in Chapter 6. It is interested to point out that the dual results to the cheap (high gain) optimal linear-quadratic control problems in the discrete-time domain are not available in the literature. However, in this chapter we present the results for the special class of discrete-time cheap optimal control problems, sampled data control systems, by following the work of (Popescu and Gajic, 1999). The dual problem to the continuous-time cheap control problem is the small measurement noise optimal continuous-time Kalman filtering problem. It is interesting to point out that the small measurement noise under certain assumptions induces the slow-fast time scale separation of the system state space variables. We present this problem according to the work of (Aganovic *et al.*, 1995), and show how to decouple exactly the corresponding pure-slow and pure-fast Kalman filters.

Chapter 7 deals with the most recent developments in the field of the Hamiltonian approach to singularly perturbed linear control systems. In a recent paper (Kecman *et al.*, 1999), the eigenvector method is introduced for simultaneous pure-fast/pure-slow block diagonalization of the Hamiltonian matrix and the solution of Chang's algebraic equations required for such a decomposition.

In Chapter 8 we discuss some additional topics related to the Hamiltonian approach to singularly perturbed linear control systems. In that respect, we present an extension of the main results of Chapter 2 to the nonstandard continuous-time singularly perturbed linear control systems (Khalil, 1989; Wang *et al.*, 1994). We also discuss the finite time (horizon) closed-loop optimization and indicate difficulties encountered in dealing with the corresponding boundary layer terms. Finally, we review the main results of (Fridman and Strygin, 1984; Sobolev 1984; Strygin *et al.*, 1985; Fridman, 1986) and follow-up work of Fridman, on slow-fast manifold theory and discuss similarities and differences with the Hamiltonian approach.

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2

Continuous-Time Linear Optimal Control Systems

In this chapter, we show how to exactly decompose the algebraic Riccati equation of continuous-time singularly perturbed control systems into two reduced-order algebraic Riccati equations corresponding to slow and fast time scales. The reduced-order algebraic Riccati equations obtained are nonsymmetric. The Newton algorithm is very efficient for solving these nonsymmetric algebraic Riccati equations since excellent initial guesses are readily available from the reduced-order, symmetric, algebraic Riccati equations that represent $O(\epsilon)$ perturbations of the nonsymmetric, reduced-order, pure-slow and pure-fast, algebraic Riccati equations. Due to complete and exact decomposition of the Riccati equation, and due to order-reduction, we have obtained an efficient parallel algorithm for solving this equation—the most important equation of the linear-quadratic optimal control and filtering theory.

The procedure used for the time-scale decomposition of the algebraic Riccati equations into the pure-slow and pure-fast algebraic Riccati equation facilitates a new insight into optimal filtering and control problems of singularly perturbed linear systems. It will be demonstrated in the subsequent sections of this chapter that corresponding reduced-order linear optimal filters and controllers are completely and exactly decoupled. The slow/fast filters and controllers work in parallel and process information independently in slow and fast time scales with the corresponding sampling rates—the slow ones with the slow sampling rate and the fast ones with the fast sampling rate.

The material presented in this chapter is based on the recent research work of the authors and their coworkers. Some of the results presented are either improvements over those already existing in the literature or appear for the first time in this book.

2.1 Exact Decomposition of the Algebraic Riccati Equation

A linear singularly perturbed control system is given by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t), \qquad && x_1(t_0) = x_{10} \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t), \qquad && x_2(t_0) = x_{20} \end{aligned} \tag{2.1}$$

where $x_i(t) \in \Re^{n_i}$, $i = 1, 2$, $u(t) \in \Re^m$ are state and control variables, respectively, and ϵ is a small positive parameter. As the parameter ϵ tends to zero, the solution behaves nonuniformly, producing the so-called singularly perturbed stiff problem (huge slope for the fast state variable at the initial time). It is the standard assumption in theory of singularly perturbed linear control systems that the fast subsystem matrix A_4 is nonsingular (Kokotovic *et al.*, 1986). Hence, the following assumption is imposed.

Assumption 2.1: The fast subsystem matrix A_4 is nonsingular.

The singularly perturbed linear systems that satisfy Assumption 2.1 are called standard singularly perturbed linear systems, in contrast to nonstandard singularly perturbed linear systems for which the fast subsystem matrix A_4 is singular. More about nonstandard singularly perturbed systems will be presented in Section 8.1. In this section, in Appendix 2.2, we will show only that the presented methodology can be extended to nonstandard singularly perturbed linear systems.

With (2.1), consider the quadratic performance criterion to be minimized by the choice of the optimal control strategy

$$\min_u J = \min_u \frac{1}{2} \int_{t_0}^{\infty} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R\, u(t) \right\} dt \tag{2.2}$$

with positive definite R and positive semidefinite Q. The open-loop optimal control problem of (2.1)-(2.2) has the solution

$$u(t) = -R^{-1}B^T p(t) \tag{2.3}$$

where $p(t) \in \Re^{n_1+n_2}$ is the costate variable satisfying (Kwakernaak and Sivan, 1972)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{2.4}$$

with

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix} \tag{2.5}$$

$$B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}, \quad S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{1}{\epsilon}Z \\ \frac{1}{\epsilon}Z^T & \frac{1}{\epsilon^2}S_2 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$S_1 = B_1 R^{-1} B_1^T, \quad S_2 = B_2 R^{-1} B_2^T, \quad Z = B_1 R^{-1} B_2^T$$

The optimal closed-loop (feedback) control law has the very-well known form

$$u(x(t)) = -R^{-1}B^T P x(t) = -Fx(t) \tag{2.6}$$

where P satisfies the algebraic regulator Riccati equation given by

$$0 = PA + A^T P + Q - PSP, \quad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix} \tag{2.7}$$

The positive semidefinite stabilizing solution of the algebraic Riccati equation (2.7) exists under the standard stabilizability-detectability conditions (Kwakernaak and Sivan, 1972). Note that the stabilizability-detectability condition is weaker than the controllability-observability condition.

Assumption 2.2: The triple $(A, B, \mathrm{Chol}(Q))$ is both stabilizable and detectable, where $\mathrm{Chol}(Q)$ denotes the Cholesky factor of the matrix Q.*

* It is common in the control literature to require that the triple $(A, B, \sqrt{Q})$ is stabilizable detectable. Note that $\sqrt{Q}$ means $Q = M^2$. Notation used in Assumption 2.2 is more rigorous since the Cholesky factor of a positive semidefinite matrix is defined by $Q = C^T C$.

Our main goal is to find the solution of (2.7) in terms of solutions of the reduced-order, pure-slow and pure-fast, algebraic Riccati equations. It is well known that the solution of the Riccati equation can be obtained from the Hamiltonian matrix. In the following, we show that for singularly perturbed systems, the Hamiltonian matrix retains the singularly perturbed form by interchanging and appropriately scaling some state and costate variables, hence it can be block diagonalized via the nonsingular transformations of (Chang, 1972; Qureshi and Gajic, 1992).

Partitioning and scaling $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_1(t) \in \Re^{n_1}$ and $p_2(t) \in \Re^{n_2}$, and interchanging second and third rows in (2.4), we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{2.8}$$

where

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$
$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \qquad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{2.9}$$

It is important to note that (2.8) retains the singular perturbation form. Also, the matrix T_4 is the Hamiltonian matrix of the fast subsystem, and it is nonsingular under stabilizability-detectability conditions imposed on the fast subsystem.

Assumption 2.3: The triple (A_4, B_2, q_2) is stabilizable-detectable.

The celebrated transformation (Chang, 1972), used for decomposition of linear singularly perturbed systems, is defined by

$$\mathbf{T}_1 = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}, \quad \mathbf{T}_1^{-1} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon LH \end{bmatrix} \tag{2.10}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{2.11}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \tag{2.12}$$

The unique solutions of (2.11) and (2.12) exist for sufficiently small values of ϵ under condition that T_4 is nonsingular, that is, under Assumption

2.3. These algebraic equations can be solved as linear algebraic equations using either the fixed-point algorithm of (Kokotovic *et al.*, 1980) or the Newton method of (Grodt and Gajic, 1988). The corresponding algorithms for solving the L-equation are given respectively by

$$\begin{aligned} L^{(i+1)} &= L^{(0)} + \epsilon T_4^{-1} L^{(i)}\left(T_1 - T_2 L^{(i)}\right) \\ L^{(0)} &= T_4^{-1} T_3, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{2.13}$$

$$\begin{gathered} D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)}, \quad L^{(0)} = T_4^{-1} T_3, \quad i = 0, 1, 2, \ldots \\ D_1^{(i)} = T_4 + \epsilon L^{(i)} T_2, \quad D_2^{(i)} = -\epsilon\left(T_1 - T_2 L^{(i)}\right) \\ Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)} \end{gathered} \tag{2.14}$$

Note that the Newton method converges quadratically, hence if it converges, it requires on average only four to five iterations. However, the fixed-point iterations converge linearly and sometimes require a lot of iterations. In addition, the L-equation can be efficiently solved by using the eigenvector method (Kecman *et al.*, 1999) and the Taylor series expansions (Derbel *et al.*, 1994). Once the solution for the L-equation is obtained, the H-equation can be solved as a linear Sylvester equation, or recursively as

$$\begin{aligned} H^{(i+1)} &= T_2 (T_4 + \epsilon L T_2)^{-1} + \epsilon (T_1 - T_2 L) H^{(i)} (T_4 + \epsilon L T_2)^{-1} \\ H^{(0)} &= T_2 T_4^{-1}, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{2.15}$$

Introduce the notation

$$\begin{bmatrix} x_1(t) \\ p_1(t) \end{bmatrix} = w(t), \qquad \begin{bmatrix} x_2(t) \\ p_2(t) \end{bmatrix} = \lambda(t) \tag{2.16}$$

The transformation (2.10) applied to (2.8) produces two completely decoupled subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t) \tag{2.17}$$

and

$$\epsilon\dot{\xi}(t) = (T_4 + \epsilon L T_2)\xi(t) \tag{2.18}$$

where

$$\begin{bmatrix} \eta(t) \\ \xi(t) \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} w(t) \\ \lambda(t) \end{bmatrix} \tag{2.19}$$

The rearrangement and modification of variables in (2.8) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ \epsilon p_2(t) \end{bmatrix} = E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{2.20}$$

Note that the inverse of E_1 can be easily obtained analytically, hence, this matrix is not numerically ill-conditioned with respect to the matrix inversion for small values of ϵ.

Combining (2.19) and (2.20), we obtain the relationship between the original and new coordinates as

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \\ \eta_2(t) \\ \xi_2(t) \end{bmatrix} = E_2^T \mathbf{T_1} E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{2.21}$$

where E_2 is a permutation matrix of the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{2.22}$$

Since at steady state $p(t) = Px(t)$, where P satisfies the algebraic Riccati equation (2.7), it follows that

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t), \quad \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(t) \tag{2.23}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for new systems (2.17) and

(2.18); that is

$$\begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \tag{2.24}$$

Then, (2.23) and (2.24) yield

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{2.25}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1}\mathbf{T}_1^{-1}E_2 = \Pi^{-1} = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} \tag{2.26}$$

where

$$E_1^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{n_2} \end{bmatrix} \tag{2.27}$$

which yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1} \tag{2.28}$$

It is shown in Appendix 2.1 that the matrix inversions defined in (2.25) and (2.28) exist for sufficiently small values of ϵ.

Partitioning (2.17) and (2.18) as

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = (T_1 - T_2 L) \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{2.29}$$

$$\epsilon \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \tag{2.30}$$

and using (2.24) yield two reduced-order, nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations

$$0 = P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s \tag{2.31}$$

$$0 = P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f \tag{2.32}$$

where

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} A_1 - A_2 L_1 + Z L_3 & -S_1 - A_2 L_2 + Z L_4 \\ -Q_1 + Q_2 L_1 + A_3^T L_3 & -A_1^T + Q_2 L_2 + A_3^T L_4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} A_4 + \epsilon(L_1 A_2 - L_2 Q_2) & -S_2 - \epsilon\big(L_1 Z + L_2 A_3^T\big) \\ -Q_3 + \epsilon(L_3 A_2 - L_4 Q_2) & -A_4^T - \epsilon\big(L_3 Z + L_4 A_3^T\big) \end{bmatrix} \tag{2.33}$$

with

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}$$

The nonsymmetric algebraic Riccati equation was studied by several researchers, see for example (Medanic, 1982) and references therein. An algorithm for solving a general nonsymmetric algebraic Riccati equation was derived in (Avramovic, 1979, see also Avramovic *et al.*, 1980).

The pure-slow algebraic Riccati equation (2.31) is nonsymmetric and it is given by

$$\begin{aligned} &P_s(A_1 - A_2 L_1 + Z L_3) + \big(A_1 - L_2^T Q_2^T - L_4^T A_3\big)^T P_s \\ &\quad + \big(Q_1 - Q_2 L_1 - A_3^T L_3\big) - P_s(S_1 + A_2 L_2 - Z L_4) P_s = 0 \end{aligned} \tag{2.34}$$

The pure-fast algebraic Riccati equation (2.32) is also nonsymmetric

$$\begin{aligned} &P_f(A_4 + \epsilon(L_1 A_2 - L_2 Q_2)) + \big(A_4^T + \epsilon\big(L_3 Z + L_4 A_3^T\big)\big) P_f \\ &+ (Q_3 - \epsilon(L_3 A_2 - L_4 Q_2)) - P_f\big(S_2 + \epsilon\big(L_1 Z + L_2 A_3^T\big)\big) P_f = 0 \end{aligned} \tag{2.35}$$

but its $O(\epsilon)$ approximation is symmetric, that is

$$P_f A_4 + A_4^T P_f + Q_3 - P_f S_2 P_f + O(\epsilon) = 0 \tag{2.36}$$

From (2.36) one can obtain an $O(\epsilon)$ approximation for P_f as

$$P_f^{(0)} A_4 + A_4^T P_f^{(0)} + Q_3 - P_f^{(0)} S_2 P_f^{(0)} = 0 \tag{2.37}$$

The unique positive semidefinite stabilizing solution of (2.37) exists under Assumption 2.3, which implies that $P_f = P_f^{(0)} + O(\epsilon)$. We can also show that (2.31) is an $O(\epsilon)$ perturbation of the first-order approximate slow algebraic Riccati equation obtained in (Chow and Kokotovic, 1976; Wang and Frank, 1992)

$$P_s^{(0)}A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)}S_s P_s^{(0)} = 0 \tag{2.38}$$

with $P_s = P_s^{(0)} + O(\epsilon)$, where A_s, Q_s, and S_s can be found either using the methodology of (Chow and Kokotovic, 1976) or from the results of (Wang and Frank, 1992) as

$$\begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} = T_1 - T_2T_4^{-1}T_3 \tag{2.39}$$

In addition, we will show in Appendix 2.2 how to obtain the matrices A_s, S_s, Q_s in terms of $S_1, S_2, Z, A_i, Q_i, i = 1,2,3,4$, matrices, which in some applications (H_∞ optimization) appears to be more convenient. The corresponding derivations are done by evaluating in an efficient manner the term $T_1 - T_2T_4^{-1}T_3$ in formula (2.39). Note that from (2.11) and (2.29) we have

$$\begin{aligned} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &= T_1 - T_2L = T_1 - T_2L^{(0)} + O(\epsilon) \\ &= T_1 - T_2T_4^{-1}T_3 + O(\epsilon) = \begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} + O(\epsilon) \end{aligned} \tag{2.40}$$

which implies

$$\begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} \tag{2.41}$$

The unique positive semidefinite stabilizing solution of the approximate slow algebraic Riccati equation (2.38) exists under the following assumption.

Assumption 2.4: The triple $(A_s, \mathrm{Chol}(S_s), \mathrm{Chol}(Q_s))$ is stabilizable-detectable.

Note that in the case when the matrix A_4 is nonsingular, Assumption 2.4 can be replaced by a simpler assumption of the form (Chow and Kokotovic, 1976).

Assumption 2.4a: The triple (A_0, B_0, q_0) is stabilizable-detectable, with $A_0 = A_1 - A_2A_4^{-1}A_3$, $B_0 = B_1 - A_2A_4^{-1}B_2$, $q_0 = q_1 - q_2A_4^{-1}A_3$.

Assumptions 2.3, 2.4, and 2.4a are the standard assumptions in theory of singularly perturbed linear control systems (Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986).

Using (2.37)-(2.38) and the implicit function theorem (Ortega and Rheinboldt, 1970), the existence of the unique solutions of (2.34) and (2.35) are guaranteed by the following lemma.

Lemma 2.1 *Let Assumptions 2.3 and 2.4 be satisfied. Then* $\exists \epsilon_0 > 0$ *such that* $\forall \epsilon \leq \epsilon_0$ *the unique solutions of (2.34) and (2.35) exist.*

$\diamond$

The proof of the above lemma is the consequence of the facts that (2.34) and (2.35) are $O(\epsilon)$ perturbations, respectively, of equations (2.38) and (2.37). Then, the direct application of the implicit function theorem provides the proof of Lemma 2.1.

Having obtained a good initial guess, the Newton algorithm can be used very efficiently for solving (2.32). The Newton algorithm is given by

$$P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)}$$
$$i = 0, 1, 2, \ldots \tag{2.42}$$

with an initial guess obtained from (2.37).

The pure-slow equation (2.31) can be solved by using the Newton algorithm also, with an initial guess obtained from (2.38). The Newton algorithm for (2.31) is given by

$$P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)}$$
$$i = 0, 1, 2, \ldots \tag{2.43}$$

It is important to notice that the total number of scalar quadratic algebraic equations in (2.34) and (2.35) is $n_1^2 + n_2^2$. On the other hand, the global algebraic Riccati equation (2.7) contains $\frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1)$ scalar algebraic equations. Thus, the proposed method can even reduce the number of equations if

$$n_1^2 + n_2^2 < \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) \tag{2.44}$$

or

$$(n_1 - n_2)^2 < n_1 + n_2 \tag{2.45}$$

which is the case when n_1 and n_2 are selected to be close to each other.

Using solutions of both pure-slow and pure-fast Riccati equations and formulas (2.24) and (2.29), we can obtain completely decoupled slow and fast subsystems in the form

$$\begin{aligned} \dot{\eta}_1(t) &= (a_1 + a_2 P_s)\eta_1(t) \\ \epsilon\dot{\xi}_1(t) &= (b_1 + b_2 P_f)\xi_1(t) \end{aligned} \tag{2.46}$$

The interpretation of the result presented by (2.46) is that the optimal processing (filtering or control) can be completely performed at the local levels (slow and fast subsystems). The global solution in the original coordinates is then obtained at any time instant by using formula (2.23), that is

$$x(t) = (\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \tag{2.47}$$

where P is given by (2.28). The use of the results given in (2.46) in optimal filtering (first of all) and control of singularly perturbed linear systems will be much more clarified in the subsequent sections of this chapter.

The quadratic performance criterion to be minimized, (2.2), in the new coordinates is given by

$$J = \frac{1}{2}\int_{t_0}^{+\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right)dt$$

$$= \frac{1}{2}\int_{t_0}^{+\infty} x^T(t)(Q + PSP)x(t)dt$$

$$= \frac{1}{2}\int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T (\Pi_1 + \Pi_2 P)^{-T}(Q + PSP)(\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt$$

$$= \frac{1}{2}\int_{t_0}^{+\infty} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}^T \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^T & \Theta_3 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} dt \tag{2.48}$$

The value of the above integral is obtained as

$$J_{opt} = J_{sopt} + \epsilon J_{fopt} = \frac{1}{2}\mathrm{tr}\left\{V\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix}\begin{bmatrix}\eta_1(t)\\ \xi_1(t)\end{bmatrix}^T\right\}$$

$$= \frac{1}{2}\mathrm{tr}\left\{\begin{bmatrix}V_1 & \epsilon V_2\\ \epsilon V_2^T & \epsilon V_3\end{bmatrix}\begin{bmatrix}\eta_1(t_0)\eta_1^T(t_0) & \eta_1(t_0)\xi_1^T(t_0)\\ \xi_1(t_0)\eta_1^T(t_0) & \xi_1(t_0)\xi_1^T(t_0)\end{bmatrix}\right\}$$

$$= \frac{1}{2}tr\{V_1\eta_1(t_0)\eta_1^T(t_0)\}$$

$$+\frac{\epsilon}{2}tr\left(V_2^T\eta_1(t_0)\xi_1^T(t_0) + V_2\xi_1(t_0)\eta_1^T(t_0) + V_3\xi_1(t_0)\xi_1^T(t_0)\right) \tag{2.49}$$

where the matrix V satisfies the algebraic Lyapunov equation

$$\begin{bmatrix}(a_1 + a_2P_1) & 0\\ 0 & \frac{1}{\epsilon}(b_1 + b_2P_2)\end{bmatrix}^T V + V\begin{bmatrix}(a_1 + a_2P_1) & 0\\ 0 & \frac{1}{\epsilon}(b_1 + b_2P_2)\end{bmatrix}$$
$$+\begin{bmatrix}\Theta_1 & \Theta_2\\ \Theta_2^T & \Theta_3\end{bmatrix} = 0$$

which implies three independent, reduced-order, Lyapunov (Sylvester) algebraic equations

$$\begin{aligned}
&(a_1 + a_2P_1)^T V_1 + V_1(a_1 + a_2P_1) + \Theta_1 = 0\\
&\epsilon(a_1 + a_2P_1)^T V_2 + V_2(b_1 + b_2P_2) + \Theta_2 = 0\\
&(b_1 + b_2P_2)^T V_3 + V_3(b_1 + b_2P_2) + \Theta_3 = 0
\end{aligned} \tag{2.50}$$

Formula (2.49) exactly decomposes slow and fast components of the optimal performance criterion. It can be concluded from (2.49) that the pure-slow component of the performance criterion is $O(1)$ and that the fast subsystem contributes only an $O(\epsilon)$ value to the performance criterion of a linear continuous-time deterministic system.

Note that the pure-slow/pure-fast decomposition of the differential Riccati equation, whose solution comprises the optimal feedback gain for the finite horizon optimization problem, appears to be much more computationally involved. That fact will be demonstrated in Section 8.2. It should be pointed that the recursive approach slow-fast decomposition of the differential Riccati equation obtained in (Grodt and Gajic, 1988) is very efficient for achieving a very high accuracy. It has been demonstrated in (Grodt and Gajic, 1988) on a real power system example that the accuracy of $O\left(\epsilon^{12}\right)$ can be easily obtained.

2.1.1 Case Study: Magnetic Tape Control

In order to illustrate the proposed method, we consider a magnetic tape control system (Chow and Kokotovic, 1976). The problem data are given by

$$A = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.345 & 0 \\ 0 & -\frac{1}{\epsilon}0.524 & -\frac{1}{\epsilon}0.465 & \frac{1}{\epsilon}0.262 \\ 0 & 0 & 0 & -\frac{1}{\epsilon} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\epsilon} \end{bmatrix}$$

$$Q = \mathrm{diag}\{1 \quad 0 \quad 1 \quad 0\}, \quad R = 1, \quad \epsilon = 0.1$$

The optimal global solution from (2.7) is

$$P_{exact} = \begin{bmatrix} 7.5400 & 6.1704 & 0.4053 & 0.1000 \\ 6.1704 & 7.4673 & 0.3951 & 0.0892 \\ 0.4053 & 0.3951 & 0.1304 & 0.0244 \\ 0.1000 & 0.0892 & 0.0244 & 0.0062 \end{bmatrix}$$

The solutions of the pure-slow and pure-fast algebraic Riccati equations (2.31) and (2.32) obtained from algorithms (2.42) and (2.43) are

$$P_s = \begin{bmatrix} 7.2437 & 5.5037 \\ 5.8884 & 6.8214 \end{bmatrix}, \quad P_f = \begin{bmatrix} 1.0411 & 0.1850 \\ 0.1785 & 0.0474 \end{bmatrix}$$

Using (2.28), the obtained solution for P is found to be identical to P_{exact} with the accuracy of 10^{-14} (MATLAB standard accuracy).

2.2 Open-Loop Singularly Perturbed Linear Control Problem

The optimal open-loop control problem is a two-point boundary value problem with the associated state-costate equations forming the Hamiltonian system. In this section, the two-point boundary value problem of linear singularly perturbed systems is transformed into the pure-slow and pure-fast, reduced-order, completely decoupled initial value problems. By doing this, the stiffness (numerical ill-conditioning) of the original singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined linear system of algebraic equations.

Consider the linear singularly perturbed control system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t), \qquad x_1(t_0) = x_{10} \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t), \qquad x_2(t_0) = x_{20} \end{aligned} \tag{2.51}$$

where $x_i(t) \in \Re^{n_i}$, $i = 1, 2$, $u(t) \in \Re^m$ are state and control variables, respectively, and ϵ is a small positive singular perturbation parameter. Let the performance criterion to be minimized be defined over a finite time period from t_0 to t_f (finite horizon optimal control problem)

$$\begin{aligned} \min_{u(t)} J = \min_{u(t)} \frac{1}{2} \Big\{ \int_{t_0}^{t_f} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T Q \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^T(t) R\, u(t) \right\} dt \\ + \frac{1}{2} \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}^T Q_f \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} \Big\} \end{aligned} \tag{2.52}$$

with positive definite R and positive semidefinite Q and Q_f.

The open-loop optimal control problem has the solution given by

$$u(t) = -R^{-1} B^T p(t) \tag{2.53}$$

where $p(t) \in \Re^{n_1+n_2}$ is a costate variable satisfying (Kwakernaak and Sivan, 1972)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{2.54}$$

with boundary conditions expressed in the standard form as

$$M \begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} + N \begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = c \tag{2.55}$$

where

$$M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ -Q_f & I_n \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix}, \quad n = n_1 + n_2 \tag{2.56}$$

for the free endpoint problem, or

$$M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ x(t_f) \end{bmatrix} \tag{2.57}$$

for the fixed endpoint problem. Since condition (2.57) leads to a two-point boundary value problem, causing both the initial and terminal boundary layers, the treatment of this section is applicable to the free end problem only.

The matrices A, Q, B, S, and Q_f in the case of singularly perturbed control systems have the forms

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}$$

$$S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{1}{\epsilon}Z \\ \frac{1}{\epsilon}Z^T & \frac{1}{\epsilon^2}S_2 \end{bmatrix}, \quad Q_f = \begin{bmatrix} Q_{f1} & \epsilon Q_{f2} \\ \epsilon Q_{f2}^T & \epsilon Q_{f3} \end{bmatrix} \tag{2.58}$$

The approximate optimal solution of the open-loop control for linear singularly perturbed systems has been studied in (Wilde and Kokotovic, 1973), where the problem order was reduced and the stiff problem was avoided successfully by using the classic approach based on the power-series expansions. The theory developed in (Wilde and Kokotovic, 1973) was based on the dichotomy transformation (Wilde and Kokotovic, 1972) which requires the positive definite and negative definite solutions of the corresponding algebraic Riccati equation. It was concluded in (Wilde and Kokotovic, 1973) that the developed method is efficient for an $O(\epsilon)$ accuracy only. In this section, the solution to the optimal open-loop control problem of singularly perturbed systems with an arbitrary order of accuracy is presented.

Let us partition and appropriately scale the costate vector $p(t)$ as $p^T(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]$ with $p_1(t) \in \Re^{n_1}$ and $p_2(t) \in \Re^{n_2}$. We know from the previous section that

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{2.59}$$

where

$$\begin{aligned} T_1 &= \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix} \\ T_3 &= \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \qquad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \end{aligned} \tag{2.60}$$

with (2.59) representing a standard singularly perturbed linear system that has nonsingular fast subsystem matrix, T_4, under Assumption 2.3.

Introduce the notation

$$\begin{bmatrix} x_1(t) \\ p_1(t) \end{bmatrix} = w(t), \qquad \begin{bmatrix} x_2(t) \\ p_2(t) \end{bmatrix} = \lambda(t) \tag{2.61}$$

The Chang transformation (2.10) applied to (2.59) produces two completely decoupled pure-slow and pure-fast subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t) \tag{2.62}$$

and

$$\epsilon\dot{\xi}(t) = (T_4 + \epsilon L T_2)\xi(t) \tag{2.63}$$

where

$$\begin{bmatrix} \eta(t) \\ \xi(t) \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} w(t) \\ \lambda(t) \end{bmatrix} \tag{2.64}$$

The boundary conditions are changed due to an interchange of $p_1(t)$ and $x_2(t)$, which modifies matrices in (2.56) as follows

$$M_1 \begin{bmatrix} w(t_0) \\ \lambda(t_0) \end{bmatrix} + N_1 \begin{bmatrix} w(t_f) \\ \lambda(t_f) \end{bmatrix} = c_1 \tag{2.65}$$

where

$$M_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_{10} \\ 0 \\ x_{20} \\ 0 \end{bmatrix}$$
$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Q_{f1} & I_{n_1} & -\epsilon Q_{f2} & 0 \\ 0 & 0 & 0 & 0 \\ -Q_{f2}^T & 0 & -Q_{f3} & In_2 \end{bmatrix} \tag{2.66}$$

The boundary condition in the new coordinates becomes

$$M_2 \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} + N_2 \begin{bmatrix} \eta(t_f) \\ \xi(t_f) \end{bmatrix} = c_1 \tag{2.67}$$

where

$$M_2 = M_1 \mathbf{T_1}^{-1}, \quad N_2 = N_1 \mathbf{T_1}^{-1} \tag{2.68}$$

Since solutions of (2.62) and (2.63) are given by

$$\eta(t) = e^{(T_1 - T_2 L)(t-t_0)}\eta(t_0) \tag{2.69}$$

$$\xi(t) = e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-t_0)}\xi(t_0) \tag{2.70}$$

we can eliminate $\eta(t_f)$ and $\xi(t_f)$ from (2.67) such that

$$\left\{M_2 + N_2 \begin{bmatrix} e^{(T_1 - T_2 L)(t_f - t_0)} & 0 \\ 0 & e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t_f - t_0)} \end{bmatrix}\right\} \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1 \tag{2.71}$$

The system of linear algebraic equations obtained, (2.71), can be represented in the form

$$\alpha(\epsilon) \begin{bmatrix} \eta(t_0) \\ \xi(t_0) \end{bmatrix} = c_1 \tag{2.72}$$

It is shown in Lemma 2.2 that $\alpha(\epsilon)$ is invertible, hence $\eta(t_0)$ and $\xi(t_0)$ can be obtained from (2.72).

Lemma 2.2 *Under Assumptions 2.3 and 2.4, the matrix $\alpha(\epsilon)$ is invertible.*

◇

Proof: Transition matrices in (2.69) and (2.70) can be denoted $\Phi(t - t_0)$ and $\Psi(t - t_0)$, respectively, and partitioned as

$$\Phi(t - t_0) = \begin{bmatrix} \Phi_{11}(t - t_0) & \Phi_{12}(t - t_0) \\ \Phi_{21}(t - t_0) & \Phi_{22}(t - t_0) \end{bmatrix} \tag{2.73}$$

$$\Psi(t - t_0) = \begin{bmatrix} \Psi_{11}(t - t_0) & \Psi_{12}(t - t_0) \\ \Psi_{21}(t - t_0) & \Psi_{22}(t - t_0) \end{bmatrix} \tag{2.74}$$

From (2.71) we have

$$\alpha(\epsilon) = \left(M_2 + N_2 \begin{bmatrix} \Phi(t_f - t_0) & 0 \\ 0 & \Psi(t_f - t_0) \end{bmatrix}\right) \tag{2.75}$$

Using expressions for M_2 and N_2, we get

$$\alpha(\epsilon) = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ * & \Phi_{22} - Q_{f1}\Phi_{12} & 0 & 0 \\ * & * & I_{n_2} & 0 \\ * & * & * & \Psi_{22} - Q_{f3}\Psi_{12} \end{bmatrix} + O(\epsilon) \tag{2.76}$$

where asterisks denote terms that are not important for nonsingularity of $\alpha(\epsilon)$. Since matrices $\Phi_{22} - Q_{f1}\Phi_{12}$ and $\Psi_{22} - Q_{f3}\Psi_{12}$ are invertible under Assumptions 2.3 and 2.4 (Kalman, 1960), the matrix $\alpha(\epsilon)$ is invertible for sufficiently small values of ϵ. Note that in the case of linear singularly perturbed systems, due to the nature of the fast subsystem transition matrix (2.70), which contains unstable modes, we can observe that $\alpha(0)$ is singular. Hence, $\alpha(\epsilon)$ is invertible for $0 < \epsilon < \epsilon_1$ with ϵ_1 sufficiently small. In other words, the stiffness of the singularly perturbed system of differential equations is carried over to the stiffness of the linear system of algebraic equations. However, the latter problem is much easier to handle.

•

Now we are able to find $\eta(t)$ and $\xi(t)$ from (2.69) and (2.70). Using (2.64), we can find $\omega(t)$ and $\lambda(t)$. Partitioning $\omega(t)$ and $\lambda(t)$ according to (2.61), we get values for $p_1(t)$ and $p_2(t)$. The costate variables $p(t)$ and the optimal open-loop control law are therefore found. Note that the optimal open-loop control is a function of time, in contrast to the optimal closed-loop (feedback) control that is a function of the state variables $x_1(t)$ and $x_2(t)$.

The only difficulty we have encountered in the procedure is to compute $\alpha(\epsilon)$ in (2.72) where an ill-defined problem occurs when ϵ is extremely small or $(t_f - t_0)$ is very large because the matrix T_4 contains both stable and unstable modes. In that case we refer to (Wilde and Kokotovic, 1973).

2.2.1 Case Study: Magnetic Tape Control

In order to illustrate efficiency of the proposed method, we consider the magnetic tape control system from (Chow and Kokotovic, 1976). Problem matrices A, B, Q, and R are given in Section 2.1.1. The system initial conditions are assumed to be

$$x^T(t_0) = [-1.3702 \quad 0.10686 \quad -0.53307 \quad 0.83467]$$

The time interval of interest is specified by $t_0 = 0$ and $t_f = 1$, and the small singular perturbation parameter is $\epsilon = 0.1$.

The approximate optimal open-loop control is defined as

$$u^{(k)}(t) = -R^{-1}B^Tp^{(k)}(t) \tag{2.77}$$

Table 2.1: Values of an approximate control at certain time instants

	$t = 0.25$	$t = 0.5$	$t = 1$
$u^{(4)}(t) = optimal$	3.1719 x 10^{-1}	3.0299 x 10^{-1}	-8.2827 x 10^{-2}
$u^{(3)}(t)$	3.1719 x 10^{-1}	3.0299 x 10^{-1}	-8.2827 x 10^{-2}
$u^{(2)}(t)$	3.1720 x 10^{-1}	3.0299 x 10^{-1}	-8.2825 x 10^{-2}
$u^{(1)}(t)$	3.1712 x 10^{-1}	3.0287 x 10^{-1}	-8.2758 x 10^{-2}
$u^{(0)}(t)$	3.3244 x 10^{-1}	3.0135 x 10^{-1}	-7.6749 x 10^{-2}

where k stands for the number of iterations used to solve recursively L, H-equations. Values for $p^{(k)}(t)$ are obtained by following steps (2.61)-(2.72). The results obtained for the approximate open-loop control $u^{(k)}(t)$ are presented in Table 2.1. It can be seen that it takes four iterations to achieve the accuracy of four decimal digits.

Note that steps (2.61)-(2.72) could have been performed by using the method of series expansions. However, that method is not recursive in its nature so that the higher order terms produce very cumbersome expressions. That method is efficient for an $O(\epsilon)$ accuracy only, as was pointed out in (Wilde and Kokotovic, 1973).

2.3 Kalman Filtering for Linear Singularly Perturbed Systems

In this section we present a method which allows complete decomposition of the optimal global Kalman filter of linear singularly perturbed systems into pure-slow and pure-fast local optimal filters both driven by system measurements. The method is based on the exact decomposition of the global singularly perturbed algebraic filter Riccati equation as presented in Section 2.1 and the duality property that exists between the linear-quadratic optimal filters and regulators.

The filtering problem of linear singularly perturbed continuous-time systems has been well documented in the control theory literature (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977;

Khalil and Gajic, 1984; Gajic, 1986; Kokotovic *et al.*, 1986; Gajic and Shen, 1993). In (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977) the suboptimal slow and fast Kalman filters were constructed producing an $O(\epsilon)$ accuracy for the estimates of the state trajectories, where the small positive singular perturbation parameter ϵ represents the separation between slow and fast phenomena. In (Khalil and Gajic, 1984; Gajic, 1986; Kokotovic *et al.*, 1986; Gajic and Shen, 1993) both the slow and fast (local) Kalman filters were obtained with an arbitrary order of accuracy, that is $O\left(\epsilon^k\right)$, where k stands for either the number of terms of the Taylor series (Khalil and Gajic, 1984) or the number of the fixed-point iterations (Gajic, 1986) used to calculate coefficients of the corresponding filters. It is important to point out that the local slow and fast filters in (Khalil and Gajic, 1984; Gajic, 1986) are driven by the innovation process so that the additional communication channels are required to form the innovation process. In the technique presented in this section, the local filters are driven by the system measurements only. In addition, the optimal filter gains are completely determined in terms of the exact pure-slow and exact pure-fast reduced-order algebraic filter Riccati equations.

Consider the linear continuous-time invariant singularly perturbed stochastic system

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + G_1w_1(t) \\ \epsilon\dot{x}_2 &= A_3x_1(t) + A_4x_2(t) + G_2w_1(t) \end{aligned} \tag{2.78}$$

with the corresponding measurements

$$y(t) = C_1x_1(t) + C_2x_2(t) + w_2(t) \tag{2.79}$$

where $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are state vectors, $w_1(t) \in \Re^r$ and $w_2(t) \in \Re^l$ are zero-mean stationary, white Gaussian noise stochastic processes with intensities $W_1 \geq 0$ and $W_2 > 0$, respectively, and $y(t) \in \Re^l$ are system measurements. In the following $A_i, G_j, C_j,\ i = 1,2,3,4,\ j = 1,2$, are constant matrices. We assume that the system under consideration has the standard singularly perturbed form, (Khalil, 1989), that is, Assumption 2.1 is imposed.

The optimal Kalman filter, corresponding to (2.78)-(2.79), driven by the innovation process is given by

$$
\begin{aligned}
\dot{\hat{x}}_1(t) &= A_1\hat{x}_1(t) + A_2\hat{x}_2(t) + K_1 v(t) \\
\epsilon\dot{\hat{x}}_2(t) &= A_3\hat{x}_1(t) + A_4\hat{x}_2(t) + K_2 v(t) \\
v(t) &= y(t) - C_1\hat{x}_1(t) - C_2\hat{x}_2(t)
\end{aligned}
\tag{2.80}
$$

where the optimal Kalman filter gains K_1 and K_2 are obtained from (Khalil and Gajic, 1984)

$$
K_1 = \left(P_{1F}C_1^T + P_{2F}C_2^T\right)W_2^{-1}, \;\; K_2 = \left(\epsilon P_{2F}^T C_1^T + P_{3F}C_2^T\right)W_2^{-1} \tag{2.81}
$$

with matrices P_{1F}, P_{2F}, and P_{3F} representing the positive semidefinite stabilizing solution matrix of the filter algebraic Riccati equation

$$
AP_F + P_F A^T - P_F S P_F + G W_1 G^T = 0 \tag{2.82}
$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \; G = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon}G_2 \end{bmatrix} \\
S &= C^T W_2^{-1} C, \; P_F = \begin{bmatrix} P_{1F} & P_{2F} \\ P_{2F}^T & \frac{1}{\epsilon}P_{3F} \end{bmatrix}
\end{aligned}
\tag{2.83}
$$

The Chang transformation (Chang, 1972) has been used in (Khalil and Gajic, 1984; Gajic, 1986) for the decomposition and approximation of the singularly perturbed Kalman filter (2.80) as

$$
\begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \epsilon H L & -\epsilon H \\ L & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{2.84}
$$

where L and H satisfy algebraic equations

$$
\begin{aligned}
A_4 L - A_3 - \epsilon L(A_1 - A_2 L) &= 0 \\
-H A_4 + A_2 - \epsilon H L A_2 + \epsilon(A_1 - A_2 L)H &= 0
\end{aligned}
\tag{2.85}
$$

The Chang transformation applied to (2.80) produces

$$
\begin{aligned}
\dot{\hat{\eta}}_1(t) &= (A_1 - A_2 L)\hat{\eta}_1(t) + (K_1 - H K_2 - \epsilon H L K_1)v(t) \\
\epsilon\dot{\hat{\eta}}_2(t) &= (A_4 + \epsilon L A_2)\hat{\eta}_2(t) + (K_2 + \epsilon L K_1)v(t)
\end{aligned}
\tag{2.86}
$$

In the new coordinates the innovation process is given by

$$
v(t) = y(t) - (C_1 - C_2 L)\hat{\eta}_1(t) - [C_2 + \epsilon(C_1 - C_2 L)H]\hat{\eta}_2(t) \tag{2.87}
$$

In (Khalil and Gajic, 1984; Gajic, 1986), the approximate reduced-order filters of (2.86)-(2.87) were defined as well.

Equations (2.85) are solvable and produce the unique solutions under Assumption 2.1. The algebraic filter Riccati equation (2.82) produces the unique stabilizing solutions for sufficiently small values of ϵ under the following assumptions.

Assumption 2.5: The triple (A_4, C_2, G_2) is stabilizable-detectable.

Assumption 2.6: The slow-subsystem triple (A_0, C_0, G_0) is both stabilizable and detectable, where the newly defined matrices are given by $A_0 = A_1 - A_2 A_4^{-1} A_3$, $C_0 = C_1 - C_2 A_4^{-1} A_3$, $G_0 = G_1 - A_2 A_4^{-1} G_2$.

In the decomposition procedure given by (2.86)-(2.87) the slow and fast filters (2.86) require some additional communication channels necessary to form the innovation process (2.87)—see Figure 2.1.

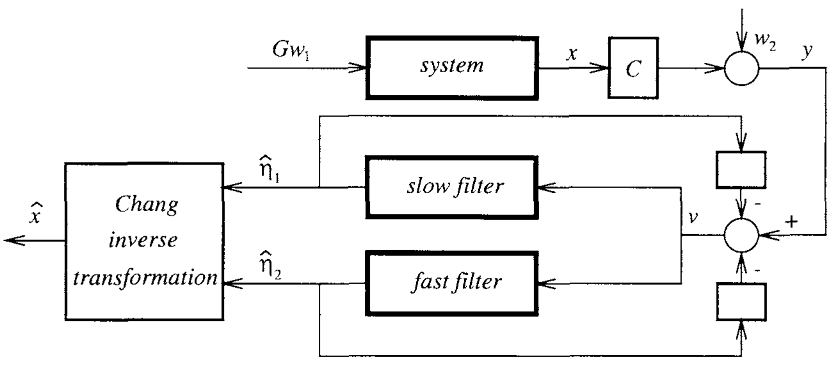

Figure 2.1: Classic filtering method for linear singularly perturbed systems

In this section, we present a decomposition scheme such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements. This method is based on the pure-slow pure-fast decomposition technique for solving the filter algebraic Riccati equation of singularly perturbed systems derived by using duality between the optimal filters and regulators and the methodology presented in Section 2.1. In that respect, we give an additional interpretation of the results presented in Section 2.1.

Using (2.5)-(2.7), the optimal regulator gain is defined by

$$F = [F_1 \quad F_2] = [R^{-1}(B_1^T P_1 + B_2^T P_2^T) \quad R^{-1}(\epsilon B_1^T P_2 + B_2^T P_3)] \tag{2.88}$$

The results of interest that we need, which can be deduced from Section 2.1, are given in the form of the following lemma.

Lemma 2.3 *Consider the optimal closed-loop linear system*

$$\begin{aligned}\dot{x}_1(t) &= (A_1 - B_1F_1)x_1(t) + (A_2 - B_1F_2)x_2(t) \\ \epsilon\dot{x}_2(t) &= (A_3 - B_2F_1)x_1(t) + (A_4 - B_2F_2)x_2(t)\end{aligned} \tag{2.89}$$

Under Assumptions 2.3 and 2.4 there exists a nonsingular transformation $\mathbf{T}$

$$\begin{bmatrix}\xi_s(t) \\ \xi_f(t)\end{bmatrix} = \mathbf{T}\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} \tag{2.90}$$

such that

$$\begin{aligned}\dot{\xi}_s(t) &= (a_1 + a_2P_s)\xi_s(t) \\ \epsilon\dot{\xi}_f(t) &= (b_1 + b_2P_f)\xi_f(t)\end{aligned} \tag{2.91}$$

where P_s and P_f are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati equations (2.31)-(2.32). The nonsingular transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\Pi_1 + \Pi_2P) \tag{2.92}$$

Even more, the global solution P can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations, that is

$$P = \left(\Omega_3 + \Omega_4\begin{bmatrix}P_s & 0 \\ 0 & P_f\end{bmatrix}\right)\left(\Omega_1 + \Omega_2\begin{bmatrix}P_s & 0 \\ 0 & P_f\end{bmatrix}\right)^{-1} \tag{2.93}$$

Known matrices Ω_i, $i = 1, 2, 3, 4$, and Π_1, Π_2 are given in terms of solutions of the Chang decoupling equations, and defined in (2.21) and (2.26).

◇

The desired slow-fast decomposition of the Kalman filter (2.80) will be obtained by producing a dual lemma to Lemma 2.3. Consider the optimal *closed-loop* Kalman filter (2.80) driven by the system measurements, that is

$$\begin{aligned}\dot{\hat{x}}_1(t) &= (A_1 - K_1C_1)\hat{x}_1(t) + (A_2 - K_1C_2)\hat{x}_2(t) + K_1y(t) \\ \epsilon\dot{\hat{x}}_2(t) &= (A_3 - K_2C_1)\hat{x}_1(t) + (A_4 - K_2C_2)\hat{x}_2(t) + K_2y(t)\end{aligned} \tag{2.94}$$

with the optimal filter gains K_1 and K_2 calculated from (2.81)-(2.83). By duality between the optimal filter and regulator, the algebraic filter Riccati equation (2.82) can be solved by using the same decomposition method for solving the algebraic regulator Riccati equation (2.7) with

$$\begin{gathered} A \rightarrow A^T, \; Q \rightarrow GW_1G^T, \; F^T = K \\ Z = BR^{-1}B^T \rightarrow S = C^TW_2^{-1}C \end{gathered} \tag{2.95}$$

By invoking results from Section 2.1, and using duality, the following matrices have to be formed

$$\begin{aligned} T_{1F} &= \begin{bmatrix} A_1^T & -C_1^TW_2^{-1}C_1 \\ -G_1W_1G_1^T & -A_1 \end{bmatrix} \\ T_{2F} &= \begin{bmatrix} A_3^T & -C_1^TW_2^{-1}C_2 \\ -G_1W_1G_2^T & -A_2 \end{bmatrix} \\ T_{3F} &= \begin{bmatrix} A_2^T & -C_2^TW_2^{-1}C_1 \\ -G_2W_1G_1^T & -A_3 \end{bmatrix} \\ T_{4F} &= \begin{bmatrix} A_4^T & -C_2^TW_2^{-1}C_2 \\ -G_2W_1G_2^T & -A_4 \end{bmatrix} \end{aligned} \tag{2.96}$$

Note that on the contrary to the results from Section 2.1, where the state-costate variables have to be partitioned and scaled as $x^T(t) = \left[x_1^T(t)\; x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\; \epsilon p_2^T(t)\right]$, in the case of the dual filter variables, we have to use the following partitions and scaling $x^T(t) = \left[x_1^T(t)\; \epsilon x_2^T(t)\right]$ and $p^T(t) = \left[p_1^T(t)\; p_2^T(t)\right]$. Since matrices T_{1F}, T_{2F}, T_{3F}, T_{4F} correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition is achieved by using the Chang decoupling equations

$$\begin{gathered} T_{4F}M - T_{3F} - \epsilon M(T_{1F} - T_{2F}M) = 0 \\ -N(T_{4F} + \epsilon MT_{2F}) + T_{2F} + \epsilon(T_{1F} - T_{2F}M)N = 0 \end{gathered} \tag{2.97}$$

By using the permutation matrices dual to those from Section 2.1 (note E_{1F} is different than the corresponding one from Section 2.1)

$$E_{1F} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & \frac{1}{\epsilon}I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix}, \quad E_{2F} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \tag{2.98}$$

we can define

$$\Pi_F = \begin{bmatrix} \Pi_{1F} & \Pi_{2F} \\ \Pi_{3F} & \Pi_{4F} \end{bmatrix} = E_{2F}^T \begin{bmatrix} I_{2n_1} - \epsilon NM & -\epsilon N \\ M & I_{2n_2} \end{bmatrix} E_{1F} \tag{2.99}$$

Then, the desired transformation is given by

$$\mathbf{T_2} = (\Pi_{1F} + \Pi_{2F} P_F) \tag{2.100}$$

The transformation $\mathbf{T_2}$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \mathbf{T_2}^{-T} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{2.101}$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = \mathbf{T_2}^{-T} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \mathbf{T_2}^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix}$$
$$+ \mathbf{T_2}^{-T} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \tag{2.102}$$

such that the complete closed-loop decomposition is achieved, that is

$$\begin{aligned} \dot{\hat{\eta}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + K_s y(t) \\ \epsilon \dot{\hat{\eta}}_f(t) &= (b_{1F} + b_{2F} P_{fF})^T \hat{\eta}_f(t) + K_f y(t) \end{aligned} \tag{2.103}$$

The matrices in (2.103) are given by

$$\begin{bmatrix} a_{1F} & a_{2F} \\ a_{3F} & a_{4F} \end{bmatrix} = (T_{1F} - T_{2F} M), \quad \begin{bmatrix} b_{1F} & b_{2F} \\ b_{3F} & b_{4F} \end{bmatrix} = (T_{4F} + \epsilon M T_{2F})$$
$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \mathbf{T_2}^{-T} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \tag{2.104}$$

and

$$\begin{aligned} 0 &= P_{sF} a_{1F} - a_{4F} P_{sF} - a_{3F} + P_{sF} a_{2F} P_{sF} \\ 0 &= P_{fF} b_{1F} - b_{4F} P_{fF} - b_{3F} + P_{fF} b_{2F} P_{fF} \end{aligned} \tag{2.105}$$

A method for solving nonsymmetric Riccati equations (2.105) is considered in Section 2.1. Note that the matrices needed for the $O(\epsilon)$ approximate slow filter algebraic Riccati equation dual to (2.38) and defined by

$$P_{sF}^{(0)} A_{sF}^T + A_{sF} P_{sF}^{(0)} + G_s W_{1s} G_s^T - P_{sF}^{(0)} C_s^T W_{2s}^{-1} C_s P_{sF}^{(0)} = 0 \quad (2.106)$$

can be obtained from (Wang and Frank, 1992)

$$\begin{bmatrix} A_{sF}^T & -C_s^T W_{2s}^{-1} C_s \\ -G_s W_{1s} G_s^T & -A_{sF} \end{bmatrix} = T_{1F} - T_{2F} T_{4F}^{-1} T_{3F} \quad (2.107)$$

Even more, the analytical expressions for $A_{sF}, C_s, G_s, W_{1s}, W_{2s}$ can be obtained by using the methodology of (Khalil and Gajic, 1984).

It is important to point out that the matrix P_F in (2.100) can be obtained in terms of P_{sF} and P_{fF} by using formula (2.93) with

$$P_s = P_{sF}, \quad P_f = P_{fF} \quad (2.108)$$

and $\Omega_1, \ \Omega_2, \ \Omega_3, \ \Omega_4$ are obtained from

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_{1F}^{-1} \begin{bmatrix} I_{2n_1} & \epsilon N \\ -M & I_{2n_2} - \epsilon M N \end{bmatrix} E_{2F}^{-T} \quad (2.109)$$

A lemma dual to Lemma 2.3 can be now formulated as follows.

Lemma 2.4 *Given the closed-loop optimal Kalman filter (2.94) of a linear singularly perturbed system. There exists a nonsingular transformation matrix (2.100), which completely decouples (2.94) into pure-slow and pure-fast local filters (2.103) both driven by the system measurements. Even more, the decoupling transformation (2.100) and the filter coefficients given in (2.104) can be obtained in terms of exact pure-slow and pure-fast reduced-order completely decoupled algebraic Riccati equations (2.105).*

◇

A comparison between the presented filtering method and the one already in use for linear singularly perturbed systems is given in Figures

2.1 and 2.2. It can be seen that the new filtering method allows complete decomposition and parallelism between pure-slow and pure-fast filters.

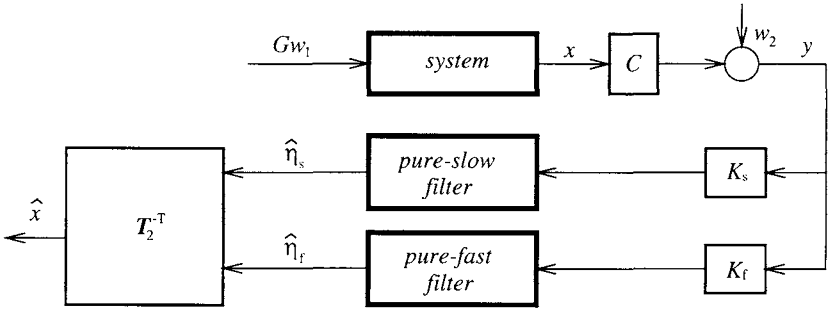

Figure 2.2: New filtering method for linear singularly perturbed systems

We can now define the corresponding approximations (in the spirit of theory of singular perturbations (Khalil and Gajic, 1984; Gajic, 1986; Kokotovic *et al.*, 1986) of the pure-slow and pure-fast filters as

$$\begin{aligned}\dot{\hat{\eta}}_s^{(k)}(t) &= \left(a_{1F}^{(k)} + a_{2F}^{(k)} P_{sF}^{(k)}\right)^T \hat{\eta}_s^{(k)}(t) + K_s^{(k)} y(t) \\ \epsilon \dot{\hat{\eta}}_f^{(k)}(t) &= \left(b_{1F}^{(k)} + b_{2F}^{(k)} P_{fF}^{(k)}\right)^T \hat{\eta}_f^{(k)}(t) + K_f^{(k)} y(t)\end{aligned} \tag{2.110}$$

where

$$\begin{aligned}\begin{bmatrix} a_{1F}^{(k)} & a_{2F}^{(k)} \\ a_{3F}^{(k)} & a_{4F}^{(k)} \end{bmatrix} &= \left(T_{1F}^{(k)} - T_{2F}^{(k)} M^{(k)}\right) \\ \begin{bmatrix} b_{1F}^{(k)} & b_{2F}^{(k)} \\ b_{3F}^{(k)} & b_{4F}^{(k)} \end{bmatrix} &= \left(T_{4F}^{(k)} + \epsilon M^{(k-1)} T_{2F}^{(k-1)}\right)\end{aligned} \tag{2.111}$$

and

$$P_{sF}^{(k)} = P_{sF} + O\left(\epsilon^k\right), \ P_{fF}^{(k)} = P_{fF} + O\left(\epsilon^k\right), \ M^{(k)} = M + O\left(\epsilon^k\right)$$

$$\begin{bmatrix} K_s^{(k)} \\ \frac{1}{\epsilon} K_f^{(k)} \end{bmatrix} = \mathbf{T}_2^{(k)^{-T}} \begin{bmatrix} K_1^{(k)} \\ \frac{1}{\epsilon} K_2^{(k)} \end{bmatrix}$$

Note that in the expression for $b_i^{(k)}$ we can use $M^{(k-1)}$ and $T_2^{(k-1)}$ since these matrices are multiplied by ϵ so that we get $b_i = b_i^{(k)} + O(\epsilon^k)$.

2.3.1 Case Study: An F-15 Aircraft

In order to demonstrate the proposed method, we study the linearized model of an F-15 aircraft example (Brumbaugh, 1994; Schomig *et al.*, 1995). For supersonic flight conditions, the aircraft's longitudinal dynamics is described by the following matrices

$$A = \begin{bmatrix} -0.00819 & -25.70839 & 0 & -32.17095 \\ -0.00019 & -1.27626 & 1 & 0 \\ 0.00069 & 1.02176 & -2.40523 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G^T = B^T = [-6.80939 \quad -0.14968 \quad -14.06111 \quad 0]$$

The eigenvalues of the matrix A are given by -0.6835, -3.0036, $-0.0013 \pm j0.1037$, which indicates the presence of two slow and two fast modes in the aircraft's dynamics. Note that the aircraft's singularly perturbed structure becomes more obvious by introducing a similarity transformation that interchanges the second and fourth state space variables. The small singular perturbation parameter ϵ is chosen as $\epsilon = 0.2$.

We assume that the matrix C is given by

$$C = [C_1 \quad C_2], \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and that the aircraft is under wind disturbances whose intensity matrices are given by

$$W_1 = 0.000315, \;\; W_2 = diag[0.000686 \;\; 40]$$

For the aircraft, we have obtained completely decoupled filters driven by the measurements $y(t)$ as

$$\dot{\hat{\eta}}_s(t) = \begin{bmatrix} -6.0542 & -32.1078 \\ 0.1171 & 0.0000 \end{bmatrix} \hat{\eta}_s(t) + \begin{bmatrix} 6.0411 & -0.0002 \\ -0.1169 & 0.0000 \end{bmatrix} y(t)$$

$$\epsilon\dot{\hat{\eta}}_f(t) = \begin{bmatrix} -2.8017 & -5.8585 \\ 0.1664 & -1.1466 \end{bmatrix} \hat{\eta}_f(t) + \begin{bmatrix} 0.2233 & 0.0000 \\ 0.0019 & 0.0000 \end{bmatrix} y(t)$$

Note that the results obtained include the initial similarity transformation that interchanges the second and fourth state space variables.

The pure-slow and pure-fast Kalman filter decomposition, and the corresponding optimal pure-slow and pure-fast estimates $\hat{\eta}_s(t)$ and $\hat{\eta}_f(t)$ can be easily realized using SIMULINK.

2.4 Optimal Linear-Quadratic Gaussian Control

In this section we present an approach for solving the linear-quadratic optimal Gaussian control problem of singularly perturbed continuous-time stochastic systems. The algorithm proposed is based on the results presented in Sections 2.1 and 2.3. It is shown that the optimal linear-quadratic Gaussian control problem takes the complete decomposition and parallelism between pure-slow and pure-fast filters and controllers.

Singularly perturbed linear-quadratic optimal control problem of stochastic continuous-time systems has been studied in the past by several researchers (Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977; Khalil and Gajic, 1984; Gajic, 1986; Kokotovic *et al.*, 1986; Gajic and Shen, 1993). In this section, we introduce a completely new approach to the stochastic control of linear singularly perturbed systems that is pretty much different than all other methods used so far in the study of the same problem. Our approach is based on a *closed-loop* decomposition technique which guarantees complete decomposition of the optimal filters and regulators and distribution of all required off-line and on-line computations. As a matter of fact, the approach combines results presented in Sections 2.1 and 2.3 and uses the separation principle for linear stochastic control (Kwakernaak and Sivan, 1972). We also show how to calculate the optimal regulator gains with respect to the optimal pure-slow and pure-fast, reduced-order, independent, Kalman filters. This decomposition allows us to design the linear controllers for slow and fast subsystems completely independently of each other and thus, to achieve the complete and exact separation for the linear-quadratic stochastic regulator problem.

Consider the singularly perturbed linear stochastic continuous-time system

$$
\begin{aligned}
\dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + B_1u(t) + G_1w(t) \\
\epsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) + B_2u(t) + G_2w(t) \\
y(t) &= C_1x_1(t) + C_2x_2(t) + w_2(t)
\end{aligned}
\tag{2.112}
$$

with the performance criterion

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{ \int_{t_0}^{t_f} \left[z^T(t) z(t) + u^T(t) R u(t) \right] dt \right\}, \quad R > 0 \quad (2.113)$$

where $x_i(t) \in \Re^{n_i}, i = 1, 2$, comprise slow and fast state vectors, respectively. $u(t) \in \Re^m$, is the control input, $y(t) \in \Re^{r_2}$, is the observed output, $w_1(t) \in \Re^{r_1}$, and $w_2(t) \in \Re^{r_2}$, are independent zero-mean stationary Gaussian mutually uncorrelated white noise processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively, and $z(t) \in \Re^s$, is the controlled output given by

$$z(t) = D_1 x_1(t) + D_2 x_2(t) \quad (2.114)$$

All matrices are of appropriate dimensions and assumed to be constant. The optimal control law for (2.112) with the performance criterion (2.113) is given by

$$u_{opt}(t) = -F_1 \hat{x}_1(t) - F_2 \hat{x}_2(t) \quad (2.115)$$

where $\hat{x}_1(t)$ and $\hat{x}_2(t)$ are the optimal estimates of the state vectors $x_1(t)$ and $x_2(t)$ obtained from the Kalman filter

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1 \hat{x}_1(t) + A_2 \hat{x}_2(t) + B_1 u(t) + K_1 v(t) \\ \epsilon \dot{\hat{x}}_2(t) &= A_3 \hat{x}_1(t) + A_4 \hat{x}_2(t) + B_2 u(t) + K_2 v(t) \\ v(t) &= y(t) - C_1 \hat{x}_1(t) - C_2 \hat{x}_2(t) \end{aligned} \quad (2.116)$$

The optimal regulator gains F_1, F_2 and filter gains K_1, K_2 are given, respectively, by (2.88) and (2.81). The required positive semidefinite stabilizing solutions of the algebraic regulator and filter Riccati equations (2.7) and (2.82) can be obtained in terms of reduced-order, pure-slow and pure-fast, regulator and filter, algebraic Riccati equations, respectively, given by (2.31)-(2.32) and (2.105).

The optimal global Kalman filter (2.116) can be put in the form in which the filter is driven by the system measurements and optimal control inputs, that is

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1 C_1) \hat{x}_1(t) + (A_2 - K_1 C_2) \hat{x}_2(t) + B_1 u(t) + K_1 y(t) \\ \epsilon \dot{\hat{x}}_2(t) &= (A_3 - K_2 C_1) \hat{x}_1(t) + (A_4 - K_2 C_2) \hat{x}_2(t) + B_2 u(t) + K_2 y(t) \end{aligned} \quad (2.117)$$

It is known from Section 2.1 that there exists a nonsingular transformation defined by (2.100) such that (2.117) is decoupled into pure-slow and pure-fast local filters both driven by system measurements and system control inputs

$$\begin{aligned}\dot{\hat{\eta}}_s(t) &= \left(a_{1F} + a_{2F}P_{sF}\right)^T \hat{\eta}_s(t) + B_s u(t) + K_s y(t) \\ \epsilon\dot{\hat{\eta}}_f(t) &= \left(b_{1F} + b_{2F}P_{fF}\right)^T \hat{\eta}_f(t) + B_f u(t) + K_f y(t)\end{aligned} \tag{2.118}$$

The pure-slow and pure-fast filter gains, K_s, K_f are defined by (2.105). The pure-slow and pure-fast system input matrices are given by

$$\begin{bmatrix} B_s \\ \frac{1}{\epsilon}B_f \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} \tag{2.119}$$

As a result, the coefficients of the optimal pure-slow filter are functions of the solution of the pure-slow algebraic Riccati equation only and those of the pure-fast filter are functions of the solution of the pure-fast algebraic Riccati equation only. Thus, these two filters can be implemented independently in the different time scales (slow and fast). It should be noted that the filtering method proposed for singularly perturbed linear stochastic systems allows complete decomposition and parallelism between pure-slow and pure-fast filters.

The optimal control in the new coordinates is given by

$$u_{opt}(t) = -F\hat{x}(t) = -F\mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = -\begin{bmatrix} F_s & F_f \end{bmatrix} \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} \tag{2.120}$$

where F_s and F_f are obtained from (Lim, 1999)

$$\begin{bmatrix} F_s & F_f \end{bmatrix} = F\mathbf{T}_2^T = R^{-1}B^T P\left(\Pi_{1F} + \Pi_{F2}P_F\right)^T \tag{2.121}$$

The optimal value of J follows from the known formula (Kwakernaak and Sivan, 1972)

$$J_{opt} = \operatorname{tr}\left\{PKW_2K^T + P_F D^T D\right\} = \operatorname{tr}\left\{PGW_1G^T + P_F F^T RF\right\} \tag{2.122}$$

In summary, the procedure to obtain the solution of the LQG control problem is given by the following algorithm.

Algorithm 2.1: Optimal LQG of Singularly Perturbed Systems.

1) Solve (2.31)-(2.32) and (2.105) to get P_s, P_f, P_{sF}, P_{fF}.

2) Compute P_F in terms of P_{sF} and P_{fF}, and P in terms of P_s and P_f.
3) Find $\mathbf{T}_2$ in terms of P_F.
4) Find the filter and regulator gains from (2.105) and (2.119).
5) Find the pure-slow and pure-fast filters in the new coordinates using (2.118).
6) Obtain J_{opt} from (2.122).

Δ

The obtained optimal control and filtering scheme is presented in Figure 2.3.

The importance of the proposed method is in the fact that it allows complete time-scale parallelism of the filtering and control tasks through the complete and exact decomposition of the optimal control and filtering problems into slow and fast time scales, which reduces both off-line and on-line required computations.

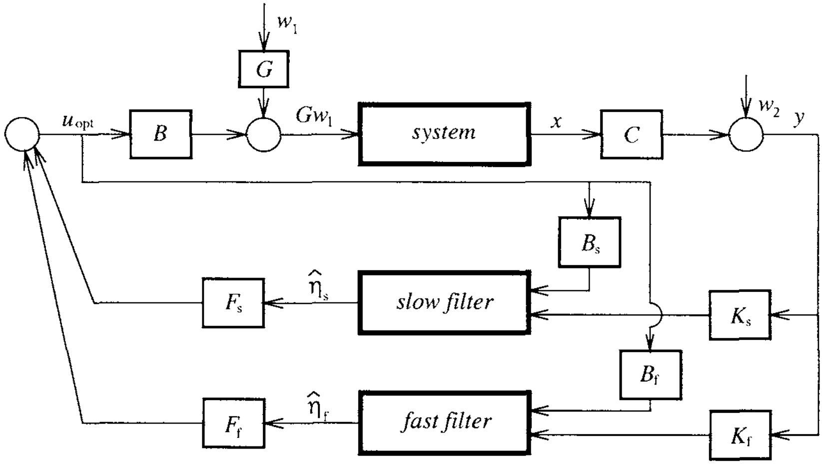

Figure 2.3: Complete parallelism and exact decomposition of the LQG regulator

2.4.1 Case Study: LQG Controller for an F-15 Aircraft

Consider the F-15 aircraft model from Section 2.3.1. The problem matrices A_1, A_2, A_3, A_4 and $G_1, G_2, B_1, B_2, W_1, W_2$ are given in Section 2.3.1. The remaining matrices are chosen as

$$D_1^T D_1 = \begin{bmatrix} 0.010000 & -0.032360 \\ -0.032360 & 0.104717 \end{bmatrix}$$

$$D_2^T D_2 = \begin{bmatrix} 0.009056 & 0.000000 \\ 0.000000 & 0.081502 \end{bmatrix}$$

$$D_1^T D_2 = \begin{bmatrix} -0.000032 & -0.000130 \\ 0.000102 & 0.000421 \end{bmatrix}, \qquad R = 1$$

The small singular perturbation parameter is $\epsilon = 0.2$.

The results obtained by using MATLAB are given below. The completely decoupled filter in the new coordinates, driven by the system measurements and control inputs are

$$\dot{\hat{\eta}}_s(t) = \begin{bmatrix} -6.0542 & -32.1708 \\ 0.1171 & 0.0000 \end{bmatrix} \hat{\eta}_s(t) + \begin{bmatrix} 32.9571 \\ -0.7216 \end{bmatrix} u(t)$$
$$+ \begin{bmatrix} 6.0411 & -0.0002 \\ -0.1169 & 0.0000 \end{bmatrix} y(t)$$

$$\epsilon\dot{\hat{\eta}}_f(t) = \begin{bmatrix} -2.8017 & -5.8585 \\ 0.1664 & -1.1466 \end{bmatrix} \hat{\eta}_f(t) + \begin{bmatrix} -0.4009 \\ -0.0027 \end{bmatrix} u(t)$$
$$+ \begin{bmatrix} 0.2333 & 0.0000 \\ 0.019 & 0.0001 \end{bmatrix} y(t)$$

The feedback control in the new coordinates is

$$u_{opt}(t) = -F\hat{x}(t) = -F\mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = -F_s\hat{\eta}_s(t) - F_f\hat{\eta}_f(t)$$
$$= -[4.6896 \quad 28.2648]\hat{\eta}_s(t) - [56.9755 \quad -680.5061]\hat{\eta}_f(t)$$

The optimal performance criterion, given by $J_{opt} = 0.8016$, is obtained by using the presented method with an arbitrary high order of accuracy (MATLAB accuracy).

2.4.2 Case Study: LQG Controller for an AIRC Aircraft

In this section we demonstrate that in the case when the system by itself is already decoupled into the slow and fast variables we can take $\epsilon = 1$ and proceed with the partitioning of the system matrices. Namely, the small singular perturbation parameter is already built-in in the system matrices, and the choice $\epsilon = 1$ is an artifice needed to complete computations and obtain the desired decomposition along the procedures presented in the previous sections of this chapter.

Consider the mathematical model of a AIRC aircraft (Hung and MacFarlane, 1982; Maciejowski, 1989) given by

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -2.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1.000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.5750 & 0 & -0.0732 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The remaining matrices are chosen as follows

$$G = B, \quad W_1 = I_3, \quad W_2 = I_3, \quad Q = I_5, \quad R = 10^{-2} I_3$$

We partition this system with $n_1 = 3$, $n_2 = 2$, and $\epsilon = 1$. Hence, the system has three slow and two fast state space variables.

The solutions for the pure-slow and pure-fast regulator algebraic Riccati equations are given by

$$P_s = \begin{bmatrix} 0.8942 & -1.8569 & 0.4585 \\ 0.0002 & 0.0851 & -0.0012 \\ 0.4329 & -2.1868 & 1.3812 \end{bmatrix}, \quad P_f = \begin{bmatrix} 0.0079 & -0.0401 \\ 0.0943 & 0.1543 \end{bmatrix}$$

which via formula (2.28) lead to

$$P = \begin{bmatrix} 1.0962 & 0.0019 & 0.6720 & 0.0670 & -0.2042 \\ 0.0019 & 0.0994 & 0.0025 & 0.0023 & -0.0008 \\ 0.6720 & 0.0025 & 1.6542 & 0.0975 & -0.2326 \\ 0.0670 & 0.0023 & 0.0975 & 0.0400 & -0.0624 \\ -0.2042 & -0.0008 & -0.2326 & -0.0624 & 0.2182 \end{bmatrix}$$

Similarly, for the solutions of the pure-slow and pure-fast filter algebraic Riccati equations, we obtain

$$P_{sF} = \begin{bmatrix} 0.8457 & 0.0796 & 0.2653 \\ 0.0150 & 0.9468 & 0.1225 \\ 0.1101 & -0.0787 & 0.4325 \end{bmatrix}, \quad P_{fF} = \begin{bmatrix} 5.2615 & 2.9041 \\ 1.4941 & 0.7786 \end{bmatrix}$$

leading to

$$P_F = \begin{bmatrix} 0.9897 & 0.0732 & -0.2507 & -0.4934 & -0.8076 \\ 0.0732 & 0.9436 & -0.0642 & -0.0485 & -0.2483 \\ -0.2507 & -0.0642 & 1.7807 & 1.6190 & 2.2215 \\ -0.4934 & -0.0485 & 1.6190 & 6.6483 & 3.9742 \\ -0.8076 & -0.2483 & 2.2215 & 3.9742 & 3.9020 \end{bmatrix}$$

Independent pure-slow and pure-fast Kalman filters in the new coordinates are given by

$$\dot{\hat{\eta}}_s(t) = \begin{bmatrix} -0.8457 & -0.0975 & 1.0219 \\ -0.0796 & -0.9956 & -0.0925 \\ -0.2653 & 0.1248 & -0.4325 \end{bmatrix} \eta_s(t)$$

$$+ \begin{bmatrix} 0.8457 & 0.0150 & 0.1101 \\ 0.0796 & 0.9468 & -0.0787 \\ 0.2653 & 0.1225 & 0.4325 \end{bmatrix} y(t)$$

$$+ \begin{bmatrix} -0.1876 & 0.0000 & 0.2065 \\ -0.0912 & 1.0000 & -0.0188 \\ -0.0674 & 0.0000 & -0.3933 \end{bmatrix} u(t)$$

and

$$\dot{\hat{\eta}}_f(t) = \begin{bmatrix} -0.8533 & -3.3665 \\ 1.0083 & -2.1823 \end{bmatrix} \eta_f(t)$$

$$+ \begin{bmatrix} -0.2725 & -0.0289 & 0.5044 \\ -0.4834 & -0.0958 & 1.3717 \end{bmatrix} y(t)$$

$$+ \begin{bmatrix} 3.9215 & 0.0368 & -1.2772 \\ 1.0016 & 0.0054 & -0.2403 \end{bmatrix} u(t)$$

The optimal feedback control obtain in terms of pure-slow and pure-fast optimal estimates is

$$u_{opt}(t) = -\begin{bmatrix} -0.0527 & -0.6489 & 2.5383 \\ 0.2335 & 9.9335 & 0.1589 \\ -9.3724 & -0.2269 & -16.5773 \end{bmatrix} \eta_s(t)$$

$$- \begin{bmatrix} 4.8221 & 26.4202 \\ 0.0494 & 0.4379 \\ -3.4356 & -2.9239 \end{bmatrix} \eta_f(t)$$

2.5 Comments

The presentation of this chapter is based on the recent research work of the authors and their coworkers. In that respect, we have followed the work of (Su *et al.*, 1992a,b; Gajic and Shen, 1993) in Sections 2.1 and 2.2. Section 2.3 is based in part on the results of (Khalil and Gajic, 1984; Gajic and Lim, 1994; Lim, 1994a, 1999), and Section 2.4 follows the presentation of (Lim, 1994b).

The results presented in this chapter also contain improvements over those already published, and at some places (for example the decomposition of the optimal quadratic performance criterion of linear singularly perturbed systems into pure-slow and pure-fast components), the results presented appear for the first time in this book. The new version of the Chang transformation, derived in (Qureshi and Gajic, 1992) could have been used in this chapter alternatively to the Chang transform. The new version of the Chang transformation is characterized by parallelism in the algebraic equations whose solutions are needed to form the Chang transformation. The transformation of (Qureshi and Gajic, 1992) is presented in Appendix 2.3. Similarly, the results of (Derbel *et al.*, 1994) could have been used for the block-diagonalization of the obtained linear singularly perturbed systems.

In this chapter we have developed a powerful fundamental technique for the pure-slow and pure-fast decomposition of optimal control and filtering tasks for linear singularly perturbed systems. Its variants will be used in the follow-up chapters of this book to solve more complex and more challenging pure-slow and pure-fast decomposition problems.

Appendix 2.1

It is easy to show that

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1}\mathbf{T}_1^{-1}E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ -L_1 & I_{n_2} & -L_2 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(\epsilon) \quad (2.123)$$

which implies

$$\Omega_1 = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 & I_{n_2} \end{bmatrix} + O(\epsilon), \qquad \Omega_2 = \begin{bmatrix} 0 & 0 \\ -L_2 & 0 \end{bmatrix} + O(\epsilon) \quad (2.124)$$

Then, the matrix

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 - L_2 P_1 & I_{n_2} \end{bmatrix} + O(\epsilon) \tag{2.125}$$

is invertible for sufficiently small values of ϵ.

Similarly, we have

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T \mathbf{T}_1 E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & -H_2 \\ L_1 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_1} & -H_4 \\ L_3 & 0 & 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix} + O(\epsilon) \tag{2.126}$$

with

$$\Pi_1 = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\epsilon), \qquad \Pi_2 = \begin{bmatrix} 0 & -H_2 \\ 0 & 0 \end{bmatrix} + O(\epsilon) \tag{2.127}$$

Hence, the matrix

$$\Pi_1 + \Pi_2 P = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\epsilon) \tag{2.128}$$

is invertible for sufficiently small values of ϵ. In this appendix, we have used the following notation for the partitioned matrix H

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \tag{2.129}$$

Appendix 2.2

In this appendix we evaluate the expressions for A_s, S_s, and Q_s using (2.39), that is

$$\begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3$$

$$= \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix} - \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix} \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix} \tag{2.130}$$

By straightforward matrix multiplication, it can be easily verified that

$$T_4^{-1} = \begin{bmatrix} I & A_4^{-1} S_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_4^{-1} & 0 \\ 0 & -\left(A_4^T + Q_3 A_4^{-1} S_2\right)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ Q_3 A_4^{-1} & I \end{bmatrix} \tag{2.131}$$

Note that under Assumptions 2.1 and 2.3 used in the chapter, the matrix $A_4^T + Q_3 A_4^{-1} S_2$ is invertible. Using (2.131) in (2.130) and performing corresponding matrix multiplications, we obtain

$$
\begin{aligned}
A_s &= A_1 - A_2 A_4^{-1} A_3 \\
&-\left(Z - A_2 A_4^{-1} S_2\right)\left(A_4^T + Q_3 A_4^{-1} S_2\right)^{-1}\left(Q_3 A_4^{-1} A_3 - Q_2^T\right)
\end{aligned}
\tag{2.132}
$$

$$
\begin{aligned}
S_s &= S_1 + A_2 A_4^{-1} Z^T \\
&+\left(Z - A_2 A_4^{-1} S_2\right)\left(A_4^T + Q_3 A_4^{-1} S_2\right)^{-1}\left(Q_3 A_4^{-1} Z^T + A_2^T\right)
\end{aligned}
\tag{2.133}
$$

$$
\begin{aligned}
Q_s &= Q_1 - Q_2 A_4^{-1} A_3 \\
&+\left(A_3^T + Q_2 A_4^{-1} S_2\right)\left(A_4^T + Q_3 A_4^{-1} S_2\right)^{-1}\left(Q_3 A_4^{-1} A_3 - Q_2^T\right)
\end{aligned}
\tag{2.134}
$$

Note that the dual formula for the filtering problem as defined in (2.107) can be similarly derived. In this case, the derivations are valid under Assumption 2.1 and 2.5.

Another set of formulas for A_s, S_s, and Q_s can be obtained by using the result of (Fridman, 1994), which requires only Assumption 2.3. The inversion of the matrix T_4 in (2.39) is evaluated by using the fact that

$$
T_4 = \begin{bmatrix} I & 0 \\ P_f^{(0)} & I \end{bmatrix} \begin{bmatrix} A_4 - S_2 P_f^{(0)} & -S_2 \\ 0 & -\left(A_4 - S_2 P_f^{(0)}\right)^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_f^{(0)} & I \end{bmatrix}
\tag{2.135}
$$

where $P_f^{(0)}$ is the positive semidefinite stabilizing solution of the approximate fast algebraic Riccati equation (2.37). Such a solution for $P_f^{(0)}$ exists under Assumption 2.3. Let us denote $A_4 - S P_f^{(0)} = \Lambda_4$. Then, the inversion of the matrix T_4 is given by

$$
T_4^{-1} = \begin{bmatrix} I & 0 \\ P_f^{(0)} & I \end{bmatrix} \begin{bmatrix} \Lambda_4^{-1} & -\Lambda_4^{-1} S_2 \Lambda_4^{-T} \\ 0 & -\Lambda_4^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_f^{(0)} & I \end{bmatrix}
\tag{2.136}
$$

Using (2.136) in (2.130) implies

$$
\begin{aligned}
A_s &= A_1 - \left(A_2 - Z P_f^{(0)}\right) \Lambda_4^{-1} A_3 \\
&-\left(A_2 - Z P_f^{(0)}\right) \Lambda_4^{-1} S_2 \Lambda_4^{-T}\left(P_f^{(0)} A_3 + Q_2^T\right) + Z \Lambda_4^{-T}\left(P_f^{(0)} A_3 + Q_2^T\right)
\end{aligned}
\tag{2.137}
$$

$$S_s = Z - \left(A_2 - ZP_f^{(0)}\right)\Lambda_4^{-1}Z^T$$

$$+\left(A_2 - ZP_f^{(0)}\right)\Lambda_4^{-1}S_2\Lambda_4^{-T}\left(P_f^{(0)}Z^T - A_2^T\right) + Z\Lambda_4^{-T}\left(P_3Z^T - A_2^T\right) \tag{2.138}$$

$$\begin{aligned} Q_s = Q_1 - \left(Q_2 + A_3^T P_f^{(0)}\right)\Lambda_4^{-1}A_3 \\ -\left(Q_2 + A_3^T P_f^{(0)}\right)\Lambda_4^{-1}S_2\Lambda_4^{-T}\left(P_f^{(0)}A_3 + Q_2^T\right) \\ -A_3^T\Lambda_4^{-T}\left(P_f^{(0)}A_3 + Q_2^T\right) \end{aligned} \tag{2.139}$$

The importance of the analytical results presented for matrices A_s, S_s, and Q_s in formulas (2.137)-(2.139) is that they do not require invertibility condition imposed on matrix A_4 (Assumption 2.1), which has been the case in corresponding formulas (2.132)-(2.134).

Similar derivations can be performed for the filtering problem defined in (2.107) using (2.106) and the corresponding fast subsystem stabilizability-detectability Assumption 2.5.

Appendix 2.3 New Version of the Chang Transformation

Consider the singularly perturbed linear time invariant system

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t), \qquad x_1(t_0) = x_{10} \\ \epsilon\dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t), \qquad x_2(t_0) = x_{20} \end{aligned} \tag{2.140}$$

where $x_i(t) \in \Re^{n_i}$, $i = 1, 2$, $u(t) \in \Re^m$ are slow and fast state variables and ϵ is a small positive parameter.

Introducing the change of variables as

$$\begin{aligned} z_1 &= x_1 - \epsilon Lx_2 \\ z_2 &= -Hx_1 + x_2 \end{aligned} \tag{2.141}$$

where L and H are constant matrices to be determined. Differentiating (2.141) we obtain

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 - \epsilon L\dot{x}_2 \\ \dot{z}_2 &= -H\dot{x}_1 + \dot{x}_2 \end{aligned} \tag{2.142}$$

Substituting for $\dot{x}_1$ and $\dot{x}_2$ from the original system, and simplifying, we get

$$\dot{z}_1 = A_{10}z_1 - \mathcal{F}_1(L)x_2 \tag{2.143}$$

where

$$A_{10} = A_1 - LA_3 \tag{2.144}$$

and

$$\mathcal{F}_1(L) = LA_4 - \epsilon A_1 L - A_2 + \epsilon LA_3 L \tag{2.145}$$

Also

$$\epsilon \dot{z}_2 = A_{40}z_2 - \mathcal{F}_2(H)x_1 \tag{2.146}$$

where

$$A_{40} = A_4 - \epsilon HA_2 \tag{2.147}$$

and

$$\mathcal{F}_2(H) = \epsilon HA_1 - A_4 H - A_3 + \epsilon HA_2 H \tag{2.148}$$

By setting $\mathcal{F}_1(L) = 0$, and $\mathcal{F}_2(H) = 0$, we get the decoupled pure-slow and pure-fast subsystems

$$\dot{z}_1 = A_{10}z_1 = (A_1 - LA_3)z_1 \tag{2.149}$$

$$\epsilon \dot{z}_2 = A_{40}z_2 = (A_4 - \epsilon HA_2)z_2 \tag{2.150}$$

where L and H can be calculated from the following two independent algebraic equations

$$0 = -LA_4 + A_2 + \epsilon(A_1 L - LA_3 L) \tag{2.151}$$

$$0 = A_4 H + A_3 - \epsilon(HA_1 + HA_2 H) \tag{2.152}$$

Note that the last two algebraic equations have the same form so that they can be solved using the same algorithm. The L-equation is similar to the corresponding L-equation of the original Chang transformation. Hence, it can be solved using either fixed point iterations or Newton method or eigenvector method.

The introduced decoupling transformation is

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L \\ -H & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.153}$$

with

$$\mathbf{T}^{-1} = \begin{bmatrix} I_{n_1} + \epsilon LNH & \epsilon LN \\ NH & N \end{bmatrix} \tag{2.154}$$

where $N = (I_{n_2} - \epsilon HL)^{-1}$. Note that for sufficiently small values of ϵ, the matrix N is invertible.

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3

Discrete-Time Linear Optimal Control Systems

Discrete-time singularly perturbed control systems have been the subject of intensive research since the early 1980s. Several researchers have produced important results on different aspects of control problems of deterministic singularly perturbed discrete-time systems such as Phillips, Blankenship, Mahmoud, Sawan, Khorasani, Naidu and their coworkers. Particularly important are the fundamental results of Khalil and Litkouhi, (Litkouhi, 1983; Litkouhi and Khalil, 1984, 1985). Along the lines of the research of Khalil and Litkouhi, in (Gajic and Shen, 1991a,b) an extension of the linear-quadratic optimal control problem of (Litkouhi and Khalil, 1984) and the formulation and the solution of the linear-quadratic Gaussian stochastic control problem are presented.

Two main structures of singularly perturbed linear discrete systems have been considered in the literature: the fast time scale version (Litkouhi and Khalil, 1984, 1985; Butuzov and Vasileva, 1971; Hoppensteadt and Miranker, 1977; Blankenship, 1981; Mahmoud, 1986; Oloomi and Sawan, 1987; Khorasani and Azimi-Sadjadi, 1987) and the slow time scale version (Phillips, 1980; Naidu and Rao, 1985). Discrete-time models of singularly perturbed linear systems, similar to those of (Phillips, 1980; Naidu and Rao, 1985), were studied also in (Othman, *et al.*, 1985;

Mahmoud *et al.*, 1986). Since the slow time scale version presupposes the asymptotic stability of the fast modes, it seems that in the design procedure of stabilizing feedback controllers, the fast time scale version is much more appropriate (Litkouhi and Khalil, 1984). An interesting approach to the design of discrete-time observers for singularly perturbed continuous-time systems has been recently proposed in (Shouse and Taylor, 1995).

In this chapter, the algebraic regulator and filter Riccati equations of singularly perturbed discrete-time control systems are completely and exactly decomposed into reduced-order *continuous-time* algebraic Riccati equations corresponding to the slow and fast time scales. That is, the exact solution of the global *discrete-time* algebraic Riccati equation is obtained in terms of the reduced-order, pure-slow and pure-fast, nonsymmetric *continuous-time* algebraic Riccati equations. In addition, the optimal global Kalman filter is decomposed into pure-slow and pure-fast local optimal filters both driven by the system measurements and the system optimal control input. It is shown that these two filters can be implemented independently in parallel in different time scales. As a result, the optimal linear-quadratic Gaussian control problem for singularly perturbed linear discrete-time systems takes the complete decomposition and parallelism between optimal pure-slow and pure-fast filters and controllers.

The approach presented in this section is based on a closed-loop decomposition technique which guarantees complete decomposition of optimal filters and regulators and distribution of all required off-line and on-line computations. In the regulation problem (optimal linear-quadratic control problem), presented in Section 3.1, we show how to decompose exactly the *numerically ill-conditioned* discrete-time singularly perturbed algebraic Riccati equation into two reduced-order, pure-slow and pure-fast, *well-conditioned* continuous-time algebraic Riccati equations. The reduced-order continuous-time algebraic Riccati equations obtained are nonsymmetric, but their $O(\epsilon)$ approximations are symmetric ones. We show that the Newton method is very efficient for their solutions since the initial guesses $O(\epsilon)$ close to the exact solutions can be easily obtained from the results already available in (Litkouhi and Khalil, 1984).

In the filtering problem, Section 3.2, in addition of using duality between the optimal Kalman filter and the optimal linear-quadratic regulator to solve the discrete-time ill-conditioned filter algebraic Riccati equation in terms of reduced-order, pure-slow and pure-fast, well-conditioned *continuous-time* algebraic Riccati equations, we have obtained completely independent pure-slow and pure-fast Kalman filters both driven by the system measurements and the system optimal control input. In the literature of linear stochastic singularly perturbed systems, it is possible to find exactly decomposed slow and fast Kalman filters (Khalil and Gajic, 1984, Gajic, 1986) for continuous-time systems, and (Gajic and Shen, 1991b) for discrete-time systems, but those filters are driven by the innovation process so that the additional communication channels have to be formed in order to construct the innovation process. In Section 3.3, we use the separation principle to solve the linear-quadratic Gaussian control problem of singularly perturbed discrete-time stochastic systems. The last section of this chapter deals with the open-loop optimal control problem of discrete-time linear singularly perturbed systems, where the ill-conditioning of the original two-point boundary value problem is replaced by a much easier problem of an ill-conditioned system of linear algebraic equations.

3.1 Linear-Quadratic Optimal Control

In this section, we present an approach for the study of the linear-quadratic optimal control problem of singularly perturbed discrete-time systems. In that direction, the discrete algebraic Riccati equation of singularly perturbed systems is completely and exactly decomposed into two reduced-order *continuous-time* pure-slow and pure-fast algebraic Riccati equations. This decomposition facilitates the design of linear optimal controllers for slow and fast subsystems completely independently of each other and hence, achieves the complete and exact separation in the computational tasks for the linear-quadratic optimal regulator problem.

Consider the singularly perturbed linear time-invariant discrete system represented in its fast time scale formulation, as described in (Litkouhi, 1983; Litkouhi and Khalil, 1984, 1985)

$$\begin{aligned} x_1(k+1) &= (I_{n_1} + \epsilon A_1)x_1(k) + \epsilon A_2 x_2(k) + \epsilon B_1 u(k), \quad x_1(0) = x_{10} \\ x_2(k+1) &= A_3 x_1(k) + A_4 x_2(k) + B_2 u(k), \quad x_2(0) = x_{20} \end{aligned} \tag{3.1}$$

with slow variables $x_1 \in \Re^{n_1}$, fast state variables $x_2 \in \Re^{n_2}$, control inputs $u \in \Re^m$, where ϵ represents a small positive singular perturbation parameter. The performance criterion of the corresponding linear-quadratic optimal control problem is given by

$$J = \frac{\epsilon}{2} \sum_{k=0}^{\infty} \left[x(k)^T Q x(k) + u(k)^T R u(k) \right] \tag{3.2}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0, \quad R > 0 \tag{3.3}$$

It is well known that the solution of the above optimal regulation problem is given by

$$u(k) = -R^{-1} B^T \lambda(k+1) = \; -\left(R + B^T P B\right)^{-1} B^T P A x(k) \tag{3.4}$$

where $\lambda(k)$ is a costate variable and P is the positive semidefinite stabilizing solution of the discrete algebraic Riccati equation (Dorato and Levis, 1971; Lewis, 1986) given by

$$P = Q + A^T P A - A^T P B \left[R + B^T P B\right]^{-1} B^T P A \tag{3.5a}$$

whose solution is properly scaled as

$$P = \begin{bmatrix} \frac{1}{\epsilon} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{3.5b}$$

The Hamiltonian form of (3.1) and (3.2) can be written as the forward recursion (Lewis, 1986)

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{3.6}$$

with

$$\mathbf{H} = \begin{bmatrix} A + B R^{-1} B^T A^{-T} Q & -B R^{-1} B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix} \tag{3.7}$$

where $\mathbf{H}$ is the symplectic matrix, which has the property that the eigenvalues of $\mathbf{H}$ are grouped into two disjoint subsets Γ_1 and Γ_2, such that for every $\lambda_c \in \Gamma_1$ there exists $\lambda_d \in \Gamma_2$, which satisfies $\lambda_c \times \lambda_d = 1$, and we can choose either Γ_1 or Γ_2 to contain only the stable eigenvalues

(Salgado *et al.*, 1988). The corresponding matrices in (3.4)-(3.7) are given by

$$A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \; B = \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix}, \; S = BR^{-1}B^T = \begin{bmatrix} \epsilon^2 S_1 & \epsilon Z \\ \epsilon Z^T & S_2 \end{bmatrix}$$

$$S_1 = B_1 R^{-1} B_1^T, \;\; S_2 = B_2 R^{-1} B_2^T, \;\; Z = B_1 R^{-1} B_2^T \tag{3.8}$$

In (3.7) the assumption that the matrix A is invertible is used. For our problem, this requires the invertibility of the matrix A_4. In that case

$$A^{-1} = \begin{bmatrix} I_{n_1} & 0 \\ -A_4^{-1} A_3 & A_4^{-1} \end{bmatrix} + O(\epsilon) \tag{3.9}$$

Hence, our presentation requires the following assumption.

Assumption 3.1: The fast subsystem matrix A_4 is nonsingular.

In the following, we show how to obtain exactly the solution of the discrete-time algebraic Riccati equation of singularly perturbed systems, (3.5), in terms of solutions of two reduced-order *continuous-time*, pure-slow and pure-fast, algebraic Riccati equations.

Partitioning the vector $\lambda(k)$ as $\lambda(k) = \left[\lambda_1^T(k) \; \lambda_2^T(k)\right]^T$ with $\lambda_1(k) \in \Re^{n_1}$ and $\lambda_2(k) \in \Re^{n_2}$, we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.10}$$

It can been shown after some algebra that the Hamiltonian matrix (3.7) has the following form (see Appendix 3.1)

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon \overline{A_1} & \epsilon \overline{A_2} & \epsilon^2 \overline{S_1} & \epsilon \overline{S_2} \\ \overline{A_3} & \overline{A_4} & \epsilon \overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I_{n_1} + \epsilon \overline{A_{11}^T} & \overline{A_{21}^T} \\ \overline{Q_3} & \overline{Q_4} & \epsilon \overline{A_{12}^T} & \overline{A_{22}^T} \end{bmatrix} \tag{3.11}$$

Note that in the remaining part of this section there is no need for the analytical expressions for bared matrices. Those matrices have to be formed by the computer in the process of calculations, which can be done easily using either MATLAB or any other corresponding computer software.

Interchanging second and third rows in (3.11) and using the following scaling $[p_1(k)\ \ p_2(k)]^T = [\epsilon\lambda_1(k)\ \ \lambda_2(k)]^T$ in (3.10) yield

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{S_1} & \epsilon\overline{A_2} & \epsilon\overline{S_2} \\ \epsilon\overline{Q_1} & I_{n_1} + \epsilon\overline{A_{11}^T} & \epsilon\overline{Q_2} & \epsilon\overline{A_{21}^T} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^T} & \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} \tag{3.12}$$

where

$$T_1 = \begin{bmatrix} \overline{A_1} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}^T} \end{bmatrix}, \quad T_2 = \begin{bmatrix} \overline{A_2} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{21}^T} \end{bmatrix}$$
$$T_3 = \begin{bmatrix} \overline{A_3} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{12}^T} \end{bmatrix}, \quad T_4 = \begin{bmatrix} \overline{A_4} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix} \tag{3.13}$$

Introducing the notation

$$U(k) = \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix}, \quad V(k) = \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} \tag{3.14}$$

we obtain the singularly perturbed discrete-time linear system

$$\begin{aligned} U(k+1) &= (I_{2n_1} + \epsilon T_1)U(k) + \epsilon T_2 V(k) \\ V(k+1) &= T_3 U(k) + T_4 V(k) \end{aligned} \tag{3.15}$$

Applying to (3.15) the discrete-time version of the Chang transformation (Chang, 1972; Shen, 1990) defined by

$$\mathbf{T_1} = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}, \quad \mathbf{T_1^{-1}} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon LH \end{bmatrix}$$
$$\begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \tag{3.16}$$

we obtain in the new coordinates two completely decoupled subsystems

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \eta(k+1) = [I_{2n_1} + \epsilon(T_1 - T_2 L)]\eta(k) \tag{3.17}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \xi(k+1) = (T_4 + \epsilon L T_2)\xi(k) \tag{3.18}$$

where the matrices L and H satisfy

$$-L + T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{3.19}$$

$$H + T_2 - H T_4 + \epsilon(T_1 - T_2 L)H - \epsilon H L T_2 = 0 \tag{3.20}$$

The unique solutions of algebraic equations (3.19) and (3.20) exist, by the implicit function theorem (Ortega and Rheinboldt, 1970), under the condition that the matrix $T_4 - I_{2n_2}$ is nonsingular. It can be shown from (3.8)–(3.13) that the matrix T_4 is given by

$$T_4 = T_4^{(0)} + O(\epsilon) = \begin{bmatrix} A_4 + S_2 A_4^{-T} Q_3 & -S_2 A_4^{-T} \\ -A_4^{-T} Q_3 & A_4^{-T} \end{bmatrix} + O(\epsilon) \tag{3.21}$$

From (3.7) we see that $T_4^{(0)}$ represents the Hamiltonian matrix of the fast subsystem. The nonsingularity of $T_4^{(0)} - I_{2n_2}$ requires the following assumption.

Assumption 3.2: The triple $(A_4, B_2, \mathrm{Chol}(Q_3))$ is stabilizable-detectable.

It follows that under Assumption 3.2, the matrix $T_4 - I_{2n_2}$ is nonsingular for sufficiently small values of ϵ.

It should be emphasized that the applicability of the Chang transformation to discrete-time singularly perturbed linear systems requires nonsingularity of the matrix $T_4 - I_{2n_2}$. On the other hand, for continuous-time linear singularly perturbed systems, the corresponding fast subsystem matrix T_4 must be nonsingular.

Algebraic equations (3.19) and (3.20) can be solved using the Newton method, similarly to the solution of the corresponding continuous-time algebraic equations (2.11)-(2.12), (Grodt and Gajic, 1988) as given in (2.14)-(2.15). The Newton method converges quadratically in the neighborhood of the sought solution, that is, its rate of convergence is

$O\left(\epsilon^{2^i}\right)$. The initial guess required for the Newton method is easily obtained with the accuracy of $O(\epsilon)$, by setting $\epsilon = 0$ in equation (3.19), that is

$$L^{(0)} = (T_4 - I)^{-1} T_3 = L + O(\epsilon) \tag{3.22}$$

The Newton algorithm can be constructed by setting $L^{(i+1)} = L^{(i)} + \Delta L^{(i)}$ and neglecting $O\left((\Delta L)^2\right)$ terms. This leads to a Lyapunov-type equation of the form

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)} \tag{3.23}$$

with

$$\begin{gathered} D_1^{(i)} = T_4 - I_{2n_1} + \epsilon L^{(i)} T_2, \quad D_2^{(i)} = -\epsilon\left(T_1 - T_2 L^{(i)}\right) \\ Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)}, \quad i = 0, 1, 2, \ldots \end{gathered} \tag{3.24}$$

where the initial condition is obtained from (3.22). The Newton sequence will be $O(\epsilon^2)$, $O(\epsilon^4)$, $O(\epsilon^8)$, ..., $O\left(\epsilon^{2^i}\right)$ close to the exact solution, respectively, in each iteration. Having found the solution for the L-equation up to the desired degree of accuracy, one can obtain the solution for the H-equation by solving directly the algebraic Lyapunov-like (Sylvester) equation of the form

$$H^{(i)} D_1^{(i)} + D_2^{(i)} H^{(i)} = T_2 \tag{3.25}$$

which implies $H^{(i)} = H + O\left(\epsilon^{2^i}\right)$.

The rearrangement and modification of variables in (3.12) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \epsilon I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{3.26}$$

From (3.14), (3.16)-(3.18), and (3.26), we obtain the relationship between the original coordinates and the new ones

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} = E_2^T \mathbf{T_1} E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \Pi \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{3.27}$$

where E_2 is a permutation matrix of the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{3.28}$$

Since at steady state $\lambda(k) = Px(k)$, where P satisfies the discrete-time algebraic Riccati equation (3.5), it follows from (3.27) that

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(k), \qquad \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(k) \tag{3.29}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same characteristic for the new systems (3.17) and (3.18), that is, at the steady state the following holds

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \tag{3.30}$$

Then (3.29) and (3.30) yield

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{3.31}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1}\mathbf{T}_1^{-1}E_2 = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1} \tag{3.32}$$

which produces

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1} \tag{3.33}$$

It can be shown, by estimating the order of quantity for the entries in matrices $\Pi_1, \Pi_2, \Omega_1, \Omega_2$, that the required matrices in (3.31) and (3.33) are invertible.

Partitioning (3.17) and (3.18) as

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} = (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L) \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \tag{3.34}$$

$$\begin{bmatrix}\xi_1(k+1)\\ \xi_2(k+1)\end{bmatrix} = \begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix} = (T_4 + \epsilon LT_2)\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix} \tag{3.35}$$

and using (3.30) yield to two reduced-order nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0 \tag{3.36}$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0 \tag{3.37}$$

It is very interesting that the algebraic Riccati equation of singularly perturbed discrete-time control systems is completely and exactly decomposed into two reduced-order nonsymmetric *continuous-time* algebraic Riccati equations (3.36)-(3.37). Note that the continuous-time algebraic Riccati equation has been thoroughly studied in the control and mathematics literature and it is nowadays a well understood equation. On the other hand, the discrete-time algebraic Riccati equation is still a challenging research topic.

The pure-fast algebraic Riccati equation (3.37) is nonsymmetric, but its $O(\epsilon)$ perturbation is a symmetric one. This can be observed from the fact that

$$\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix} = (T_4 + \epsilon LT_2) = \begin{bmatrix}b_1^{(0)} & b_2^{(0)}\\ b_3^{(0)} & b_4^{(0)}\end{bmatrix} + O(\epsilon) = T_4^{(0)} + O(\epsilon) \tag{3.38}$$

with $T_4^{(0)}$ given in (3.21). The coefficients of the Hamiltonian matrix $T_4^{(0)}$ imply the following approximate, fast subsystem, discrete-time algebraic Riccati equation

$$P_f^{(0)} = A_4^T P_f^{(0)} A_4 + Q_3 - A_4^T P_f^{(0)} B_2 \left(R + B_2^T P_f^{(0)} B_2\right)^{-1} B_2^T P_f^{(0)} A_4 \tag{3.39}$$

such that $P_f = P_f^{(0)} + O(\epsilon)$. Note that the positive semidefinite stabilizing solution of (3.39) exists under Assumption 3.2. Equation (3.39) is identical to the approximate fast discrete-time algebraic Riccati equation of (Litkouhi and Khalil, 1984, 1985).

In order to establish that an $O(\epsilon)$ approximation of the pure-slow algebraic Riccati equation (3.36) is symmetric we use the following arguments. It follows from (3.34) and (3.22) that

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = I_{2n_1} + \epsilon(T_1 - T_2 L) = I_{2n_1} + \epsilon T_s$$

$$= I_{2n_1} + \epsilon\left(T_1 - T_2 L^{(0)}\right) + O(\epsilon) = I_{2n_1} + \epsilon T_s^{(0)} + O(\epsilon)$$

$$= \begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} + O(\epsilon) = I_{2n_1} + \epsilon\left(T_1 - T_2(T_4 - I_{2n_2})^{-1}T_3\right) + O(\epsilon) \tag{3.40}$$

On the other hand, the approximate slow continuous-time algebraic Riccati equation can be obtained from

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} \tag{3.41}$$

by using the methodology of (Litkouhi and Khalil, 1984, 1985) and assuming that the fast variables $x_2(k)$ and $p_2(k)$ are at the steady state. Using the fact that at the steady state $x_2(k+1) = x_2(k)$ and $p_2(k+1) = p_2(k)$ we get from (3.41)

$$\begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} = (I_{2n_2} - T_4)^{-1} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \tag{3.42}$$

and

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \end{bmatrix} = \left\{I_{2n_1} + \epsilon\left(T_1 - T_2(T_4 - I_{2n_2})^{-1}T_3\right)\right\} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix}$$

$$= \left(I_{2n_1} + \epsilon T_s^{(0)} + O\left(\epsilon^2\right)\right) \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \tag{3.43}$$

The matrix $T_s^{(0)}$ determines the coefficients for the approximate slow continuous-time algebraic Riccati equation of (Litkouhi and Khalil, 1984). It can be observed from (3.40) and (3.43) that

$$\begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} = I_{2n_1} + \epsilon T_s^{(0)}$$

$$= I_{2n_1} + \epsilon\left(T_1^{(0)} - T_2^{(0)}\left(T_4^{(0)} - I_{2n_3}\right)^{-1} T_3^{(0)}\right) \tag{3.44}$$

$$= \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon B_s R_s^{-1} B_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix} = \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon S_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix}$$

The corresponding approximate continuous-time algebraic Riccati equation is given by

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0 \tag{3.45}$$

such that $P_s = P_s^{(0)} + O(\epsilon)$. The matrices defined in (3.44) can be also found in (Litkouhi and Khalil, 1984). The unique positive semidefinite stabilizing solution of the slow approximate continuous-time algebraic Riccati equation exists under the assumption that the approximate slow subsystem is stabilizable-detectable.

Assumption 3.3: The approximate slow subsystem determined by $T_s^{(0)}$ is stabilizable-detectable. Let $\mathrm{Chol}(S_s)$ and $\mathrm{Chol}(Q_s)$ represent the Cholesky factors of the corresponding matrices. Then, the slow subsystem is stabilizable-detectable if the corresponding triple $(A_s, \mathrm{Chol}(S_s), \mathrm{Chol}(Q_s))$ is stabilizable-detectable.

We have established that $O(\epsilon)$ perturbations of (3.36) and (3.37) lead to the symmetric reduced-order approximate slow and fast algebraic Riccati equations obtained in (Litkouhi and Khalil, 1984). The solutions of these equations, (Litkouhi and Khalil, 1984), can be used as very good initial guesses for the Newton method for solving the obtained nonsymmetric Riccati equations (3.36) and (3.37).

The Newton algorithm for (3.36) is given by

$$P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)}$$
$$i = 0, 1, 2, \ldots \tag{3.46}$$

with the initial guess $P_s^{(0)}$ obtained from the continuous-time approximate slow algebraic Riccati equation (3.45). The Newton algorithm for (3.37) is similarly obtained as

$$P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)}$$
$$i = 0, 1, 2, ... \tag{3.47}$$

with the initial guess $P_f^{(0)}$ found by solving the discrete-time approximate fast algebraic Riccati equation (3.39).

The proposed method is very suitable for parallel computations since it allows complete parallelism. In addition, due to complete and exact decomposition of the discrete algebraic Riccati equation, the optimal control at steady state can be performed independently and in parallel in both slow and fast time scales. The pure-slow and pure-fast subsystems in the new coordinates are, respectively, given by

$$\eta_1(k+1) = (a_1 + a_2 P_s)\eta_1(k) \tag{3.48}$$

$$\xi_1(k+1) = (b_1 + b_2 P_f)\xi_1(k) \tag{3.49}$$

In summary, the optimal strategy and the optimal performance value are obtained by using the following algorithm.

Algorithm 3.1: Discrete-Time Singularly Perturbed Optimal Regulator.
Step 1: Solve Chang decoupling equations (3.19)-(3.20).
Step 2: Find coefficients a_i, b_i, $i = 1, 2, 3, 4$ by using (3.34)-(3.35).
Step 3: Solve the reduced-order exact pure-slow and pure-fast algebraic Riccati equations (3.36)-(3.37) which leads to P_s and P_f.
Step 4: Find the global solution of the algebraic Riccati equation in terms of P_s and P_f by using (3.33).
Step 5: Find the optimal regulator gain from (3.4) and the optimal performance criterion as $J_{opt} = \epsilon \times 0.5 x^T(t_0) P x(t_0)$.

Δ

3.1.1 Case Study: Discrete Model of an F-15 Aircraft

In order to demonstrate the efficiency of the proposed method, a discrete model of an F-15 aircraft introduced in Section 2.3.1 is considered. The continuous-time problem matrices are given in Section 2.3.1. The small perturbation parameter ϵ is chosen as $\epsilon = 0.2$. This model is discretized by using the sampling period $T = 1$ leading to the following discrete-time matrices

$$A = \begin{bmatrix} 0.9901 & -32.0281 & -13.2566 & -18.3436 \\ 0.0002 & 0.9978 & 0.4207 & 0.1785 \\ 0.0003 & -0.0056 & 0.1645 & 0.1975 \\ 0.0000 & 0.0005 & 0.1968 & 0.3887 \end{bmatrix}$$

$$B^T = [68.8772 \quad -3.8453 \quad -5.9435 \quad -2.5701]^T$$

The eigenvalues of the matrix A_4 are $0.0542, 0.4990$. Hence, the fast subsystem is asymptotically stable.

The linear-quadratic optimal control problem is solved for weighting matrices $R = 1$ and $Q = 0.01I_4$. The optimal global solution of the discrete-time algebraic Riccati equation is obtained as

$$P_{exact} = \begin{bmatrix} 0.0143 & -0.1009 & -0.0345 & -0.0613 \\ -0.1009 & 2.3820 & 0.8142 & 1.4319 \\ -0.0345 & 0.8142 & 0.2909 & 0.4917 \\ -0.0613 & 1.4319 & 0.4917 & 0.8799 \end{bmatrix}$$

Solutions of the pure-slow and pure fast algebraic Riccati equations obtained from (3.46) and (3.47) are

$$P_s = \begin{bmatrix} 0.0015 & -0.0016 \\ 0.0125 & 0.0350 \end{bmatrix}, \quad P_f = \begin{bmatrix} 0.0116 & 0.0034 \\ -0.0020 & 0.0100 \end{bmatrix}$$

Using formula (3.33), the solution for P is found to be identical to P_{exact}. The error between the solution of the proposed method and the exact one, obtained by using the classical global method for solving the algebraic Riccati equation, is given by $P_{exact} - P = O(10^{-15})$, which is the standard accuracy of MATLAB used in this book for numerical computations.

Assuming that the system initial conditions are given by $x^T(0) = [1 \quad 1 \quad 1 \quad 1]^T$, we find the optimal performance value, by using formula (3.4), as $J_{opt} = 0.5\epsilon x^T(0)Px(0) = 4.3247\epsilon$.

3.2 Kalman Filtering for Discrete Singularly Perturbed Systems

The singularly perturbed discrete-time Kalman filter has been studied in (Rao and Naidu, 1984; Gajic and Shen, 1991b; Lim, 1994; Gajic *et al.*, 1995; Kando, 1997). The approaches of Rao and Naidu (1984) and Kando (1997) are based on the power series expansions, and hence they are not efficient for achieving high accuracy for the filter estimation error. The recursive approach of Gajic and Shen (1991b), based on the fixed-point iterations to the discrete-time filtering of singularly perturbed systems, achieves high accuracy for the estimation error, but the slow-fast filters obtained are driven by the innovation process so that the additional communication channels have to be used in order to construct the innovation process.

In this section, we improve the results of Gajic and Shen (1991b) and derive the pure-slow and pure-fast, reduced-order, independent Kalman filters driven by the system measurements. The presented method is based on the exact decomposition of the global singularly perturbed algebraic filter Riccati equation into the pure-slow and pure-fast local algebraic filter Riccati equations. The optimal filter gain is completely determined in terms of exact pure-slow and exact pure-fast, reduced-order, continuous-time, algebraic filter Riccati equations, obtained by using the duality property between the optimal linear-quadratic filters and regulators. The methodology presented follows the results of (Lim, 1994) and (Gajic *et al.*, 1995).

Consider the linear, time-invariant, singularly perturbed, discrete-time stochastic system

$$x_1(k+1) = (I_{n_1} + \epsilon A_1)x_1(k+1) + \epsilon A_2 x_2(k) + \epsilon G_1 w_1\ (k)$$
$$x_2(k+1) = A_3 x_1(k) + A_4 x_2(k) + G_2 w_1(k)$$
$$x_1(0) = x_{10}, \quad x_2(0) = x_{20} \tag{3.50}$$

with the corresponding measurements

$$y(k) = C_1 x_1(k) + C_2 x_2(k) + w_2(k) \tag{3.51}$$

where $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are, respectively, slow and fast state vectors, $w_1(k) \in \Re^r$ and $w_2(k) \in \Re^l$ are zero-mean, stationary, white Gaussian noise stochastic processes with intensities $W_1 \geq 0$ and

$W_2 > 0$, respectively, and $y \in \Re^l$ are system measurements. In the following $A_i, G_j, C_j,\ i = 1, 2, 3, 4,\ j = 1, 2$, are constant matrices.

The optimal Kalman filter driven by the innovation process is given by (Kwakernaak and Sivan, 1972)

$$\hat{x}(k+1) = A\hat{x}(k) + K[y(k) - C\hat{x}(k)] \tag{3.52}$$

where

$$A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \ C = [C_1 \ \ C_2], \ K = \begin{bmatrix} \epsilon K_1 \\ K_2 \end{bmatrix} \tag{3.53}$$

The optimal filter gain K that minimizes the variance of the estimation error is obtained from

$$K = AP_F C^T \left(W_2 + CP_F C^T\right)^{-1} \tag{3.54}$$

where P_F is the positive semidefinite stabilizing solution of the discrete-time filter algebraic Riccati equation given by

$$P_F = AP_F A^T - AP_F C^T \left(W_2 + CP_F C^T\right)^{-1} CP_F A^T + GW_1 G^T \tag{3.55}$$

where

$$G = \begin{bmatrix} \epsilon G_1 \\ G_2 \end{bmatrix} \tag{3.56}$$

Due to the singularly perturbed structure of the problem matrices the required solution P_F in the fast time scale version has the form

$$P_F = \begin{bmatrix} \epsilon P_{F1} & \epsilon P_{F2} \\ \epsilon P_{F2}^T & P_{F3} \end{bmatrix} \tag{3.57}$$

Partitioning the discrete-time filter Riccati equation (3.55), in the sense of the singular perturbation methodology (Naidu and Rao, 1985; Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986), will produce a lot of terms and make the corresponding problem numerically inefficient, even though the problem order-reduction is achieved.

Using the decomposition procedure for the discrete-time algebraic regulator Riccati equation presented in the previous section and the duality property between the optimal linear-quadratic filters and regulators, we will obtain an efficient decomposition scheme such that the slow and fast filters of singularly perturbed discrete-time linear systems are completely decoupled and both of them are driven by the system measurements. The results of interest, from Section 3.1 that are needed for this section, are summarized in the form of the following lemma.

Lemma 3.1 *Consider the optimal closed-loop linear discrete system*

$$
\begin{aligned}
x_1(k+1) &= (I + \epsilon A_1 - \epsilon B_1 F_1)x_1(k) + \epsilon(A_2 - B_1 F_2)x_2(k) \\
x_2(k+1) &= (A_3 - B_2 F_1)x_1(k) + (A_4 - B_2 F_2)x_2(k)
\end{aligned}
\tag{3.58}
$$

There exists a nonsingular transformation $\mathbf{T}$

$$
\begin{bmatrix} \xi_s(k) \\ \xi_f(k) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
\tag{3.59}
$$

such that

$$
\begin{aligned}
\xi_s(k+1) &= (a_1 + a_2 P_s)\xi_s(k) \\
\xi_f(k+1) &= (b_1 + b_2 P_f)\xi_f(k)
\end{aligned}
\tag{3.60}
$$

where P_s and P_f are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic regulator Riccati equations (3.36)-(3.37). The nonsingular transformation $\mathbf{T}$ is given by

$$
\mathbf{T} = (\Pi_1 + \Pi_2 P)
\tag{3.61}
$$

Even more, the global solution P can be obtained from the reduced-order exact pure-slow and pure-fast algebraic regulator Riccati equations, that is

$$
P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1}
\tag{3.62}
$$

Known matrices $\Omega_i,\ i = 1, 2, 3, 4$ and $\Pi_1,\ \Pi_2$ are determined in terms of solutions of the Chang decoupling equations as given in (3.27)-(3.32).

◇

The desired slow-fast decomposition of the Kalman filter (3.52) will be obtained by using duality between the optimal filter and regulator, and the decomposition method developed in the previous section. Consider the optimal closed-loop singularly perturbed Kalman filter (3.52) driven by the system measurements, that is

$$\hat{x}_1(k+1) = (I + \epsilon A_1 - \epsilon K_1 C_1)\hat{x}_1(k)$$

$$+\epsilon(A_2 - K_1 C_2)\hat{x}_2(k) + \epsilon K_1 y(k)$$

$$\hat{x}_2(k+1) = (A_3 - K_2 C_1)\hat{x}_1(k) + (A_4 - K_2 C_2)\hat{x}_2(k) + K_2 y(k) \tag{3.63}$$

with the optimal Kalman filter gains K_1 and K_2 obtained from (3.53)-(3.54). By duality between the optimal filter and regulator, that is

$$\begin{gathered} A \to A^T, \;\; Q \to GW_1G^T, \;\; B \to C^T \\ BR^{-1}B^T \to C^T W_2^{-1} C \end{gathered} \tag{3.64}$$

the filter "state-costate equation" can be defined as

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \overline{\mathbf{H}} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{3.65}$$

where

$$\overline{\mathbf{H}} = \begin{bmatrix} A^T + C^T W_2^{-1} C A^{-1} G W_1 G^T & -C^T W_2^{-1} C A^{-1} \\ -A^{-1} G W_1 G^T & A^{-1} \end{bmatrix} \tag{3.66}$$

Partitioning $\lambda(k)$ as $\lambda(k) = \left[\lambda_1^T(k) \;\; \lambda_2^T(k)\right]^T$ with $\lambda_1(k) \in \Re^{n_1}$ and $\lambda_2(k) \in \Re^{n_2}$, (3.65) can be rewritten as follows (see Appendix 3.2)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{n_1} + \epsilon \overline{A_1^T} & \overline{A_3^T} & \overline{S_1} & \overline{S_2} \\ \epsilon \overline{A_2^T} & \overline{A_4^T} & \overline{S_3} & \overline{S_4} \\ \epsilon^2 \overline{Q_1} & \epsilon \overline{Q_2} & I_{n_1} + \epsilon \overline{A_{11}} & \epsilon \overline{A_{12}} \\ \epsilon \overline{Q_3} & \overline{Q_4} & \overline{A_{21}} & \overline{A_{22}} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.67}$$

Interchanging second and third rows in (3.67) and introducing partitioning and scaling as $x(k) = \left[\epsilon x_1^T(k) \;\; x_2^T(k)\right]^T$ yield

$$\begin{aligned} \begin{bmatrix} \epsilon x_1(k+1) \\ \lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1) \end{bmatrix} &= \begin{bmatrix} I_{n_1} + \epsilon \overline{A_1^T} & \epsilon \overline{S_1} & \epsilon \overline{A_3^T} & \epsilon \overline{S_2} \\ \epsilon \overline{Q_1} & I_{n_1} + \epsilon \overline{A_{11}} & \epsilon \overline{Q_2} & \epsilon \overline{A_{12}} \\ \overline{A_2^T} & \overline{S_3} & \overline{A_4^T} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{21}} & \overline{Q_4} & \overline{A_{22}} \end{bmatrix} \begin{bmatrix} \epsilon x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} \\ &= \begin{bmatrix} I + \epsilon T_{1F} & \epsilon T_{2F} \\ T_{3F} & T_{4F} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} \end{aligned} \tag{3.68}$$

where

$$T_{1F} = \begin{bmatrix} \overline{A_1^T} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}} \end{bmatrix}, \quad T_{2F} = \begin{bmatrix} \overline{A_3^T} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{12}} \end{bmatrix}$$

$$T_{3F} = \begin{bmatrix} \overline{A_2^T} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{21}} \end{bmatrix}, \quad T_{4F} = \begin{bmatrix} \overline{A_4^T} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}} \end{bmatrix} \tag{3.69}$$

These matrices comprise the system matrix of a standard singularly perturbed discrete linear system, namely

$$\begin{bmatrix} I + \epsilon T_{1F} & \epsilon T_{2F} \\ T_{3F} & T_{4F} \end{bmatrix}$$

The slow-fast decomposition can be achieved by applying the Chang transformation to (3.68), which yields two completely decoupled subsystems

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{1F} & a_{2F} \\ a_{3F} & a_{4F} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}$$
$$= (I + \epsilon(T_{1F} - T_{2F}L_F)) \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \tag{3.70}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \begin{bmatrix} b_{1F} & b_{2F} \\ b_{3F} & b_{4F} \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} = (T_{4F} + \epsilon L_F T_{2F}) \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \tag{3.71}$$

Note that the decoupling transformation has the form of (3.16) with H_F and L_F matrices obtained from (3.19)-(3.20) with T_{iF}'s taken from (3.69). By duality and Lemma 3.1 the following reduced-order nonsymmetric algebraic Riccati equations exist

$$P_{sF}a_{1F} - a_{4F}P_{sF} - a_{3F} + P_{sF}a_{2F}P_{sF} = 0 \tag{3.72}$$

$$P_{fF}b_{1F} - b_{4F}P_{fF} - b_{3F} + P_{fF}b_{2F}P_{fF} = 0 \tag{3.73}$$

The assumptions dual to Assumptions 3.2 and 3.3, in the case of the filter algebraic Riccati equation are given as follows.

Assumption 3.4: The triple (A_4, C_2, G_2) is stabilizable-detectable.

Assumption 3.5: The triple corresponding to the slow filter, that is, $\left(A_s, \mathrm{Chol}\left(C_s^T W_{2s}^{-1} C_s\right), \mathrm{Chol}\left(G_s W_{1s}^{-1} G_s^T\right)\right)$ is stabilizable-detectable.

The matrices introduced in Assumption 3.5 are obtained using duality with (3.44) as

$$\begin{aligned} T_{sF}^{(0)} &= T_{1F}^{(0)} - T_{2F}^{(0)}\left(T_{4F}^{(0)} - I_{2n_2}\right)^{-1} T_3^{(0)} \\ &= \begin{bmatrix} I + \epsilon A_s^T & -\epsilon C_s^T W_{2s}^{-1} C_s \\ -\epsilon G_s W_{1s} G_s^T & I - \epsilon A_s \end{bmatrix} \end{aligned} \tag{3.74}$$

Under Assumption 3.4 the unique solution of the corresponding Chang decoupling algebraic equations exists for sufficiently small values of the singular perturbation parameter ϵ. Assumptions 3.4 and 3.5 guarantee the existence of the unique positive semidefinite stabilizing solution of the filter algebraic Riccati equation (3.55) and the existence of the unique solutions for the pure-slow and pure-fast filter algebraic Riccati equations (3.72)-(3.73), both for sufficiently small values of the singular perturbation parameter ϵ.

By using the permutation matrices

$$\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} = E_{1F} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.75}$$

with (note that E_{1F} is different than the corresponding one for the regulator case)

$$E_{1F} = \begin{bmatrix} \epsilon I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix}, \quad E_{2F} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \tag{3.76}$$

we can define

$$\Pi_F = \begin{bmatrix} \Pi_{1F} & \Pi_{2F} \\ \Pi_{3F} & \Pi_{4F} \end{bmatrix} = E_{2F}^T \begin{bmatrix} I_{2n_1} - \epsilon H_F L_F & -\epsilon H_F \\ L_F & I_{2n_2} \end{bmatrix} E_{1F} \tag{3.77}$$

Then, the desired transformation is given by

$$\mathbf{T_2} = (\Pi_{1F} + \Pi_{2F} P_F) \tag{3.78}$$

The transformation $\mathbf{T_2}$ applied to the filter variables (3.63) as

$$\begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix} = \mathbf{T_2}^{-T} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{3.79}$$

produces

$$\begin{bmatrix}\hat{\eta}_s(k+1)\\ \hat{\eta}_f(k+1)\end{bmatrix} =$$

$$\mathbf{T}_2^{-T}\begin{bmatrix} I_{n_1}+\epsilon A_1-\epsilon K_1C_1 & \epsilon(A_2-K_1C_2)\\ A_3-K_2C_1 & A_4-K_2C_2\end{bmatrix}\mathbf{T}_2^T\begin{bmatrix}\hat{\eta}_s(k)\\ \hat{\eta}_f(k)\end{bmatrix} \tag{3.80}$$

$$+\mathbf{T}_2^{-T}\begin{bmatrix}\epsilon K_1\\ K_2\end{bmatrix}y(k)$$

such that the complete *closed-loop* decomposition is achieved, that is

$$\begin{aligned}\hat{\eta}_s(k+1) &= (a_{1F}+a_{2F}P_{sF})^T\hat{\eta}_s(k)+K_sy(k)\\ \hat{\eta}_f(k+1) &= (b_{1F}+b_{2F}P_{fF})^T\hat{\eta}_f(k)+K_fy(k)\end{aligned} \tag{3.81}$$

where

$$\begin{bmatrix}K_s\\ K_f\end{bmatrix} = \mathbf{T}_2^{-T}\begin{bmatrix}\epsilon K_1\\ K_2\end{bmatrix} \tag{3.82}$$

It is important to point out that the matrix P_F in (3.78) can be obtained in terms of P_{sF} and P_{fF} using formula (3.62) with Ω_{1F}, Ω_{2F}, Ω_{3F}, Ω_{4F} obtained from the following expression

$$\Omega_F = \begin{bmatrix}\Omega_{1F} & \Omega_{2F}\\ \Omega_{3F} & \Omega_{4F}\end{bmatrix} = E_{1F}^{-1}\begin{bmatrix} I_{n_1} & \epsilon H_F\\ -L_F & I_{n_2}-\epsilon L_FH_F\end{bmatrix}E_{2F} \tag{3.83}$$

A lemma dual to Lemma 3.1 can be now formulated as follows.

Lemma 3.2 *Given the closed-loop optimal Kalman filter (3.63) of a linear discrete-time singularly perturbed system. Then there exists a nonsingular transformation matrix (3.78), which completely decouples (3.63) into pure-slow and pure-fast local filters (3.81) both driven by the system measurements. Even more, the decoupling transformation (3.78) and the filter coefficients given in (3.81) can be obtained in terms of exact pure-slow and pure-fast reduced-order completely decoupled algebraic filter Riccati equations (3.72) and (3.73).*

◇

It should be noted that the filtering method presented facilitates complete decomposition and parallelism between pure-slow and pure-fast filters. The complete solution to the filtering problem of singularly perturbed discrete-time linear systems is summarized by the following algorithm.

Algorithm 3.2: Discrete-Time Singularly Perturbed Optimal Filter.
Step 1: Find T_{1F}, T_{2F}, T_{3F}, and T_{4F} from (3.69).
Step 2: Calculate L_F and H_F from (3.19)–(3.20) with the coefficient matrices obtained in Step 1.
Step 3: Find a_{iF}, b_{iF}, for $i = 1, 2, 3, 4$ from (3.70)-(3.71).
Step 4: Solve for P_{sF} and P_{fF} from (3.72) and (3.73).
Step 5: Find $\mathbf{T_2}$ from (3.78) with P_F obtained from (3.62).
Step 6: Calculate K_s and K_f from (3.82).
Step 7: Find the pure-slow and pure-fast filter system matrices by using (3.81).

Δ

The design of the observer-based controllers, a deterministic version of the problem studied in this section, has been considered in several papers (Oloomi and Sawan, 1987; Wang *et al.*, 1993; Li and Li, 1995; Shouse and Taylor, 1995). The use of the delta operator via a unified approach (Middleton and Goodwin, 1990) for discrete-time filtering of linear-singularly perturbed stochastic systems has been recently proposed in (Shim and Sawan, 1999). The unified delta operator approach to linear-quadratic optimal regulator of discrete-time singularly perturbed systems has been considered in (Shim and Sawan, 1998). It seems from the results presented in the above papers that the unified approach is a promising technique for studying singularly perturbed linear systems.

3.3 Linear-Quadratic Optimal Gaussian Control Problem

The discrete-time linear-quadratic Gaussian control problem of singularly perturbed systems has been studied for the full state feedback in (Gajic and Shen, 1991b) and for the output feedback in (Qureshi *et al.*, 1992) by using the recursive approach based on the fixed-point iterations. Here, we solve the linear-quadratic full state Gaussian optimal control problem by using the Hamiltonian approach whose main results are obtained in Sections 3.1 and 3.2.

Consider the singularly perturbed discrete linear stochastic system represented in the fast time scale formulation (Litkouhi and Khalil, 1984, 1985; Gajic and Shen, 1991b):

$$
\begin{aligned}
x_1(k+1) &= (I_{n_1} + \epsilon A_1)x_1(k) + \epsilon A_2 x_2(k) + \epsilon B_1 u(k) + \epsilon G_1 w_1(k) \\
x_2(k+1) &= A_3 x_1(k) + A_4 x_2(k) + B_2 u(k) + G_2 w_1(k) \\
y(k) &= C_1 x_1(k) + C_2 x_2(k) + w_2(k)
\end{aligned}
\tag{3.84}
$$

with the performance criterion

$$
J = \frac{\epsilon}{2} E\left\{ \sum_{k=0}^{\infty} \left[z^T(k) z(k) + u^T(k) R u(k) \right] \right\}, \quad R > 0 \tag{3.85}
$$

where $x_i(k) \in \Re^{n_i}$, $i = 1, 2$, comprise slow and fast state vectors respectively. $u(k) \in \Re^m$ is the control input, $y(k) \in \Re^l$ is the observed output, $w_1(k) \in \Re^r$ and $w_2(k) \in \Re^l$ are independent zero-mean stationary Gaussian mutually uncorrelated white noise processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively. $z(k) \in \Re^s$ is the controlled output given by

$$
z(k) = D_1 x_1(k) + D_2 x_2(k) \tag{3.86}
$$

All matrices are of appropriate dimensions and assumed to be constant.

The optimal control law of the system (3.84) with performance criterion (3.85) is given by (Kwakernaak and Sivan, 1972)

$$
u(k) = -F\hat{x}(k) \tag{3.87}
$$

with the time-invariant filter

$$
\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K[y(k) - C\hat{x}(k)] \tag{3.88}
$$

where

$$
A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ \ C_2], \quad K = \begin{bmatrix} \epsilon K_1 \\ K_2 \end{bmatrix} \tag{3.89}
$$

The regulator gain F and filter gain K are obtained from

$$
F = \left(R + B^T P_R B\right)^{-1} B^T P_R A \tag{3.90}
$$

$$
K = A P_F C^T \left(W_2 + C P_F C^T\right)^{-1} \tag{3.91}
$$

where P_R and P_F are, respectively, the positive semidefinite stabilizing solutions of the discrete-time algebraic regulator and filter Riccati equations (Dorato and Levis, 1971), respectively given by

$$P_R = D^T D + A^T P_R A - A^T P_R B \left(R + B^T P_R B\right)^{-1} B^T P_R A \tag{3.92}$$

$$P_F = A P_F A^T - A P_F C^T \left(W_2 + C P_F C^T\right)^{-1} C P_F A^T + G W_1 G^T \tag{3.93}$$

where

$$D = [D_1 \ \ D_2], \ \ G = \begin{bmatrix} \epsilon G_1 \\ G_2 \end{bmatrix} \tag{3.94}$$

The required solutions P_R and P_F in the fast time scale version have the forms

$$P_R = \begin{bmatrix} P_{R1}/\epsilon & P_{R2} \\ P_{R2}^T & P_{R3} \end{bmatrix}, \ \ P_F = \begin{bmatrix} \epsilon P_{F1} & \epsilon P_{F2} \\ \epsilon P_{F2}^T & P_{F3} \end{bmatrix} \tag{3.95}$$

The exact decomposition method of the discrete algebraic regulator and filter Riccati equations from Section 3.1 and 3.2 produces two sets of two reduced-order nonsymmetric, pure-slow and pure-fast, algebraic Riccati equations, that is, for the regulator

$$P_1 a_1 - a_4 P_1 - a_3 + P_1 a_2 P_1 = 0 \tag{3.96}$$

$$P_2 b_1 - b_4 P_2 - b_3 + P_2 b_2 P_2 = 0 \tag{3.97}$$

and for the filter

$$P_s a_{1F} - a_{4F} P_s - a_{3F} + P_s a_{2F} P_s = 0 \tag{3.98}$$

$$P_f b_{1F} - b_{4F} P_f - b_{3F} + P_f b_{2F} P_f = 0 \tag{3.99}$$

where the unknown coefficients are obtained in the previous sections of this chapter. The Newton algorithm can be used efficiently in solving the reduced-order nonsymmetric Riccati equations (3.96)-(3.99).

It was shown in Section 3.2 that the optimal global Kalman filter, based on the exact decomposition technique, is decomposed into pure-slow and pure-fast local optimal filters both driven by the system measurements. As a result, the coefficients of the optimal pure-slow filter are functions of the solution of the pure-slow Riccati equation only and those of the pure-fast filter are functions of the solution of the pure-fast Riccati equation only. Thus, these two filters can be implemented independently in the different time scales (slow and fast). The pure-slow and pure-fast filters are, respectively, given by

$$\begin{aligned}\hat{\eta}_s(k+1) &= (a_{1F} + a_{2F}P_s)^T \hat{\eta}_s(k) + K_s y(k) + B_s u(k) \\ \hat{\eta}_f(k+1) &= (b_{1F} + b_{2F}P_f)^T \hat{\eta}_f(k) + K_f y(k) + B_f u(k)\end{aligned} \tag{3.100}$$

where

$$\begin{bmatrix} B_s \\ B_f \end{bmatrix} = \mathbf{T}_2^{-T} B = (\Pi_{1F} + \Pi_{2F}P_F)^{-T} B \tag{3.101}$$

It should be noted that the filtering method proposed for singularly perturbed linear discrete-time systems facilitates complete decomposition and parallelism between pure-slow and pure-fast filters.

The optimal control in the new coordinates has been obtained as

$$u(k) = -F\hat{x}(k) = -F\mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(k) \\ \hat{\eta}_f(k) \end{bmatrix} = -[F_s \quad F_f] \begin{bmatrix} \hat{\eta}_s(k) \\ \hat{\eta}_f(k) \end{bmatrix} \tag{3.102}$$

where F_s and F_f are obtained from

$$[F_s \quad F_f] = F\mathbf{T}_2^T = \left(R + B^T P_R B\right)^{-1} B^T P_R A (\Pi_{1F} + \Pi_{2F}P_F)^T \tag{3.103}$$

The optimal value of J is given by the very well-known form (Kwakernaak and Sivan, 1972)

$$J_{opt} = \frac{\epsilon}{2} tr\left[D^T D P_F + P_R K \left(C P_F C^T + W_2\right) K^T\right] \tag{3.104}$$

where F, K, P_R, and P_F are obtained from (3.90)-(3.94). Note that the full-order regulator and filter algebraic Riccati equations (3.92)-(3.93) should be solved in terms of solutions of reduced-order pure-slow and pure-fast algebraic Riccati equations (3.96)-(3.99) by using the corresponding formula (3.33). Hence, the complete problem can be solved in terms of the quantities obtained from the reduced-order problems.

The proposed scheme, presented in this section, for the solution of the linear-quadratic optimal Gaussian control problem of singularly perturbed discrete-time systems in terms of exact reduced-order, parallel, pure-slow and pure-fast, controllers and filters can be represented by the same block diagram as the one given in Figure 2.3 with the continuous-time signals being replaced by the corresponding discrete-time signals. That block diagram can be easily realized using the SIMULINK package.

3.3.1 Case Study: A Steam Power System

In order to demonstrate the efficiency of the proposed method, we consider a fifth-order discrete model of a steam power system (Mahmoud, 1982). The system matrices are given by

$$A = \begin{bmatrix} 0.9150 & 0.0510 & 0.0380 & 0.0150 & 0.0380 \\ -0.0300 & 0.8890 & -0.0005 & 0.0460 & 0.1110 \\ -0.0060 & 0.4680 & 0.2470 & 0.0140 & 0.0480 \\ -0.7150 & -0.0220 & -0.0211 & 0.2400 & -0.0240 \\ -0.1480 & -0.0030 & -0.0040 & 0.0900 & 0.0260 \end{bmatrix}$$

$$B^T = [0.0098 \quad 0.1220 \quad 0.0360 \quad 0.5620 \quad 0.1150]$$

The remaining matrices are chosen as

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad D^T D = \mathrm{diag}\{5 \quad 5 \quad 5 \quad 5 \quad 5\}, \quad R = I$$

It is assumed that $G = B$ and that the white noise processes are independent with intensities

$$W_1 = 5, \quad W_2 = \mathrm{diag}\{5 \quad 5\}$$

It is shown (Mahmoud, 1982) that this model possesses the singularly perturbed structure with $n_1 = 2$, $n_2 = 3$, and $\epsilon = 0.264$.

The completely decoupled filters driven by measurements $y(k)$ are given as

$$\hat{\eta}_s(k+1) = \begin{bmatrix} 0.8804 & 0.0428 \\ -0.0481 & 0.7824 \end{bmatrix} \hat{\eta}_s(k) + \begin{bmatrix} 0.1045 & 0.0643 \\ 0.1717 & 0.2780 \end{bmatrix} y(k)$$
$$+ \begin{bmatrix} 0.0629 \\ 0.3650 \end{bmatrix} u(k)$$

$$\hat{\eta}_f(k+1) = \begin{bmatrix} 0.2606 & -0.0112 & -0.0158 \\ -0.0533 & 0.1822 & -0.0585 \\ -0.0224 & 0.0662 & 0.0069 \end{bmatrix} \hat{\eta}_f(k)$$

$$+ \begin{bmatrix} -0.0044 & -0.0163 \\ 0.0164 & 0.0741 \\ 0.0067 & 0.0296 \end{bmatrix} y(k) + \begin{bmatrix} -0.0458 \\ 0.5590 \\ 0.1157 \end{bmatrix} u(k)$$

The feedback control in the new coordinates is

$$u(k) = [0.1407 \quad -0.3068]\hat{\eta}_s(k) - [0.1918 \quad 0.3705 \quad 0.1019]\hat{\eta}_f(k)$$

The difference of the performance criterion between the optimal value, J_{opt}, and the one of the proposed method, J, is given by

$$J_{opt} = \epsilon \times 6.73495$$
$$J - J_{opt} = 0.7727 \times 10^{-13}$$

3.4 Open-Loop Discrete Singularly Perturbed Control Problem

The optimal open-loop control problem is a two-point boundary value problem with the associated state-costate equations forming the Hamiltonian system. For singularly perturbed system, the Hamiltonian matrix retains the singularly perturbed form by interchanging and scaling some state and costate variables so that it can be block diagonalized via the nonsingular transformations of (Chang, 1972; Qureshi and Gajic, 1992). In this section, the original two-point boundary value problem is transformed into the pure-slow and pure-fast reduced-order completely decoupled initial value problems. By doing this, the stiffness of the singularly perturbed two-point boundary value problem is converted into the problem of an ill-defined linear system of algebraic equations. The proposed method is very suitable for parallel computations since it allows complete parallelism in both slow and fast time scales.

A singularly perturbed linear discrete-time system is represented by (Litkouhi and Khalil, 1984)

$$\begin{aligned} x_1(k+1) &= (I_{n_1} + \epsilon A_1)x_1(k) + \epsilon A_2 x_2(k) + \epsilon B_1 u(k) \\ x_2(k+1) &= A_3 x_1(k) + A_4 x_2(k) + B_2 u(k) \\ x_1(0) &= x_{10}, \qquad x_2(0) = x_{20} \end{aligned} \tag{3.105}$$

with slow state variables $x_1 \in \Re^{n_1}$, fast state variables $x_2 \in \Re^{n_2}$, and control inputs $u \in \Re^m$, where ϵ is a small positive singular perturbation parameter. The performance criterion of the corresponding linear-quadratic optimal control problem is defined by

$$J = \frac{1}{2}x^T(k_f)Q_f x(k_f) + \frac{1}{2}\sum_{k=0}^{k_f-1}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \quad (3.106)$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0$$

$$Q_f = \begin{bmatrix} \frac{1}{\epsilon}Q_{f1} & Q_{f2} \\ Q_{f2}^T & Q_{f3} \end{bmatrix} \geq 0, \qquad R > 0 \quad (3.107)$$

The open-loop optimal control problem has the solution given by

$$u(k) = -R^{-1}B^T\lambda(k+1) \quad (3.108)$$

where $\lambda(k)$ is the costate variable. The Hamiltonian form of (3.105)-(3.106) can be written as the forward recursion (Lewis, 1986)

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H}\begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \quad (3.109)$$

where

$$\mathbf{H} = \begin{bmatrix} A + BR^{-1}B^TA^{-T}Q & -BR^{-1}B^TA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \quad (3.110)$$

with boundary conditions expressed in the standard form as

$$M_1\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + N_1\begin{bmatrix} x(k_f) \\ \lambda(k_f) \end{bmatrix} = c \quad (3.111)$$

Note that

$$M_1 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ -Q_f & I_n \end{bmatrix}, \quad c = \begin{bmatrix} x(0) \\ 0 \end{bmatrix}, \quad n = n_1 + n_2 \quad (3.112)$$

for the free ending problem, or

$$M_1 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}, \quad c = \begin{bmatrix} x(0) \\ x(k_f) \end{bmatrix} \quad (3.113)$$

for the fixed endpoint problem.

For the singularly perturbed discrete system the matrices A and S have the forms

$$A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix} \tag{3.114}$$

$$S = BR^{-1}B^T = \begin{bmatrix} \epsilon^2 S_1 & \epsilon Z \\ \epsilon Z^T & S_2 \end{bmatrix}$$
$$S_1 = B_1 R^{-1} B_1^T, \quad S_2 = B_2 R^{-1} B_2^T, \quad Z = B_1 R^{-1} B_2^T \tag{3.115}$$

The approximate optimal solution of the open-loop control for linear singularly perturbed systems has been studied in (Naisu and Rao, 1985; Naidu, 1988), where the problem order was reduced and the stiff problem was avoided successfully by using the classic approach based on the power-series expansions. The developed method (Naidu and Rao, 1985; Naidu, 1988) is efficient for an $O(\epsilon)$ accuracy only. In the method presented in this section, an arbitrary order of the accuracy is easily obtained.

Partitioning vector $\lambda(k)$ as $\lambda(k) = [\lambda_1^T(k) \quad \lambda_2^T(k)]^T$ with $\lambda_1(k) \in \Re^{n_1}$ and $\lambda_2(k) \in \Re^{n_2}$, we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.116}$$

where

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{A_2} & \epsilon^2\overline{S_1} & \epsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \epsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I_{n_1} + \epsilon\overline{A_{11}^T} & \overline{A_{21}^T} \\ \overline{Q_3} & \overline{Q_4} & \epsilon\overline{A_{12}^T} & \overline{A_{22}^T} \end{bmatrix} \tag{3.117}$$

(see Appendix 3.1).

The standard singularly perturbed structure of (3.116) that can be further block diagonalized using the discrete-time version of the Chang transform (Gajic and Shen, 1993, Chapter 3, see also Appendix 3.4) can be obtained by interchanging the second and third rows in (3.117), which produces

$$\begin{bmatrix} x_1(k+1) \\ \epsilon\lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{S_1} & \epsilon\overline{A_2} & \epsilon\overline{S_2} \\ \epsilon\overline{Q_1} & I_{n_1} + \epsilon\overline{A_{11}^T} & \epsilon\overline{Q_2} & \epsilon\overline{A_{21}^T} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^T} & \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \epsilon\lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \epsilon\lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.118}$$

where

$$T_1 = \begin{bmatrix} \overline{A_1} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}^T} \end{bmatrix}, \; T_2 = \begin{bmatrix} \overline{A_2} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{21}^T} \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \overline{A_3} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{12}^T} \end{bmatrix}, \; T_4 = \begin{bmatrix} \overline{A_4} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix} \tag{3.119}$$

Introducing the notation

$$U(k) = \begin{bmatrix} x_1(k) \\ \epsilon\lambda_1(k) \end{bmatrix}, \qquad V(k) = \begin{bmatrix} x_2(k) \\ \lambda_2(k) \end{bmatrix} \tag{3.120}$$

we get the singularly perturbed discrete system under new notation

$$\begin{aligned} U(k+1) &= (I_{2n_1} + \epsilon T_1)U(k) + \epsilon T_2 V(k) \\ V(k+1) &= T_3 U(k) + T_4 V(k) \end{aligned} \tag{3.121}$$

Applying the Chang transformation, that is

$$\mathbf{T_1} = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}, \quad \mathbf{T_1}^{-1} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon LH \end{bmatrix}$$

$$\begin{bmatrix} U(k) \\ V(k) \end{bmatrix} = \mathbf{T_1} \begin{bmatrix} \overline{U}(k) \\ \overline{V}(k) \end{bmatrix} \tag{3.122}$$

to (3.121), we obtain two completely decoupled subsystems

$$\begin{aligned} \overline{U}(k+1) &= (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L)\overline{U}(k), \\ \overline{V}(k+1) &= (T_4 + \epsilon L T_2)\overline{V}(k) \end{aligned} \tag{3.123}$$

where the matrices L and H satisfy

$$\begin{aligned} 0 &= H + T_2 - HT_4 + \epsilon(T_1 - T_2 L)H - \epsilon HLT_2 \\ 0 &= -L + T_4 L - T_3 - \epsilon L(T_1 - T_2 L) \end{aligned} \tag{3.124}$$

Expanding (3.119) by using the partitioned matrices given by (3.107) and (3.114)-(3.115), and identifying the terms for the matrix T_4, we obtain

$$T_4 = \begin{bmatrix} A_4 + S_2 A_4^{-T} Q_3 & -S_2 A_4^{-T} \\ -A_4^{-T} Q_3 & A_4^{-T} \end{bmatrix} + O(\epsilon) \tag{3.125}$$

which is an $O(\epsilon)$ perturbation of the Hamiltonian matrix of the fast subsystem. Under Assumption 3.2, the matrix T_4 has no eigenvalues on the unit circle, so that $T_4 - I_{2n_2}$ is a nonsingular matrix, which implies the existence of the unique solutions for L and H in (3.124). The matrices L and H can be obtained by using the Newton recursive algorithm (3.22)-(3.24) with the rate of convergence is $O\left(\epsilon^{2^j}\right)$, where j is the number of iterations used to solve the L-equation.

The boundary conditions are changed due to an interchange of $\lambda_1(k)$ and $x_2(k)$, which modifies matrices in (3.112) as follows

$$M_2 \begin{bmatrix} U(0) \\ V(0) \end{bmatrix} + N_2 \begin{bmatrix} U(k_f) \\ V(k_f) \end{bmatrix} = c_1 \tag{3.126}$$

where

$$\begin{aligned} M_2 &= \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_1(0) \\ 0 \\ x_2(0) \\ 0 \end{bmatrix} \\ N_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Q_{f1} & I_{n_1} & -\epsilon Q_{f2} & 0 \\ 0 & 0 & 0 & 0 \\ -Q_{f2}^T & 0 & -Q_{f3} & I_{n_2} \end{bmatrix} \end{aligned} \tag{3.127}$$

The nonsingular transformation (3.122) applied to (3.126) produces

$$M_3 \begin{bmatrix} \overline{U}(0) \\ \overline{V}(0) \end{bmatrix} + N_3 \begin{bmatrix} \overline{U}(k_f) \\ \overline{V}(k_f) \end{bmatrix} = c_1 \tag{3.128}$$

where

$$M_3 = M_2\mathbf{T_1}, \qquad N_3 = N_2\mathbf{T_1} \tag{3.129}$$

The solutions of (3.121) are then given by

$$\begin{aligned} \overline{U}(k) &= (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L)^k \overline{U}(0) \\ \overline{V}(k) &= (T_4 + \epsilon L T_2)^k \overline{V}(0) \end{aligned} \tag{3.130}$$

We can eliminate $\overline{U}(k_f)$ and $\overline{V}(k_f)$ from (3.128) such that

$$\alpha(\epsilon)\begin{bmatrix}\overline{U}(0)\\ \overline{V}(0)\end{bmatrix} = c_1 \tag{3.131}$$

where

$$\alpha(\epsilon) = \left\{ M_3 + N_3 \begin{bmatrix} (I + \epsilon T_1 - \epsilon T_2 L)^{k_f} & 0 \\ 0 & (T_4 + \epsilon L T_2)^{k_f} \end{bmatrix} \right\} \tag{3.132}$$

$\overline{U}(0)$ and $\overline{V}(0)$ can be obtained from (3.131) provided the matrix $\alpha(\epsilon)$ is invertible. It is shown in Appendix 3.3 that the matrix $\alpha(\epsilon)$ is invertible for sufficiently small values of ϵ. Thus, we are able to find $\overline{U}(k)$ and $\overline{V}(k)$ from (3.130). Using (3.122), we can find $U(k)$ and $V(k)$.

After getting the solutions of $U(k)$ and $V(k)$, we can use the following relations to get the values for $\lambda_1(k)$ and $\lambda_2(k)$.

$$\begin{bmatrix} x_1(k) \\ \epsilon\lambda_1(k) \end{bmatrix} = \begin{bmatrix} U_1(k) \\ U_2(k) \end{bmatrix} = U(k), \quad \begin{bmatrix} x_2(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} V_1(k) \\ V_2(k) \end{bmatrix} = V(k) \tag{3.133}$$

The only difficulty we may encounter in the procedure to compute $\alpha(\epsilon)$ in (3.131) is when an ill-defined problem occurs due to presence of unstable modes in T_4 giving rise to large value of $(T_4 + \epsilon L T_2)^{k_f}$ for large values of k_f. In such a case we refer to the $O(\epsilon)$ solution as given in (Naidu, 1988).

The approximate optimal open-loop control, in view of (3.108), can be defined by

$$u^{(j)}(k) = -R^{-1}B^T\lambda^{(j)}(k+1) \tag{3.134}$$

where $\lambda^{(j)}(k+1)$ denotes the corresponding approximation for the optimal costate variable, where j is the number of iterations used for solving the L-equation.

3.4.1 Case Study: An F-8 Aircraft Control Problem

In order to demonstrate the proposed method, we study the linearized model of an F-8 aircraft. The original problem matrices can be found in (Elliot, 1977). By a proper scaling this model is presented in the standard singularly perturbed form as a linear continuous-time system (fast time scale representation) with

$$A_{cf} = \begin{bmatrix} -0.015 & -0.0805 & -0.0011666 & 0 \\ 0 & 0 & 0 & 0.03333 \\ -2.28 & 0 & -0.84 & 1 \\ 0.6 & 0 & -4.8 & -0.49 \end{bmatrix}$$

and

$$B_{cf} = \begin{bmatrix} -0.000916 & 0.0007416 \\ 0 & 0 \\ -0.11 & 0 \\ -8.7 & 0 \end{bmatrix}$$

The small elements in the first two rows of the above matrices indicate two slow variables (fast time scale representation). The small singular perturbation parameter is chosen as $\epsilon = 1/30$. This model is discretized in (Litkouhi, 1983) by using the sampling period $T = 1$, leading to

$$A = \begin{bmatrix} 0.98475 & -0.079903 & 0.0009054 & -0.0010765 \\ 0.041588 & 0.99899 & -0.035855 & 0.012684 \\ -0.54662 & 0.044916 & -0.32991 & 0.19318 \\ 2.6624 & -0.10045 & -0.92455 & -0.26325 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.0037112 & 0.00073610 \\ -0.087051 & 0.0000093411 \\ -1.19844 & -0.00041378 \\ -3.1927 & 0.00092535 \end{bmatrix}$$

The eigenvalues of the matrix A_4 are $\lambda_{1,2} = -0.297 \pm j0.442$, which indicates that the fast subsystem is asymptotically stable. The linear-quadratic optimal open-loop control problem is solved for the following choice of the weighting matrices $R = I_2, Q = 10^{-2} I_4$. The system initial condition is chosen as $x^T(0) = [1 \quad 1 \quad 1 \quad 1]$ and the terminal penalty matrix is assumed to be $Q_f = \mathrm{diag}[0.5 \quad 0.5 \quad 0.01 \quad 0.01]$. The terminal time is $k_f = 9$.

The approximate and optimal values of the performance criterion are presented in Table 3.1. Table 3.2 shows the approximate and optimal values of the control input $u(k)$ obtained by using formulas (3.108) and (3.134).

Table 3.1: Values of the performance criterion

Number of iterations j	$J^{(j)}$	$J_{opt} - J^{(j)}$
1	2.6070	0.2787
2	2.3292	0.0009
3	2.3285	0.0002
4	2.3283	< 0.000001

We can see that the approximate optimal control $u(k)$ and the approximate optimal performance criterion converge very rapidly to the optimum values. It can be seen that the error of the performance criterion reduces with the rate of $O\left(\epsilon^2\right)$, which is consistent with the analytical results. On the other hand, the approximate optimal control improves by an $O(\epsilon)$ per iteration.

3.5 Comments

The presentation of this chapter is mostly based on the recent research work of the authors and their coworkers. In Sections 3.1–3.3, we follow the works of (Lim, 1994; Lim *et al.*, 1995; Gajic *et al.*, 1995). Section 3.4 is based on the results of (Qureshi *et al.*, 1991; Qureshi, 1992). In some sections the previously published results are improved and presented in a more systematic manner.

The methodology presented for the optimal linear-quadratic Gaussian control gives the maximum that one can expect from the slow-fast time separation, namely, it gives the exact decomposition and perfect parallelism of the optimal control and filtering tasks, which very efficiently facilitates both off-line and on-line computational and implementational requirements.

Extensions of the results presented to other classes of linear-quadratic optimal control and filtering problems of discrete-time singularly perturbed linear systems, for example discrete-time high gain and cheap

Table 3.2: Approximate and optimal values of $u(k)$

k	$u^{(0)}(k)$	$u^{(1)}(k)$	$u^{(2)}(k)$	$u^{(3)}(k) = u_{opt}(k)$
0	0.3838 -0.0063	0.3950 -0.0063	0.3948 -0.0063	0.3947 -0.0063
1	0.4876 -0.0060	0.4977 -0.0060	0.4973 -0.0060	0.4973 -0.0060
2	0.5120 -0.0056	0.5217 -0.0056	0.5214 -0.0056	0.5214 -0.0056
3	0.5495 -0.0052	0.5601 -0.0052	0.5599 -0.0053	0.5599 -0.0053
4	0.5664 -0.0048	0.5769 -0.0048	0.5767 -0.0048	0.5767 -0.0048
5	0.6250 -0.0044	0.6358 -0.0044	0.6357 -0.0044	0.6357 -0.0044
6	0.6713 -0.0039	0.6825 -0.0039	0.6825 -0.0039	0.6825 -0.0039
7	0.5265 -0.0035	0.5359 -0.0034	0.5359 -0.0034	0.5359 -0.0034
8	0.8580 -0.0029	0.8695 -0.0029	0.8694 -0.0029	0.8694 -0.0029
9	0.8929 -0.0021	0.9055 -0.0021	0.9060 -0.0021	0.9059 -0.0021

control problems, are possible future research topics. In Chapter 6, we will present the results to the cheap control optimal problem of a special class of discrete-time linear singularly perturbed systems (sampled data systems).

Appendix 3.1

Here, we verify the structure of the Hamiltonian matrix $\mathbf{H}$ introduced in (3.11). From formula (3.7), we have

$$\mathbf{H} = \begin{bmatrix} A + BR^{-1}B^TA^{-T}Q & -BR^{-1}B^TA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \tag{3.135}$$

Since A^{-T} has the same structure as A^T, that is

$$A^{-T} = \begin{bmatrix} I_{n_1} + O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix} \tag{3.136}$$

then, we have the following estimates of the order of the particular elements

$$\begin{aligned} A^{-T}Q &= \begin{bmatrix} I_{n_1} + O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix} \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \\ BR^{-1}B^TA^{-T} &= \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} \begin{bmatrix} I_{n_1} + O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix} \\ &= \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} \\ A + BR^{-1}B^TA^{-T}Q &= \begin{bmatrix} I + O(\epsilon) & O(\epsilon) \\ O(1) & O(1) \end{bmatrix} \end{aligned} \tag{3.137}$$

The above estimates of the entries of the matrix $\mathbf{H}$ used in (3.135) produce the desired result, that is

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{A_2} & \epsilon^2\overline{S_1} & \epsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \epsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & \overline{I_{n_1} + \epsilon A_{11}^T} & \overline{A_{21}^T} \\ \overline{Q_3} & \overline{Q_4} & \overline{\epsilon A_{12}^T} & \overline{A_{22}^T} \end{bmatrix}$$

Appendix 3.2

In this appendix we verify the structure of the matrix $\overline{\mathbf{H}}$ introduced in formula (3.68). Formula (3.66) is given by

$$\overline{\mathbf{H}} = \begin{bmatrix} A^T + C^T W_2^{-1} C A^{-1} G W_1 G^T & -C^T W_2^{-1} C A^{-1} \\ -A^{-1} G W_1 G^T & A^{-1} \end{bmatrix} \tag{3.138}$$

From the structure of A^{-1} it is easy to see that the matrix A^T has the following form

$$A^T = \begin{bmatrix} I_{n_1} + O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix} \tag{3.139}$$

so that

$$C^T W_2^{-1} C A^{-1} G W_1 G^T$$

$$= \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} I_{n_1} + O(\epsilon) & O(\epsilon) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} \tag{3.140}$$

$$= \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} = \begin{bmatrix} O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix}$$

Similarly

$$A^T + C^T W_2^{-1} C A^{-1} G W_1 G^T = \begin{bmatrix} I_{n_1} + O(\epsilon) & O(1) \\ O(\epsilon) & O(1) \end{bmatrix}$$

$$\begin{aligned} C^T W_2^{-1} C A^{-1} &= \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} I_{n_1} + O(\epsilon) & O(\epsilon) \\ O(1) & O(1) \end{bmatrix} \\ &= \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \end{aligned}$$

$$\tag{3.141}$$

$$\begin{aligned} A^{-1} G W_1 G^T &= \begin{bmatrix} I_{n_1} + O(\epsilon) & O(\epsilon) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} \\ &= \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & O(1) \end{bmatrix} \end{aligned}$$

From the above estimates of the entries of the matrix $\overline{\mathbf{H}}$, we obtain

$$\overline{\mathbf{H}} = \begin{bmatrix} I_{n_1} + \epsilon \overline{A_1^T} & \overline{A_3^T} & \overline{S_1} & \overline{S_2} \\ \epsilon \overline{A_2^T} & \overline{A_4^T} & \overline{S_3} & \overline{S_4} \\ \epsilon^2 \overline{Q_1} & \epsilon \overline{Q_2} & I_{n_1} + \epsilon \overline{A_{11}} & \overline{A_{12}} \\ \epsilon \overline{Q_3} & \overline{Q_4} & \overline{A_{21}} & \overline{A_{22}} \end{bmatrix} \tag{3.142}$$

Appendix 3.3

In this appendix we prove the following lemma.

Lemma 3.3 *Under stabilizability-detectability assumption imposed on (3.105)-(3.107), the matrix $\alpha(\epsilon)$ defined in (3.131) is invertible.*

$\diamond$

Proof: The matrix $\alpha(\epsilon)$ can be written as

$$\alpha(\epsilon) = M_3 + N_3 \begin{bmatrix} I & 0 \\ 0 & T_4^{k_f} \end{bmatrix} + O(\epsilon) \tag{3.143}$$

Let

$$T_4^{k_f} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{3.144}$$

then by using expressions for M_3 and N_3 given by (3.129), we obtain

$$\alpha(\epsilon) = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ * & I_{n_1} & 0 & 0 \\ * & * & I_{n_2} & 0 \\ * & * & * & \phi_{22} - Q_{f3}\phi_{12} \end{bmatrix} + O(\epsilon) \tag{3.145}$$

where asterisks denote terms which are not important for the nonsingularity of $\alpha(\epsilon)$.

Note that from (3.145), assuming that the matrix $\phi_{22} - Q_{f3}\phi_{12}$ is invertible, then the matrix $\alpha(\epsilon)$ is also invertible for a sufficiently small value of ϵ. It will be shown in the following that the invertibility of $\phi_{22} - Q_{f3}\phi_{12}$ follows from the assumption that the system is stabilizable-detectable. From (3.122) and (3.130), we can write

$$\begin{bmatrix} U(k_f) \\ V(k_f) \end{bmatrix} = \mathbf{T}_1^{-1} \begin{bmatrix} I + O(\epsilon) & 0 \\ 0 & T_4^{k_f} + O(\epsilon) \end{bmatrix} \mathbf{T}_1 \begin{bmatrix} U(0) \\ V(0) \end{bmatrix} \tag{3.146}$$

By using the values of $\mathbf{T}_1$ and $\mathbf{T}_1^{-1}$ from (3.122), we obtain

$$\begin{aligned} U(k_f) &= U(0) + O(\epsilon) \\ V(k_f) &= \left(T_4^{k_f} L - L\right) U(0) + T_4^{k_f} V(0) + O(\epsilon) \end{aligned} \tag{3.147}$$

Let

$$T_4^{k_f} L - L = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \tag{3.148}$$

then by using (3.122) and (3.148) in (3.147) yields

$$
\begin{aligned}
x_2(k_f) &= \psi_{11}x_1(0) + \phi_{11}x_2(0) + \phi_{12}\lambda_2(0) + O(\epsilon) \\
\lambda_2(k_f) &= \psi_{21}x_1(0) + \phi_{21}x_2(0) + \phi_{22}\lambda_2(0) + O(\epsilon)
\end{aligned}
\tag{3.149}
$$

From the boundary condition $\lambda(k_f) = Q_f x(k_f)$, we have

$$
\lambda_2(k_f) = Q_{f2}^T x_1(k_f) + Q_{f3}x_2(k_f) \tag{3.150}
$$

Since $U(k_f) = U(0) + O(\epsilon)$, therefore, $x_1(k_f) = x_1(0) + O(\epsilon)$. Using this fact and substituting the value of $x_2(k_f)$ from (3.149) into (3.150), we obtain

$$
\begin{aligned}
(\phi_{22} - Q_{f3}\phi_{12})\lambda_2(0) &= (Q_{f3}\phi_{11} - \phi_{21})x_2(0) \\
&+ (Q_{f2}^T - \phi_{21} + Q_{f3}\phi_{11})x_1(0) + O(\epsilon)
\end{aligned}
\tag{3.151}
$$

Since the system is stabilizable-detectable, the control $u(0)$, and hence $\lambda_2(0)$ exist, which concludes that $(\phi_{22} - Q_{f3}\phi_{12})$ must be invertible. Thus, for sufficiently small ϵ, the matrix $\alpha(\epsilon)$ is invertible.

•

Appendix 3.4 New Version of the Chang Transformation

Consider the linear discrete-time time-invariant singularly perturbed system represented in its fast time scale formulation

$$
\begin{aligned}
x_1(n+1) &= (I + \epsilon A_1)x_1(n) + \epsilon A_2 x_2(n) \\
x_2(n+1) &= A_3 x_1(n) + A_4 x_2(n)
\end{aligned}
\tag{3.152}
$$

Let us apply the following transformation to (3.152)

$$
\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\epsilon L \\ -H & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{3.153}
$$

where L and H satisfy decoupled algebraic matrix equations

$$
L(I - A_4) + \epsilon A_1 L - \epsilon L A_3 L + A_2 = 0 \tag{3.154}
$$

$$
(A_4 - I)H - \epsilon H A_1 - \epsilon H A_2 H + A_3 = 0 \tag{3.155}
$$

Note that unique solutions of (3.154)-(3.155) exist for sufficiently small values of ϵ under the assumption that the matrix A_4 has no eigenvalues at −1, which is the standard condition imposed on discrete-time singularly perturbed linear systems. It is easy to show that in the new coordinates, we have completely decoupled pure-slow and pure-fast subsystems given by

$$\begin{aligned} z_1(k+1) &= (I_{n_1} + \epsilon A_1 - \epsilon L A_3) z_1(k) \\ z_2(k+1) &= (A_4 - \epsilon H A_2) z_2(k) \end{aligned} \tag{3.156}$$

Note that the inverse of this transformation is

$$\mathbf{T}^{-1} = \begin{bmatrix} I_{n_1} + \epsilon LNH & \epsilon LN \\ NH & N \end{bmatrix} \tag{3.157}$$

where $N = (I_{n_2} - \epsilon HL)^{-1}$. It is obvious that the matrix N is nonsingular for sufficiently small values of ϵ so that the inverse decoupling transformation exists for sufficiently small values of the singular perturbation parameter.

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4

Optimal Control and Filtering of Multimodeling Structures

In this chapter we show how to exactly decompose the algebraic Riccati equations of deterministic and stochastic multimodeling in terms of one pure-slow and two pure-fast algebraic Riccati equations. The algebraic Riccati equations obtained are of reduced-order and nonsymmetric. However, their $O(\epsilon)$ perturbations (where $\epsilon = \left\| \begin{matrix} \epsilon_1 \\ \epsilon_2 \end{matrix} \right\|$ and ϵ_1, ϵ_2 are small positive singular perturbation parameters) are symmetric. The Newton method is perfectly suited for solving the nonsymmetric reduced-order pure-slow and pure-fast algebraic Riccati equations since excellent initial guesses are available from their $O(\epsilon)$ perturbed reduced-order symmetric algebraic Riccati equations that can be solved rather easily. Derivations are done in detail for the regulator type algebraic Riccati equation. We use duality between optimal linear filtering and regulation in order to derive the corresponding decomposition for the filter type algebraic Riccati equation. In addition, we show how to completely decompose the optimal Kalman filter of the multimodeling structures in terms of one pure-slow and two pure-fast well-defined reduced-order, independent Kalman filters. The 9-th order model of a power control system and the 8-th order model of a passenger car are used to demonstrate efficiency of the proposed techniques. The proposed decomposition schemes might facilitate new approaches to filtering and control multimodeling problems that are conceptually simpler and numerically more efficient than the ones previously used to solve corresponding multimodeling problems.

The concept of multimodeling was introduced to the control audience in (Khalil and Kokotovic, 1978). Since then, the deterministic and stochastic multimodeling control and filtering problems have been studied by several researchers (Ozguner, 1979; Khalil, 1980; Kokotovic, 1981; Saksena and Cruz, 1981a,b; Saksena and Basar, 1982; Saksena *et al.*, 1983; Gajic and Khalil, 1986; Gajic, 1988a,b; Vaz and Davison, 1990; Zhuang and Gajic, 1991). The multimodeling problems arise in large scale dynamic systems that have multiple decision makers and multiple information channels (structures). Large scale systems are composed of several subsystems and are characterized by the presence of slow and fast dynamics and weak and strong interconnections among state variables. It is known from (Khalil and Kokotovic, 1978) that theory of singular perturbations is very well suited to capture the multimodeling structure of interconnected large scale systems displaying slow and fast dynamics.

The optimal solution of the multimodeling deterministic and stochastic linear-quadratic optimal control and filtering problems requires the solution of the regulator and filter singularly perturbed algebraic Riccati equations. In Section 4.1, we show how to exactly decompose the regulator algebraic Riccati equation in terms of independent one slow and two fast, reduced-order, algebraic Riccati equations. In Section 4.2, we use duality between the optimal regulator and optimal filtering problems to exactly decompose the filter algebraic Riccati equation into independent, reduced-order, pure-slow and pure-fast algebraic Riccati equations. Section 4.3 presents two case studies: design of the optimal controller for a power system and the Kalman filter for a passenger car under road disturbances.

The results presented in this chapter represent very powerful tools for simplifying derivations of the optimal multimodeling control and filtering strategies. In that respect, the results of (Ozguner, 1979; Gajic, 1988b, Zhuang and Gajic, 1991) can be obtained with perfect accuracy by performing only minor modifications of the original works. The extension to the Pareto multimodeling strategies of (Khalil and Kokotovic, 1978; Gajic and Khalil, 1986) will require a generalization of the results presented in this chapter to the Pareto game algebraic Riccati equation. The extension to the multimodeling team problems (Saksena and Basar,

1982) will require much more work along the lines considered in this chapter. The Nash multimodeling strategies of (Khalil, 1980; Saksena and Cruz, 1981a) can be similarly studied under the assumption that the results of this chapter can be applied to the coupled Nash algebraic Riccati equations. Even more, by using the results of this chapter, the desired multimodeling strategies could have been implemented with perfect accuracy. It is known that the multimodeling is an $O(\epsilon)^*$ approximate strategy. Several examples done in (Gajic *et al.*, 1989; Skataric and Gajic, 1992; Gajic and Shen, 1993; Mizukami and Suzumura, 1993) indicate that an $O(\epsilon)$ accuracy is very often not sufficient. Hence, the development of more accurate techniques for singularly perturbed control and filtering systems is mandatory.

4.1 Decomposition of the Regulator Algebraic Riccati Equation

The multimodeling structure is defined by a linear dynamic system that has one slow and two fast subsystems. The fast subsystems are strongly connected to the slow subsystem and weakly connected (or not connected) among themselves. Such large scale systems describe dynamics of several real physical systems, for example, power systems (Khalil and Kokotovic, 1978) and automobiles (Salman *et al.*, 1990; Zhuang and Gajic, 1991). The corresponding multimodeling representation of (Khalil and Kokotovic, 1978) is defined by

$$\begin{aligned}
\dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t) + B_{01}u_1(t) + B_{02}u_2(t)\\
\epsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \epsilon_3 A_{12}x_2(t)\\
&\quad + B_{11}u_1(t) + \epsilon_3 B_{02}u_2(t)\\
\epsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \epsilon_3 A_{21}x_1(t) + A_{22}x_2(t)\\
&\quad + \epsilon_3 B_{21}u_1(t) + B_{22}u_2(t)
\end{aligned} \tag{4.1}$$

where $x_0 \in \Re^{n_0}$ are slow state variables, $x_1 \in \Re^{n_1}, x_2 \in \Re^{n_2}$ are fast state variables, and $u_1 \in \Re^{m_1}$, $u_2 \in \Re^{m_2}$ are control inputs. ϵ_3 is a

* $O(\epsilon^i)$ is defined by $O(\epsilon^i) < c\epsilon^i$, where c is a bonded constant and i is a real number. In this chapter $\epsilon = \left\| \begin{smallmatrix} \epsilon_1 \\ \epsilon_2 \end{smallmatrix} \right\|$.

small weak coupling parameter, and ϵ_1 and ϵ_2 are small positive singular perturbation parameters of the same order of magnitude, that is (Gajic and Khalil, 1986)

$$0 < k_1 \leq \frac{\epsilon_2}{\epsilon_1} = \alpha \leq k_2 < \infty$$

In addition, it is assumed that the following limit exists (Gajic, 1988a)

$$\alpha_0 = \lim_{\epsilon_1 \to 0, \epsilon_2 \to 0,} \alpha = \lim_{\epsilon_1 \to 0, \epsilon_2 \to 0,} \left(\frac{\epsilon_2}{\epsilon_1} \right)$$

In order to simplify derivations, without loss of generality, we assume that the fast state variables are not connected among themselves, that is, we set the weak coupling parameter ϵ_3 to zero.

In the deterministic optimal control of the above multimodeling structure, the quadratic performance criterion has to be minimized by the proper choice of the control variables $u_1(t)$ and $u_2(t)$. The performance criterion is given by

$$J = \frac{1}{2} \int_0^{+\infty} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt, \quad Q = Q^T \geq 0, \; R = R^T > 0 \tag{4.2}$$

where

$$x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix} = q^T q = \begin{bmatrix} q_{01} & q_{11} & 0 \\ q_{02} & 0 & q_{22} \end{bmatrix}^T \begin{bmatrix} q_{01} & q_{11} & 0 \\ q_{02} & 0 & q_{22} \end{bmatrix}$$

$$= \begin{bmatrix} q_{01}^T q_{01} + q_{02}^T q_{02} & q_{01}^T q_{11} & q_{02}^T q_{22} \\ q_{11}^T q_{01} & q_{11}^T q_{11} & 0 \\ q_{22}^T q_{02} & 0 & q_{22}^T q_{22} \end{bmatrix} \tag{4.3}$$

In a more general multimodeling case, all zero-elements in matrices R and Q can be replaced by $O(\epsilon_3)$ elements.

In the multimodeling problem one proceeds with constructing two different models of (4.1), obtained by setting $\epsilon_1 = 0$, which leads to the first model for the first controller, and by setting $\epsilon_2 = 0$, which produces the second model for the second controller. The rational for this is the fact that each controller "sees" the slow dynamics of both subsystems and only its own fast dynamics. Thus, the fast dynamics of the other subsystem is approximated by an algebraic equation (the corresponding ϵ_i is set to zero). The same approximation is done for the performance criterion (4.2), hence two performance criteria are obtained, which leads to a multicriteria optimization problem. Depending on the actual problem set up, very often described by differential games, the two controllers find their own optimal strategies and apply such strategies to the global system defined by (4.1). In such a way obtained, the multimodeling strategy is well posed if the performance criterion under the multimodeling strategy is $O(\epsilon)$ close to the global optimal performance criterion obtained by performing direct optimization on the original system using the original performance criterion.

In this chapter, we present a method for the exact decomposition of the optimal control associated with (4.1) and (4.2) such that the optimal solution is obtained in terms of three independent, reduced-order algebraic Riccati equations, representing one slow and two fast subsystems. The idea presented in this chapter will allow the development of new techniques for new setups and more efficient solutions of the corresponding multimodeling problems.

The optimal feedback solution to (4.1)-(4.2) is given by

$$u_{opt}(t) = -R^{-1}B^TPx(t) \tag{4.4}$$

where P is the positive semidefinite stabilizing solution of the algebraic Riccati equation

$$A^TP + PA + Q - PSP = 0, \qquad S = BR^{-1}B^T \tag{4.5}$$

where

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \frac{1}{\epsilon_1}A_{10} & \frac{1}{\epsilon_1}A_{11} & 0 \\ \frac{1}{\epsilon_2}A_{20} & 0 & \frac{1}{\epsilon_2}A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{01} & B_{02} \\ \frac{1}{\epsilon_1}B_{11} & 0 \\ 0 & \frac{1}{\epsilon_2}B_{22} \end{bmatrix}$$

$$P = \begin{bmatrix} P_{00} & \epsilon_1 P_{01} & \epsilon_2 P_{02} \\ \epsilon_1 P_{01}^T & \epsilon_1 P_{11} & \sqrt{\epsilon_1\epsilon_2}P_{12} \\ \epsilon_2 P_{02}^T & \sqrt{\epsilon_1\epsilon_2}P_{12}^T & \epsilon_2 P_{22} \end{bmatrix}$$

$$S=\begin{bmatrix} S_{00} & \frac{1}{\epsilon_1}S_{01} & \frac{1}{\epsilon_2}S_{02} \\ \frac{1}{\epsilon_1}S_{01}^T & \frac{1}{\epsilon_1^2}S_{11} & 0 \\ \frac{1}{\epsilon_2}S_{02}^T & 0 & \frac{1}{\epsilon_2^2}S_{22} \end{bmatrix}, \quad \begin{matrix} S_{00}=B_{01}R^{-1}B_{01}^T+B_{02}R^{-1}B_{02}^T \\ S_{0i}=B_{0i}R^{-1}B_{ii}^T, \ \ i=1,2 \\ S_{ii}=B_{ii}R^{-1}B_{ii}^T, \ \ i=1,2 \end{matrix} \tag{4.6}$$

The scaling of the matrix P is done according to the nature of the solution of (4.5) as discussed in (Gajic and Khalil, 1986; Gajic, 1988a). The required solution of the algebraic Riccati equation (4.5) exists under the standard assumption (Kwakernaak and Sivan, 1972).

Assumption 4.1: The triple (A, B, q) is stabilizable-detectable.

Note that the multimodeling optimal control problem is studied under the following assumption (Khalil and Kokotovic, 1978).

Assumption 4.2: The triples (A_s, B_s, q_s) and $(A_{ii}, B_{ii}, q_{ii}), i = 1, 2,$ are stabilizable-detectable.

The matrices A_s, B_s, q_s are given by (Khalil and Kokotovic, 1978; Gajic, 1988a)

$$A_s = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$$

$$B_s = [B_{1s} \quad B_{2s}], \qquad B_{is} = B_{0i} - A_{0i}A_{ii}^{-1}B_{io}, \quad i = 1,2$$

$$Q_s = q_s^T q_s = q_{1s}^T q_{1s} + q_{2s}^T q_{2s}, \quad q_{is} = q_{0i} - q_{ii}A_{ii}^{-1}A_{i0}, \quad i = 1,2$$

In this chapter, the A_s, B_s, Q_s matrices will be redefined later on as a part of the proposed design methodology. Note that for sufficiently small values of $\epsilon = \left\| \begin{matrix} \epsilon_1 \\ \epsilon_2 \end{matrix} \right\|$, Assumption 4.2 is equivalent to Assumption 4.1, (Khalil and Kokotovic, 1978).

The derivations that follow will require Assumption 4.2. Consider the Hamiltonian matrix corresponding to (4.1) and (4.2)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \mathbf{H}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.7}$$

where $p(t)$ represents the so-called costate system variables compatibly partitioned as $p^T(t) = [p_0^T(t) \quad \epsilon_1 p_1^T(t) \quad \epsilon_2 p_2^T(t)]$. Let E_1 be the permutation matrix defined by

$$E_1 = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\epsilon_1} I_{n_1} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon_2} I_{n_2} \end{bmatrix} \tag{4.8}$$

The similarity transformation E_1 applied to (4.7) produces

$$\begin{aligned} \begin{bmatrix} \dot{x}_0 \\ \dot{p}_0 \\ \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} &= E_1 \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} E_1^{-1} \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \\ &= \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ \frac{1}{\epsilon_1} T_{10} & \frac{1}{\epsilon_1} T_{11} & 0 \\ \frac{1}{\epsilon_2} T_{20} & 0 & \frac{1}{\epsilon_2} T_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} = T \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \end{aligned} \tag{4.9}$$

where

$$\begin{gathered} T_{00} = \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix}, \quad T_{01} = \begin{bmatrix} A_{01} & -S_{01} \\ -Q_{01} & -A_{10}^T \end{bmatrix} \\ T_{02} = \begin{bmatrix} A_{02} & -S_{02} \\ -Q_{02} & -A_{20}^T \end{bmatrix}, \quad T_{12} = 0, \quad T_{21} = 0 \\ T_{10} = \begin{bmatrix} A_{10} & -S_{01}^T \\ -Q_{01}^T & -A_{01}^T \end{bmatrix}, \quad T_{11} = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} \\ T_{20} = \begin{bmatrix} A_{20} & -S_{02}^T \\ -Q_{02}^T & -A_{02}^T \end{bmatrix}, \quad T_{22} = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} \end{gathered} \tag{4.10}$$

Note that the above transformation combines in pairs the slow state/costate and fast state/costate variables such that (4.9) has the singularly perturbed structure. It should be pointed out that due to the second part of Assumption 4.2, the fast Hamiltonian matrices T_{11} and T_{22} are nonsingular (Kwakernaak and Sivan, 1972). In addition, the first part of Assumption 4.2 implies that the slow Hamiltonian matrix given by

$$T_s = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} = T_{00} - T_{01}T_{11}^{-1}T_{10} - T_{02}T_{22}^{-1}T_{20} \tag{4.11}$$

is nonsingular. Note that T_s is obtained from (4.9) by extracting the slow subsystem, that is, by multiplying the fast derivatives respectively by ϵ_1 and ϵ_2 and setting them to zero. This expression also gives new definitions for matrices A_s, Q_s, S_s, with $S_s = B_s R_s^{-1} B_s^T$. The procedure for obtaining independently R_s can be found in (Khalil and Kokotovic, 1978). For the purpose of this chapter we need only S_s. Due to the fact that R_s is invertible, it follows that stabilizability of (A_s, B_s) is equivalent to stabilizability of $(A_s, \mathrm{Chol}(S_s))$.

The singularly perturbed system defined in (4.9) can be block-diagonalized by using the generalized Chang transformation (Chang, 1972; Ladde and Rajalakshmi, 1985) given by

$$K = \begin{bmatrix} K_{11} & K_{12} & \epsilon_2(H_3 + \epsilon_1 H_1 H_2) \\ L_1 - \epsilon_2 H_2 L_2 & I_{n_1} - \epsilon_2 H_2 L_3 & -\epsilon_2 H_2 \\ L_2 & L_3 & I_{n_2} \end{bmatrix}$$

$$K_{11} = I_{n_0} - \epsilon_1 H_1 L_1 + \epsilon_1 \epsilon_2 H_1 H_2 L_2 + \epsilon_2 H_3 L_2 \tag{4.12}$$

$$K_{12} = -\epsilon_1 H_1 + \epsilon_1 \epsilon_2 H_1 H_2 L_3 + \epsilon_2 H_3 L_2$$

The corresponding inverse transformation is

$$K^{-1} = \begin{bmatrix} I_{n_0} & \epsilon_1 H_1 & -\epsilon_2 H_3 \\ -L_1 & I_{n_1} - \epsilon_1 H_1 L_1 & \epsilon_2(H_2 + H_3 L_1) \\ -L_2 + L_1 L_3 & \epsilon_1 H_1 (L_1 L_3 - L_2) - L_3 & K_{33I} \end{bmatrix}$$

$$K_{33I} = I_{n_2} + \epsilon_2(H_3 L_3 - H_2 L_3 - H_3 L_3 L_1) \tag{4.13}$$

In the above transformation the matrices $H_j, L_j, j = 1, 2, 3$, satisfy

$$
\begin{aligned}
0 &= T_{11}L_1 - T_{10} - \epsilon_1 L_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1) \\
0 &= T_{22}L_2 - \alpha L_3 T_{10} - T_{20} - \epsilon_2 L_2(T_{00} - T_{02}L_2) \\
0 &= T_{22}L_3 - \alpha L_3 T_{11} - \epsilon_2 L_2(T_{01} - T_{02}L_3) \\
0 &= -H_1T_{11} - \epsilon_1 H_1 L_1(T_{01} - T_{02}L_3) + (T_{01} - T_{02}L_3) \\
&\quad + \epsilon_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_1 \\
0 &= -H_2T_{22} + \alpha T_{11}H_2 + \epsilon_2 L_1(T_{01} - T_{02}L_3)H_2 + (L_1 - \epsilon_2 H_2 L_2)T_{02} \\
0 &= -H_3T_{22} - \epsilon_2 H_3 L_2 T_{02} - \epsilon_2(T_{01} - T_{02}L_3)H_2 - T_{02} \\
&\quad + \epsilon_2(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_3 \\
& 0 < k_1 \le \frac{\epsilon_2}{\epsilon_1} = \alpha \le k_2 < \infty
\end{aligned}
\tag{4.14}
$$

Even though the above algebraic equations are nonlinear, it can be noticed that all nonlinear terms are multiplied by the small singular perturbation parameters. Hence, an $O(\epsilon)$ perturbation of (4.14) produces a set of linear algebraic equations. An $O(\epsilon)$ perturbation of (4.14) is given by

$$
\begin{aligned}
0 &= T_{11}L_1^{(0)} - T_{10} \quad \Rightarrow \quad L_1^{(0)} = T_{11}^{-1}T_{10} \\
0 &= T_{22}L_2^{(0)} - \alpha_0 L_3^{(0)}T_{10} - T_{20} \quad \Rightarrow \quad L_2^{(0)} = T_{22}^{-1}T_{20} \\
0 &= T_{22}L_3^{(0)} - \alpha_0 L_3^{(0)}T_{11} \quad \Rightarrow \quad L_3^{(0)} = 0 \\
0 &= -H_1^{(0)}T_{11} + \left(T_{01} - T_{02}L_3^{(0)}\right) \quad \Rightarrow \quad H_1^{(0)} = T_{01}T_{11}^{-1} \\
0 &= -H_2^{(0)}T_{22} + \alpha_0 T_{11}H_2^{(0)} + L_1^{(0)}T_{02} \\
0 &= -H_3^{(0)}T_{22} - T_{02} \quad \Rightarrow \quad H_3^{(0)} = -T_{02}T_{22}^{-1} \\
\alpha_0 &= \lim_{\epsilon_1 \to 0, \epsilon_2 \to 0,} \alpha = \lim_{\epsilon_1 \to 0, \epsilon_2 \to 0,} \left(\frac{\epsilon_2}{\epsilon_1}\right)
\end{aligned}
\tag{4.15}
$$

It can be seen that these linear algebraic equations can be solved rather easily due to their decoupled structure and the fact that the Hamiltonian matrices T_{11} and T_{22} are nonsingular, which is the consequence of Assumption 4.2. Note that the equations for $L_3^{(0)}$ and $H_2^{(0)}$ are the Sylvester linear algebraic equations. The unique solutions of these equations exist under the following assumption, (Gajic and Qureshi, 1995).

Assumption 4.3: The Hamiltonian matrices T_{22} and $\alpha_0 T_{11}$ have no eigenvalues in common.

Note that due to the above assumption, the existence of $L_1^{(0)}, L_2^{(0)}, L_3^{(0)}$ is not uniform in α. Since the unique solutions for $L_1^{(0)}, L_2^{(0)}, L_3^{(0)}$ exist under Assumption 4.3, then by the Implicit Function Theorem (Ortega and Rheinboldt, 1970) the unique solutions L_1, L_2, L_3 exist for sufficiently small values of ϵ. Solutions of the set of linear algebraic equations (4.15) represent excellent initial conditions for the fixed point algorithm to be used for solving (4.14) since $L_j = L_j^{(0)} + O(\epsilon)$, $H_j = H_j^{(0)} + O(\epsilon)$, $j = 1, 2, 3$. The fixed point algorithm for solving (4.14) is given by

$$
\begin{aligned}
T_{11}L_1^{(i+1)} &= T_{10} + \epsilon_1 L_1^{(i)}\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right) \\
T_{22}L_2^{(i+1)} - \alpha L_3^{(i+1)}T_{10} &= T_{20} + \epsilon_2 L_2^{(i)}\left(T_{00} - T_{02}L_2^{(i)}\right) \\
T_{22}L_3^{(i+1)} - \alpha L_3^{(i+1)}T_{11} &= \epsilon_2 L_2^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right)
\end{aligned}
\tag{4.16a}
$$

$$
\begin{aligned}
H_1^{(i+1)}T_{11} &= -\epsilon_1 H_1^{(i)}L_1^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right) + \left(T_{01} - T_{02}L_3^{(i)}\right) \\
&\quad + \epsilon_1\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right)H_1^{(i)} \\
H_2^{(i+1)}T_{22} - \alpha T_{11}H_2^{(i+1)} - L_1^{(i+1)}T_{02} & \\
&= \epsilon_2 L_1^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right)H_2^{(i)} - \epsilon_2 H_2^{(i)}L_2^{(i)}T_{02}
\end{aligned}
\tag{4.16b}
$$

$$H_3^{(i+1)}T_{22} = -\epsilon_2 H_3^{(i)} L_2^{(i)} T_{02} - \epsilon_2\left(T_{01} - T_{02}L_3^{(i)}\right)H_2^{(i)} - T_{02}$$

$$+\epsilon_2\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right)H_3^{(i)}$$

A theorem is proved in Appendix 4.1 that establishes that the above fixed point algorithm has the rate of convergence of $O(\epsilon)$, that is

$$\begin{aligned}
&\left\|L_j^{(i+1)} - L_j^{(i)}\right\| = O(\epsilon), \quad j = 1,2,3; \quad i = 0,1,2,...\\
&\left\|H_j^{(i+1)} - H_j^{(i)}\right\| = O(\epsilon), \quad j = 1,2,3; \quad i = 0,1,2,...
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\left\|L_j - L_j^{(i)}\right\| = O\left(\epsilon^{i+1}\right), \quad j = 1,2,3; \quad i = 0,1,2,...\\
&\left\|H_j - H_j^{(i)}\right\| = O\left(\epsilon^{i+1}\right), \quad j = 1,2,3; \quad i = 0,1,2,...
\end{aligned} \tag{4.18}$$

Note that the L-equations can be also solved by using the Newton method with the solutions of (4.15) playing the roles of the initial conditions (see Appendix 4.2).

By applying the transformation K to (4.9), the system is transformed into the new coordinates with completely decoupled slow and fast dynamics

$$\begin{bmatrix} \dot{\eta}_{01}(t) \\ \dot{\eta}_{02}(t) \\ \epsilon_1\dot{\eta}_{11}(t) \\ \epsilon_1\dot{\eta}_{12}(t) \\ \epsilon_2\dot{\eta}_{21}(t) \\ \epsilon_2\dot{\eta}_{22}(t) \end{bmatrix} = \begin{bmatrix} D_0 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \begin{bmatrix} \eta_{01}(t) \\ \eta_{02}(t) \\ \eta_{11}(t) \\ \eta_{12}(t) \\ \eta_{21}(t) \\ \eta_{22}(t) \end{bmatrix} \tag{4.19}$$

with

$$\begin{aligned}
D_0 &= T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1 \triangleq \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\\
D_1 &= T_{11} + \epsilon_1 L_1(T_{01} - T_{02}L_3) \triangleq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\\
D_2 &= T_{22} + \epsilon_2 L_2 T_{02} \triangleq \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}
\end{aligned} \tag{4.20}$$

In (4.19) $\eta_{01}, \eta_{11}, \eta_{21}$ represent the state variables and $\eta_{02}, \eta_{12}, \eta_{22}$ are the costate variables. At steady state the state and costate variables are related by

$$\begin{aligned}\eta_{02}(t) &= P_s \eta_{01}(t)\\ \eta_{12}(t) &= P_{f1} \eta_{11}(t)\\ \eta_{22}(t) &= P_{f2} \eta_{21}(t)\end{aligned} \tag{4.21}$$

where P_s, P_{f1}, P_{f2} satisfy the independent, reduced-order, pure-slow and pure-fast, algebraic Riccati equations. The algebraic Riccati equations are derived from (4.19)-(4.21) as

$$\begin{aligned}P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s &= 0\\ P_{f1} b_1 - b_4 P_{f1} - b_3 + P_{f1} b_2 P_{f1} &= 0\\ P_{f2} c_1 - c_4 P_{f2} - c_3 + P_{f2} c_2 P_{f2} &= 0\end{aligned} \tag{4.22}$$

The pure-slow and pure-fast algebraic Riccati equations obtained are nonsymmetric. However, their $O(\epsilon)$ perturbations are symmetric ones, that is

$$P_s = P_0 + O(\epsilon), \quad P_{f1} = P_1 + O(\epsilon), \quad P_{f2} = P_2 + O(\epsilon) \tag{4.23}$$

with

$$\begin{aligned}P_0 A_s + A_s^T P_0 + Q_s - P_0 S_s P_0 &= 0\\ P_1 A_{11} + A_{11}^T P_1 + Q_{11} - P_1 S_{11} P_1 &= 0\\ P_2 A_{22} + A_{22}^T P_2 + Q_{22} - P_2 S_{22} P_2 &= 0\end{aligned} \tag{4.24}$$

where matrices A_s, Q_s, S_s are defined in (4.11). The second and third statement in (4.23) follows directly by examining coefficients $b_j, c_j, j = 1, 2, 3, 4$. Namely, the coefficients of the corresponding algebraic Riccati equations in (4.22) and (4.24) are $O(\epsilon)$ apart, that is

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = D_{11} = T_{11} + O(\epsilon) = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} + O(\epsilon)$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = D_{22} = T_{22} + O(\epsilon) = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} + O(\epsilon)$$

The first statement in (4.23) is based on the fact that from (4.20) we have

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_{00} - T_{01}L_1^{(0)} - T_{02}L_2^{(0)} + T_{02}L_3^{(0)}L_1^{(0)} + O(\epsilon)$$

Since from (4.15)

$$L_1^{(0)} = T_{11}^{-1}T_{10},\ L_2^{(0)} = T_{22}^{-1}T_{20},\ L_3^{(0)} = 0$$

we obtain

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_{00} - T_{01}T_{11}^{-1}T_{10} - T_{02}T_{22}^{-1}T_{20} + O(\epsilon)$$

which by (4.11) implies

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} + O(\epsilon)$$

The unique positive semidefinite stabilizing solutions of the algebraic Riccati equations defined in (4.24) exist under Assumption 4.2. Then, in view of (4.23) and by the Implicit Function Theorem, the unique solutions of the algebraic Riccati equations (4.22) exist. These solutions can be obtained by using the Newton method since equations (4.24) produce excellent initial guesses. It is known that the Newton method converges quadratically and that for good initial guesses it requires only four to five iterations. The Newton method for solving the nonsymmetric algebraic Riccati equations (4.22) is given by

$$\begin{gathered} P_s^{(i+1)}\left(a_1 + a_2P_s^{(i)}\right) - \left(a_4 - P_s^{(i)}a_2\right)P_s^{(i+1)} \\ = a_3 + P_s^{(i)}a_2P_s^{(i)},\quad P_s^{(0)} = P_0 \\ P_{f1}^{(i+1)}\left(b_1 + b_2P_{f1}^{(i)}\right) - \left(b_4 - P_{f1}^{(i)}b_2\right)P_{f1}^{(i+1)} \\ = b_3 + P_{f1}^{(i)}b_2P_{f1}^{(i)},\quad P_{f1}^{(0)} = P_1 \\ i = 0, 1, 2, \ldots \end{gathered} \tag{4.25}$$

$$
P_{f2}^{(i+1)}\left(c_1 + c_2 P_{f2}^{(i)}\right) - \left(c_4 - P_{f2}^{(i)} c_2\right) P_{f2}^{(i+1)}
$$

$$
= c_3 + P_{f2}^{(i)} c_2 P_{f2}^{(i)}, \quad P_{f2}^{(0)} = P_2
$$

$$
i = 0, 1, 2, \ldots
$$

In the following we establish the relation between the new and original coordinates and the relation between the solution of the global algebraic Riccati equation (4.5) and the solutions of the pure-slow and pure-fast, reduced-order, independent, algebraic Riccati equations (4.22).

The relationship between the original and new coordinates can be established as follows. Define the permutation matrix as

$$
E_2 = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{4.26}
$$

Then, the new state/costate variables are related to the old ones by

$$
\begin{bmatrix} \eta_{01}(t) \\ \eta_{02}(t) \\ \eta_{11}(t) \\ \eta_{12}(t) \\ \eta_{21}(t) \\ \eta_{22}(t) \end{bmatrix} = E_2^T K E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{4.27}
$$

In order to establish the relationship between the solutions of the global and local Riccati equations, we first observe that due to the fact that $p(t) = Px(t)$, it follows from (4.27) that

$$
\begin{bmatrix} \eta_{01}(t) \\ \eta_{11}(t) \\ \eta_{21}(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t), \quad \begin{bmatrix} \eta_{02}(t) \\ \eta_{12}(t) \\ \eta_{22}(t) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(t) \tag{4.28}
$$

Since

$$
\begin{bmatrix} \eta_{02}(t) \\ \eta_{12}(t) \\ \eta_{22}(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{1f} & 0 \\ 0 & 0 & P_{2f} \end{bmatrix} \begin{bmatrix} \eta_{01}(t) \\ \eta_{11}(t) \\ \eta_{21}(t) \end{bmatrix} \tag{4.29}
$$

The last two formulas imply

$$\begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{1f} & 0 \\ 0 & 0 & P_{2f} \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{4.30}$$

It is shown in Appendix 4.3 that the matrix inversion in (4.30) exists for small values of singular perturbation parameters. Similarly, we can express P in terms of P_s, P_{f1}, P_{f2}

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f1} & 0 \\ 0 & 0 & P_{f2} \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f1} & 0 \\ 0 & 0 & P_{f2} \end{bmatrix} \right)^{-1} \tag{4.31}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1} \tag{4.32}$$

Invertibility of the matrices in (4.30) and (4.31) is established for small values of singular perturbation parameters in Appendix 4.4. Invertibility of the matrix Π can be easily shown.

Example 4.1: Consider the following third-order example

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 50 & -20 & 0 \\ 20 & 0 & -30 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 10 & 0 \\ 0 & 30 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$R = I_2, \quad \epsilon_1 = \epsilon_2 = 0.1$$

The solutions of L, H equations (4.14) are given by

$$L_1 = \begin{bmatrix} -1.1634 & 0.0206 \\ -3.0287 & 0.1362 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.0642 & 1.2163 \\ -0.2570 & 0.2590 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 0.1477 & 0.0595 \\ 0.0262 & 0.0494 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1294 & 0.0238 \\ 2.9764 & -1.1767 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.1910 & 0.7912 \\ -0.9738 & 4.9768 \end{bmatrix}, \quad H_3 = \begin{bmatrix} -0.2493 & 1.1710 \\ -0.2474 & -0.0618 \end{bmatrix}$$

The coefficients of the pure-slow and pure-fast algebraic Riccati equations (4.22) are

$$a_1 = -3.4104, \quad a_2 = -7.0102, \quad a_3 = -20.0442, \quad a_4 = 3.4104$$

$$b_1 = -2.2011, \quad b_2 = -0.9074, \quad b_3 = -5.5301, \quad b_4 = 2.2011$$

$$c_1 = -3.1024, \quad c_2 = -9.2818, \quad c_3 = -3.1030, \quad c_4 = 3.1024$$

The solutions of the algebraic Riccati equations (4.22) are obtained as

$$P_s = 1.2731, \qquad P_{f1} = 1.0353, \qquad P_{f2} = 0.3336$$

The solution of the global algebraic Riccati equation obtained by using (4.31) is

$$P = \begin{bmatrix} 1.1542 & 0.1725 & 0.0397 \\ 0.1725 & 0.1024 & 0.0018 \\ 0.0397 & 0.0018 & 0.0339 \end{bmatrix}$$

This solution is $O\left(10^{-14}\right)$ close to the exact solution of the corresponding algebraic Riccati equation.

4.2 Decomposition of the Optimal Kalman Filter

The multimodeling structure corresponding to the optimal Kalman filtering problem is given by (Gajic and Khalil, 1986)

$$\begin{aligned} \dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t) + G_{01}w_1(t) + G_{02}w_2(t) \\ \epsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \epsilon_3 A_{12}x_2(t) \\ &\quad + G_{11}w_1(t) + \epsilon_3 G_{11}w_2(t) \\ \epsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \epsilon_3 A_{11}x_1(t) + A_{22}x_2(t) \\ &\quad + \epsilon_3 G_{21}w_1(t) + G_{22}w_2(t) \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} y_1(t) &= C_{10}x_0(t) + C_{11}x_1(t) + \epsilon_3 C_{12}x_2(t) + v_1(t) \\ y_2(t) &= C_{20}x_0(t) + \epsilon_3 C_{21}x_1(t) + C_{22}x_2(t) + v_2(t) \end{aligned} \tag{4.34}$$

where $y_i(t) \in \Re^{l_i}, i = 1,2$, are the system measurements, and $w_i(t) \in \Re^{r_i}$, $v_i(t) \in \Re^{l_i}, i = 1,2$, are zero-mean stationary, Gaussian, mutually uncorrelated, white noise stochastic processes with intensities $W_i \geq 0$ and $V_i > 0$.

In the following, for the sake of simplicity, we assume that the fast state variables are decoupled in the sense that they do not interact directly ($\epsilon_3 = 0$), but they are connected through the slow state space variables, (Gajic and Khalil, 1986).

Our goal is to determine the optimal estimates of the state variables $x_0(t), x_1(t), x_2(t)$ in terms of completely independent, reduced-order, pure-slow and pure-fast Kalman filters. The global Kalman filter driven by the innovation process, corresponding to the above filtering problem, is given by (Kwakernaak and Sivan, 1972)

$$\begin{bmatrix} \dot{\hat{x}}_0(t) \\ \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \frac{1}{\epsilon_1}A_{10} & \frac{1}{\epsilon_1}A_{11} & 0 \\ \frac{1}{\epsilon_2}A_{20} & 0 & \frac{1}{\epsilon_2}A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_0(t) \\ \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} K_{01} & K_{02} \\ \frac{1}{\epsilon_1}K_{11} & \frac{1}{\epsilon_1}K_{12} \\ \frac{1}{\epsilon_2}K_{21} & \frac{1}{\epsilon_2}K_{22} \end{bmatrix} \begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix} \tag{4.35}$$

and

$$\begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} C_{01} & C_{11} & 0 \\ C_{02} & 0 & C_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_0(t) \\ \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{4.36}$$

The optimal Kalman filter gain is

$$\begin{gathered} \begin{bmatrix} K_{01} & K_{02} \\ \frac{1}{\epsilon_1}K_{11} & \frac{1}{\epsilon_1}K_{12} \\ \frac{1}{\epsilon_2}K_{21} & \frac{1}{\epsilon_2}K_{22} \end{bmatrix} = K = P_F C^T V^{-1}, \quad V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \\ C = \begin{bmatrix} C_{01} & C_{11} & 0 \\ C_{02} & 0 & C_{22} \end{bmatrix} \end{gathered} \tag{4.37}$$

where P_F satisfies the algebraic filter Riccati equation

$$AP_F + P_F A^T - P_F C^T V^{-1} C P_F + GWG^T = 0 \tag{4.38}$$

with

$$P_F = \begin{bmatrix} P_{F00} & P_{F01} & P_{F02} \\ P_{F01}^T & \frac{1}{\epsilon_1}P_{F11} & \frac{1}{\sqrt{\epsilon_1\epsilon_2}}P_{F12} \\ P_{F02}^T & \frac{1}{\sqrt{\epsilon_1\epsilon_2}}P_{F12}^T & \frac{1}{\epsilon_2}P_{F22} \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

$$G = \begin{bmatrix} G_{01} & G_{02} \\ \frac{1}{\epsilon_1}G_{11} & 0 \\ 0 & \frac{1}{\epsilon_2}G_{22} \end{bmatrix} \tag{4.39}$$

The scaling for P_F is discussed in (Gajic and Khalil, 1986). Note that according to the results of (Gajic and Khalil, 1986), it is known that $P_{F12} = O(\epsilon)$, which implies $K_{12} = O(\epsilon)$ and $K_{21} = O(\epsilon)$.

Under the standard stabilizability-detectability assumption given below, the positive semidefinite stabilizing solution of the algebraic filter Riccati equation (4.38) exists.

Assumption 4.4: The triple (A, G, C) is stabilizable-detectable.

For sufficiently small values of singular perturbation parameters, Assumption 4.4 can be replaced by the corresponding subsystem stabilizability-detectability assumptions (Gajic and Khalil, 1986).

Assumption 4.5: The triples (A_{sF}, G_s, C_s) and (A_{ii}, G_{ii}, C_{ii}) are stabilizable-detectable.

The matrices A_{sF}, G_s, C_s can be derived by using the methodology of (Gajic and Khalil, 1986). They will be also defined later on in this chapter.

The optimal Kalman filtering problem for the multimodeling structure defined in (4.33)-(4.34) will be solved by using duality with the corresponding optimal regulator problem considered in the previous chapter. In that respect, we set

$$A^T \to A, \quad GWG^T \to Q, \quad C^TV^{-1}C \to S = BR^{-1}B^T \tag{4.40}$$

where $\to$ stands for "replace." It can be shown after lengthy algebra that the corresponding dual filter state/costate equations have the same form as in (4.9) with the following partitioning of the dual "state/costate" variables $x^T = [x_0^T \quad \epsilon_1 x_1^T \quad \epsilon_2 x_2^T]$, $p^T = [p_0^T \quad p_1^T \quad p_2^T]$ and with

$$T_{00F} = \begin{bmatrix} A_{00}^T & t_{12} \\ t_{21} & -A_{00} \end{bmatrix}, \quad \begin{aligned} t_{12} &= -\left(C_{01}^T V_1^{-1} C_{01} + C_{02}^T V_2^{-1} C_{02}\right) \\ t_{21} &= -\left(G_{01} W_1 G_{01}^T + G_{02} W_2 G_{02}^T\right) \end{aligned}$$

$$T_{01F} = \begin{bmatrix} A_{10}^T & -C_{01}^T V_1^{-1} C_{11} \\ -G_{01} W_1 G_{11}^T & -A_{01} \end{bmatrix}$$

$$T_{02F} = \begin{bmatrix} A_{20}^T & -C_{02}^T V_2^{-1} C_{22} \\ -G_{02} W_2 G_{22}^T & -A_{02} \end{bmatrix}$$

$$T_{10F} = \begin{bmatrix} A_{01}^T & -C_{11}^T V_1^{-1} C_{01} \\ -G_{11} W_1 G_{01}^T & -A_{10} \end{bmatrix}$$

$$T_{11F} = \begin{bmatrix} A_{11}^T & -C_{11}^T V_1^{-1} C_{11} \\ -G_{11} W_1 G_{11}^T & -A_{11} \end{bmatrix}$$

$$T_{20F} = \begin{bmatrix} A_{02}^T & -C_{22}^T V_2^{-1} C_{02} \\ -G_{22} W_2 G_{02}^T & -A_{20} \end{bmatrix}$$

$$T_{22F} = \begin{bmatrix} A_{22}^T & -C_{22}^T V_2^{-1} C_{22} \\ -G_{22} W_2 G_{22}^T & -A_{22} \end{bmatrix}$$

$$T_{12F} = 0, \qquad T_{21F} = 0 \tag{4.41}$$

The slow-subsystem matrices used in Assumption 4.5 are obtained from

$$\begin{bmatrix} A_{sF}^T & -C_s^T V_s^{-1} C_s \\ -G_s W_s G_s^T & -A_{sF} \end{bmatrix} = T_{00F} - T_{01F} T_{11F}^{-1} T_{10F} - T_{02F} T_{22F}^{-1} T_{20F} \tag{4.42}$$

With the above defined T_{ijF} matrices, we solve L, H equations of (4.14) by using the fixed point algorithm (4.16). The solutions obtained for L_{jF}, H_{jF} are used in (4.12) in order to get the corresponding filter decoupling transformation K_F.

Using the solutions obtained for $L_{jF}, j = 1, 2, 3$, we form the matrices $D_{kF}, k = 0, 1, 2$, as defined in (4.20) with $T'_{ij}s$ calculated from

(4.41) and obtain the coefficients

$$\begin{bmatrix} a_{1F} & a_{2F} \\ a_{3F} & a_{4F} \end{bmatrix} = D_{0F}, \quad \begin{bmatrix} b_{1F} & b_{2F} \\ b_{3F} & b_{4F} \end{bmatrix} = D_{1F}, \quad \begin{bmatrix} c_{1F} & c_{2F} \\ c_{3F} & c_{4F} \end{bmatrix} = D_{2F} \tag{4.43}$$

Note that all partitioned matrices introduced in (4.43) are square matrices. The pure-slow and pure-fast, independent, reduced-order, algebraic filter Riccati equations are given by

$$\begin{gathered} P_{sF}a_{1F} - a_{4F}P_{sF} - a_{3F} + P_{sF}a_{2F}P_{sF} = 0 \\ P_{f1F}b_{1F} - b_{4F}P_{f1F} - b_{3F} + P_{f1F}b_{2F}P_{f1F} = 0 \\ P_{f2F}c_{1F} - c_{4F}P_{f2F} - c_{3F} + P_{f2F}c_{2F}P_{f2F} = 0 \end{gathered} \tag{4.44}$$

The obtained pure-slow and pure-fast algebraic filter Riccati equations are nonsymmetric. However, their $O(\epsilon)$ perturbations are symmetric ones, that is

$$P_{sF} = P_{0F} + O(\epsilon), \quad P_{f1F} = P_{1F} + O(\epsilon), \quad P_{f2F} = P_{2F} + O(\epsilon) \tag{4.45}$$

with

$$\begin{gathered} P_{0F}A_{sF}^T + A_{sF}P_{0F} + G_sW_sG_s^T - P_{0F}C_s^TV_s^{-1}C_sP_{0F} = 0 \\ P_{1F}A_{11}^T + A_{11}P_{1F} + G_{11}W_1G_{11}^T - P_{1F}C_{11}^TV_1^{-1}C_{11}P_{1F} = 0 \\ P_{2F}A_{22} + A_{22}^TP_{2F} + G_{22}W_2G_{22}^T - P_{2F}C_{22}^TV_2^{-1}C_{22}P_{2F} = 0 \end{gathered} \tag{4.46}$$

Formulas (4.45) and equations (4.46) can be justified by using dual arguments to those presented for the regulator problem and given below (4.24).

Nonsymmetric algebraic Riccati equations can be solved by using the Newton algorithm (4.25) with the initial conditions obtained from (4.46). The solution of the global filter algebraic equation (4.38) can be obtained in terms of pure-slow, P_{sF}, and pure-fast solutions, P_{f1F}, P_{f2F}, by using formula (4.31). It can be shown that the corresponding filter permutation matrices are given by

$$E_{1F} = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & \frac{1}{\epsilon_1}I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & \frac{1}{\epsilon_2}I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix}, \quad E_{2F} = E_2 \tag{4.47}$$

and that

$$\Omega_F = \begin{bmatrix} \Omega_{1F} & \Omega_{2F} \\ \Omega_{3F} & \Omega_{4F} \end{bmatrix} = E_2^T K_F E_{1F} \tag{4.48}$$

In addition to obtaining the solution of the global algebraic filter Riccati equation (an ill-conditioned equation due to the presence of small singular perturbation parameters) in terms of solutions of reduced-order well-defined algebraic Riccati equations, in the following we decouple the global singularly perturbed Kalman filter (4.35)-(4.36). We first write the closed-loop form of the Kalman filter as

$$\begin{bmatrix} \dot{\hat{x}}_0(t) \\ \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = (A - KC) \begin{bmatrix} \hat{x}_0(t) \\ \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + K \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{4.49}$$

Note that in the previous section we have shown that the similarity transformation defined in (4.28) completely decouples the closed-loop system in the new coordinates. The corresponding similarity transformation for the Kalman filter is given by

$$\begin{bmatrix} \hat{\eta}_0 \\ \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} = (\Pi_{1F} + \Pi_{2F} P_F) \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{4.50}$$

where

$$\Pi_F = \Omega_F^{-1} = E_{1F}^{-1} K_F^{-1} E_2 \tag{4.51}$$

Applied to (4.49) as

$$\begin{bmatrix} \dot{\hat{\eta}}_0(t) \\ \dot{\hat{\eta}}_1(t) \\ \dot{\hat{\eta}}_2(t) \end{bmatrix} = \mathbf{T}^{-T}(A - KC)\mathbf{T}^T \begin{bmatrix} \hat{\eta}_0(t) \\ \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} + \mathbf{T}^{-T} K y(t), \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{4.52}$$

this similarity transformation produces in the new coordinates the reduced-order, independent, pure-slow and pure-fast Kalman filters of the form

$$\begin{aligned} \dot{\hat{\eta}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{\eta}_s(t) + K_s y(t) \\ \epsilon_1 \dot{\hat{\eta}}_{f1}(t) &= (b_{1F} + b_{2F} P_{f1F})^T \hat{\eta}_{f1}(t) + K_{f1} y(t) \\ \epsilon_2 \dot{\hat{\eta}}_{f2}(t) &= (c_{1F} + c_{2F} P_{f2F})^T \hat{\eta}_{f2}(t) + K_{f2} y(t) \end{aligned} \tag{4.53}$$

where

$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon_1} K_{f1} \\ \frac{1}{\epsilon_2} K_{f2} \end{bmatrix} = \mathbf{T}^{-T} K \tag{4.54}$$

Note that filters (4.53) work in parallel, hence each one can be implemented with the corresponding sampling rate—the slow filter with the slow sampling rate and the fast filters with the fast sampling rates. In the original Kalman filtering scheme defined in (4.35), due to coupling of the slow and fast variables, all three filters (for $\hat{x}_0, \hat{x}_1, \hat{x}_2$) have to be implemented with the fast sampling rate. Hence, the slow-fast signal processing obtained in (4.53) is computationally very efficient.

Example 4.2: Consider the following third-order filtering problem

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 50 & -20 & 0 \\ 20 & 0 & -30 \end{bmatrix}, \ G = \begin{bmatrix} 1 & 1 \\ 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$V = W = I_2, \ \epsilon_1 = \epsilon_2 = 0.1$$

The solutions of the corresponding L, H equations are given by

$$L_{1F} = \begin{bmatrix} -0.4692 & 1.1685 \\ -0.3297 & -1.5924 \end{bmatrix}, \quad L_{2F} = \begin{bmatrix} -0.9716 & 0.6866 \\ -0.6117 & -0.6727 \end{bmatrix}$$

$$L_{3F} = \begin{bmatrix} 0.4728 & 0.0569 \\ 0.0129 & 0.1923 \end{bmatrix}, \quad H_{1F} = \begin{bmatrix} -1.2218 & -0.9568 \\ 0.0718 & -0.3117 \end{bmatrix}$$

$$H_{2F} = \begin{bmatrix} 2.4662 & -1.1544 \\ -0.7359 & 5.4287 \end{bmatrix}, \quad H_{3F} = \begin{bmatrix} 0.5617 & 0.5734 \\ -0.5108 & 0.8113 \end{bmatrix}$$

The coefficients of the pure-slow and pure-fast algebraic Riccati equations (4.44) are

$$a_{1F} = 1.9360, \quad a_{2F} = -10.2660, \quad a_{3F} = -5.4862, \quad a_{4F} = -1.9360$$

$$b_{1F} = -2.2479, \quad b_{2F} = -1.1164, \quad b_{3F} = -1.0563, \quad b_{4F} = 2.2479$$

$$c_{1F} = -3.2630, \quad c_{2F} = -1.1088, \quad c_{3F} = -1.0551, \quad c_{4F} = 3.2630$$

The solutions of algebraic Riccati equations (4.44) are obtained as

$$P_{sF} = 0.9436, \quad P_{f1F} = 0.2226, \quad P_{f2F} = 0.1575$$

The solution of the global algebraic Riccati equation obtained by using (4.38) is

$$P_F = \begin{bmatrix} 1.0342 & 2.1466 & 0.8726 \\ 2.1466 & 5.9482 & 1.2009 \\ 0.8726 & 1.2009 & 2.0358 \end{bmatrix}$$

Completely decoupled pure-slow and pure-fast, reduced-order, Kalman filters are given by (4.53)-(4.54) as

$$\dot{\hat{\eta}}_s(t) = -7.7505\hat{\eta}_s(t) + [2.7758 \quad 1.8458]y(t)$$

$$\epsilon_1\dot{\hat{\eta}}_{f1}(t) = -2.4965\hat{\eta}_{f1}(t) + [0.0176 \quad -0.0133]y(t)$$

$$\epsilon_2\dot{\hat{\eta}}_{f2}(t) = -3.4376\hat{\eta}_{f2}(t) + [0.0090 \quad 0.0170]y(t)$$

The transformation that relates the optimal filter estimates in the original and new coordinates is obtained as

$$\mathbf{T} = \begin{bmatrix} 1.1016 & 2.0067 & 0.4922 \\ 0.7710 & 11.0142 & -1.4173 \\ -0.1392 & 6.5401 & 10.6675 \end{bmatrix}$$

4.3 Case Studies

In order to demonstrate efficiency of the proposed optimal control and filtering schemes we use the model of two real physical systems, a power control system of order nine (Khalil and Kokotovic, 1978), and a passenger car (Salman *et al.*, 1990) filtering system of order eight.

4.3.1 A Power Plant Control System

Consider the optimal control problem of the power control system from (Khalil and Kokotovic, 1978). The problem matrices are given by

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}, \quad A_{11} = A_{22} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}$$

$$A_{00} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & -0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix}$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix}$$

$$A_{12} = A_{21} = 0^{2\times 2}$$

$$B_{01}^T = B_{02}^T = [0 \quad 0 \quad 0 \quad 0 \quad 0], \; B_{11}^T = B_{22}^T = [0 \quad 0.1]$$

$$B_{12}^T = B_{21}^T = 0^{1\times 2}$$

$$\epsilon_1 = \epsilon_2 = 0.01, \quad R_{11} = R_{22} = 20, \;\; Q_{00} = \mathrm{diag}\{2, 2, 2, 2, 2\}$$

$$Q_{11} = Q_{22} = \mathrm{diag}\{1, 1\}, \; Q_{01} = Q_{02} = 0^{2\times 5}, \;\; Q_{10} = Q_{20} = Q_{01}^T$$

With the accuracy of $O(10^{-14})$, which is the standard MATLAB accuracy, we have obtained the following solutions for the pure-slow and pure-fast algebraic regulator Riccati equations (4.22)

$$P_s = \begin{bmatrix} 8.2262 & 1.3518 & 59.4923 & -1.2889 & 0.3714 \\ 1.3518 & 8.2262 & -1.2889 & 59.4923 & -0.3714 \\ 65.2196 & 0.4501 & 997.4386 & -478.6628 & 6.2122 \\ 0.4501 & 65.2196 & -478.6628 & 997.4386 & -6.2122 \\ 0.6162 & -0.6162 & 12.0812 & -12.0812 & 2.9062 \end{bmatrix}$$

$$P_{f1} = \begin{bmatrix} 10.3890 & 3.6730 \\ 3.4831 & 6.7633 \end{bmatrix}, \quad P_{f2} = \begin{bmatrix} 9.8632 & 3.5953 \\ 3.0153 & 6.6886 \end{bmatrix}$$

Using formula (4.31), we have obtained the global solution with the accuracy of $O(10^{-14})$ as

$$P =$$

$$\begin{bmatrix} 8.8636 & 1.3617 & 74.8676 & -3.5042 & 0.5543 & 1.3609 & 0.6325 & -0.0222 & 0.0000 \\ 1.3617 & 8.8636 & -3.5042 & 74.8676 & -0.5543 & -0.0222 & 0.0000 & 1.3609 & 0.6325 \\ 74.8676 & -3.5042 & 1245.1750 & -610.6820 & 10.2609 & 21.2746 & 9.1612 & -9.3790 & -3.7397 \\ -3.5042 & 74.8676 & -610.6820 & 1245.1750 & -10.2609 & -9.3790 & -3.7387 & 21.2746 & 9.1612 \\ 0.5543 & -0.5543 & 10.2609 & -10.2609 & 2.8794 & -0.1003 & -0.1060 & 0.1003 & 0.1060 \\ 1.3609 & -0.0222 & 21.2746 & -9.3790 & -0.1003 & 0.4993 & 0.2157 & -0.1713 & -0.0756 \\ 0.6325 & 0.0000 & 9.1612 & -3.7387 & -0.1060 & 0.2157 & 0.1518 & 0.0757 & -0.0352 \\ -0.0222 & 1.3609 & -9.3790 & 21.2746 & 0.1003 & -0.1713 & 0.0757 & 0.4994 & 0.2157 \\ 0.0000 & 0.6325 & -3.7397 & 9.1612 & 0.1060 & -0.0756 & -0.0352 & 0.2157 & 0.1518 \end{bmatrix}$$

4.3.2 Filtering Problem for an Automobile

The mathematical model of a passenger car under unevenness of the road disturbances is derived in (Salman *et al.*, 1990). Here, we solve the Kalman filtering problem for the given model in terms of reduced-order pure-slow and pure-fast Kalman filters. The problem data are (Salman *et al.*, 1990; Zhuang and Gajic, 1991)

$$A_{00} = \begin{bmatrix} 0 & 0 & 1 & 0.8755 \\ 0 & 0 & 1 & -1.79 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{10} = A_{20} = 0^{2\times 4}, \ A_{11} = A_{22} = \begin{bmatrix} 0 & 6.0435 \\ -6.0435 & 0 \end{bmatrix}$$

$$A_{12} = A_{21} = 0^{2\times 2}, \qquad \epsilon_1 = \epsilon_2 = 0.1$$

$$G_{01}^T = G_{02}^T = [0 \quad 0 \quad 0 \quad 0], \ G_{11}^T = G_{22}^T = [-0.1 \quad 0]$$

$$G_{12}^T = G_{21}^T = 0^{1\times 2}$$

The remaining problem matrices are chosen as

$$C_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad C_{02} = \begin{bmatrix} 0.2 & 0 & 1 & 0.1 \\ 0 & 0 & 0.2 & 1 \end{bmatrix}$$

$$C_{11} = \mathrm{diag}\{1, 1\}, \quad C_{22} = \mathrm{diag}\{2, 2\}$$

$$V_1 = \mathrm{diag}\{1, 1\}, \quad V_2 = \mathrm{diag}\{2, 2\}, \quad W_1 = W_2 = 1$$

With the accuracy of $O\left(10^{-14}\right)$, we have obtained the following solutions for the pure-slow and pure-fast algebraic filter Riccati equations (4.44)

$$P_{sF} = \begin{bmatrix} 0.0164 & 0.0001 & 0 & 0 \\ 0.0001 & 0.0165 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{f1F} = \begin{bmatrix} 0.0713 & -0.0004 \\ -0.0004 & 0.0713 \end{bmatrix}, \quad P_{f2F} = \begin{bmatrix} 0.0500 & -0.0004 \\ -0.0004 & 0.0500 \end{bmatrix}$$

These solutions lead to

$$P_F = \begin{bmatrix} 0.0166 & 0.0001 & 0.0000 & 0.0000 & -0.0118 & 0.0000 & 0.0000 & 0.0001 \\ 0.0001 & 0.0167 & 0.0000 & 0.0000 & -0.0002 & 0.0000 & -0.0083 & -0.0002 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0118 & -0.0002 & 0.0000 & 0.0000 & 0.7130 & -0.0042 & 0.0004 & 0.0017 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0042 & 0.7129 & 0.0017 & 0.0004 \\ 0.0000 & -0.0083 & 0.0000 & 0.0000 & 0.0004 & 0.0017 & 0.5000 & -0.0041 \\ 0.0001 & -0.0002 & 0.0000 & 0.0000 & -0.0017 & 0.0004 & -0.0041 & 0.4999 \end{bmatrix}$$

The pure-slow and pure-fast, independent, Kalman filters are given by

$$\dot{\hat{\eta}}_s = \begin{bmatrix} -0.0167 & 0.0001 & 0.9984 & 0.8752 \\ -0.0001 & -0.0165 & 1.0000 & -1.7732 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \hat{\eta}_s$$

$$+ \begin{bmatrix} 0.0164 & -0.0001 & 0.0016 & 0.0000 \\ 0.0001 & 0.0165 & 0.0000 & 0.0003 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} y(t)$$

$$\epsilon_1 \dot{\hat{\eta}}_{f1}(t) = \begin{bmatrix} -0.0701 & 6.0439 \\ -6.0431 & -0.0701 \end{bmatrix} \hat{\eta}_{f1}(t)$$

$$+ \begin{bmatrix} 0.0070 & 0.0000 & 0.0000 & 0.0001 \\ 0.0000 & 0.0071 & 0.0001 & 0.0000 \end{bmatrix} y(t)$$

$$\epsilon_2 \dot{\hat{\eta}}_{f2}(t) = \begin{bmatrix} -0.1000 & 6.0443 \\ -6.0427 & -0.1000 \end{bmatrix} \hat{\eta}_{f2}(t)$$

$$+ \begin{bmatrix} 0.0000 & 0.0001 & 0.0050 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0050 \end{bmatrix} y(t)$$

The transformation that relates the Kalman filter estimates in the new and original coordinates as defined by (4.50) is

$$
\mathrm{T} = \begin{bmatrix}
1.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0001 & 0.0116 & 0.0000 & 0.0017 \\
0.0002 & 1.0000 & 0.0000 & 0.0000 & -0.0118 & -0.0002 & 0.0000 & -0.0001 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & -0.0002 & -0.0002 & -0.0017 & 0.0082 \\
-0.0002 & 0.0001 & 0.0000 & 1.0000 & 0.0116 & 0.0005 & -0.0083 & 0.0007 \\
-0.1654 & -0.0030 & 0.0000 & 0.0000 & 9.9993 & 0.0000 & 0.0190 & -0.0813 \\
0.0027 & -0.0013 & 0.0000 & 0.0000 & -0.0019 & 9.9993 & 0.0812 & 0.0183 \\
0.0002 & -0.1654 & 0.0000 & 0.0000 & -0.0125 & -0.0815 & 10.0000 & 0.0000 \\
-0.0008 & -0.0001 & 0.0000 & 0.0000 & 0.0813 & -0.0134 & 0.0000 & 10.0000
\end{bmatrix}
$$

4.4 Comments

The results presented in this chapter are based on the recent research work of (Coumarbatch and Gajic, 1998, 2000; Coumarbatch, 2000). The extension of the results obtained to the Pareto multimodeling strategies (Khalil and Kokotovic, 1978) and the quasi-decentralized (Gajic, 1987) multimodel estimation of (Gajic, 1988b) are under way by the same authors.

Appendix 4.1

Theorem 4.1: *The fixed-point algorithm used for solving equations (4.14) has the rate of convergence of $O(\epsilon)$, that is*

$$
\begin{aligned}
\left\| L_j^{(i+1)} - L_j^{(i)} \right\| &= O(\epsilon), \quad j = 1,2,3; \quad i = 0,1,2,... \\
\left\| H_j^{(i+1)} - H_j^{(i)} \right\| &= O(\epsilon), \quad j = 1,2,3; \quad i = 0,1,2,...
\end{aligned}
\tag{4.55}
$$

which implies

$$
\begin{aligned}
\left\| L_j - L_j^{(i)} \right\| &= O\left(\epsilon^{i+1}\right), \quad j = 1,2,3; \quad i = 0,1,2,... \\
\left\| H_j - H_j^{(i)} \right\| &= O\left(\epsilon^{i+1}\right), \quad j = 1,2,3; \quad i = 0,1,2,...
\end{aligned}
\tag{4.56}
$$

◇

Proof: Let us first observe that by nonsingularity of the matrices T_{11} and T_{22} and under Assumption 4.3, the unique bounded $O(1)$-solutions $L_1^{(0)}, L_2^{(0)}, L_3^{(0)}$ defined by (4.15) exist. Also, by the Implicit Function Theorem, the unique $O(1)$-bounded solutions for L_1, L_2, L_3 defined by (4.14) exist.

The matrix L_1 and its first order approximation $L_1^{(0)}$ satisfy

$$\begin{gathered} T_{11}L_1 = T_{10} + \epsilon_1 L_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1) \\ T_{11}L_1^{(0)} = T_{10} \end{gathered} \tag{4.57}$$

It follows from these equations that

$$T_{11}\left(L_1 - L_1^{(0)}\right) = O(\epsilon) \tag{4.58}$$

which by nonsingularity of T_{11} and $O(1)$-boundness of L_1 and $L_1^{(0)}$ implies

$$\left\|L_1 - L_1^{(0)}\right\| = O(\epsilon) \tag{4.59}$$

Since $L_3^{(0)} = 0$ and $L_3 = O(\epsilon)$ it follows that

$$\left\|L_3 - L_3^{(0)}\right\| = O(\epsilon) \tag{4.60}$$

The equations for L_2 and its approximation $L_2^{(0)}$ are given by

$$\begin{gathered} T_{22}L_2 = T_{20} + \alpha L_3 T_{10} + \epsilon_2 L_2(T_{00} - T_{02}L_2) \\ T_{22}L_2^{(0)} = T_{02} \end{gathered} \tag{4.61}$$

Using the fact that $L_3 = O(\epsilon)$, and by nonsingularity of T_{22} and $O(1)$ boundness of the solution matrices L_2 and $L_2^{(0)}$ we obtain

$$T_{22}\left(L_2 - L_2^{(0)}\right) = O(\epsilon) \Rightarrow \left\|L_2 - L_2^{(0)}\right\| = O(\epsilon) \tag{4.62}$$

In the following we use induction to establish the general result. Let us first prove that the statement

$$\left\|L_j - L_j^{(i)}\right\| = O\left(\epsilon^{i+1}\right), \ j = 1,2,3; \quad i = 0,1,2,... \tag{4.63}$$

holds for $i = 1$. From the first equations in (4.14) and (4.16) we have

$$T_{11}\left(L_1 - L_1^{(1)}\right) = \epsilon_1 L_1 (T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)$$
$$-\epsilon_1 L_1^{(0)}\left(T_{00} - T_{01}L_1^{(0)} - T_{02}L_2^{(0)} + T_{02}L_3^{(0)}L_1^{(0)}\right) \tag{4.64}$$

Using the results established in (4.59)-(4.60), and (4.62) we have

$$T_{11}\left(L_1 - L_1^{(1)}\right) = \epsilon_1\left(L_1^{(0)} + O(\epsilon)\right)[T_{00} - T_{01}\left(L_1^{(0)} + O(\epsilon)\right)$$
$$-T_{02}\left(L_2^{(0)} + O(\epsilon)\right) + T_{02}\left(L_3^{(0)} + O(\epsilon)\right)\left(L_1^{(0)} + O(\epsilon)\right)] \tag{4.65}$$
$$-\epsilon_1 L_1^{(0)}\left(T_{00} - T_{01}L_1^{(0)} - T_{02}L_2^{(0)} + T_{02}L_3^{(0)}L_1^{(0)}\right)$$

After the cancellation takes place, we obtain

$$T_{11}\left(L_1 - L_1^{(1)}\right) = O\left(\epsilon^2\right) \;\Rightarrow\; \left\|L_1 - L_1^{(1)}\right\| = O\left(\epsilon^2\right) \tag{4.66}$$

Similarly, we have

$$\left\|L_2 - L_2^{(1)}\right\| = O\left(\epsilon^2\right) \tag{4.67}$$

$$\left\|L_3 - L_3^{(1)}\right\| = O\left(\epsilon^2\right) \tag{4.68}$$

Assume now that

$$\left\|L_1 - L_1^{(i-1)}\right\| = O\left(\epsilon^i\right) \;\Rightarrow\; L_1 = L_1^{(i-1)} + O\left(\epsilon^i\right) \tag{4.69}$$

$$\left\|L_2 - L_2^{(i-1)}\right\| = O\left(\epsilon^i\right) \;\Rightarrow\; L_2 = L_2^{(i-1)} + O\left(\epsilon^i\right) \tag{4.70}$$

$$\left\|L_3 - L_3^{(i-1)}\right\| = O\left(\epsilon^i\right) \;\Rightarrow\; L_3 = L_3^{(i-1)} + O\left(\epsilon^i\right) \tag{4.71}$$

It follows from (4.14) and (4.16)

$$T_{11}\left(L_1 - L_1^{(i)}\right) = \epsilon_1 L_1 (T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)$$
$$-\epsilon_1 L_1^{(i-1)}\left(T_{00} - T_{01}L_1^{(i-1)} - T_{02}L_2^{(i-1)} + T_{02}L_3^{(i-1)}L_1^{(i-1)}\right) \tag{4.72}$$

Using the assumptions (4.69)-(4.71), we obtain

$$T_{11}\left(L_1 - L_1^{(i)}\right) = \epsilon_1\left(L_1^{(i-1)} + O\left(\epsilon^i\right)\right)[T_{00} - T_{01}\left(L_1^{(i-1)} + O\left(\epsilon^i\right)\right)$$

$$-T_{02}\left(L_2^{(i-1)} + O\left(\epsilon^i\right)\right) + T_{02}\left(L_3^{(i-1)} + O\left(\epsilon^i\right)\right)\left(L_1^{(i-1)} + O\left(\epsilon^i\right)\right)]$$

$$-\epsilon_1 L_1^{(i-1)}\left(T_{00} - T_{01}L_1^{(i-1)} - T_{02}L_2^{(i-1)} + T_{02}L_3^{(i-1)}L_1^{(i-1)}\right) \tag{4.73}$$

By cancelling the appropriate terms we have

$$T_{11}\left(L_1 - L_1^{(i)}\right) = O\left(\epsilon^{i+1}\right) \Rightarrow \left\|L_1 - L_1^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \tag{4.74}$$

A similar proof will show that

$$\left\|L_2 - L_2^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \tag{4.75}$$

and

$$\left\|L_3 - L_3^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \tag{4.76}$$

In addition, it follows directly from the first and second equations of (4.16) that

$$T_{11}\left(L_1^{(i+1)} - L_1^{(i)}\right) = O(\epsilon) \Rightarrow \left\|L_1^{(i+1)} - L_1^{(i)}\right\| = O(\epsilon) \tag{4.77}$$

and

$$T_{22}\left(L_2^{(i+1)} - L_2^{(i)}\right) = O(\epsilon) \Rightarrow \left\|L_2^{(i+1)} - L_2^{(i)}\right\| = O(\epsilon) \tag{4.78}$$

The third equation from (4.16) implies

$$T_{22}\left(L_3^{(i+1)} - L_3^{(i)}\right) - \alpha T_{11}\left(L_3^{(i+1)} - L_3^{(i)}\right) = O(\epsilon) \tag{4.79}$$

which by Assumptions 4.2 and 4.3 implies that

$$\left\|L_3^{(i+1)} - L_3^{(i)}\right\| = O(\epsilon) \tag{4.80}$$

This completes the proof of the first part of the theorem concerned with the fixed-point algorithm for L-equations.

In the following, we establish the convergence result for the H-equations. Note that H-equations are linear algebraic matrix equations. The unique solutions of this system of linear matrix equations exist under Assumption 4.3 and by nonsingularity assumptions imposed on the matrices T_{11} and T_{22}, which follow from Assumption 4.2.

From (4.14)-(4.15), (4.59)-(4.60), and (4.62), we have

$$\left(H_1 - H_1^{(0)}\right)T_{11} = O(\epsilon) \Rightarrow \left\|H_1 - H_1^{(0)}\right\| = O(\epsilon) \tag{4.81}$$

$$\left(H_3 - H_3^{(0)}\right)T_{22} = O(\epsilon) \Rightarrow \left\|H_3 - H_3^{(0)}\right\| = O(\epsilon) \tag{4.82}$$

$$\begin{aligned} -\left(H_2 - H_2^{(0)}\right)T_{22} + \alpha T_{11}\left(H_2 - H_2^{(0)}\right) &= O(\epsilon) \\ \Rightarrow \left\|H_2 - H_2^{(0)}\right\| &= O(\epsilon) \end{aligned} \tag{4.83}$$

In the last result, Assumption 4.3 was used. Using the same procedure in (4.14) and (4.16) and utilizing the results established in (4.74)-(4.76) produce for $i = 1, 2, 3, ...$

$$\left(H_1 - H_1^{(i)}\right)T_{11} = O\left(\epsilon^{i+1}\right) \Rightarrow \left\|H_1 - H_1^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \tag{4.84}$$

$$\left(H_3 - H_3^{(i)}\right)T_{22} = O\left(\epsilon^{i+1}\right) \Rightarrow \left\|H_3 - H_3^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \tag{4.85}$$

$$\begin{aligned} &-\left(H_2 - H_2^{(i)}\right)T_{22} + \alpha T_{11}\left(H_2 - H_2^{(i)}\right) \\ &= O\left(\epsilon^{i+1}\right) \Rightarrow \left\|H_2 - H_2^{(i)}\right\| = O\left(\epsilon^{i+1}\right) \end{aligned} \tag{4.86}$$

Similarly from (4.16), (4.74)-(4.76), we have for every $i = 0, 1, 2, 3, ...$

$$\left(H_1^{(i+1)} - H_1^{(i)}\right)T_{11} = O(\epsilon) \Rightarrow \left\|H_1^{(i+1)} - H_1^{(i)}\right\| = O(\epsilon) \tag{4.87}$$

$$\left(H_3^{(i+1)} - H_3^{(i)}\right)T_{22} = O(\epsilon) \Rightarrow \left\|H_3^{(i+1)} - H_3^{(i)}\right\| = O(\epsilon) \tag{4.88}$$

$$
\begin{aligned}
&-\left(H_2^{(i+1)} - H_2^{(i)}\right)T_{22} + \alpha T_{11}\left(H_2^{(i+1)} - H_2^{(i)}\right) \\
&= O(\epsilon) \;\Rightarrow\; \left\| H_2^{(i+1)} - H_2^{(i)} \right\| = O(\epsilon)
\end{aligned}
\tag{4.89}
$$

•

Appendix 4.2

In this appendix we derive the Newton method for solving the L-equations. Note that the L-equations, in addition to quadratic nonlinearities multiplied by small singular perturbation parameters, also contain a cubic nonlinearity multiplied by a small singular perturbation parameter. These equations are linearized by the Newton method as follows.

$$
\begin{gathered}
D_1^{(i)} L_1^{(i+1)} - \epsilon_1 L_1^{(i+1)} D_2^{(i)} - \epsilon_1 L_1^{(i)} T_{02} L_2^{(i+1)} - \epsilon_1 L_1^{(i)} T_{02} L_3^{(i+1)} = Q_1^{(i)} \\
i = 0, 1, 2, \ldots
\end{gathered}
\tag{4.90}
$$

$$
\begin{gathered}
\left(T_{22} - \epsilon_2 L_2^{(i)} T_{02}\right) L_2^{(i+1)} - \epsilon_2 L_2^{(i+1)} \left(T_{00} + T_{02} L_2^{(i)}\right) \\
-\frac{\epsilon_2}{\epsilon_1} L_3^{(i+1)} T_{10} = Q_2^{(i)}, \qquad i = 0, 1, 2, \ldots
\end{gathered}
\tag{4.91}
$$

$$
\begin{gathered}
\left(T_{22} + \epsilon_2 L_2^{(i)} T_{02}\right) L_3^{(i+1)} - \frac{\epsilon_2}{\epsilon_1} L_3^{(i+1)} T_{11} \\
-\epsilon_2 L_2^{(i+1)} \left(T_{01} - T_{02} L_2^{(i)}\right) = \epsilon_2 Q_3^{(i)}, \qquad i = 0, 1, 2, \ldots
\end{gathered}
\tag{4.92}
$$

where

$$
D_1^{(i)} = T_{11} - \epsilon_1 L_1^{(i)} T_{01} - \epsilon_1 L_1^{(0)} T_{02} L_3^{(0)} \tag{4.93}
$$

$$
D_2^{(i)} = T_{00} + T_{01} L_1^{(0)} + T_{02} L_2^{(0)} + T_{02} L_3^{(0)} L_1^{(0)} \tag{4.94}
$$

$$
Q_1^{(i)} = T_{10} + \epsilon_1 L_1^{(0)} \left(T_{01} L_1^{(0)} + T_{02} L_2^{(0)} + T_{02} L_3^{(0)} L_1^{(0)} + T_{02} L_3^{(0)} L_1^{(0)}\right) \tag{4.95}
$$

$$Q_2^{(i)} = T_{20} + \epsilon_2 L_2^{(i)} T_{02} L_2^{(i)}, \quad Q_3^{(i)} = L_2^{(i)} T_{02} L_3^{(i)} \tag{4.96}$$

Equations (4.105)-(4.107) represent a system of coupled algebraic Sylvester equations. Finding an efficient algorithm for their solution is an open research problem, see (Gajic and Qureshi, 1995).

The initial conditions for the Newton method that are $O(\epsilon)$ close to the exact solutions are obtained from (4.15), that is

$$L_1^{(0)} = T_{11}^{-1} T_{10}, \quad L_2^{(0)} = T_{22}^{-1} T_{20}, \quad L_3^{(0)} = 0 \tag{4.97}$$

It is known that under the assumption that the initial conditions are good enough, the Newton method converges with the quadratic rate of convergence, which implies

$$\begin{gathered} \left\| L_j - L_j^{(1)} \right\| = O\left(\epsilon^2\right), \quad \left\| L_j - L_j^{(2)} \right\| = O\left(\epsilon^4\right) \\ \left\| L_j - L_j^{(3)} \right\| = O\left(\epsilon^8\right), \quad \ldots \qquad j = 1, 2, 3 \end{gathered} \tag{4.98}$$

Appendix 4.3

Here we show that the matrix

$$\Pi_1 + \Pi_2 P \tag{4.99}$$

is invertible. Note that

$$E_2^T K E_1 = \Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \tag{4.100}$$

with matrices E_1, E_2 respectively defined in (4.8) and (4.26). From (4.12) we have

$$K = \begin{bmatrix} I_{n_0} + O(\epsilon) & O(\epsilon) & O(\epsilon) \\ L_1 + O(\epsilon) & I_{n_1} + O(\epsilon) & O(\epsilon) \\ L_2 & L_3 & I_{n_2} \end{bmatrix} \tag{4.101}$$

Let the matrices L_1, L_2, L_3 be partitioned as (with dimensions compatible to (4.103)-(4.105))

$$L_1 = \begin{bmatrix} L_1^{11} & L_1^{12} \\ L_1^{21} & L_1^{22} \end{bmatrix}, \quad L_2 = \begin{bmatrix} L_2^{11} & L_2^{12} \\ L_2^{21} & L_2^{22} \end{bmatrix}, \quad L_3 = \begin{bmatrix} L_3^{11} & L_3^{12} \\ L_3^{21} & L_3^{22} \end{bmatrix} \tag{4.102}$$

Then, after some lengthy algebra, it can be shown that

$$\Pi_1 = \begin{bmatrix} I_{n_0} & 0 & 0 \\ L_1^{11} & I_{n_0} & 0 \\ L_2^{11} & L_3^{21} & I_{n_1} \end{bmatrix} + O(\epsilon), \quad \Pi_2 = \begin{bmatrix} 0 & 0 & 0 \\ L_1^{21} & 0 & 0 \\ L_2^{12} & L_3^{12} & 0 \end{bmatrix} + O(\epsilon) \tag{4.103}$$

It follows from (4.6) that the solution matrix P has the following order

$$P = \begin{bmatrix} P_{00} & O(\epsilon) & O(\epsilon) \\ O(\epsilon) & O(\epsilon) & O(\epsilon) \\ O(\epsilon) & O(\epsilon) & O(\epsilon) \end{bmatrix} = \begin{bmatrix} P_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + O(\epsilon) \tag{4.104}$$

Hence, we have

$$\Pi_1 + \Pi_2 P = \begin{bmatrix} I_{n_0} & 0 & 0 \\ L_1^{11} + L_1^{12} P_{00} & I_{n_0} & 0 \\ L_2^{11} + L_2^{12} P_{00} & L_3^{21} & I_{n_0} \end{bmatrix} + O(\epsilon) \tag{4.105}$$

Due to the fact that we got a lower triangular matrix, it follows that the required matrix is invertible for small values of ϵ.

Appendix 4.4

In this appendix we establish invertibility of the matrix

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f_1} & 0 \\ 0 & 0 & P_{f_2} \end{bmatrix} \tag{4.106}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} K^{-1} E_2^{-T} \tag{4.107}$$

It can be shown from (4.8) that

$$E_1^{-1} = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon_2 I_{n_2} \end{bmatrix} \tag{4.108}$$

From (4.26) we have

$$E_2^{-T} = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix} = E_2 \tag{4.109}$$

From (4.13) we get

$$K^{-1} = \begin{bmatrix} I_{n_0} & O(\epsilon) & O(\epsilon) \\ -L_1 & I_{n_1} & O(\epsilon) \\ -L_2 + L_1 L_3 & -L_3 + O(\epsilon) & O(\epsilon) \end{bmatrix} \tag{4.110}$$

Using the same notation for the partitioned L_1, L_2, L_3 matrices as in Appendix 4.3, and defining

$$L_1 L_3 = \begin{bmatrix} L_{1,3}^{11} & L_{1,3}^{12} \\ L_{1,3}^{21} & L_{1,3}^{22} \end{bmatrix} \tag{4.111}$$

we obtain

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} I_{n_0} & 0 & 0 \\ -L_1^{11} & I_{n_0} & 0 \\ -L_2^{11} + L_{1,3}^{11} & -L_3^{21} & I_{n_1} \end{bmatrix} + O(\epsilon) \\ \Omega_2 &= \begin{bmatrix} 0 & 0 & 0 \\ -L_1^{21} & 0 & 0 \\ -L_2^{12} + L_{1,3}^{12} & -L_3^{12} & 0 \end{bmatrix} + O(\epsilon) \end{aligned} \tag{4.112}$$

The order estimates from the previous formulas imply

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f_1} & 0 \\ 0 & 0 & P_{f_2} \end{bmatrix}$$

$$= \begin{bmatrix} I_{n_0} & 0 & 0 \\ -L_1^{11} - L_1 P_s & I_{n_0} & 0 \\ (-L_2^{12} + L_{1,3}^{12}) P_s - L_2^{11} + L_{1,3}^{11} & -L_3^{21} - L_3^{21} P_{f_1} & I_{n_1} \end{bmatrix} + O(\epsilon) \tag{4.113}$$

which indicates that the required matrix is invertible for small values of the singular perturbation parameter ϵ.

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5

H_∞ Optimal Control and Filtering

During the past twenty years the H_∞ optimization became one of the most interesting and challenging areas of optimal control and filtering theories and their applications. The main advantage of the H_∞ optimization comes from the fact that such obtained controllers and filters are robust with respect to internal and external disturbances. H_∞ controllers and filters are optimized in such a manner that they give the optimal system performance under worst case system and measurement disturbances.

Singularly perturbed H_∞ linear-quadratic optimal control and filtering problems have been studied by several researchers (Khalil and Chen, 1992; Pan and Basar, 1993, 1994, 1996; Vian and Sawan, 1994; Oloomi and Sawan, 1996; Shen and Deng, 1996; Fridman, 1995, 1996, 1999; Fridman and Shaked, 2000; Tuan and Hosoe, 1997, 1999; Tan *et al.*, 1998; Mukaidani *et al.*, 1999; Singh *et al.*, 1999). Related problems for singularly perturbed differential games and disturbance attenuation and systems with markovian jump parameters have been considered in (Dragan, 1993, 1996; Pan and Basar, 1995; Xu and Muzikami, 1997; Dragan *et al.*, 1999).

It is well known (Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986) that the singularly perturbed algebraic Riccati equation is ill-conditioned. In this chapter, we show how to exactly decouple the algebraic Riccati equation of H_∞ optimal control problem of singularly perturbed systems in terms of pure-slow and pure-fast, reduced-order, *well-conditioned*, H_∞ algebraic Riccati equations. The results are obtained by generalizing the results of (Su *et al.*, 1992; Gajic and Shen, 1993) to the corresponding H_∞ optimization problem. We also establish conditions that allow such a decomposition, and formulate the corresponding algorithm. Even though, the obtained reduced-order H_∞ algebraic Riccati equations are nonsymmetric, they are efficiently solved in terms of Lyapunov iterations by using the Newton method. The iterative algorithm of (Li and Gajic, 1995), also given in terms of Lyapunov iterations, is used to obtain numerical solutions of the corresponding reduced-order, slow and fast, symmetric, H_∞ algebraic Riccati equations whose solutions produce excellent initial guesses for the Newton method. It should be emphasized that the H_∞ algebraic Riccati equation, in contrast to the standard algebraic Riccati equation, contains an indefinite coefficient matrix in the quadratic term—hence, it is much more difficult for solving and analyzing.

Another approach to decomposition of the algebraic Riccati equation for the same class of systems, based on a transformation derived in (Sobolev, 1984), was studied in (Fridman, 1995, 1996). That approach leads to valuable results referring to decoupling of slow and fast phenomena in the corresponding singularly perturbed linear control system. The results are particularly important for finite horizon H_∞ optimization. The essence of the approach due to Sobolev and Fridman will be presented in Chapter 8. The problem of deriving an algorithm for solving the corresponding algebraic Riccati equation, which is the main topic of this chapter, is not addressed in (Fridman, 1995, 1996).

In the second part of this chapter we present a method that allows complete time-scale separation and parallelism of the H_∞ optimal filtering problem for linear systems with slow and fast modes. The algebraic Riccati equation of singularly perturbed H_∞ filtering problem is decoupled into two completely independent, reduced-order, pure-slow and pure-fast, H_∞ algebraic Riccati equations. The corresponding H_∞ filter is decoupled into independent reduced-order, well-defined, pure-slow and pure-fast, H_∞ filters driven by the system measurements. The proposed exact closed-loop decomposition technique produces a lot of savings in both on-line and off-line computations and allows parallel processing of information with different sampling rates for slow and fast signals.

Numerical examples, corresponding to real physical control systems, are included in order to demonstrate the efficiency of the proposed algorithms, which can be implemented independently in slow and fast time scales. The results obtained in this chapter allow exact and complete slow-fast time scale decomposition of H_∞ optimal control and filtering tasks of singularly perturbed systems, and reduced-order parallel processing of all off-line and on-line computational requirements.

5.1 Basic $\mathbf{H_\infty}$ Controllers of Linear Systems

H_∞ optimization of linear control systems originated in the work of (Zames, 1981) and became one of the most interesting and challenging research areas of control engineering during the 1980s and 1990s (Francis, 1987; Zhou *et al.*, 1996; Zhou and Doyle, 1998). In this section, we will give only elementary definitions and results used in H_∞ optimization.

Consider a linear time invariant control system under a disturbance defined by the following equations

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\
z(t) &= C_{11}x(t) + D_{12}u(t) \\
y(t) &= C_{22}x(t) + D_{21}u(t)
\end{aligned}
\tag{5.1}
$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the system input, $w(t) \in \Re^r$ is the system disturbance, $z(t) \in \Re^q$ is the system controlled output, and $y(t) \in \Re^p$ is the system measured output. We assume that the system disturbance is bounded. No other assumptions, such as disturbance statistics, knowledge of the disturbance upper bound are needed. Matrices $A, B, D, C_{11}, C_{22}, D_{12}, D_{21}$ are constant and of appropriate dimensions.

Let $G_{wz}(s)$ represents the transfer function from the system disturbance to the system controlled output. The basic optimal H_∞ controller minimizes the H_∞ norm (the infinity norm in the Hardy space—from the linear system point of view, it is the space of stable proper transfer functions) of the closed-loop time invariant system transfer function from the system disturbance to the system controlled output, see Figure 5.1.

The general basic H_∞ optimal control problem can be defined as: Find a dynamic controller $K(s)$ such that the feedback control

$U(s) = K(s)Y(s)$ minimizes the H_∞ norm of $G_{wz}(s)$, that is

$$\min_{K(s)} \{||G_{zw}(s)||_\infty\} \tag{5.2}$$

The reason for the choice of Hardy's norm is the fact that it represents the maximal system gain (for single input-single output systems the H_∞ norm is represented by the peak of the Bode magnitude diagram). For multi-input multi-output systems the H_∞ norm represents the supremum over all frequencies of the maximal singular value of the system transfer function. In general, it is not a simple task to find the H_∞ norm.

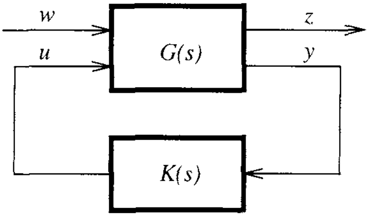

Figure 5.1: Basic feedback configuration.

The general basic H_∞ optimization problem as defined by (5.2) is very complicated. Even more, it does not have the unique solution (Francis, 1987; Zhou *et al.*, 1996; Zhou and Doyle, 1998). The general basic *suboptimal* H_∞ optimization control problem can be defined as follows: Find a dynamic controller $K(s)$ such that the feedback control $U(s) = K(s)Y(s)$ makes the H_∞-norm of $G_{wz}(s)$ smaller than a certain prescribed value, that is

$$\text{Find } K(s) \text{ such that } ||G_{zw}(s)||_\infty < \gamma \tag{5.3}$$

The solution to the optimization problem defined in (5.3) is observer-based and given in terms of solutions of two algebraic Riccati equations, which correspond independently to the full state controller problem and the output estimation problem. The complete solution is given by

$$\begin{gathered} u_{opt}(t) = F_{opt}\hat{x}(t) = -B^T P_c \hat{x}(t) \\ \frac{d}{dt}\hat{x}(t) = \left(A + \frac{1}{\gamma^2} DD^T P_c - BB^T P_o + Z_\infty L_\infty C_{22}\right)\hat{x}(t) \\ -Z_\infty L_\infty y(t) \\ L_\infty = -P_o C_{22}^T, \quad Z_\infty = \left(I - \frac{1}{\gamma^2} P_o P_c\right)^{-1} \end{gathered} \tag{5.4}$$

where the matrices P_c and P_o are the unique positive semidefinite stabilizing solutions of the H_∞ algebraic controller and observer Riccati matrix equations, respectively given by

$$\begin{aligned} A^T P_c + P_c A + C_{11}^T C_{11} + P_c\left(\frac{1}{\gamma^2} BB^T - DD^T\right) P_c = 0 \\ AP_o + P_o A^T + BB^T + P_o\left(\frac{1}{\gamma^2} C_{11} C_{11}^T - C_{22} C_{22}^T\right) P_o = 0 \end{aligned} \tag{5.5}$$

It follows from the invertibility requirement of the matrix Z_∞ that the positive parameter γ should satisfy

$$\gamma^2 > \lambda_{max}(P_o P_c) \tag{5.6}$$

Note that the H_∞ algebraic Riccati equations have indefinite quadratic terms, which make these equations much more difficult for studying than the standard algebraic Riccati equations with positive semidefinite quadratic terms. The H_∞ algebraic Riccati equation is still not very well understood and it has been the subject of intensive research since the beginning of the 1990s, see for example (Hewer, 1993).

Even more, the solution to the basic suboptimal H_∞ control problem as defined by (5.3)-(5.6) is not unique. An arbitrary stable strictly proper transfer function, $K_{stable}(s)$, of appropriate dimensions whose H_∞ norm is smaller than γ can be put around the suboptimal controller $K_{subopt}(s)$ as demonstrated in Figure 5.2 by the dashed lines. The problem formulation of the suboptimal H_∞ optimization problem can lead to the unique solution if we assume that $K_{stable}(s) = 0$, which is a common procedure in practice. Such suboptimal H_∞ controller is called the *central controller.*

Many variants of the basic suboptimal H_∞ optimization problem exists in the control literature. Those variants deal also with optimization of quadratic performance criteria as well as with H_∞ optimization filtering issues. For detailed coverage of the essence of the H_∞ optimization, the interested reader is referred to one of the most comprehensive papers on this subject (Doyle *et al.*, 1989) and the recent text book (Zhou and Doyle, 1998). In the case of multiple objectives, a game type H_∞ problems comes into the picture (Khargonekar and Rotea, 1991; Limebeer *et al.*, 1994; Shen and Deng, 1997). In the following, we will only present the results for H_∞ full state suboptimal control and filtering of linear time invariant singularly perturbed systems. Note that similarly to the suboptimal H_∞ control problem defined in (5.3)-(5.6), the solution to the suboptimal H_∞ filtering problem is given in terms of positive semidefinite stabilizing solution of the H_∞ filter algebraic Riccati equation, which is similar to the H_∞ observer Riccati equation defined in (5.5).

It should be emphasized that since it is customary in the control literature to call the *suboptimal* H_∞ problem the *optimal* H_∞ problem, which we will do very often in the remaining parts of this chapter.

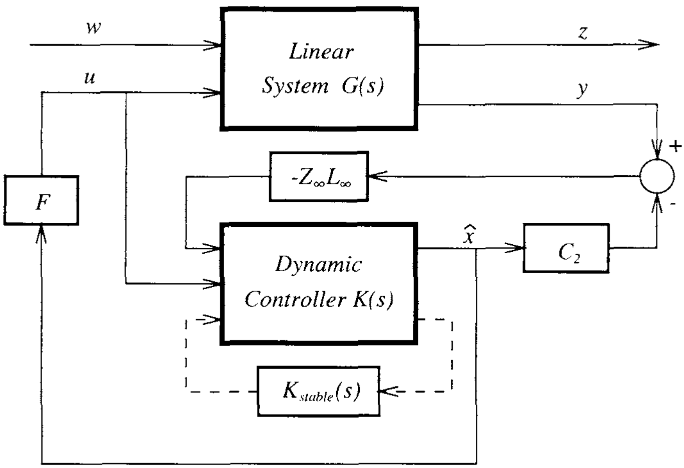

Figure 5.2: Block diagram for the suboptimal H_∞ controller.

5.2 Singularly Perturbed Optimal H_∞ Control Problem

The linear singularly perturbed time invariant control system under disturbances is represented by

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w(t) \tag{5.7}$$

where $x_1(t) \in \Re^{n_1}, x_2(t) \in \Re^{n_2}$ are, respectively, system slow and fast state space variables, $u(t) \in \Re^m$ is the control input, $w(t) \in \Re^p$ is the system disturbance, and ϵ is a small positive singular perturbation parameter. The performance criterion to be minimized is given by

$$J = \frac{1}{2} \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt, \quad Q \geq 0, \quad R > 0 \tag{5.8}$$

The H_∞ optimal control problem associated with (5.7) and (5.8) has a solution given in terms of positive semidefinite stabilizing solution of the following algebraic Riccati equation (Zhou and Khargonekar, 1987; Bernstein and Haddad, 1989; Basar and Bernhard, 1991)

$$A^T P + PA + Q - P\left(S - \frac{1}{\gamma^2}Z\right)P = 0 \tag{5.9}$$

where

$$\begin{gathered}
A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & \frac{1}{\epsilon}S_2 \\ \frac{1}{\epsilon}S_2^T & \frac{1}{\epsilon^2}S_3 \end{bmatrix} \geq 0 \\
Z = \begin{bmatrix} Z_1 & \frac{1}{\epsilon}Z_2 \\ \frac{1}{\epsilon}Z_2^T & \frac{1}{\epsilon^2}Z_3 \end{bmatrix} \geq 0 \\
Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} = \begin{bmatrix} q_1 q_1^T & q_1 q_2^T \\ q_2 q_1^T & q_2 q_2^T \end{bmatrix} \\
S_1 = B_1 R^{-1} B_1^T, \quad S_2 = B_1 R^{-1} B_2^T, \quad S_3 = B_2 R^{-1} B_2^T \\
Z_1 = D_1 D_1^T, \quad Z_2 = D_1 D_2^T, \quad Z_3 = D_2 D_2^T
\end{gathered} \tag{5.10}$$

and γ is a real positive parameter that represents an optimal disturbance attenuation level in the sense

$$\inf_{u(t)} \sup_{w(t)} \left\{ \frac{\sqrt{J}}{\|w(t)\|} \right\} < \gamma \tag{5.11}$$

The optimal controller that guarantees the γ level of optimality is given by

$$u_{opt}(t) = -R^{-1} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} Px(t) \tag{5.12}$$

where P is the positive semidefinite stabilizing solution of (5.9). The existence of such a solution of (5.9) has been studied in (Hewer, 1993).

An efficient numerical algorithm for solving (5.9) is presented in (Li and Gajic, 1995).

The algebraic Riccati equation (5.9) with an indefinite coefficient matrix in the quadratic term appears also in zero-sum differential games (Basar, 1991), stabilization of uncertain systems (Peterson and Hollot, 1986; Peterson, 1988), disturbance attenuation problems (Peterson, 1987), and decentralized stabilization (Mageirou and Ho, 1977).

5.3 Solution of the Singularly Perturbed H_∞ Algebraic Riccati Equation

The Hamiltonian form corresponding to the H_∞ algebraic Riccati equation (5.9) will be used in further analysis. This form is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -\left(S - \frac{1}{\gamma^2}Z\right) \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{5.13}$$

with

$$p(t) = Px(t) \tag{5.14}$$

Our goal is to find the solution of (5.9) in terms of solutions of the reduced-order, pure-slow and pure-fast, H_∞ algebraic Riccati equations by following the methodology of (Su *et al.*, 1992). In addition, we establish conditions for such a decomposition, and formulate the corresponding algorithm.

By partitioning the costate vector $p(t)$ as $p(t) = [p_1(t) \quad \epsilon p_2(t)]$ with $p_1(t) \in \Re^{n_1}$, $p_2(t) \in \Re^{n_2}$ and interchanging the second and third rows in (5.13) we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{5.15}$$

where

$$T_1 = \begin{bmatrix} A_1 & -\left(S_1 - \frac{1}{\gamma^2}Z_1\right) \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -\left(S_2 - \frac{1}{\gamma^2}Z_2\right) \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -\left(S_2 - \frac{1}{\gamma^2}Z_2\right)^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -\left(S_3 - \frac{1}{\gamma^2}Z_3\right) \\ -Q_3 & -A_4^T \end{bmatrix} \tag{5.16}$$

It is important to notice that (5.15) retains the singularly perturbed form. In the following, in order to be able to apply the Chang transformation (Chang, 1972) to (5.15), we will need nonsingularity of the fast subsystem matrix T_4. It is established in (Fridman, 1995, 1996) that the matrix T_4 is nonsingular under the following assumption.

Assumption 5.1: The triple $(A_4, B_2, \mathrm{Chol}(Q_3))$ is controllable-observable.

Applying the Chang transformation (Chang, 1972) to (5.15) we obtain in the new coordinates two independent pure-slow and pure-fast subsystems

$$\dot{\eta}(t) = (T_1 - T_2 L)\eta(t) \tag{5.17}$$

$$\epsilon\dot{\zeta}(t) = (T_4 + \epsilon L T_2)\zeta(t) \tag{5.18}$$

where the matrix L is obtained from the Chang transformation equations

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{5.19}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \tag{5.20}$$

Note that one can also apply to (5.15) a new version of the Chang transformation (Qureshi and Gajic, 1992) that produces complete independence between the L and H equations. However, in that case, the H equation is a weakly nonlinear one. The unique solutions of equations (5.19)-(5.20) exist for sufficiently small values of ϵ under the assumption that the matrix T_4 is nonsingular (by the Implicit Function Theorem). The relationship between the new and old state variables is determined by the Chang transformation as

$$\begin{bmatrix} \eta(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} I - \epsilon H L & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \mathrm{T} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{5.21}$$

The coupled algebraic equations (5.19)–(5.20) can be solved efficiently for ϵ sufficiently small by using either the Newton method or fixed point iterations (Kokotovic *et al.*, 1980; Grodt and Gajic, 1988). For example, the Newton method applied to (5.19) leads to Sylvester iterations

$$
\begin{gathered}
D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = D_3^{(i)} \\
L^{(0)} = T_4^{-1} T_3 + O(\epsilon), \quad i = 0, 1, 2, \ldots \\
D_1^{(i)} = T_4 + \epsilon L^{(i)}, \quad D_2^{(i)} = -\epsilon\Big(T_1 - T_2 L^{(i)}\Big) \\
D_3^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)}
\end{gathered}
\tag{5.22}
$$

Having obtained the solution for L, equation (5.20) can be solved either directly as the Sylvester equation or iteratively like a system of linear equations as

$$
H^{(i+1)} = \Big[T_2 + \epsilon(T_1 - T_2 L) H^{(i)}\Big](T_4 + \epsilon L T_2)^{-1} \tag{5.23}
$$

The relationship between the original and new coordinates is given by

$$
\begin{aligned}
\begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \\ \eta_2(t) \\ \zeta_2(t) \end{bmatrix} &= E_2^T \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = E_2^T \mathbf{T} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ \epsilon p_2(t) \end{bmatrix} \\
&= E_2^T \mathbf{T} E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}
\end{aligned}
\tag{5.24}
$$

where the permutation matrices have forms

$$
E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix}, \quad E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}
\tag{5.25}
$$

Since at steady state $p(t) = Px(t)$, where P satisfies the H_∞ algebraic Riccati equation (5.9), it follows from (5.24) that

$$
\begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P) x(t), \quad \begin{bmatrix} \eta_2(t) \\ \zeta_2(t) \end{bmatrix} = (\Pi_3 + \Pi_4 P) x(t) \tag{5.26}
$$

In the new coordinates, the state and costate equations are related by

$$
\begin{bmatrix} \eta_2(t) \\ \zeta_2(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \end{bmatrix} \tag{5.27}
$$

Hence

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{5.28}$$

Also, we can find P in terms of P_s and P_f as

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1} \tag{5.29}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathbf{T}^{-1} E_2 = \Pi^{-1} \tag{5.30}$$

It can be shown that the inversions defined in (5.28) and (5.29) exist for sufficiently small values of the singular perturbation parameter ϵ since the corresponding matrices are equal to $I + O(\epsilon)$.

By partitioning system matrices in (5.17) and (5.18) as

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = (T_1 - T_2 L) \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{5.31}$$

$$\epsilon \begin{bmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \tag{5.32}$$

and using (5.27) we get two reduced-order nonsymmetric, pure-slow and pure-fast, H_∞ algebraic Riccati equations, respectively given by

$$\begin{aligned} 0 &= P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s \\ 0 &= P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f \end{aligned} \tag{5.33}$$

These reduced-order algebraic nonsymmetric Riccati equations can be solved by using the eigenvector method (Kwakernaak and Sivan, 1972), in terms of eigenvectors spanning the stable subspace. Another approach for solving equations (5.33), which is more in the spirit of theory of singular perturbations, is given below.

It is interesting to point out that H_∞ algebraic Riccati equations (5.33) are nonsymmetric, but their $O(\epsilon)$ perturbations are symmetric. Namely, by closely examining the coefficients in (5.32), the pure-fast H_∞ algebraic Riccati equation is represented by

$$P_f A_4 + A_4^T P_f + Q_3 - P_f \left(S_3 - \frac{1}{\gamma^2} Z_3 \right) P_f + O(\epsilon) = 0 \tag{5.34}$$

This follows from the fact that

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon L T_2)$$

$$b_1 = A_4 + \epsilon(L_1 A_2 - L_2 Q_2)$$

$$b_2 = -\left(S_3 - \frac{1}{\gamma^2} Z_3\right) - \epsilon L_1 \left(S_2 - \frac{1}{\gamma^2} Z_2\right) - \epsilon L_2 A_3^T \tag{5.35}$$

$$b_3 = -Q_3 + \epsilon(L_3 A_2 - L_4 Q_2)$$

$$b_4 = -A_4^T - \epsilon L_3 \left(S_2 - \frac{1}{\gamma^2} Z_2\right) - \epsilon L_4 A_3^T$$

where

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \tag{5.36}$$

It can be observed from (5.29) that

$$P = \begin{bmatrix} P_s & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) \tag{5.37}$$

It is known from (Pan and Basar, 1993) that the nature of the solution matrix of (5.9) is

$$P = \begin{bmatrix} P_1 + O(\epsilon) & \epsilon(P_{12} + O(\epsilon)) \\ \epsilon(P_{12}^T + O(\epsilon)) & \epsilon(P_2 + O(\epsilon)) \end{bmatrix}, \quad P_1 = P_1^T, \quad P_2 = P_2^T \tag{5.38}$$

where P_1 satisfies the symmetric slow H_∞ algebraic Riccati equation. It follows from (Pan and Basar, 1993), (5.37) and (5.38) that

$$P_s A_s + A_s^T P_s + Q_s - P_s \left(S_s - \frac{1}{\gamma^2} Z_s\right) P_s + O(\epsilon) = 0 \tag{5.39}$$

From (5.34) and (5.39) one can obtain $O(\epsilon)$ approximations for P_s and P_f equations by solving the following H_∞ algebraic Riccati equations

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} \left(S_s - \frac{1}{\gamma^2} Z_s\right) P_s^{(0)} = 0 \tag{5.40}$$

$$P_f^{(0)} A_4 + A_4^T P_f^{(0)} + Q_3 - P_f^{(0)} \left(S_3 - \frac{1}{\gamma^2} Z_3\right) P_f^{(0)} = 0 \tag{5.41}$$

The unique positive semidefinite stabilizing solution of (5.41) exists under Assumption 5.1, (Li and Gajic, 1995). The unique positive semidefinite stabilizing solution of (5.40) exists under the following assumption, (Li and Gajic, 1995).

Assumption 5.2: The triple $(A_s, B_s, \mathrm{Chol}(Q_s))$ is stabilizable-detectable.

Matrices A_s, B_s, Q_s, can be derived either by using the methodology of (Pan and Basar, 1993, 1994). These matrices can be also obtained numerically in a simpler manner using the results of (Wang and Frank, 1992) as

$$\begin{bmatrix} A_s & -\left(S_s - \frac{1}{\gamma^2} Z_s\right) \\ -Q_s & -A_s^T \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3 \tag{5.42}$$

An important feature of equations (5.40)-(5.41), which distinguishes these equations from the standard algebraic Riccati equation of the linear-quadratic optimal control problem is that the quadratic terms in (5.40)-(5.41) have indefinite coefficient matrices. The algorithm of (Li and Gajic, 1995), given in terms of Lyapunov iterations, converges globally to the positive semidefinite stabilizing solution of (5.40)-(5.41) under Assumptions 5.1 and 5.2. It has been demonstrated in (Gajic and Qureshi, 1995) that the Lyapunov iterations are efficient numerical tool for solving many nonlinear algebraic equations arising in optimal control and filtering problems. Using the algorithm of (Li and Gajic, 1995), equation (5.41) is solved by performing the following Lyapunov iterations

$$\begin{aligned} &P_f^{(0)^{(i+1)}} \left(A_4 - S_3 P_f^{(0)^{(i)}}\right) + \left(A_4 - S_3 P_f^{(0)^{(i)}}\right)^T P_f^{(0)^{(i+1)}} \\ &= -\left(Q_3 + P_f^{(0)^{(i)}} S_3 P_f^{(0)^{(i)}} + \frac{1}{\gamma^2} P_f^{(0)^{(i)}} Z_3 P_f^{(0)^{(i)}}\right) \end{aligned} \tag{5.43}$$

with the initial condition obtained from the standard algebraic Riccati equation

$$P_f^{(0)^{(0)}} A_4 + A_4^T P_f^{(0)^{(0)}} + Q_3 - P_f^{(0)^{(0)}} S_3 P_f^{(0)^{(0)}} = 0 \tag{5.44}$$

This choice of the initial condition is an interesting feature of the algorithm of (Li and Gajic, 1995), and it is important for the efficiency of the overall algorithm for solving the singularly perturbed H_∞ algebraic Riccati equation. Having obtained the approximate solution $P_f^{(0)} = P_f + O(\epsilon)$, we can implement the Newton method for solving the pure-fast algebraic Riccati equation given in (5.33) since a good initial guess is available. The Newton method leads to the following Lyapunov-like (Sylvester) iterations

$$P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(0)} b_2\right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)} \quad (5.45)$$

and converges in only a few iterations.

Similarly, the algorithm of (Li and Gajic, 1995) is applied for solving (5.40) as

$$\begin{aligned} &P_s^{(0)^{(i+1)}}\left(A_s - S_s P_s^{(0)^{(i)}}\right) + \left(A_s - S_s P_s^{(0)^{(i)}}\right)^T P_s^{(0)^{(i+1)}} \\ &\quad = -\left(Q_s + P_s^{(0)^{(i)}} S_s P_s^{(0)^{(i)}} + \frac{1}{\gamma^2} P_s^{(0)^{(i)}} Z_s P_s^{(0)^{(i)}}\right) \end{aligned} \quad (5.46)$$

with the initial condition obtained from the standard slow algebraic Riccati equation

$$P_s^{(0)^{(0)}} A_s + A_s^T P_s^{(0)^{(0)}} + Q_s - P_s^{(0)^{(0)}} S_s P_s^{(0)^{(0)}} = 0 \quad (5.47)$$

Having obtained the approximate solution $P_s^{(0)} = P_s + O(\epsilon)$, we can implement the Newton method for solving the corresponding pure-slow algebraic Riccati equation defined in (5.33) since a good initial guess is available. The Newton methods leads to the following Lyapunov-like (Sylvester) iterations

$$P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(0)} a_2\right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)} \quad (5.48)$$

which converge quadratically to the required solution.

From the presentation in this section, we conclude that under Assumptions 5.1 and 5.2, the positive semidefinite stabilizing solution of the singularly perturbed H_∞ algebraic equation can be found (assuming that such a solution exits) by using the following algorithm.

Algorithm 5.1: Solution of the H_∞ Algebraic Riccati Equation

Step 1: Form matrices $T_i, i = 1, 2, 3, 4$, and solve equations (5.19) and (5.20) for L and H.

Step 2: Calculate the coefficients $\Omega_i, a_i, b_i, i = 1, 2, 3, 4$, from (5.30)-(5.32).

Step 3: Solve the standard slow and fast algebraic Riccati equations (5.44) and (5.47) to get initial conditions $P_f^{(0)^{(0)}}$ and $P_s^{(0)^{(0)}}$.

Step 4: Use the initial guesses obtained in Step 3 for Sylvester iterations (5.43) and (5.46) to find $P_f^{(0)}$ and $P_s^{(0)}$.

Step 5: Run Sylvester iterations (5.45) and (5.48) with the initial guesses obtained in Step 4 to get the solutions for pure-fast and pure-slow H_∞ algebraic Riccati equation, P_s and P_f, respectively.

Step 6: Use the solutions P_s and P_f obtained in Step 4 in formula (5.29) in order to calculate the required solution P of equation (5.9).

Δ

Note that in Steps 1 and 2 we calculate the coefficient matrices needed for Steps 3–6. All calculations in Steps 3–6 can be done independently and in parallel for slow and fast subsystems.

In the next section, we solve an example in order to demonstrate the presented algorithm.

5.3.1 Case Study: H_∞ Optimal Control of an F-8 Aircraft

Consider a model of longitudinal motion of an F-8 aircraft whose system and penalty needed for H_∞ optimization are given by (Vian and Sawan, 1994)

$$A = \begin{bmatrix} -0.015 & -0.0519 & -0.0226 & 0 \\ 0 & 0 & 0 & 1 \\ -0.1178 & 0 & -0.84 & 1 \\ 0.031 & 0 & -4.8 & -0.490 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0018 \\ 0 \\ -0.11 \\ -8.7 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 & 0 \\ 0.921022 & -0.161179 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad Z = \text{diag}\{0, 0, 0, 1\}$$

$$R = 1, \quad \epsilon = 0.0336, \quad \gamma = 1$$

The algorithm of (Li and Gajic, 1995) is used to find the global positive semidefinite stabilizing solution of (5.9). The global solution is given by

$$P = P_{global} = \begin{bmatrix} 14.9230 & -1.3819 & 0.3224 & -0.0867 \\ -1.3819 & 1.9354 & -0.1358 & 0.1258 \\ 0.3224 & -0.1358 & 0.7690 & 0.0383 \\ -0.0867 & 0.1258 & 0.0383 & 0.1706 \end{bmatrix}$$

The solutions of the pure-fast and pure-slow algebraic Riccati equations (5.33) obtained from (5.45) and (5.48), respectively, with the initial guesses obtained from (5.40)-(5.41) are

$$P_s = \begin{bmatrix} 15.0150 & -1.5667 \\ -1.3997 & 1.9817 \end{bmatrix}, \quad P_f = \begin{bmatrix} 20.8626 & 3.2012 \\ 1.3227 & 4.9225 \end{bmatrix}$$

The corresponding L and H matrices from (5.19) and (5.20) are given by

$$L = \begin{bmatrix} 0.0465 & 0.3256 & -0.0132 & 0.1407 \\ -0.0469 & 0.1129 & -0.0036 & 0.0951 \\ 1.6543 & -20.4707 & 0.5873 & 20.2251 \\ -0.1191 & -0.3510 & 0.0230 & -0.5469 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.4141 & 0.0187 & 0.0100 & 0.0024 \\ 15.3360 & -0.4148 & -0.1067 & -0.0727 \\ -1.2880 & 0.0920 & 0.0351 & -0.0525 \\ 15.5234 & 0.2660 & 0.2469 & 0.0856 \end{bmatrix}$$

Note that L and H matrices are obtained by solving linear algebraic equations.

Using formula (5.29) the solution obtained for P is identical to P_{global}.

5.4 H_∞ Filtering

The main advantage of the H_∞ optimization is due to the fact that such obtained controllers and filters are robust with respect to internal and external disturbances. In the case of filtering, the additional advantage of the H_∞ filter over the standard Kalman filter is that the former one does not require knowledge of the system and measurement noise intensity matrices—data hardly exactly known. Even more, the system and measurement disturbances are not required to be Gaussian white noise stochastic processes. The only requirement that is imposed on the system disturbances is that the disturbances are bounded. In the 1990s the H_∞ filter replaced the Kalman filter in many applications.

The Kalman filter is probably the most important result of modern control theory due to its vast applications in many diverse engineering and scientific fields. In addition to its many control engineering applications for various dynamic systems such as aircrafts, robots, cars, ships, nuclear and chemical reactors, and so on, the Kalman filter has been used for navigation, weather forecasting, stock market prediction, in hydrology, economy, and sociology. The Kalman filter has been extensively used since the early 1980s in signal processing, see for example (Azimi-Sadjadi 1991; Azimi-Sadjadi *et al.*, 1991; Galatsanos and Chin, 1991; Iiguni *et al.*, 1992; Belaifa and Schwartz, 1992; Wu and Kundu, 1992; Chen and Chiang, 1994; Chen *et al.*, 1995; La Scala *et al.*, 1996; Shen and Deng, 1996, 1997; and references therein), image processing (for example, Citrin and Azimi-Sadjadi, 1992; Burl, 1993; Namazi *et al.*, 1994; Koch *et al.*, 1995), and communications (Yasuhara and Yasumoto, 1984; Merritt, 1989; Aghamohammadi *et al.*, 1989; Iltis, 1990; Fuxjaeger and Iltis, 1994).

The standard Kalman filter for singularly perturbed linear systems was previously studied in (Khalil and Gajic, 1984; Gajic, 1986; Gajic and Shen, 1993; Gajic and Lim, 1994). It is known that the singularly perturbed Kalman filter is numerically ill-conditioned due to coupling of the slow and fast modes (signals). Hence, the main goal in theory of singular perturbations is to decouple (separate) the slow and fast signals and process them independently.

The H_∞ filter has recently become popular in signal processing (see for example, (Shen and Deng, 1996; 1997) and references therein). In this section we study the H_∞ filter for linear systems with slow and fast modes (singularly perturbed linear systems). Difficulties encountered with the full-order H_∞ filter of singularly perturbed linear systems are in the facts that the corresponding algebraic filter Riccati equation is ill-conditioned and it has an indefinite coefficient matrix multiplying the quadratic term (which makes this equation much more difficult for studying than the corresponding one of standard singularly perturbed optimal filtering problems).

The main result of H_∞ filtering can be presented briefly as follows. Consider a linear time invariant dynamic system:

$$\dot{x}(t) = Ax(t) + \Gamma w(t)$$
$$y(t) = Cx(t) + v(t) \tag{5.49}$$

where $x(t) \in \Re^n$ is the state vector, $w(t) \in \Re^r$ and $v(t) \in \Re^p$ are the system and measurement disturbances respectively, and $y(t) \in \Re^p$ are the system measurements. It is assumed that the disturbances are bounded. The H_∞ suboptimal filtering problem looks for a solution in terms of a dynamic system whose state variables $\hat{x}(t)$ satisfy

$$\sup_{w(t),v(t)} \left\{ \frac{\|Gx(t) - G\hat{x}(t)\|_R^2}{\|w(t)\|_{W^{-1}}^2 + \|v(t)\|_{V^{-1}}^2} \right\} < \gamma^2 \tag{5.50}$$

where G is an arbitrary matrix of appropriate dimensions, and R, W, V are weighting matrices and γ is a positive number. It has been shown (Nagpal and Khargonekar, 1991; Zhou *et al.*, 1996; Zhou and Doyle, 1998) that the H_∞ suboptimal filter has the same structure as the Kalman filter, namely, it is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t)) \tag{5.51}$$

with the H_∞ filter gain given by

$$L = -PC^T V^{-1} \tag{5.52}$$

where P satisfies the H_∞ algebraic Riccati equation

$$AP + PA^T + \Gamma W \Gamma^T + P\left(\frac{1}{\gamma^2} G^T R G - C^T V^{-1} C\right) P = 0 \tag{5.53}$$

In Section 5.3 the algebraic regulator Riccati equation of H_∞ optimal linear-quadratic regulator problem is decomposed into reduced-order, pure-slow and pure-fast, algebraic regulator Riccati equations. In this section, we extend the results of Section 5.3 to the decomposition of the corresponding H_∞ algebraic filter Riccati equation and use those results to decompose the H_∞ singularly perturbed filter into independent, well-defined, reduced-order, H_∞ filters. The filters obtained are completely independent and work in parallel. Each of them can process information with a different sampling rate—the fast filter requires a small sampling period and the slow one can process information with a relatively large sampling period.

5.5 H_∞ Filter for Singularly Perturbed Systems

Consider the linear singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + D_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + D_2 w(t) \end{aligned} \tag{5.54}$$

with the corresponding measurements

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + v(t) \tag{5.55}$$

where $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are slow and fast state variables, respectively, $y(t) \in \Re^p$ are system measurements, $w(t) \in \Re^r$ and $v(t) \in \Re^p$ are respectively system and measurement disturbances. A_i, D_j, C_j, $i = 1,2,3,4$, $j = 1,2$, are constant matrices of appropriate dimensions. ϵ is a small positive singular perturbation parameter, which indicates system separation into slow and fast time scales.

In this section we design a filter to estimate system states $x_1(t)$ and $x_2(t)$. The states to be estimated are given by a linear combination

$$z(t) = G_1 x_1(t) + G_2 x_2(t) \tag{5.56}$$

The H_∞ filtering (estimation) problem is to obtain an estimate $\hat{z}(t)$ of $z(t) \in \Re^q$ using the measurements $y(t)$, when the performance measure for the infinite horizon estimation problem is defined by a disturbance attenuation functional (Nagpal and Khargonekar, 1991; Shaked and Theodor, 1992; Shen and Deng, 1996; Zhou *et al*., 1996),

$$J = \frac{\int_0^\infty \|z(t) - \hat{z}(t)\|_R^2 dt}{\int_0^\infty \left(\|w(t)\|_{W^{-1}}^2 + \|v(t)\|_{V^{-1}}^2 \right) dt} \tag{5.57}$$

where $R \geq 0, W > 0, V > 0$ are the weighting matrices to be chosen by designers. The H_∞ suboptimal filter is to ensure that the energy gain from the disturbances to the estimation errors, $z(t) - \hat{z}(t)$, is less than a prespecified level γ^2. That is,

$$\sup_{w,v} J < \gamma^2 \tag{5.58}$$

where "sup" stands for supremum and γ^2 is a prescribed level of noise attenuation.

The H_∞ filter associated with singularly perturbed linear systems, driven by the innovation process has the structure (Gajic and Lim, 1994; Shen and Deng, 1996)

$$\begin{aligned}\dot{\hat{x}}_1(t) &= A_1\hat{x}_1(t) + A_2\hat{x}_2(t) + K_1\nu(t)\\ \epsilon\dot{\hat{x}}_2(t) &= A_3\hat{x}_1(t) + A_4\hat{x}_2(t) + K_2\nu(t)\\ \nu(t) &= y(t) - C_1\hat{x}_1(t) - C_2\hat{x}_2(t)\end{aligned} \tag{5.59}$$

where the filter gains K_1 and K_2 are obtained from

$$K_1 = \left(P_1C_1^T + P_2C_2^T\right)V^{-1}, \quad K_2 = \left(\epsilon P_2^TC_1^T + P_3C_2^T\right)V^{-1} \tag{5.60}$$

with matrices P_1, P_2, and P_3 representing the positive semidefinite stabilizing solution of the following algebraic Riccati equation (Nagpal and Khargonekar, 1991; Shen and Deng, 1996)

$$AP + PA^T - P\left(C^TV^{-1}C - \frac{1}{\gamma^2}G^TRG\right)P + DWD^T = 0 \tag{5.61}$$

where

$$\begin{aligned}A &= \begin{bmatrix} A_1 & A_2\\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4\end{bmatrix}, \quad D = \begin{bmatrix} D_1\\ \frac{1}{\epsilon}D_2\end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_2\\ P_2^T & \frac{1}{\epsilon}P_3\end{bmatrix}\\ C &= \begin{bmatrix} C_1 & C_2\end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2\end{bmatrix}\end{aligned} \tag{5.62}$$

In order to form the innovation process defined in (5.59), communications of the filter estimates are required, thus additional communication channels are necessary. In the following, we will achieve the slow-fast H_∞ filter decomposition in which both filters are directly driven by the system measurements and thus, we will eliminate the need for communication of estimates. The problem of solving the H_∞ singularly perturbed algebraic filter Riccati equation (5.61) will be solved by using duality between the optimal filters and regulators and the efficient algorithm of (Hsieh and Gajic, 1998) for solving (5.61) in terms of reduced-order, pure-slow and pure-fast, H_∞ algebraic Riccati equations.

5.5.1 Decomposition of the H_∞ Filter Algebraic Riccati Equation

In this section we first summarize the main results of Section 5.3 for solving the algebraic Riccati equation of H_∞ linear-quadratic optimal control problem in terms of two completely independent, pure-slow and pure-fast, H_∞ algebraic regulator Riccati equations. These results are needed for the decomposition of the H_∞ filter algebraic Riccati equation.

The singularly perturbed H_∞ linear-quadratic optimal control problem is defined by (5.7)-(5.11). The optimal controller that guarantees the γ level of optimality is given by (5.11), that is

$$u(t) = -R_1^{-1}\begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}^T P_r x(t) = -Fx(t) = -F_1 x_1(t) - F_2 x_2(t) \quad (5.63)$$

where the regulator gains F_1 and F_2 are given by

$$F_1 = -R_1^{-1}\left(B_1^T P_{r1} + B_2^T P_{r2}^T\right), \quad F_2 = R_1^{-1}\left(\epsilon B_1^T P_{r2} + B_2^T P_{r3}\right) \quad (5.64)$$

In Section 5.3 the ill-defined singularly perturbed algebraic regulator Riccati equation is solved in terms of well-defined, pure-slow and pure-fast, reduced-order, algebraic regulator Riccati equations. The solution of the full-order, ill-conditioned, singularly perturbed, H_∞ regulator algebraic Riccati equation is obtained in Section 5.3 as

$$P_r = \left(\Omega_{3r} + \Omega_{4r}\begin{bmatrix} P_{sr} & 0 \\ 0 & P_{fr} \end{bmatrix}\right)\left(\Omega_{1r} + \Omega_{2r}\begin{bmatrix} P_{sr} & 0 \\ 0 & P_{fr} \end{bmatrix}\right)^{-1} \quad (5.65)$$

where P_1 and P_2 are the unique solutions of well-conditioned, pure-slow and pure-fast, completely decoupled, H_∞ algebraic Riccati equations

$$\begin{aligned} P_{sr}a_{1r} - a_{4r}P_{sr} - a_{3r} + P_{sr}a_{2r}P_{sr} &= 0 \\ P_{fr}b_{1r} - b_{4r}P_{fr} - b_{3r} + P_{fr}b_{2r}P_{fr} &= 0 \end{aligned} \quad (5.66)$$

Matrices $a_{ir}, b_{ir}, \Omega_{ir}, i = 1,2,3,4$, are defined in (5.30)-(5.32).

The desired decomposition of the H_∞ filter (5.59) will be obtained by first producing dual results to (5.64)-(5.66). Consider the H_∞ optimal closed-loop filter (5.59) driven by the system measurements

$$\begin{aligned} \dot{\hat{x}}_1(t) &= (A_1 - K_1C_1)\hat{x}_1(t) + (A_2 - K_1C_2)\hat{x}_2(t) + K_1 y(t) \\ \epsilon\dot{\hat{x}}_2(t) &= (A_3 - K_2C_1)\hat{x}_1(t) + (A_4 - K_2C_2)\hat{x}_2(t) + K_2 y(t) \end{aligned} \quad (5.67)$$

with the optimal filter gains K_1 and K_2 calculated from (5.60)-(5.62). By duality between the optimal filter and regulator, the filter algebraic

Riccati equation (5.61) can be solved by using the same decomposition method as the one used for solving the H_∞ algebraic regulator Riccati equation with

$$A \to A^T, \quad Q \to DWD^T, \quad F^T \to K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix}$$
$$S = BR_1^{-1}B^T \to C^T V^{-1} C, \quad DD^T \to G^T RG \tag{5.68}$$

By invoking results from (Gajic and Lim, 1994; Hsieh and Gajic, 1998), and using duality between the optimal linear-quadratic controllers and optimal filters, the following matrices have to be formed

$$T_1 = \begin{bmatrix} A_1^T & -\left(C_1^T V^{-1} C_1 - \frac{1}{\gamma^2} G_1^T RG_1\right) \\ -D_1 W D_1^T & -A_1 \end{bmatrix}$$
$$T_2 = \begin{bmatrix} A_3^T & -\left(C_1^T V^{-1} C_2 - \frac{1}{\gamma^2} G_1^T RG_2\right) \\ -D_1 W D_2^T & -A_2 \end{bmatrix}$$
$$T_3 = \begin{bmatrix} A_2^T & -\left(C_2^T V^{-1} C_1 - \frac{1}{\gamma^2} G_2^T RG_1\right) \\ -D_2 W D_1^T & -A_3 \end{bmatrix} \tag{5.69}$$
$$T_4 = \begin{bmatrix} A_4^T & -\left(C_2^T V^{-1} C_2 - \frac{1}{\gamma^2} G_2^T RG_2\right) \\ -D_2 W D_2^T & -A_4 \end{bmatrix}$$

It can be shown after some algebra that matrices (T_1, T_2, T_3, T_4) comprise the system matrix of a standard singularly perturbed system, namely

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{1}{\epsilon}T_3 & \frac{1}{\epsilon}T_4 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{5.70}$$

Note that in contrast to the results of Section 5.3, where the state-costate variables have to be partitioned as $x^T = \left[x_1^T \; x_2^T\right]$ and $p^T = \left[p_1^T \; \epsilon p_2^T\right]$, in the case of the dual filter variables, we have to use the following partitions $x^T = \left[x_1^T \; \epsilon x_2^T\right]$ and $p^T = \left[p_1^T \; p_2^T\right]$. Since matrices T_1, T_2, T_3 , and T_4 correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition of (5.77) can be achieved by using the Chang decoupling equations (Chang, 1972) of the form

$$\begin{aligned} T_4M - T_3 - \epsilon M(T_1 - T_2M) &= 0 \\ -N(T_4 + \epsilon MT_2) + T_2 + \epsilon(T_1 - T_2M)N &= 0 \end{aligned} \tag{5.71}$$

The unique solutions of algebraic equations (5.71) exist, by the implicit function theorem, for ϵ sufficiently small, under the assumption that the matrix T_4 is nonsingular. The solutions of the above equations can be easily obtained in terms of linear algebraic equations by using either the Newton method or the fixed point iterations with the initial conditions given by $M^{(0)} = T_4^{-1}T_3$ and $N^{(0)} = T_2T_4^{-1}$. Using the results of (Fridman, 1995, 1996) and the duality relationships between optimal linear-quadratic regulators and filters given in (5.68), it follows that the matrix T_4 is nonsingular under the following assumption.

Assumption 5.3: The triple (A_4, C_2, D_2) is controllable-observable.

The Chang decoupling transformation corresponding to (5.69)-(5.70) is given by (Chang, 1972)

$$\mathbf{T} = \begin{bmatrix} I - \epsilon NM & -\epsilon N \\ M & I \end{bmatrix} \tag{5.72}$$

Then, by duality, from (5.29), we have

$$P = \left(\Omega_3 + \Omega_4\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)\left(\Omega_1 + \Omega_2\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right)^{-1} \tag{5.73}$$

where, the pure-slow and pure-fast, well-conditioned, reduced-order, algebraic H_∞ filter Riccati equations are given by

$$\begin{aligned} P_sa_1 - a_4P_s - a_3 + P_sa_2P_s &= 0 \\ P_fb_1 - b_4P_f - b_3 + P_fb_2P_f &= 0 \end{aligned} \tag{5.74}$$

with

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2M), \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon MT_2) \tag{5.75}$$

The $\Omega_i, i = 1, 2, 3, 4$, matrices in (5.73) are defined by

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1}\mathbf{T}^{-1}E_2 = E_1^{-1}\begin{bmatrix} I & \epsilon N \\ -M & I - \epsilon MN \end{bmatrix}E_2 \tag{5.76}$$

The permutation matrices dual to those from Section 5.3, (note that E_1 is different than the corresponding one from Section 5.3) are given by

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & \frac{1}{\epsilon} I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}, \quad E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{5.77}$$

It can be shown that one can obtain $O(\epsilon)$ approximations for P_s and P_f by solving the following H_∞ symmetric algebraic filter Riccati equations

$$\begin{aligned} & P_s^{(0)} A_s^T + A_s P_s^{(0)} + D_s W_s D_s^T \\ & - P_s^{(0)} \left(C_s^T V_s^{-1} C_s - \frac{1}{\gamma^2} G_s^T R_s G_s \right) P_s^{(0)} = 0 \end{aligned} \tag{5.78}$$

$$P_f^{(0)} A_4^T + A_4 P_f^{(0)} + D_2 W D_2^T - P_f^{(0)} \left(C_2^T V_s^{-1} C_2 - \frac{1}{\gamma^2} G_2^T R G_2 \right) P_f^{(0)} = 0 \tag{5.79}$$

The newly defined matrices appearing in (5.78) are obtained from

$$\begin{bmatrix} A_s^T & -\left(C_s^T V_s^{-1} C_s - \frac{1}{\gamma^2} G_s^T R_s G_s \right) \\ -D_s W_s D_s^T & -A_s \end{bmatrix} = T_1 - T_2 T_4^{-1} T_3 \tag{5.80}$$

An important feature of equations (5.78)-(5.79), which distinguishes these equations from the standard algebraic filter Riccati equation, is that the quadratic terms have indefinite coefficient matrices.

The algorithm of (Li and Gajic, 1995), developed for solving the H_∞ algebraic Riccati equations in terms of Lyapunov iterations, converges globally to the positive semidefinite stabilizing solution of (5.78)-(5.79) under the following stabilizability-detectability assumptions.

Assumption 5.4: The slow subsystem defined by the triple $\left(A_s, \mathrm{Chol}\left(C_s V_s^{-1} C_s^T\right), \mathrm{Chol}\left(D_s W_s D_s^T\right)\right)$ is stabilizable-detectable.

Assumption 5.5: The triple (A_4, C_2, D_2) is stabilizable-detectable.

Note that Assumption 5.5 is weaker than Assumption 5.3, hence, it is sufficient to use in this section only Assumptions 5.3 and 5.4. Also, Assumption 5.4 can be written in a simpler form requiring that the triple (A_s, C_s, D_s) is stabilizable-detectable. However, in that case one has to find C_s and D_s matrices explicitly. This can be done by using the procedure of (Pan and Basar, 1993, 1994) for forming the reduced-order slow approximate system.

Using the algorithm of (Li and Gajic, 1995), equation (5.79) is solved by performing the following Lyapunov iterations

$$\begin{aligned} & P_f^{(0)^{(i+1)}}\left(A_4 - C_2^T V^{-1} C_2 P_f^{(0)^{(i)}}\right)^T \\ & + \left(A_4 - C_2^T V^{-1} C_2 P_f^{(0)^{(i)}}\right) P_f^{(0)^{(i+1)}} \\ = & -\left(D_2 W D_2^T + P_f^{(0)^{(i)}} C_2^T V^{-1} C_2 P_f^{(0)^{(i)}} + \frac{1}{\gamma^2} P_f^{(0)^{(i)}} G_2^T R G_2 P_f^{(0)^{(i)}}\right) \end{aligned} \tag{5.81}$$

with the initial condition obtained from the standard algebraic filter Riccati equation

$$P_f^{(0)^{(0)}} A_4^T + A_4 P_f^{(0)^{(0)}} + D_2 W D_2^T - P_f^{(0)^{(0)}} C_2^T V^{-1} C_2 P_f^{(0)^{(0)}} = 0 \tag{5.82}$$

This choice of the initial condition is an interesting feature of the algorithm of (Li and Gajic, 1995), and it is important for the efficiency of the overall algorithm for solving the singularly perturbed H_∞ algebraic Riccati equation. Having obtained an approximate solution $P_f^{(0)} = P_f + O(\epsilon)$, we can implement the Newton method for solving the pure-fast algebraic Riccati equation given in (5.74) since a good initial guess is available. The Newton method leads to the following Lyapunov-like (Sylvester) iterations

$$P_f^{(i+1)}\left(b_1 + b_2 P_f^{(i)}\right) - \left(b_4 - P_f^{(i)} b_2\right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)} \tag{5.83}$$

with $P_f^{(0)}$ obtained in (5.81), and converges in only a few iterations.

Similarly, the algorithm of (Li and Gajic, 1995) is applied for solving (5.85) as

$$\begin{aligned} & P_s^{(0)^{(i+1)}}\left(A_s - C_s^T V_s^{-1} C_s P_s^{(0)^{(i)}}\right)^T \\ & + \left(A_s - C_s^T V_s^{-1} C_s P_s^{(0)^{(i)}}\right) P_s^{(0)^{(i+1)}} \\ = & -\left(D_s W_s D_s^T + P_s^{(0)^{(i)}} C_s^T V_s^{-1} C_s P_s^{(0)^{(i)}} + \frac{1}{\gamma^2} P_s^{(0)^{(i)}} G_s^T R_s G_s P_s^{(0)^{(i)}}\right) \end{aligned} \tag{5.84}$$

with the initial condition obtained from the slow algebraic Riccati equation

$$P_s^{(0)^{(0)}} A_s^T + A_s P_s^{(0)^{(0)}} + D_s W_s D_s^T - P_s^{(0)^{(0)}} C_s^T V_s^{-1} C_s P_s^{(0)^{(0)}} = 0 \quad (5.85)$$

Having obtained an approximate solution $P_s^{(0)} = P_s + O(\epsilon)$, we can implement the Newton method for solving the corresponding slow Riccati equation defined in (5.74) since a good initial guess is available. The Newton method leads to the following Lyapunov-like (Sylvester) iterations

$$P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)} \quad (5.86)$$

with $P_s^{(0)}$ obtained from (5.84). The iterative scheme (5.86) converges quadratically to the required solution.

5.5.2 Decomposition of the Singularly Perturbed H_∞ Filter

It is interesting to point out that for the standard (classical) filtering of linear singularly perturbed systems (see Section 2.3), the transformation that relates the old and new coordinates defined by

$$\Gamma = (\Pi_1 + \Pi_2 P) \quad (5.87)$$

where

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix} E_1 \quad (5.88)$$

is used to decouple both the algebraic filter Riccati equation and the Kalman filter into independent pure-slow and pure-fast components (Gajic and Lim, 1994). However, in the case of the H_∞ filtering the similarity transformation

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \quad (5.89)$$

does not produce in the new coordinates the optimal pure-slow and optimal pure-fast filters, that is

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} &= \Gamma^{-1} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \Gamma \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} \\ &\quad + \Gamma^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \end{aligned} \quad (5.90)$$

does not lead to a block diagonal filter matrix in the new coordinates. The reason for this is inconsistency lies in the fact that the "closed-loop H_∞ filtering problem matrix" is

$$A - P\left(C^T V^{-1} C - \frac{1}{\gamma^2} G^T R G\right) = A - KC - \frac{1}{\gamma^2} P G^T R G \quad (5.91)$$

This matrix is indeed block diagonalized by the similarity transformation Γ. However, the H_∞ optimal filter defined in (5.67) has the feedback matrix given by

$$A - P C^T V^{-1} C = A - KC = \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon}(A_3 - K_2 C_1) & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \quad (5.92)$$

This singularly perturbed matrix can be diagonalized by using another Chang transformation of the form

$$\mathbf{T}_{\mathrm{F}} = \begin{bmatrix} I - \epsilon H L & -\epsilon H \\ L & I \end{bmatrix}, \quad \mathbf{T}_{\mathrm{F}}^{-1} = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon L H \end{bmatrix} \quad (5.93)$$

where L and H matrices satisfy the Chang decoupling equations

$$\begin{gathered} (A_4 - K_2 C_2) L - (A_3 - K_2 C_1) \\ -\epsilon[(A_1 - K_1 C_1) - (A_2 - K_1 C_2) L] = 0 \\ -H(A_4 - K_2 C_2) + (A_2 - K_1 C_2) \\ -\epsilon H L (A_2 - K_1 C_2) + \epsilon[(A_1 - K_1 C_1) - (A_2 - K_1 C_2) L] H = 0 \end{gathered} \quad (5.94)$$

The unique solutions of these equations exist under the assumption that the matrix $A_4 - K_2 C_2$ is nonsingular. Note that based on the theory of singular perturbations (Kokotovic *et al.*, 1986) the matrix $A_4 - P_3 C_2^T V^{-1} C_2 - \frac{1}{\gamma^2} P_3 G_2^T R G_2$ is nonsingular since it represents the fast feedback matrix. By the result from (Zhou *et al.*, 1996), the stability of the matrix $A_4 - P_3 C_2^T V^{-1} C_2 - \frac{1}{\gamma^2} P_3 G_2^T R G_2$ implies that the matrix $A_4 - P_3 C_2^T V^{-1} C_2$ is nonsingular also. Using (5.60), we observe that $A_4 - K_2 C_2 + O(\epsilon)$ is a stable matrix. Thus, the matrix $A_4 - K_2 C_2$ is stable for sufficiently small values of the small singular perturbation parameter ϵ. The unique solutions of equations (5.94) can be easily obtained either by using the Newton method or the fixed point iterations starting with the following initial conditions

$$
\begin{aligned}
L^{(0)} &= (A_4 - K_2C_2)^{-1}(A_3 - K_2C_1) \\
M^{(0)} &= (A_2 - K_1C_2)(A_4 - K_2C_2)^{-1}
\end{aligned}
\tag{5.95}
$$

Hence, the optimal H_∞ filter obtained by applying the following similarity transformation

$$
\begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} = \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}
\tag{5.96}
$$

produces in the new coordinates the optimal pure-slow and optimal pure-fast, reduced-order, H_∞ filters, that is

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{\zeta}}_s(t) \\ \dot{\hat{\zeta}}_f(t) \end{bmatrix} &= \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} A_1 - K_1C_1 & A_2 - K_1C_2 \\ \frac{1}{\epsilon}(A_3 - K_2C_1) & \frac{1}{\epsilon}(A_4 - K_2C_2) \end{bmatrix} \mathbf{T}_{\mathrm{F}} \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} \\
&+ \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} y(t) = \begin{bmatrix} a_s & 0 \\ 0 & \frac{1}{\epsilon}a_f \end{bmatrix} \begin{bmatrix} \hat{\zeta}_s(t) \\ \hat{\zeta}_f(t) \end{bmatrix} + \begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} y(t)
\end{aligned}
\tag{5.97}
$$

where the pure-slow and pure-fast H_∞ filter gains are given by

$$
\begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} = \mathbf{T}_{\mathrm{F}}^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix}
\tag{5.98}
$$

Using the expression for the similarity transformation defined in (5.93) we can obtain analytical expressions for a_s, a_f, K_s, K_f as follows

$$
\begin{aligned}
a_s &= (A_1 - K_1C_1) - (A_2 - K_1C_2)L \\
a_f &= (A_4 - K_2C_2) + \epsilon L(A_2 - K_1C_2) \\
K_s &= K_1 - HK_2 - \epsilon HLK_1 \\
K_f &= K_2 + \epsilon LK_1
\end{aligned}
\tag{5.99}
$$

The reduced-order, independent, pure-slow and pure-fast, H_∞ filters defined in (5.97) represent the main result of this section. Due to complete independence of the slow and fast H_∞ filters, the slow and fast signals can now be processed with different sampling rates. In contrast, the original, full-order, H_∞ filter (5.67) requires the fast sampling rate for processing of both the slow and fast signals.

5.5.3 Case Study: $\mathbf{H_\infty}$ Filter for an F-8 Aircraft

In order to demonstrate efficiency of the proposed method, a linearized model of an F-8 aircraft system from (Shen and Deng, 1996) is considered. The system matrices are given by

$$A_1 = \begin{bmatrix} 0.278386 & -0.965256 \\ 0.089833 & -0.290700 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.074210 & 0.016017 \\ 0.012815 & -0.001398 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.001815 & 0.005873 \\ 0.002850 & -0.009223 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -0.030344 & 0.075024 \\ -0.075092 & -0.016777 \end{bmatrix}$$

$$G_1^T G_1 = \begin{bmatrix} 0.010000 & -0.032360 \\ -0.032360 & 0.104717 \end{bmatrix}$$

$$G_1^T G_2 = \begin{bmatrix} -0.000032 & -0.000130 \\ 0.000102 & 0.000421 \end{bmatrix}$$

$$G_2^T G_2 = \begin{bmatrix} 0.009056 & 0.000000 \\ 0.000000 & 0.081502 \end{bmatrix}$$

$$W = 0.000315, \quad V = \mathrm{diag}\{0.000686, 40\}$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 1 & -3.236 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.005000 \\ -0.003152 & 0.013020 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -46.626960 \\ 7.858776 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -18.210002 \\ -45.049998 \end{bmatrix}, \quad R = I_2$$

The singular perturbation parameter ϵ is equal to $\epsilon = 0.025$, and the parameter γ is chosen as $\gamma = 2$. The results obtained by using MATLAB, are presented below.

The solutions of the pure-slow and pure-fast algebraic H_∞ filter Riccati equations (5.81) are given by

$$P_s = \begin{bmatrix} 1.4987 & 0.4311 \\ 0.4332 & 0.1339 \end{bmatrix}, \quad P_f = \begin{bmatrix} 5.8710 & 1.3760 \\ 1.3506 & 4.2770 \end{bmatrix}$$

These solutions used in (5.80) produce the solution for the global algebraic H_∞ filter Riccati equation as

$$P = \begin{bmatrix} 1.6255 & 0.4151 & 5.0396 & 3.6326 \\ 0.4151 & 0.1379 & -0.1977 & -0.8348 \\ 5.0396 & -0.1977 & 234.2476 & 54.3790 \\ 3.6326 & -0.8348 & 54.3790 & 171.4752 \end{bmatrix}$$

The completely decoupled, pure-slow and pure-fast, reduced-order H_∞ filters in the new coordinates, driven by the system measurements are obtained from (5.104) as

$$\dot{\hat{\zeta}}_s(t) = \begin{bmatrix} 0.2754 & -0.9556 \\ 0.0904 & -0.2926 \end{bmatrix} \hat{\zeta}_s(t) + \begin{bmatrix} -1.1483 & 0.0021 \\ -0.2138 & -0.0000 \end{bmatrix} y(t)$$

$$\epsilon \dot{\hat{\zeta}}_f(t) = \begin{bmatrix} -0.0307 & 0.0533 \\ -0.0749 & -0.1728 \end{bmatrix} \hat{\zeta}_f(t) + \begin{bmatrix} 10.0682 & 0.0036 \\ 31.1923 & 0.0052 \end{bmatrix} y(t)$$

The estimates of the original state variables can be recovered by using the transformation defined in (5.103).

5.6 Conclusions

In this chapter we have presented an algorithm for solving the H_∞ algebraic Riccati equation and establish conditions (assumptions) under which the algorithm is applicable. The algorithm is based on the exact slow-fast decomposition of the corresponding linear-quadratic optimal control problem. The results obtained, with certain modifications, are extended to the dual problem of H_∞ filtering. The completely independent, reduced-order, pure-slow and pure-fast, H_∞ filters driven by the system measurements are obtained. The proposed method allows independent and parallel processing of information in slow and fast time scales. Presentation of this chapter mostly follows the works of (Hsieh and Gajic, 1998) and (Lim and Gajic, 2000). Other variants of H_∞ linear-quadratic continuous-time optimization of singularly perturbed systems can be formulated and solved by using results reported in the previous sections of this chapter.

Extensions of the presented results to discrete-time H_∞ filtering (Grimble and Sayed, 1990; Shaked and Theodor, 1992; Shen and Deng, 1997) and control represent interesting research problems.

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6

High Gain, Cheap Control, and Small Noise Problems

In this chapter we present a unified methodology to study three classes of independent control theory and its application problems: high gain optimal linear-quadratic control problem, optimal linear-quadratic cheap control problem, and small measurement noise problem of linear stochastic systems (small measurement noise Kalman filter). All three problems influence system separation into slow and fast time scales. The theory of singular perturbations seems to be a natural tool for studying the above problems. The studies are performed in the continuous-time domain. In the last section of this chapter we study a special class of discrete-time systems in the framework of an optimal cheap control problem.

In Section 6.1 we present essentials of optimal control formulations for high gain and cheap control problems. Namely, we show that the corresponding optimal control problems are equivalent since they lead to the same algebraic Riccati equation. Section 6.2 studies the open-loop optimal cheap control and high gain problems and presents the corresponding pure-slow and pure-fast decompositions of the optimal state and costate equations. The problem of complete pure-slow/pure-fast decomposition of the cheap control/high gain algebraic Riccati equation into the reduced-order pure-slow and pure-fast Riccati equations is considered in Section 6.3. In Section 6.4, we generalize the results obtained in Sections 6.2 and 6.3 to pure-slow and pure-fast decomposition for the small measurement noise Kalman filtering problem, and show how to avoid ill-conditioning and solve this problem in terms of pure-slow and pure-fast reduced-order Kalman filters.

Section 6.5 studies the discrete-time cheap control problem for sampled data systems using theory of singular perturbations. In that section we show that this problem can be solved in terms of two reduced order subproblems for which computations can be done in parallel, thus increasing computational speed. Similarly to the continuous time case, the singular perturbation approach enables the decomposition of the corresponding algebraic Riccati equation into two reduced order pure-slow and pure-fast continuous-time algebraic Riccati equations.

6.1 Linear-Quadratic Optimal Continuous-Time High Gain and Cheap Control Problems

A primary goal in studying optimal control of high gain and cheap control problems, like with any optimal control problem, is to determine the control which minimizes the value of a certain quadratic performance criterion. In general, high gain linear systems are those in which the norm of the feedback control matrix has high magnitude, usually one or more orders of magnitude greater than that of the norm of the system matrix. A cheap control problem is characterized by a small penalty factor applied to the control term in the quadratic performance criterion, usually one or more orders of magnitude smaller that the state penalty term. The difference in magnitudes can be quantitatively described by a small positive parameter ϵ.

High gain and cheap control problems have been studied extensively by a number of researchers (Jameson and O'Malley, 1975; O'Malley and Jameson, 1975, 1977; O'Malley, 1976; Young *et al.*, 1977; Francis and Glover, 1978; Francis, 1979; Sannuti, 1983; Sannuti and Wason, 1985; O'Reilly, 1983; Priel and Shaked, 1983; Saberi and Sannuti, 1986, 1987; Kokotovic *et al.*, 1986; Peterson, 1986; Murata *et al.*, 1990). The modern approach to the analysis of high gain and cheap control problems involves the use of singular perturbation method. The singular perturbation technique offers an intuitive understanding into the behavior of high gain and cheap control problems.

In general, the application of the singular perturbation method involves a suitable representation and partitioning of the problem matrices, and explicitly introduces a small positive parameter ϵ. The role of the parameter ϵ varies with the type of system under investigation; however, once introduced, its exact nature is unimportant. The solution is found for $\epsilon = 0$, and a Taylor series expansion is then taken about this zero-order solution to find a higher order solution to any prescribed degree of accuracy. Several problems arise as a result of the application of this technique. The problem matrices must be analytical functions of ϵ, and for certain systems this may not be so. A solution of higher order requires an enormous number of terms. Furthermore, when the value of ϵ is not small enough, the obtained solution may fail to yield an accurate solution.

The structures of the problem matrices and the methodology used in this chapter do not allow either impulsive behavior or singular controls in the problems under considerations. These two undesired limiting phenomena appear very often in the cases when the classical singular perturbation approach is used to study the cheap control and high gain feedback problems. As a matter of fact, we study these two problems for ϵ small and positive — which is physical reality, but not for $\epsilon \to 0$—which is mathematical artifice that produces impulsive behavior and singular controls.

We study the optimal control problem of high gain and cheap control problems at steady state. These results can be extended to the finite continuous-time optimization problems by using results on the slow-fast time scale decomposition of the differential Riccati equation from (Grodt and Gajic, 1988). An extension of the presented results to the discrete-time cheap control problems (Priel and Shaked, 1983; Sen and Datta, 1992) might be an interesting area for future research.

6.1.1 High Gain Optimal Feedback Control

Consider a system given by

$$\dot{x}(t) = Ax(t) + \widehat{B}u(t) \tag{6.1}$$

which can be partitioned into subsystems as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\epsilon} B_2 \end{bmatrix} u(t) \tag{6.2}$$

where $x_1(t) \in \Re^n$, $x_2(t) \in \Re^m$ are state variables, $u(t) \in \Re^m$ is the control input, and ϵ is a small positive parameter. $A_{ij}, B_i, i, j = 1, 2$, are constant matrices of appropriate dimensions. No loss of generality is incurred, since the system model can always be transformed into (6.2) provided that $B_2 \in \Re^{m \times m}$ is of the full rank m (Jameson and O'Malley, 1975). Hence, the problem is studied under the following standard assumption (Kokotovic *et al.*, 1986).

Assumption 6.1: The square matrix B_{22} has the full rank, that is $det\ B_2 \neq 0$.

The scalar cost functional associated with (6.2), defined by

$$J(\epsilon) = \frac{1}{2} \int_0^\infty \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt$$
$$Q = Q^T \geq 0, \quad R = R^T > 0 \tag{6.3}$$

is minimized along the trajectories of (6.1)-(6.2) by the well-known optimal control law

$$u(t) = -R^{-1} \widehat{B}^T K x(t) = -\frac{1}{\epsilon} R^{-1} B^T K x(t), \qquad B = \epsilon \widehat{B} \tag{6.4}$$

where K is the positive semidefinite symmetric solution of the quadratic matrix algebraic Riccati equation

$$KA + A^T K + Q = \frac{1}{\epsilon^2} K B R^{-1} B^T K \tag{6.5}$$

Matrices K and Q are partitioned as

$$K = \begin{bmatrix} K_{11} & \epsilon K_{12} \\ \epsilon K_{12}^T & \epsilon K_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{6.6}$$

Due to the presence of $O\left(\frac{1}{\epsilon}\right)$ term in (6.2), which multiplies the control input $u(t)$, this problem is known in the literature as the high gain feedback control problem.

Another assumption that is commonly used in theory of high gain optimal feedback control (Kokotovic *et al.*, 1986) will be needed for Section 6.2 of this chapter. That assumption is stated as follows.

Assumption 6.2: The penalty matrix Q_{22} is positive definite.

6.1.2 Optimal Cheap Control Problem

Consider a linear dynamic system model represented by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t) \tag{6.7}$$

where $x_1(t) \in \Re^n$, $x_2(t) \in \Re^m$, $u(t) \in \Re^m$. $A_{ij}, B_i, i, j = 1, 2$, are constant matrices of appropriate dimensions. In addition, it is assumed that B_2 is a nonsingular $m \times m$ constant matrix (B_2 satisfies Assumption 6.1).

The scalar cost functional associated with (6.7), which defines the optimal cheap control problem, is given by

$$J(\epsilon) = \frac{1}{2} \int_0^\infty \left[x^T(t) Q x(t) + \epsilon^2 u^T(t) R u(t) \right] dt \tag{6.8}$$

This functional has to be minimized by selecting the m-dimensional control vector u. Q and R are symmetric positive semidefinite and positive definite matrices, respectively, and ϵ is a small positive parameter. In addition, it is assumed that the matrix Q can be partitioned consistently to (6.6) with Q_{22} satisfying Assumption 6.2.

The feedback control law for the optimal cheap control problem defined by (6.7)-(6.8) is given by

$$u(t) = -\frac{1}{\epsilon^2} R^{-1} B^T K x(t) \tag{6.9}$$

where K is the positive semidefinite symmetric solution of the quadratic matrix algebraic Riccati equation defined in (6.5) with

$$\begin{gathered} A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ S = B R^{-1} B^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_2 R^{-1} B_2^T \end{bmatrix} \end{gathered} \tag{6.10}$$

Although the initial problem statements may differ, we can notice that the forms of the Riccati equations are identical for both the high gain and cheap control problems, assuming that the problem matrices are defined consistently. The parallel algorithm for finding the solution of this Riccati equation in terms of the reduced-order problems is presented in Section 6.3.

6.2 Open-Loop Continuous-Time Cheap Control and High Gain Problems

In this section, the singular perturbation approach is used to obtain an alternate and more efficient method of solving the two-point boundary value problem for the optimal open-loop cheap control and high feedback gain) problems. The original two-point boundary value problem is transformed into completely decoupled initial value problems. The solution obtained in this manner clearly exhibits both the singular arc and the fast transients, separately.

Consider the cheap control problem defined by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t), \quad x_0 = x(t_0) \tag{6.11}$$

where $x_1(t) \in \Re^n$, $x_2(t) \in \Re^m$, $u(t) \in \Re^m$, and B_2 is a nonsingular $m \times m$ matrix. The performance criterion of the cheap control problem, over a finite time optimization interval, is defined as

$$J = \frac{1}{2} x^T(T) F x(T) + \frac{1}{2} \int_{t_0}^{T} \left[x^T(t) Q x(t) + \epsilon^2 u^T(t) R u(t) \right] dt \tag{6.12}$$

The performance criterion penalty matrices are chosen such that $R = R^T > 0$, $Q = Q^T \geq 0$, and $F = F^T \geq 0$. In addition, the matrix B_2 satisfies Assumption 6.1, and the matrix Q is partitioned consistently to (6.6) satisfying Assumption 6.2. It is obvious that for $\epsilon = 0$, the optimal control problem (6.11)-(6.12) is singular.

The open-loop optimal cheap control problem has the solution given by

$$u(t) = -\frac{1}{\epsilon^2} R^{-1} B^T p(t), \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{6.13}$$

where $p(t) \in \Re^{n+m}$ is a costate variable satisfying (Jameson and O'Malley, 1975)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{6.14}$$

The boundary conditions are expressed in the standard form as

$$M\begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} + N\begin{bmatrix} x(T) \\ p(T) \end{bmatrix} = c \tag{6.15}$$

where

$$M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ -F & I \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix} \tag{6.16}$$

for the free endpoint problem; and

$$M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad c = \begin{bmatrix} x(t_0) \\ x(T) \end{bmatrix} \tag{6.17}$$

for the fixed endpoint problem.

Matrices A, Q, B, S, and F, and vectors $x(t)$ and $p(t)$, respectively, have the partitions

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$S = \frac{1}{\epsilon^2} BR^{-1}B^T = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\epsilon^2} B_{22}R^{-1}B_{22}^T \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & \epsilon F_3 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad p(t) = \begin{bmatrix} p_1(t) \\ \epsilon p_2(t) \end{bmatrix}$$

The main purpose of this section is to obtain the solution of the open-loop cheap control problem by singular perturbation approach, so that the solution can clearly exhibit both the singular arc and the fast transients away from it.

If we partition $p(t) = \left[p_1^T(t)\; \epsilon p_2^T(t)\right]^T$ with $p_1(t) \in \Re^n$ and $p_2(t) \in \Re^m$, and interchange the second and third rows in (6.14) we will get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \epsilon\dot{x}_2(t) \\ \epsilon\dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{6.18}$$

where

$$
T_1 = \begin{bmatrix} A_{11} & 0 \\ -Q_{11} & -A_{11}^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_{12} & 0 \\ -Q_{12} & -\epsilon A_{21}^T \end{bmatrix}
$$

$$
T_3 = \begin{bmatrix} \epsilon A_{21} & 0 \\ -Q_{12}^T & -A_{12}^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} \epsilon A_{22} & -B_2 R^{-1} B_2^T \\ -Q_{22} & -\epsilon A_{22}^T \end{bmatrix} \tag{6.19}
$$

The boundary conditions are changed by interchanging $p_1(t)$ and $x_2(t)$. The modified matrices in (6.15) are

$$
M_1 \begin{bmatrix} x_1(t_0) \\ p_1(t_0) \\ x_2(t_0) \\ p_2(t_0) \end{bmatrix} + N_1 \begin{bmatrix} x_1(T) \\ p_1(T) \\ x_2(T) \\ p_2(T) \end{bmatrix} = c_1 \tag{6.20}
$$

where

$$
M_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -F_{11} & I_n & -\epsilon F_{12} & 0 \\ 0 & 0 & 0 & 0 \\ -F_{12}^T & 0 & -F_{22} & I_m \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_1(t_0) \\ 0 \\ x_2(t_0) \\ 0 \end{bmatrix} \tag{6.21}
$$

for the free ending problem; and

$$
M_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} x_1(t_0) \\ x_1(T) \\ x_2(t_0) \\ x_2(T) \end{bmatrix} \tag{6.22}
$$

for the fixed ending problem.

Note that equation (6.18) has the singular perturbation form and the matrix T_4 is the Hamiltonian matrix of the fast subsystem. It can be observed that under Assumptions 6.1 and 6.2, the matrix T_4 is nonsingular for sufficiently small values of parameter ϵ since its $O(\epsilon)$ perturbation is nonsingular.

It can be deduced from analogies outlined in Sections 6.1.1 and 6.1.2 between the cheap control and high gain problems that the high gain optimal open-loop control problem satisfies the same set of state-costate equations as the one obtained in (6.18)-(6.19) for the cheap control optimal open-loop problem.

In the sequel, we use the following transformation (Chang, 1972) defined by

$$\mathbf{T}_1 = \begin{bmatrix} I_{2n} - \epsilon HL & -\epsilon H \\ L & I_{2m} \end{bmatrix}, \quad \mathbf{T}_1^{-1} = \begin{bmatrix} I_{2n} & \epsilon H \\ -L & I_{2m} - \epsilon LH \end{bmatrix} \tag{6.23}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{6.24}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \tag{6.25}$$

Equations (6.21) and (6.22) have unique solutions under condition that T_4 is nonsingular at $\epsilon = 0$, which is true under Assumptions 6.1 and 6.2. These equations can be solved by using any of the algorithms presented in Chapter 2. The transformation (6.20) is then applied to (6.18) to produce two completely decoupled subsystems

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\xi}_1(t) \end{bmatrix} = (T_1 - T_2 L) \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \tag{6.26}$$

$$\epsilon \begin{bmatrix} \dot{\eta}_2(t) \\ \dot{\xi}_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} \tag{6.27}$$

where

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \\ \eta_2(t) \\ \xi_2(t) \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} \tag{6.28}$$

The nonsingular transformation (6.23) applied to (6.20) produces

$$M_2 \begin{bmatrix} \eta_1(t_0) \\ \xi_1(t_0) \\ \eta_2(t_0) \\ \xi_2(t_0) \end{bmatrix} + N_2 \begin{bmatrix} \eta_1(T) \\ \xi_1(T) \\ \eta_2(T) \\ \xi_2(T) \end{bmatrix} = c_1 \tag{6.29}$$

where

$$M_2 = M_1 \mathbf{T}_1^{-1}, \quad N_2 = N_1 \mathbf{T}_1^{-1} \tag{6.30}$$

Since solutions of (6.26) and (6.27) are given by

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} = e^{(T_1 - T_2 L)(t - t_0)} \begin{bmatrix} \eta_1(t_0) \\ \xi_1(t_0) \end{bmatrix} \tag{6.31}$$

$$
\begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = e^{\frac{1}{\epsilon}(T_4+\epsilon LT_2)(t-t_0)} \begin{bmatrix} \eta_2(t_0) \\ \xi_2(t_0) \end{bmatrix} \tag{6.32}
$$

we can eliminate $\eta_1(T)$, $\xi_1(T)$, $\eta_2(T)$, and $\xi_2(T)$ from (6.29), which yields

$$
\left\{ M_2 + N_2 \begin{bmatrix} e^{(T_1-T_2L)(T-t_0)} & 0 \\ 0 & e^{\frac{1}{\epsilon}(T_4+\epsilon LT_2)(T-t_0)} \end{bmatrix} \right\} \begin{bmatrix} \eta_1(t_0) \\ \xi_1(t_0) \\ \eta_2(t_0) \\ \xi_2(t_0) \end{bmatrix} = c_1 \tag{6.33}
$$

Equation (6.33) can be represented in the form

$$
\beta(\epsilon) \begin{bmatrix} \eta_1(t_0) \\ \xi_1(t_0) \\ \eta_2(t_0) \\ \xi_2(t_0) \end{bmatrix} = c_1 \tag{6.34}
$$

Using the same set of arguments as in Lemma 2.2, a dual lemma that guarantees invertibility of the matrix $\beta(\epsilon)$ can be established. Hence, equation (6.34) can be solved to obtain $\eta_1(t_0)$, $\xi_1(t_0)$, $\eta_2(t_0)$, and $\xi_2(t_0)$.

Formula (6.31) gives the solution of singular arc $\eta_1(t)$, and formula (6.32) gives the solution of fast transient $\eta_2(t)$ of the cheap control problem.

Having obtained the solutions of (6.31) and (6.32), using (6.22) we obtain values for $x_1(t)$, $x_2(t)$, $p_1(t)$, and $p_2(t)$. The costate variable $p(t)$ and the optimal control law $u(t)$ are therefore found.

6.3 Slow-Fast Decoupling of the Cheap Control/High Gain Continuous-Time Algebraic Riccati Equation

In this section, we study the linear-quadratic regulator problem of cheap control problem by using the approach presented in Chapter 2. The ill-defined algebraic Riccati equation of cheap control (and high gain) problem is completely and exactly decomposed into two reduced-order nonsymmetric well-defined algebraic Riccati equations, and the corresponding solution of the Riccati equation is obtained in terms of the reduced-order problems.

In equation (6.5), we have obtained the form of the algebraic Riccati equation for the cheap control and high gain feedback problems

$$
PA + A^T P + Q = \frac{1}{\epsilon^2} PBR^{-1}B^T P
$$

The relationship between the state and costate variables at steady state is given by $p(t) = Px(t)$, where P is a constant matrix. Partitioning $p(t)$ as $p(t) = [p_1^T(t) \quad \epsilon p_2^T(t)]^T$ with $p_1(t) \in \Re^n$, $p_2(t) \in \Re^m$ and interchanging second and third rows in (6.14), we can get (6.18), that is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \\ \epsilon\dot{x}_2(t) \\ \epsilon\dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix}$$

where $T_i's$ are defined in (6.19).

The transformation (6.23) applied to (6.18) produces two completely decoupled subsystems (6.26)-(6.27)

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\xi}_1(t) \end{bmatrix} = (T_1 - T_2L) \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix}$$

$$\epsilon \begin{bmatrix} \dot{\eta}_2(t) \\ \dot{\xi}_2(t) \end{bmatrix} = (T_4 + \epsilon LT_2) \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix}$$

with the corresponding transformation is defined by (6.23) and

$$\begin{bmatrix} \eta_1(t) \\ \xi_1(t) \\ \eta_2(t) \\ \xi_2(t) \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix}$$

In order to find the optimal solution of the cheap control problem in terms of the reduced-order subsystems, we have to find the relations between the full-order Riccati equation (6.5) and the decomposed reduced-order Riccati equations corresponding to subsystems (6.26) and (6.27).

The rearrangement and modification of variables in (6.18) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(t) \\ p_1(t) \\ x_2(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ \epsilon p_2(t) \end{bmatrix} = E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{6.35}$$

Combining (6.28) and (6.35), we obtain the relationship between the original coordinates and the new ones

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \xi_1(t) \\ \xi_2(t) \end{bmatrix} = E_2^T \mathbf{T}_1 E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{6.36}$$

where E_2 is a permutation matrix in the form

$$E_2 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \tag{6.37}$$

Since at steady state $p(t) = Px(t)$, where P satisfies the algebraic Riccati equation (6.5), it follows that

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t), \quad \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = (\Pi_3 + \Pi_4 P)x(t) \tag{6.38}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for the new systems (6.26) and (6.27); that is

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{6.39}$$

Then, (6.23) and (6.24) yield

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{6.40}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1}\mathbf{T}_1^{-1}E_2 = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} \tag{6.41}$$

with

$$E_1^{-1} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_m \end{bmatrix} \tag{6.42}$$

which yields to

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)^{-1} \tag{6.43}$$

It can be shown using similar proofs as in Appendix 2.1 that the required matrices in (6.40) and (6.43) are invertible for sufficiently small values of ϵ.

Partitioning (6.26) and (6.27) as

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\xi}_1(t) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} = (T_1 - T_2 L) \begin{bmatrix} \eta_1(t) \\ \xi_1(t) \end{bmatrix} \tag{6.44}$$

$$\epsilon \begin{bmatrix} \dot{\eta}_2(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \eta_2(t) \\ \xi_2(t) \end{bmatrix} \tag{6.45}$$

and using (6.39) yield to two reduced-order nonsymmetric algebraic Riccati equations

$$0 = P_1 a_1 - a_4 P_1 - a_3 + P_1 a_2 P_1 \tag{6.46}$$

$$0 = P_2 b_1 - b_4 P_2 - b_3 + P_2 b_2 P_2 \tag{6.47}$$

where

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12} L_1 & -A_{12} L_2 \\ -Q_{11} + Q_{12} L_1 + \epsilon A_{21}^T L_3 & -A_{11}^T + Q_{12} L_2 + \epsilon A_{21}^T L_4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} \epsilon A_{22} + \epsilon (L_1 A_{12} - L_2 Q_{12}) & -B_2 R^{-1} B_2^T - \epsilon^2 L_2 A_{21}^T \\ -Q_{22} + \epsilon (L_3 A_{12} - L_4 Q_{12}) & -\epsilon A_{22}^T - \epsilon^2 L_4 A_{21}^T \end{bmatrix} \tag{6.48}$$

with

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \tag{6.49}$$

The reduced-order algebraic Riccati equation (6.46) is nonsymmetric and it is given by

$$\begin{aligned} & P_1 (A_{11} - A_{12} L_1) + \left(A_{11}^T - Q_{12} L_2 - \epsilon A_{21}^T L_4\right)^T P_1 \\ & \quad + \left(Q_{11} - Q_{12} L_1 - \epsilon A_{21}^T L_3\right) - P_1 A_{12} L_2 P_1 = 0 \end{aligned} \tag{6.50}$$

The reduced-order algebraic Riccati equation (6.47) is also nonsymmetric

$$\begin{aligned} & \epsilon P_2 (A_{22} + L_1 A_{12} - L_2 Q_{12}) + \epsilon \left(A_{22}^T + \epsilon L_4 A_{21}^T\right) P_2 \\ & + [Q_{22} - \epsilon (L_3 A_{12} - L_4 Q_{12})] - P_2 \left(B_2 R^{-1} B_2^T + \epsilon^2 L_2 A_{21}^T\right) P_2 = 0 \end{aligned} \tag{6.51}$$

but its $O(\epsilon)$ approximation is a symmetric one, that is

$$\bar{P}_2 B_2 R^{-1} B_2^T \bar{P}_2 - Q_{22} = 0 \tag{6.52}$$

In the literature on optimal linear-quadratic cheap control problems for singularly perturbed systems (Kokotovic *et al.*, 1986) Assumptions 6.1 and 6.2 are used to guarantee the existence of the positive definite solution for $\bar{P}_2$. Note that Assumption 6.1 is stronger than the controllability assumption imposed on the fast subsystem pair (A_{22}, B_2). On the other hand, Assumption 6.2 is stronger than the observability assumption imposed on the pair $(A_{22}, \mathrm{Chol}(Q_{22}))$.

Under Assumptions 6.1 and 6.2, the unique positive definite solution of equation (6.52) is given by

$$\bar{P}_2 = S_{22}^{-\frac{1}{2}} \left(S_{22}^{\frac{1}{2}} Q_{22} S_{22}^{\frac{1}{2}} \right)^{\frac{1}{2}} S_{22}^{-\frac{1}{2}} \tag{6.53}$$

where

$$S_{22} = B_2 R^{-1} B_2^T$$

It can be shown that an $O(\epsilon)$ of (6.50) can be obtained by solving the following algebraic Riccati equation

$$\begin{aligned} &\bar{P}_1 \left(A_{11} - A_{12} Q_{22}^{-1} Q_{12}^T \right) + \left(A_{11} - A_{12} Q_{22}^{-1} Q_{12}^T \right)^T \bar{P}_1 \\ &- \bar{P}_1 A_{12} Q_{22}^{-1} A_{12}^T \bar{P}_1 + \left(Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T \right) = 0 \end{aligned} \tag{6.54}$$

which is the slow approximate algebraic Riccati equation of the cheap control problem (Kokotovic *et al.*, 1986). To show this result, we have to set $\epsilon = 0$ in (6.19) and (6.24) and find $L(0)$, which leads to

$$\begin{aligned} L(0) = T_4^{-1}(0) T_3(0) &= \begin{bmatrix} 0 & -B_2 R^{-1} B_2^T \\ -Q_{22} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -Q_{12}^T & -A_{12}^T \end{bmatrix} \\ &= \begin{bmatrix} 0 & -Q_{22}^{-1} \\ -\left(B_2 R^{-1} B_2^T \right)^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -Q_{12}^T & -A_{12}^T \end{bmatrix} \\ &= \begin{bmatrix} Q_{22}^{-1} Q_{12}^T & Q_{22}^{-1} A_{12}^T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} L_1(0) & L_2(0) \\ L_3(0) & L_4(0) \end{bmatrix} \end{aligned}$$

Using the values for $L_i(0), i = 1, ..., 4$, is (6.50), we obtain the result stated in (6.54).

The unique positive semidefinite stabilizing solution of the algebraic Riccati equation (6.54) exists under the assumption that the triple (A_0, B_0, Q_0) is stabilizable detectable, where

$$A_0 = A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T, \quad B_0B_0^T = A_{12}Q_{22}^{-1}A_{21}^T$$
$$Q_0Q_0^T = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T$$

Using the results of (Wonham, 1968) the above assumption is equivalent to the following assumption.

Assumption 6.3: The pair (A_{11}, A_{12}) is stabilizable and the pair (Q_0, A_{11}) is detectable.

Even more, it has been shown in (Kokotovic *et al.*, 1986) that Assumption 6.3 is implied by the following assumption.

Assumption 6.3a: The pair $(A,\ B)$ is stabilizable and the pair $(Q_0,\ A_{11})$ is detectable.

Using the facts that the unique equations (6.53) and (6.54) exist, and that these equations are obtained by perturbing the original equations (6.50)-(6.51) by an $O(\epsilon)$, the existence of the unique solutions of (6.50) and (6.51) is guaranteed by the following lemma.

Lemma 6.1 *Under Assumptions 6.1, 6.2, and 6.3*, it exists $\epsilon_0 > 0$ *such that* $\forall \epsilon \leq \epsilon_0$ *unique solutions of (6.50) and (6.51) exist.*

◇

Proof: It follows by the direct application of the implicit function theorem (Ortega and Rheinboldt, 1970) and by the facts that the corresponding Jacobians of (6.50)-(6.51) are nonsingular at $\epsilon = 0$.

●

Solutions of equations (6.52) and (6.54) represent very good choices of the initial conditions for the Newton method to be used for solving the original equations (6.50) and (6.51).

It can be shown, as in Chapter 2, that the Newton algorithm in this case is given by

$$P_1^{(i+1)}\left(a_1 + a_2P_1^{(i)}\right) - \left(a_4 - P_1^{(i)}a_2\right)P_1^{(i+1)} = a_3 + P_1^{(i)}a_2P_1^{(i)}$$
$$P_1^{(0)} = \bar{P}_1, \quad i = 0, 1, 2, ... \tag{6.55a}$$

$$P_2^{(i+1)}\left(b_1 + b_2 P_2^{(i)}\right) - \left(b_4 - P_2^{(i)} b_2\right) P_2^{(i+1)} = b_3 + P_2^{(i)} b_2 P_2^{(i)}$$
$$P_2^{(0)} = \bar{P}_2, \quad i = 0, 1, 2, ... \tag{6.55b}$$

The fast subsystem algebraic Riccati equation can also be solved by using the fixed point iterations as

$$\begin{gathered} P_2^{(i+1)} = S_{22}^{-\frac{1}{2}} \left(S_{22}^{\frac{1}{2}} Z^{(i)} S_{22}^{\frac{1}{2}}\right)^{\frac{1}{2}} S_{22}^{-\frac{1}{2}} \\ P_2^{(0)} = \bar{P}_2 \end{gathered} \tag{6.56}$$

where

$$Z^{(i)} = P_2^{(i)} b_1 - b_4 P_2^{(i)} - b_3^{(i)} - \epsilon^2 P_2^{(i)} L_2 A_{21}^T P_2^{(i)}$$

The algorithm (6.56) converges to the desired solution with the rate of convergence of $O(\epsilon)$. Hence, it is efficient for very small values of ϵ.

Note that equations (6.46)-(6.47) are nonsymmetric algebraic Riccati equations, so that we need to solve twice n^2 equations in (6.55) in order to get the solutions for P_1 and P_2. Iterating (6.56) we can also get the solution for P_2, which is efficient when ϵ is small. The global solution P is obtained from (6.43) using information about P_1 and P_2.

Using solutions of both Riccati equations (6.50) and (6.51), and formulas (6.39), (6.44), and (6.45), we can get completely decoupled slow and fast subsystems in the new coordinates as

$$\dot{\eta}_1(t) = (a_1 + a_2 P_1)\eta_1(t) \tag{6.57}$$

$$\epsilon\dot{\eta}_2(t) = (b_1 + b_2 P_2)\eta_2(t) \tag{6.58}$$

The interpretation of the result presented by (6.57) and (6.58) is that the optimal processing (control and/or filtering) might be completely performed at the subsystem levels. In addition, considerable reduction in computational requirements is achieved, since we only need to solve the reduced-order equations independently.

6.4 Small Measurement Noise Continuous-Time Kalman Filter

In this section we present a method which produces complete decomposition of the optimal global Kalman filter for linear stochastic systems with small measurement noise into exact pure-slow and pure-fast local reduced-order optimal filters both driven by the system measurements. The method is based on the exact decomposition of the global small measurement noise algebraic Riccati equation into exact pure-slow and pure-fast local algebraic Riccati equations. Several examples are included in order to demonstrate the proposed method.

Several authors have studied the limiting properties of the optimal Kalman filter (Friedland, 1971; Doyle, 1978; Halevi, 1986; Braslavsky *et al.*, 1999). In several papers the filtering problem with perfect system measurements is considered (Moylan, 1974; Haas, 1984; Shaked, 1986; Soroka and Shaked, 1988). Singular measurement noise (when the noise intensity matrix is singular) is studied in (Bernstein and Hyland, 1985; Haddad and Bernstein, 1987; Halevi, 1989). The filtering of linear stochastic systems with small measurement noise, which is an important problem for several engineering areas such as signal processing, communications, and control theory, has been studied, to the best of our knowledge, only in (Sachs, 1980, 1981; Brigo, 1995, 1996) and very recently in (Braslavsky *et al.*, 1999).

Our approach to the filtering problem with small measurement noise employs the singular perturbation technique (Kokotovic *et al.*, 1986; Gajic and Shen, 1993; Gajic and Lim, 1994). The use of singular perturbation method for this problem is also demonstrated in (Sachs, 1980; 1981). We have obtained the exact expressions for the optimal local filters which are of reduced-order and driven by the system measurements. In addition, the optimal filter gains are completely determined in terms of the exact pure-slow and pure-fast reduced-order algebraic Riccati equations. Thus, we get the complete reduction in both off-line and on-line computations, so that the optimal filtering can be completely done at the local levels.

Consider the linear continuous-time invariant stochastic system

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) + A_2x_2(t) + G_1w(t) \\ \dot{x}_2(t) &= A_3x_1(t) + A_4x_2(t) + G_2w(t) \end{aligned} \tag{6.59}$$

with the corresponding measurements corrupted by small noise (the noise smallness is represented by a small noise intensity matrix)

$$y(t) = C_2x_2(t) + v(t) \tag{6.60}$$

where $x_1(t) \in \Re^n$ and $x_2(t) \in \Re^m$ are state vectors, $w(t) \in \Re^r$ is a zero-mean, stationary, white Gaussian noise stochastic process with intensity $W > 0$, and $y(t) \in \Re^m$ are the system measurements. The measurements are corrupted by small zero-mean, stationary, Gaussian white noise $v \in \Re^m$ whose intensity matrix is assumed to be $\epsilon^2 V > 0$ with ϵ being a small positive parameter. No loss of generality is incurred in (6.60) provided that the measurement matrix has full rank m, (Jameson and O'Malley, 1975)—this fact is also demonstrated in Example 6.1 and Appendix 6.1. Thus, the problem is studied under the following assumption.

Assumption 6.4: The square matrix C_2 is nonsingular, that is $detC_2 \neq 0$.

In the following $A_i, G_j, C_2,\ i = 1,2,3,4,\ j = 1,2$ are constant matrices.

The optimal Kalman filter, corresponding to (6.59)-(6.60), driven by the innovation process is given by (Kwakernaak and Sivan, 1972)

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1\hat{x}_1(t) + A_2\hat{x}_2(t) + K_1\nu(t) \\ \dot{\hat{x}}_2(t) &= A_3\hat{x}_1(t) + A_4\hat{x}_2(t) + K_2\nu(t) \\ \nu(t) &= y(t) - C_2\hat{x}_2(t) \end{aligned} \tag{6.61}$$

with the optimal filter gain obtained from

$$K = PC^TV^{-1} = \begin{bmatrix} P_2C_2^TV^{-1} \\ \frac{1}{\epsilon}P_3C_2^TV^{-1} \end{bmatrix} = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 \\ C_2^T \end{bmatrix}^T \tag{6.62}$$

where P is the positive semidefinite stabilizing solution of the algebraic Riccati equation

$$AP + PA^T - \frac{1}{\epsilon^2}PSP + GWG^T = 0 \tag{6.63}$$

with

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},\ S = C^TV^{-1}C,\ P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & \frac{1}{\epsilon}P_3 \end{bmatrix} \tag{6.64}$$

It can be easily seen, by observing the form of the algebraic Riccati equation (6.63), that the above problem is dual to the corresponding cheap control problem (Jameson and O'Malley, 1975; Francis, 1979; Kokotovic *et al.*, 1986, Huey *et al.*, 1993) with $\epsilon^2 V$ playing the role of $\epsilon^2 R$, where

R is the input penalty matrix so that the filtering problem with small measurement noise can be studied as a singularly perturbed system.

The filtering problem of singularly perturbed systems has been studied in (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977; Khalil and Gajic, 1984; Gajic, 1986; Gajic and Lim, 1994). The results of (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977) produce only $O(\epsilon)$ accuracy, whereas, the results of (Khalil and Gajic, 1984; Gajic, 1986, Gajic and Lim, 1994) produce an arbitrary order of accuracy. In the small measurement noise problem we can not be pleased with $O(\epsilon)$ accuracy.

For the decomposition and approximation of the singularly perturbed Kalman filter (6.61) the Chang transformation (Chang, 1972) has been used in (Khalil and Gajic, 1984; Gajic, 1986)

$$\begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} I - \epsilon H L & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{6.65}$$

where L and H satisfy algebraic equations

$$\begin{aligned} A_4 L - A_3 - \epsilon L(A_1 - A_2 L) = 0 \\ -H A_4 + A_2 - \epsilon H L A_2 + \epsilon(A_1 - A_2 L) H = 0 \end{aligned} \tag{6.66}$$

The Chang transformation applied to (6.61) produces independent slow ($\hat{\eta}_1$) and fast ($\hat{\eta}_2$) filters driven by the innovation process

$$\begin{aligned} \dot{\hat{\eta}}_1(t) &= (A_1 - A_2 L)\hat{\eta}_1(t) + (K_1 - H K_2 - \epsilon H L K_1) v(t) \\ \epsilon \dot{\hat{\eta}}_2(t) &= (A_4 + \epsilon L A_2)\hat{\eta}_2(t) + (K_2 + \epsilon L K_1) v(t) \end{aligned} \tag{6.67}$$

In the new coordinates the innovation process is given by

$$v(t) = y(t) + C_2 L \hat{\eta}_1(t) - (C_2 + \epsilon C_2 L H)\hat{\eta}_2(t) \tag{6.68}$$

The existence for the positive semidefinite stabilizing solution of the filter small noise algebraic Riccati equation (6.63) is guaranteed by Assumption 6.4 (dual to Assumption 6.1) and assumptions dual to Assumptions 6.2 and 6.3 that can be formulated as follows.

Assumption 6.5: The matrix $G_2 W G_2^T$ has full rank.

Assumption 6.6: The slow filter (subsystem) is both stabilizable and detectable.

Note that Assumptions 6.4 and 6.5 combined are stronger than the controllability-observability condition of the fast filter (subsystem).

Equations (6.66) are solvable and produce the unique solutions under the following assumptions.

Assumption 6.7: The matrix A_4 is invertible.

6.4.1 Exact Local Filter Decomposition

In the filter decomposition procedure presented in (6.67) the slow and fast filters require additional communication channels necessary to form the innovation process (6.68). In addition, the filter gains K_1 and K_2 are given in terms of solution of the global algebraic Riccati equation (6.63). Here, we propose a decomposition scheme following the results of (Gajic and Lim, 1994), presented in Section 2.3, such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements and the corresponding filter coefficients are obtained from the local reduced-order exact slow and fast algebraic Riccati equations.

The method is based on the pure-slow pure-fast decomposition technique for solving the cheap control algebraic Riccati equation of singularly perturbed systems presented in Section 6.3, and the slow-fast decomposition technique of (Gajic and Lim, 1994) derived for general singularly perturbed systems. Here, we give first a brief summary and an additional interpretation of the results from Section 2.3, which are needed for the purpose of this section.

Consider the linear-quadratic optimal cheap control problem of (6.7)-(6.8), that is

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t)$$

$$\dot{x}_2(t) = A_3 x_1(t) + A_4 x_2(t) + B_2 u(t)$$

$$J = \int_0^\infty \left[\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T Q \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \epsilon^2 u^T(t) R u(t) \right] dt, \quad Q \geq 0, \quad R > 0 \tag{6.69}$$

where the control vector, $u(t) \in \Re^m$, has to be chosen such that the performance criterion, J, is minimized. The very well-known solution to this problem is given by

$$u(t) = -R^{-1} B^T P_r x(t) = -F_1 x_1(t) - F_2 x_2(t) \tag{6.70}$$

where P_r is the positive semidefinite solution of the regulator algebraic Riccati equation

$$A^T P_r + P_r A + Q - \frac{1}{\epsilon^2} P_r Z P_r = 0 \tag{6.71}$$

with

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \; Z = BR^{-1}B^T, \; B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; P_r = \begin{bmatrix} P_{1r} & \epsilon P_{2r} \\ \epsilon P_{2r}^T & \epsilon P_{3r} \end{bmatrix} \tag{6.72}$$

Note that B_2 is a square nonsingular matrix.

The optimal regulator gains F_1 and F_2 are given by

$$F_1 = R^{-1} B_2^T P_{2r}^T, \quad F_2 = R^{-1} B_2^T P_{3r} \tag{6.73}$$

The results of interest that we need, which can be deduced from Section 6.3 are given in the form of the following lemma.

Lemma 6.2 *Consider the optimal closed-loop linear system*

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) \\ \epsilon \dot{x}_2(t) &= (A_3 - B_2 F_1) x_1(t) + (A_4 - B_2 F_2) x_2(t) \end{aligned} \tag{6.74}$$

then there exists a nonsingular transformation $\mathbf{T}_1$

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{6.75}$$

such that

$$\begin{aligned} \dot{\xi}_1(t) &= (a_1 + a_2 P_{rs}) \xi_1(t) \\ \epsilon \dot{\xi}_2(t) &= (b_1 + b_2 P_{rf}) \xi_2(t) \end{aligned} \tag{6.76}$$

where P_{rs} and P_{rf} are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic Riccati equations

$$\begin{aligned} 0 &= P_{rs} a_1 - a_4 P_{rs} - a_3 + P_{rs} a_2 P_{rs} \\ 0 &= P_{rf} b_1 - b_4 P_{rf} - b_3 + P_{rf} b_2 P_{rf} \end{aligned} \tag{6.77}$$

Matrices $a_i, \; b_i, \; i = 1, 2, 3, 4$, are defined in (6.48). The nonsingular transformation $\mathbf{T}_1$ is given by

$$\mathbf{T}_1 = (\Pi_1 + \Pi_2 P_r) \tag{6.78}$$

Even more, the global solution P_r can be obtained from the solutions of the reduced-order exact pure-slow and pure-fast algebraic Riccati equations as

$$P_r = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix}\right)^{-1} \tag{6.79}$$

Known matrices $\Omega_i,\ i = 1,2,3,4$ and $\Pi_1,\ \Pi_2$ are given in terms of the solutions of the Chang decoupling equations.

◇

The desired slow-fast decomposition of the Kalman filter (6.61) will be obtained by producing a dual lemma to Lemma 6.2. Consider the optimal closed-loop Kalman filter (6.61) driven by the system measurements, that is

$$\begin{aligned} \dot{\hat{x}}_1(t) &= A_1\hat{x}_1(t) + (A_2 - K_1C_2)\hat{x}_2(t) + K_1y(t) \\ \epsilon\dot{\hat{x}}_2(t) &= A_3\hat{x}_1(t) + (A_4 - K_2C_2)\hat{x}_2(t) + K_2y(t) \end{aligned} \tag{6.80}$$

with the optimal filter gains K_1 and K_2 calculated from (6.62)-(6.64). By duality between the optimal filter and regulator, the filter Riccati equation (6.63) can be solved by using the same decomposition method for solving (6.71) with

$$\begin{aligned} &A \to A^T,\ \ Q \to GWG^T,\ \ F^T = K \\ &Z = BR^{-1}B^T \to S = C^TV^{-1}C \end{aligned} \tag{6.81}$$

$$K_1 = F_1^T, \qquad K_2 = F_2^T, \qquad F = [F_1 \quad F_2]$$

Consider the state-costate equations of the following system

$$\begin{aligned} \dot{x}(t) &= A^Tx(t) - \frac{1}{\epsilon^2}C^TV^{-1}Cp(t) \\ \dot{p}(t) &= -GWG^Tx(t) - Ap(t) \end{aligned} \tag{6.82}$$

Partitioning the filter state-costate variables as $x(t) = \left[x_1^T(t)\ x_2^T(t)\right]^T$ and $p(t) = \left[p_1^T(t)\ \epsilon p_2^T(t)\right]^T$, we obtain the standard singularly perturbed system of the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{p}_1(t) \end{bmatrix} &= T_1 \begin{bmatrix} x_1(t) \\ p_1(t) \end{bmatrix} + T_2 \begin{bmatrix} x_2(t) \\ p_2(t) \end{bmatrix} \\ \epsilon \begin{bmatrix} \dot{x}_2(t) \\ \dot{p}_2(t) \end{bmatrix} &= T_3 \begin{bmatrix} x_1(t) \\ p_1(t) \end{bmatrix} + T_4 \begin{bmatrix} x_2(t) \\ p_2(t) \end{bmatrix} \end{aligned} \tag{6.83}$$

where the matrices $T_1,\ T_2,\ T_3,\ T_4$ are given by

$$T_1 = \begin{bmatrix} A_1^T & 0 \\ -G_1 W G_1^T & -A_1 \end{bmatrix},\ T_2 = \begin{bmatrix} A_3^T & 0 \\ -G_1 W G_2^T & -\epsilon A_2 \end{bmatrix}$$
$$T_3 = \begin{bmatrix} \epsilon A_2^T & 0 \\ -G_2 W G_1^T & -A_3 \end{bmatrix},\ T_4 = \begin{bmatrix} \epsilon A_4^T & -C_2^T V^{-1} C_2 \\ -G_2 W G_2^T & -\epsilon A_4 \end{bmatrix} \tag{6.84}$$

For the singularly perturbed linear system (6.83), the slow-fast decomposition is achieved by using the Chang decoupling equations

$$\begin{gathered} T_4 M - T_3 - \epsilon M(T_1 - T_2 M) = 0 \\ -N(T_4 + \epsilon M T_2) + T_2 + \epsilon(T_1 - T_2 M)N = 0 \end{gathered} \tag{6.85}$$

These equations can be efficiently solved by either using the Newton method or the fixed point iterations as discussed in (Gajic and Shen, 1993). Also equations (6.85) can be solved by using the eigenvector approach (see Avramovic *et al.*, 1980; Medanic, 1982; Kecman *et al.*, 1999).

Applying the Chang decomposition transformation to (6.83), we obtain

$$\begin{gathered} \begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\zeta}_1(t) \end{bmatrix} = (T_1 - T_2 M) \begin{bmatrix} \eta_1(t) \\ \zeta_1(t) \end{bmatrix} \\ \epsilon \begin{bmatrix} \dot{\eta}_2(t) \\ \dot{\zeta}_2(t) \end{bmatrix} = (T_4 + \varepsilon M T_2) \begin{bmatrix} \eta_2(t) \\ \zeta_2(t) \end{bmatrix} \end{gathered} \tag{6.86}$$

By using the permutation matrices

$$E_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon} I_m \end{bmatrix},\ E_2 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \tag{6.87}$$

we can relate variables in the new and old coordinates by

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon N M & -\epsilon N \\ M & I \end{bmatrix} E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{6.88}$$

where $p(t) = Px(t)$. Introducing the notation

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^T & \Pi_4 \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon NM & -\epsilon N \\ M & I \end{bmatrix} E_1 \tag{6.89}$$

we obtain

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = [\Pi_1 \quad \Pi_2] \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = (\Pi_1 + \Pi_2 P)x(t) = \mathbf{T}_2 x(t) \tag{6.90}$$

The desired transformation is given by

$$\mathbf{T}_2 = (\Pi_1 + \Pi_2 P) \tag{6.91}$$

This transformation block diagonalizes the closed loop filter system matrix $(A - KC)^T$, that is

$$\mathbf{T}_2 (A - KC)^T \mathbf{T}_2^{-1}$$

is block diagonal, which follows by duality, see Lemma 6.2 and formulas (6.75)-(6.76).

The similarity transformation $\mathbf{T}_2$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{6.92}$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s(t) \\ \dot{\hat{\eta}}_f(t) \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} A_1 & A_2 - K_1 C_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon}(A_4 - K_2 C_2) \end{bmatrix} \mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s(t) \\ \hat{\eta}_f(t) \end{bmatrix} + \mathbf{T}_2^{-T} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} y(t) \tag{6.93}$$

such that the complete closed-loop decomposition is achieved, that is

$$\begin{aligned} \dot{\hat{\eta}}_s(t) &= (a_1 + a_2 P_s)^T \hat{\eta}_s(t) + K_s y(t) \\ \epsilon \dot{\hat{\eta}}_f(t) &= (b_1 + b_2 P_f)^T \hat{\eta}_f(t) + K_f y(t) \end{aligned} \tag{6.94}$$

Note that the transposes of the system matrices in (6.94) come from the fact that the derivations have been performed by using duality. The matrices in (6.94) are given by

$$\begin{gathered} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2 M), \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon M T_2) \\ \begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \end{gathered} \tag{6.95}$$

$$
\begin{aligned}
0 &= P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s \\
0 &= P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f
\end{aligned}
\tag{6.96}
$$

The Newton method for solving nonsymmetric Riccati equations (6.96) is considered in Chapter 2. For this particular case, it has the form of (6.55) with $a_i, b_i, i = 1, 2, 3, 4$, matrices obtained from (6.95) and with the initial conditions obtained from the algebraic equations dual to (6.53) and (6.54).

It is important to point out that the matrix P in (6.91) can be obtained in terms of P_s and P_f by using (6.79) with

$$
P_{rs} = P_s, \quad P_{rf} = P_f \tag{6.97}
$$

and $\Omega_1, \; \Omega_2, \; \Omega_3, \; \Omega_4$ obtained from

$$
\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \begin{bmatrix} I & \epsilon N \\ -M & I - \epsilon M N \end{bmatrix} E_2^{-T} \tag{6.98}
$$

A lemma dual to Lemma 6.2 can now be formulated as.

Lemma 6.3 *Given the closed-loop optimal Kalman filter (6.80) of a linear singularly perturbed system, then there exists a nonsingular transformation matrix (6.91), which completely decouples (6.80) into pure-slow and pure-fast local filters (6.94) both driven by the system measurements. Even more, the decoupling transformation (6.91) and the filter coefficients given in (6.94) can be obtained in terms of exact pure-slow and pure-fast reduced-order completely decoupled algebraic Riccati equations (6.96).*

$\diamond$

Note that the actual procedure for computing the filter decomposition can be done completely at the local levels. In the proposed method we have complete separation for both off-line (coefficient calculation) and on-line (filtering process itself) computations. The procedure is summarized in the form of the following algorithm.

Algorithm 6.1: Small Noise Kalman Filtering.

Step 1: Solve Chang decoupling equations (6.85).

Step 2: Find coefficients $a_i, b_i, \; i = 1, 2, 3, 4$ by using (6.95).

Step 3: Solve the reduced-order algebraic Riccati equations (6.96).

Step 4: Form the transformation (6.96) by using results from (6.79), (6.89), and (6.97)-(6.98).

Step 5: Calculate the local filter gains by using (6.62) and (6.95), where the matrix P is calculated from (6.79), (6.96)-(6.98).

Δ

Example 6.1: Consider the following fourth-order linear stochastic system with corresponding partitioned matrices given by

$$A_1 = \begin{bmatrix} -1 & 2 \\ 0.2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.3 & 0.4 \\ -0.1 & -0.2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 1.2 & 1 \\ -2 & 1.3 \end{bmatrix}, \; G_2 = \begin{bmatrix} -3 & 0 \\ 0.5 & 2 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0.4 \\ 0.2 & 1 \end{bmatrix}$$

$$W = 2I_2, \quad V = I_2, \quad \epsilon = 0.1$$

The solutions of the nonsymmetric algebraic Riccati equations, obtained by using the Newton method from Appendix 6.2, are given by

$$P_1 = \begin{bmatrix} 0.0070 & 0.0069 \\ 0.0099 & -0.0024 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.6761 & -1.4857 \\ -1.4437 & 2.9853 \end{bmatrix}$$

We have obtained completely decoupled filters driven by the system measurements $y(t)$ as

$$\dot{\hat{\eta}}_s(t) = \begin{bmatrix} -0.8530 & 2.2296 \\ 0.0303 & -3.1834 \end{bmatrix} \hat{\eta}_s(t) + \begin{bmatrix} 2.7055 & -0.0720 \\ -1.9165 & 1.0218 \end{bmatrix} y(t)$$

$$\epsilon \dot{\hat{\eta}}_f(t) = \begin{bmatrix} -4.08826 & -1.07443 \\ -0.27273 & -2.78067 \end{bmatrix} \hat{\eta}_f(t)$$

$$+ \begin{bmatrix} -4.09963 & -0.51166 \\ -0.29086 & 2.68738 \end{bmatrix} y(t)$$

Δ

In summary, we have exactly solved the small measurement noise problem by using the singular perturbation methodology. Two completely independent slow and fast reduced-order filters are obtained. In addition to the practical importance of the solved problem, we hope that the obtained results will bring deeper understanding of the effect of small measurement noise in the Kalman filtering since the slow and fast phenomena are now completely and exactly separated. Also we believe that the limiting perfect measurement noise case can be studied by using the results of this book as an approximation for $\epsilon = 0$. It is our opinion that the discrete-time version of this problem might be an interesting area for future research.

6.4.2 Case Study: Kalman Filtering for a System Positioning Problem

In order to demonstrate the proposed method we solve the control system positioning problem from (Kwakernaak and Sivan, 1972). The problem matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -4.6 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad W = 10$$

We assume that the measurements of angular displacement, x_1, and the angular velocity, x_2, are given as a linear combination corrupted by white measurement noise, that is

$$y(t) = [1 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + v(t)$$

with the measurement noise intensity taken from (Kwakernaak and Sivan, 1972) as $V = \epsilon^2 \times 0.1$ with $\epsilon = 0.001$. In order to obtain the measurement matrix form considered in this paper we apply the transformation from Appendix 6.1 with

$$x(t) = M_1 z(t), \qquad M_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0.5 \end{bmatrix}, \qquad N_1 = 1$$

which leads to the following matrices in the new coordinates (see Appendix 6.1)

$$\mathbf{A} = \begin{bmatrix} -2.8 & 1.4 \\ 3.6 & -1.8 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} -0.05 \\ 0.1 \end{bmatrix}$$

The measurement equation is now given by

$$y(t) = [0 \quad 1] z(t) + v_{new}(t), \qquad int\{v_{new}(t)\} = N_1 V N_1 = 0.1\epsilon^2$$

The following pure-slow and pure-fast filters are obtained

$$\dot{\hat{\eta}}_1(t) = -\hat{\eta}_1(t) + y(t)$$

$$\varepsilon \dot{\hat{\eta}}_2(t) = -\hat{\eta}_2(t) + 0.9982 y(t)$$

We have run the same problem with different values for a small parameter with the following results

$$\dot{\hat{\eta}}_1(t) = -0.999\hat{\eta}_1(t) + 0.999y(t)$$

$$\epsilon\dot{\hat{\eta}}_2(t) = -1.001\hat{\eta}_2(t) + 0.981y(t), \qquad \epsilon = 0.01$$

$$\dot{\hat{\eta}}_1(t) = -0.9116\hat{\eta}_1(t) + 0.934y(t)$$

$$\epsilon\dot{\hat{\eta}}_2(t) = -1.0970\hat{\eta}_2(t) + 0.769y(t), \qquad \epsilon = 0.1$$

It is important to note that the actual value of the small positive parameter ϵ is also dependent on the system noise input matrix G and the intensity matrix of the system noise. It seems from our experience that the best estimate for ϵ is the "signal to noise ratio," (Anderson and Moore, 1979), that is

$$\epsilon^2 = \frac{\|GWG^T\|}{\|V\|}$$

where $\|.\|$ is any norm and W and V are constant matrices.

6.5 Cheap Control Problem for Sampled Data Linear Systems

The continuous-time optimal cheap control problem of linear systems as introduced in Section 6.1.2 has been studied by many researchers. The discrete-time optimal cheap control problem of linear systems has not been completely solved in the direction of removing its ill-conditioned caused by multiple time scale phenomena and decomposing the problem into slow and fast time scales. The first attempt in that direction can be found in (Oloomi and Sawan, 1996), where an approximate solution has been presented. In this section, we present the complete and exact decomposition into pure-slow and pure-fast dynamics of a class of a discrete-time cheap control problem considered in (Oloomi and Sawan, 1996). That class of linear systems, known as sampled data linear systems, is obtained by sampling continuous-time linear systems. The results presented eliminate ill-conditioning of the original problem and increase computational speed due to full parallelism between the slow and fast subsystems.

Consider a sampled data linear system obtained by uniformly sampling a continuous-time linear system with a small sampling period ϵ by using Euler's approximation

$$x(k+1) = (I + \epsilon A)x(k) + \epsilon Bu(k) \tag{6.99}$$

It is assumed that the matrices A and B are constant and that the dimensions of the system state variables $x(t)$ and the system control variables $u(t)$ are respectively given by n and n_2. In addition, it is assumed that the matrix B has the structure

$$B^{n \times n_2} = \begin{bmatrix} 0 \\ B_2^{n_2 \times n_2} \end{bmatrix}, \qquad \det(B_2) \neq 0 \tag{6.100}$$

Hence, the matrix B_2 satisfies Assumption 6.1. The system (6.99) can be consistently partitioned according to (6.100) as

$$\begin{aligned} x_1(k+1) &= (I_{n_1} + \epsilon A_1)x_1(k) + \epsilon A_2 x_2(k) \\ x_2(k+1) &= \epsilon A_3 x_1(k) + (I_{n_2} + \epsilon A_4)x_2(k) + \epsilon B_2 u(k) \end{aligned} \tag{6.101}$$

With (6.99), a performance criterion that defines the cheap control problem is associated

$$J = \frac{1}{2} \sum_{k=0}^{+\infty} \left[x^T(k)Qx(k) + \epsilon^2 u^T(k)Ru(k) \right] \tag{6.102}$$

The penalty matrices in (6.101) have to be chosen such as

$$R = R^T > 0, \qquad Q = Q^T = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0, \;\; Q_3 > 0 \tag{6.103}$$

Note that $Q_3 > 0$ indicates that Assumption 6.2 is satisfied. Without loss of generality, we use the same small parameter ϵ to indicate both the small sampling period and the small control penalty in the performance criterion. This can be done due to the fact that the matrix R is an arbitrary positive definite matrix.

It should be emphasized that the cheap control problem for sampled data linear control systems is studied in this section under the following assumption.

Assumption 6.8: The square $n_2 \times n_2$ matrices B_2 and Q_3 have full ranks.

Optimization of (6.101) along trajectories of (6.99) leads to the following expression for the optimal control

$$u(k) = -\frac{1}{\epsilon} R^{-1} B^T \lambda(k+1) \tag{6.104}$$

where $\lambda(k)$ is a costate variable that satisfies the following difference equation

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{6.105}$$

$\mathbf{H}$ is the discrete-time Hamiltonian matrix

$$\mathbf{H} = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix} \tag{6.106}$$

where

$$A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ \epsilon A_3 & I_{n_2} + \epsilon A_4 \end{bmatrix} \tag{6.107}$$

The more detailed structure of the Hamiltonian matrix $\mathbf{H}$ will be needed for the purpose of this section. It is given by

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 & 0 & 0 \\ \epsilon A_3 + B_2 R^{-1} B_2^T \bar{Q}_3 & h_{22} & -\epsilon B_2 R^{-1} B_2^T \bar{A}_3 & -h_{24} \\ -\bar{Q}_1 & -\bar{Q}_2 & I_{n_1} + \epsilon \bar{A}_1 & \epsilon \bar{A}_2 \\ -\bar{Q}_3 & -\bar{Q}_4 & \epsilon \bar{A}_3 & I_{n_1} + \epsilon \bar{A}_4 \end{bmatrix}$$

$$h_{22} = I_{n_2} + \epsilon A_4 + B_2 R^{-1} B_2^T \bar{Q}_4$$

$$h_{24} = B_2 R^{-1} B_2^T \left(I_{n_2} + \epsilon \bar{A}_4 \right) \tag{6.108}$$

The following notation will be used in the remaining part of this section

$$A^{-T} = \begin{bmatrix} I_{n_1} + \epsilon \bar{A}_1 & \epsilon \bar{A}_2 \\ \epsilon \bar{A}_3 & I_{n_1} + \epsilon \bar{A}_4 \end{bmatrix}, \quad A^{-T} Q = \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \\ \bar{Q}_3 & \bar{Q}_4 \end{bmatrix} \tag{6.109}$$

which is consistent with the corresponding mathematical operations.

Partitioning and appropriately scaling the costate variables as $[\epsilon\lambda_1^T(k) \quad \lambda_2^T(k)]^T = [p_1^T(k) \quad p_2^T(k)]^T$ and using the coordinate transformation

$$\begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \epsilon I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{6.110}$$

we obtain in the new coordinates the singularly perturbed system

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ p_1(k+1) \end{bmatrix} &= (I_{2n_1} + \epsilon T_1) \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} + \epsilon T_2 \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} \\ \begin{bmatrix} x_2(k+1) \\ p_2(k+1) \end{bmatrix} &= T_3 \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} + T_4 \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} \end{aligned} \tag{6.111}$$

where

$$\begin{gathered} T_1 = \begin{bmatrix} A_1 & 0 \\ -\bar{Q}_1 & \bar{A}_1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & 0 \\ -\bar{Q}_2 & \epsilon \bar{A}_2 \end{bmatrix} \\ T_3 = \begin{bmatrix} \epsilon A_3 + B_2 R^{-1} B_2^T \bar{Q}_3 & B_2 R^{-1} B_2^T \bar{A}_3 \\ -\bar{Q}_3 & \bar{A}_3 \end{bmatrix} \\ T_4 = \begin{bmatrix} I_{n_2} + \epsilon A_4 + B_2 R^{-1} B_2^T \bar{Q}_4 & -B_2 R^{-1} B_2^T (I_{n_2} + \epsilon \bar{A}_4) \\ -\bar{Q}_4 & I_{n_2} + \epsilon \bar{A}_4 \end{bmatrix} \end{gathered} \tag{6.112}$$

Since the matrices $T_i, i = 1, 2, 3, 4$, are all $O(1)$, the difference equation (6.111) represents the standard singularly perturbed system (Litkouhi and Khali, 1984, 1985).

The discrete-time linear singularly perturbed system can be block diagonalized via the use of the Chang transform defined by

$$\mathbf{T} = \begin{bmatrix} I_{2n_1} - \epsilon H L & -\epsilon H \\ L & I_{2n_2} \end{bmatrix} \tag{6.113}$$

where the matrices H and L satisfy the following algebraic equations

$$\begin{gathered} (I - T_4)L + T_3 + \epsilon L(T_1 - T_2 L) = 0 \\ H(I - T_4) + T_2 + \epsilon(T_1 - T_2 L)H + \epsilon H L T_2 = 0 \end{gathered} \tag{6.114}$$

It is important to observe that the matrix T_4 that represents the Hamiltonian matrix of the fast subsystem, has no eigenvalues on the unit circle under stabilizability-detectability assumptions imposed on the fast subsystem. The stabilizability-detectability condition is guaranteed by Assumption 6.8 so that the matrix $I - T_4$ is nonsingular, which guarantees the existence of the unique solutions for the algebraic equations defined in (6.114). Methods for solving (6.114) are considered in (Gajic and Shen, 1993).

Using the change of the coordinates as

$$\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} \tag{6.115}$$

we obtained decoupled slow state-costate and fast state-costate variables as

$$\begin{aligned} \begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \end{bmatrix} &= \left(I_{2n_1} + \epsilon(T_1 - T_2 L)\right) \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \\ \begin{bmatrix} \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} &= \left(T_4 + \epsilon L T_2\right) \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} \end{aligned} \tag{6.116}$$

In the original coordinates, at steady state, the state and costate variables are related by

$$\lambda(k) = P x(k) \tag{6.117}$$

where P satisfies the discrete-time algebraic Riccati equation

$$P = Q + A^T P A - A^T P B \left(R + B P B^T\right)^{-1} B P A \tag{6.118}$$

In the new coordinates, at steady state, we have

$$\xi_1(k) = P_1 \eta_1(k), \qquad \xi_2(k) = P_2 \eta_2(k) \tag{6.119}$$

where the matrices P_1 and P_2 satisfy, respectively, pure-slow and pure-fast discrete-time algebraic Riccati equations to be derived later.

The relation between the new and original coordinates is given by

$$\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \\ \xi_1(k) \\ \xi_2(k) \end{bmatrix} = \Pi \begin{bmatrix} x_1(k) \\ x_2(k) \\ p_1(k) \\ p_2(k) \end{bmatrix} \tag{6.120}$$

where

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2 \mathbf{T} E_1 \tag{6.121}$$

with

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{6.122}$$

Using (6.117) in (6.120)-(6.122), we obtain

$$\begin{gathered} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} = \Pi_1 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \Pi_2 \begin{bmatrix} \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = (\Pi_1 + \Pi_2 P) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} = \Pi_3 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \Pi_4 \begin{bmatrix} \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = (\Pi_3 + \Pi_4 P) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{gathered} \tag{6.123}$$

It follows from (6.119) and (6.123) that

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{6.124}$$

Reversing the order of arguments, we can also find the matrix P in terms of matrices P_1 and P_2. Defining the matrix

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathbf{T}^{-1} E_2^T = \Pi^{-1} \tag{6.125}$$

where

$$\mathbf{T}^{-1} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon L H \end{bmatrix} \tag{6.126}$$

we obtain

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right)^{-1} \tag{6.127}$$

It is shown in Appendix 6.2 that the matrix inversions introduced in (6.124) and (6.127) exist for sufficiently small values of the parameter ϵ.

The pure-slow and pure-fast discrete-time algebraic Riccati equations can be derived from (6.116) and (6.119), that is

$$\begin{gathered} \begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \end{bmatrix} = (I_{2n_1} + \epsilon(T_1 - T_2 L)) \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \\ \begin{bmatrix} \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} = (T_4 + \epsilon L T_2) \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} \end{gathered} \tag{6.128}$$

where the square matrices $a_i, b_i, i = 1,2,3,4$, are given by

$$
\begin{aligned}
a_1 &= I_{n_1} + \epsilon(A_1 - A_2L_1), \quad a_2 = -\epsilon A_2 L_2 \\
a_3 &= \epsilon\left(-\bar{Q}_1 + \bar{Q}_2 L_1 - \epsilon A_2 L_3\right) \\
a_4 &= I_{n_1} + \epsilon\left(\bar{A}_1 + \bar{Q}_2 L_2 - \epsilon \bar{A}_2 L_4\right)
\end{aligned}
\tag{6.129}
$$

and

$$
\begin{aligned}
b_1 &= I_{n_2} + \epsilon A_4 + B_2 R^{-1} B_2^T \bar{Q}_4 + \epsilon\left(L_1 A_2 - L_3 \bar{Q}_2\right) \\
b_2 &= -B_2 R^{-1} B_2^T \left(I_{n_2} + \epsilon \bar{A}_4\right) + \epsilon^2 L_2 \bar{A}_2 \\
b_3 &= -\bar{Q}_4 + L_3 A_2 - L_4 \bar{Q}_2, \quad b_4 = I_{n_2} + \epsilon \bar{A}_4 + \epsilon^2 L_4 \bar{A}_2
\end{aligned}
\tag{6.130}
$$

with

$$
L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}
\tag{6.131}
$$

Using (6.119) in (6.128), we derive the pure-slow and pure-fast discrete-time algebraic Riccati equations, respectively as

$$
\begin{aligned}
&P_1(A_1 - A_2 L_1) - \left(\bar{A}_1 + \bar{Q}_2 L_2 - \epsilon \bar{A}_2 L_4\right) P_1 + \bar{Q}_1 - \bar{Q}_2 L_1 \\
&\quad + \epsilon A_2 L_3 - P_1 A_2 P_1 = 0
\end{aligned}
\tag{6.132}
$$

and

$$
\begin{aligned}
&P_2\left[B_2 R^{-1} B_2^T \bar{Q}_4 + \epsilon\left(A_4 + L_1 A_2 - L_2 \bar{Q}_2\right)\right] - \epsilon\left(\bar{A}_4 + \epsilon L_4 \bar{A}_2\right) P_2 \\
&\quad + P_2\left[-B_2 R^{-1} B_2^T \left(I_{n_2} + \epsilon \bar{A}_4\right) + \epsilon^2 L_2 \bar{A}_2\right] P_2 \\
&\quad + \left(\bar{Q}_4 - L_3 A_2 + L_4 \bar{Q}_2\right) = 0
\end{aligned}
\tag{6.133}
$$

It is shown in Appendix 6.3 that $O(\epsilon)$ perturbations of (6.132) and (6.133) produce symmetric algebraic Riccati equations

$$
\begin{aligned}
&P_1^{(0)}\left(A_1 - A_2 Q_3^{-1} Q_2^T\right) + \left(A_1 - A_2 Q_3^{-1} Q_2^T\right)^T P_1^{(0)} \\
&\quad + \left(Q_1 - Q_2 Q_3^{-1} Q_2^T\right) - P_1^{(0)} A_2 Q_3^{-1} A_2^T P_1^{(0)} = 0
\end{aligned}
\tag{6.134}
$$

and

$$
P_2^{(0)} = I_{n_2} P_2^{(0)} I_{n_2} + Q_3 - P_2^{(0)} B_2 \Big(R + B_2^T P_2^{(0)} B_2\Big)^{-1} B_2^T P_2^{(0)} \tag{6.135}
$$

Note that the slow approximate algebraic Riccati equation is the continuous-time type algebraic Riccati equation and that the fast approximate algebraic Riccati equation is the discrete-time algebraic Riccati equation.

The unique positive definite stabilizing solution of the approximate fast algebraic Riccati equation (6.135) exists under Assumption 6.8. The unique positive semidefinite stabilizing solution of the approximate slow algebraic Riccati equation (6.134) exists under the following assumption.

Assumption 6.9: The triple (A_1, A_2) is stabilizable and the triple (Q_0, A_1) is detectable, where $Q_0 = \mathrm{Chol}(Q_1 - Q_2Q_3^{-1}Q_2^T)$.

Having obtained the solutions for $P_1^{(0)}$ and $P_2^{(0)}$, that are $O(\epsilon)$ apart from the corresponding exact solutions, the Newton method can be used for solving the nonsymmetric pure-slow and pure-fast algebraic Riccati equations (6.132) and (6.133). The global solution of the corresponding algebraic Riccati equation (6.118) can be obtained in terms of P_1 and P_2 using formula (6.127).

The optimal control to the optimization problem defined by (6.99)-(6.103) is given by

$$u(k) = -\frac{1}{\epsilon}\left(R + B^TPB\right)^{-1} BPAx(k) \tag{6.136}$$

This optimal control can be also expressed in terms of the new variables as

$$\begin{aligned} u(k) &= -\frac{1}{\epsilon}\left(R + B^TPB\right)^{-1} BPA(\Pi_1 + \Pi_2P)\begin{bmatrix}\eta_1(k)\\ \eta_2(k)\end{bmatrix} \\ &= -\frac{1}{\epsilon}F_1\eta_1(k) - \frac{1}{\epsilon}F_2\eta_2(k) \end{aligned} \tag{6.137}$$

where $\eta_1(k)$ and $\eta_2(k)$ represent pure-slow and pure-fast closed-loop systems, respectively given by

$$\begin{aligned} \eta_1(k+1) &= (a_1 + a_2P_1)\eta_1(k) \\ \eta_2(k+1) &= (b_1 + b_2P_2)\eta_2(k) \end{aligned} \tag{6.138}$$

The optimal gains F_1 and F_2 are obtained by an appropriate partition of the gain matrix defined in (6.137).

It should be emphasized that the transformation that relates the original and new coordinates is given by

$$\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} = (\Pi_1 + \Pi_2 P)\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{6.139}$$

6.6 Comments

Presentation of this chapter follows closely recent research work of Professor Gajic and his doctoral students. In that direction we have presented the main results of (Huey *et al.*, 1993; Gajic and Shen, 1993; Aganovic *et al.*, 1995, Popescu and Gajic, 1999). Several future research problems can emerge from the presented material; of special interest will be the solutions of the corresponding problems in the discrete-time domain.

Appendix 6.1

The following result is known from (Wilkinson, 1965). Given any matrix C of dimension $m \times n$ such that

$$rank\, C^{m\times n} = m$$

then there exist two nonsingular matrices $N_1^{m\times m}, M_1^{n\times n}$ such that

$$N_1^{m\times m} C^{m\times n} M_1^{n\times n} = [0 \quad I_m\,]^{m\times n}$$

Given a linear stochastic system of the form

$$\dot{x}(t) = Ax(t) + Gw(t)$$

$$y(t) = Cx(t) + v(t)$$

with constant problem matrices $A^{n\times n}, G^{n\times m}, C^{m\times n}$ and constant noise intensity matrices $int(w) = W^{m\times m}, int(v) = V^{m\times m}$. The following transformation

$$x = M_1 z$$

maps the given system into

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{G}w(t)$$

$$y_{new}(t) = [0 \quad I_r\,]z(t) + v_{new}(t)$$

where

$$\mathbf{A} = M_1^{-1} A M_1, \quad \mathbf{G} = M_1^{-1} G,$$

$$y_{new}(t) = N_1 y(t), \qquad v_{new}(t) = N_1 v(t) \Rightarrow int\{v_{new}(t)\} = N_1 V N_1^T$$

Appendix 6.2

In this appendix we show that the matrix inversions used in (6.124) and (6.127) exist. From (6.121) we have

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2 \mathbf{T} E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ L_1 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L_3 & 0 & 0 & I_{n_2} \end{bmatrix} + O(\epsilon)$$

More precise analysis reveals that

$$\Pi_1 = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\epsilon), \qquad \Pi_2 = \begin{bmatrix} O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & 0 \end{bmatrix}$$

Using the fact that the solution of the considered algebraic Riccati equation (6.118) has the form (Oloomi and Sawan, 1996)

$$P = \begin{bmatrix} \frac{1}{\epsilon} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \begin{bmatrix} O\left(\frac{1}{\epsilon}\right) & O(1) \\ O(1) & O(1) \end{bmatrix}$$

we conclude that

$$\Pi_1 + \Pi_2 P = \begin{bmatrix} I_{n_1} & 0 \\ L_1 + L_2 P_{11} & I_{n_2} \end{bmatrix} + O(\epsilon)$$

which indicates that the matrix $\Pi_1 + \Pi_2 P$ is invertible for sufficiently small values of ϵ.

On the other hand

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathbf{T}^{-1} E_2^T = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ -L_1 & I_{n_2} & -L_2 & 0 \\ 0 & H_3 & \frac{1}{\epsilon} I_{n_1} & H_4 \\ -L_3 & 0 & -L_4 & I_{n_2} \end{bmatrix} + O(\epsilon)$$

which implies

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 - L_2 P_1 & I_{n_2} \end{bmatrix} + O(\epsilon) \tag{6.140}$$

This indicates that the corresponding matrix is also nonsingular for sufficiently small values of the small parameter $\epsilon > 0$.

Appendix 6.3

It can be shown from (6.109) that $O(\epsilon)$ perturbations of matrices $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4$ are given by

$$\bar{A}_{1\epsilon} = -A_1^T, \quad \bar{A}_{2\epsilon} = A_3^T, \quad \bar{A}_{3\epsilon} = A_2^T, \quad \bar{A}_{4\epsilon} = -A_4^T$$

where the subscript ϵ indicates an $O(\epsilon)$ perturbation.

It is also easy to show from (6.109) that $O(\epsilon)$ perturbations of matrices $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \bar{Q}_4$ are given by

$$\bar{Q}_{1\epsilon} = Q_1, \quad \bar{Q}_{2\epsilon} = Q_2, \quad \bar{Q}_{3\epsilon} = Q_2^T, \quad \bar{Q}_{4\epsilon} = Q_3$$

The $O(\epsilon)$ perturbations of the Chang transformation and its inverse, defined respectively in (6.123) and (6.126), are given by

$$\mathbf{T}_\epsilon = \begin{bmatrix} I_{2n_1} & 0 \\ L_\epsilon & I_{2n_2} \end{bmatrix}, \quad \mathbf{T}_\epsilon^{-1} = \begin{bmatrix} I_{2n_1} & 0 \\ -L_\epsilon & I_{2n_2} \end{bmatrix}$$

where

$$L_\epsilon = (T_{4\epsilon} - I)^{-1} T_{3\epsilon}$$

with

$$T_{3\epsilon} = \begin{bmatrix} B_2 R^{-1} B_2^T Q_2^T & -B_2 R^{-1} B_2^T A_2^T \\ -Q_2^T & A_2^T \end{bmatrix}$$

$$T_{4\epsilon} - I = \begin{bmatrix} B_2 R^{-1} B_2^T Q_3 & -B_2 R^{-1} B_2^T \\ -Q_3 & 0 \end{bmatrix}$$

$$(T_{4\epsilon} - I)^{-1} = \begin{bmatrix} 0 & -Q_3^{-1} \\ -B_2^{-T} R B_2^{-1} & -I \end{bmatrix}$$

Note that in the process of finding the matrix inversion we have used Assumption 6.8. The $O(\epsilon)$ perturbation for L_ϵ is

$$L_\epsilon = (T_{4\epsilon} - I)^{-1} T_{3\epsilon} = \begin{bmatrix} Q_3^{-1} Q_2^T & Q_3^{-1} A_2^T \\ 0 & 0 \end{bmatrix}$$

At this point we are ready to find the $O(\epsilon)$ approximation of the pure-slow algebraic Riccati equation (6.132). It can be easily shown that the coefficient matrices in (6.132) can be approximated with an $O(\epsilon)$ accuracy as follows

$$A_1 - A_2L_1 = A_1 - A_2L_{1\epsilon} + O(\epsilon) = A_1 - A_2Q_3^{-1}Q_2^T + O(\epsilon)$$

$$\bar{A}_1 + \bar{Q}_2L_2 + O(\epsilon) = \left(A_1 - A_2Q_3^{-1}Q_2^T\right)^T + O(\epsilon)$$

$$A_2L_2 = A_2Q_3^{-1}A_2^T + O(\epsilon)$$

Using these approximations in (6.132) we obtain the symmetric approximate slow algebraic Riccati equation as defined in (6.134).

The pure-fast algebraic Riccati equation is directly approximated with accuracy of $O(\epsilon)$ by

$$P_2^{(0)}B_2R^{-1}B_2^T\bar{Q}_{4\epsilon} - P_2^{(0)}B_2R^{-1}B_2^TP_2^{(0)}$$
$$+ \left(\bar{Q}_{4\epsilon} - L_{3\epsilon}A_2 + L_{4\epsilon}\bar{Q}_2\right) = 0$$

Since $L_{3\epsilon} = 0$, $L_{\epsilon 4} = 0$, $\bar{Q}_{4\epsilon} = Q_3$, we have

$$P_2^{(0)}B_2R^{-1}B_2^TQ_3 - P_2^{(0)}B_2R^{-1}B_2^TP_2^{(0)} + Q_3 = 0$$

From this equation it follows that

$$Q_3 = \left(I + P_2^{(0)}B_2R^{-1}B_2^T\right)^{-1}P_2^{(0)}B_2R^{-1}B_2^TP_2^{(0)}$$

The last equation can be transformed using standard matrix algebra into

$$Q_3 - P_2^{(0)}B_2\left(R + B_2^TP_2^{(0)}B_2\right)^{-1}B_2^TP_2^{(0)} = 0$$

which is the discrete-time approximate fast algebraic Riccati equation derived in (Oloomi and Sawan, 1996). Adding $P_2^{(0)}$ to the left- and right-hand sides of this equation, we obtain equation (6.135).

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7
Eigenvector Approach for Slow-Fast Decoupling

In this chapter we show how to decompose the singularly perturbed algebraic Riccati equation and the corresponding linear-quadratic optimal control problem in terms of the reduced-order slow-fast problems by using the eigenvector approach. The proposed methodology is also applied, under certain assumptions, to the reduced-order decomposition of regular (standard) linear-quadratic optimal control problems. Real-world control system numerical examples are solved in order to demonstrate efficiency of the procedures presented. The results obtained can be extended to the reduced-order decomposition of the Kalman filtering problems.

The research of this chapter is motivated by the existence of a transformation for the exact slow-fast decomposition of the singularly perturbed algebraic Riccati equation and the corresponding linear-quadratic optimal control and filtering problems (Su *et al.*, 1992b; Gajic and Shen, 1993; Gajic and Lim, 1994). That transformation is valid for relatively small values of a singular perturbation parameter for which the techniques are based on either fixed point iterations or the Newton method (Su *et al.*, 1992b; Gajic and Shen, 1993) or Taylor series and asymptotic expansions (Derbel *et al.*, 1994) produce solutions of certain algebraic equations—these solutions comprise the required transformation. A similar problem in solving the transformation equations appears if one uses the subspace iteration approach of (Avramovic, 1979), which is based on an iterative method developed in (Stewart, 1976) for computing the dominant eigenspace. Avramovic's algorithm after k iterations produce the accuracy of $O\left(\epsilon^k\right)$, where ϵ is a small positive singular perturbation parameter, hence it is efficient for small values of ϵ. The algebraic equations comprising the desired transformation have the structure of general nonsquare Riccati equations, which for sufficiently small values of a singular perturbation parameter can be solved by performing iterations on systems of linear algebraic equations. However, if the singular perturbation parameter is not small enough the above methods will not produce the desired solutions, that is, they will not provide the desired decomposition. Even more, the upper bound of the small singular perturbation parameter for which the corresponding algebraic equations can be solved by the above iterative methods is problem dependent.

This chapter is organized as follows. In Section 7.1, we summarize, in the extent needed for the purpose of this chapter, the transformation for the reduced-order decomposition of the optimal singularly perturbed linear-quadratic control problem considered in Chapter 2. The eigenvector method for solving general nonsquare algebraic Riccati equations is considered in Section 7.2. The eigenvector method is applied to both the singularly perturbed (Section 7.3) and regular linear-quadratic optimal control problems (Section 7.4). In Section 7.5, we present real-world control system examples in order to demonstrate the proposed procedures.

7.1 Exact Slow-Fast Decomposition of Singularly Perturbed Systems: A Summary

In (Su *et al.*, 1992b) a powerful transformation for the exact slow-fast decomposition of the algebraic Riccati equation of singularly perturbed systems is obtained so that the optimal control and filtering tasks can be solved *exactly* and *performed independently* in slow and fast time scales, (Gajic and Shen, 1993; Gajic and Lim, 1994; Lim, 1999). Before the results of (Su *et al.*, 1992b) became available, the control engineers were able to decompose *exactly* only *linear* singularly perturbed systems by using the celebrated Chang transformation, (Chang, 1972). In (Chow and Kokotovic, 1976) the *nonlinear* algebraic Riccati equation was decomposed into slow and fast algebraic Riccati equations *with the accuracy of* $O(\epsilon)$. Several real world examples done in (Gajic and Shen, 1993) indicate that very often an $O(\epsilon)$ accuracy is not satisfactory. The results of (Su *et al.*, 1992b) are as a matter of fact the extended and improved results of (Chow and Kokotovic, 1976). It can be said that the results of (Su *et al.*, 1992b) achieve the same goal as the results of (Chow and Kokotovic, 1976), but with *perfect accuracy*.

The approach taken in (Su *et al.*, 1992b) was based on block diagonalization of the singularly perturbed Hamiltonian matrix. Slow-fast decomposition of the Hamiltonian matrix has been previously used in (Grodt and Gajic, 1988) for the exact solution of the differential singularly perturbed Riccati equation and in (Su *et al.*, 1992a) for the *exact* slow-fast decomposition of the open-loop singularly perturbed linear-quadratic optimal control problem. In the recent papers (Fridman, 1995, 1996), the slow-fast decomposition of the H^{∞}–optimal linear quadratic singularly perturbed control problem is obtained by using the results of the integral manifold theory for singularly perturbed linear systems (Sobolev, 1984).

The singularly perturbed linear control system under consideration is given by

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_2 x_2 + B_1 u \\ \epsilon \dot{x}_2 &= A_3 x_1 + A_4 x_2 + B_2 u \end{aligned} \tag{7.1}$$

where $x_1 \in \Re^{n_1}$ are slow and $x_2 \in \Re^{n_2}$ are fast system state space variables, $u \in \Re^m$ is the vector input, and ϵ is the small positive singular perturbation parameter. Matrices A_i, $i = 1, ..., 4$, and B_j, $j = 1, 2$, are constant and of appropriate dimensions with A_4 *being nonsingular*. Nonsingularity of A_4 indicates the so-called standard singularly perturbed linear control system, in contrast to the case when the matrix A_4 is singular when we have the nonstandard singularly perturbed control system (Khalil, 1984, 1989).

With (7.1) a quadratic performance criterion to be minimized is associated

$$J = \frac{1}{2} \int_0^{\infty} \left(x^T Q x + u^T R u\right) dt, \qquad Q \geq 0, \quad R > 0 \tag{7.2}$$

Let P denote the solution of the algebraic Riccati equation corresponding to the standard singularly perturbed control system. This equation is given by

$$A^T P + PA + Q - PSP = 0, \qquad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix} \tag{7.3}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}$$

$$S = \begin{bmatrix} S_1 & \frac{1}{\epsilon}Z \\ \frac{1}{\epsilon}Z^T & \frac{1}{\epsilon^2}S_2 \end{bmatrix} = BR^{-1}B^T, \quad B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} \tag{7.4}$$

The optimal control is represented in terms of P as

$$u = -R^{-1}B^TPx = -F_1x_1 - F_2x_2, \quad x^T = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$$

$$F_1 = R^{-1}\left(B_1^TP_1 + B_2^TP_2^T\right), \quad F_2 = R^{-1}\left(\epsilon B_1^TP_2 + B_2^TP_3\right) \tag{7.5}$$

For the optimal control problem defined by (7.1)-(7.2) and for the *standard singularly perturbed control system* (A_4 nonsingular) the exact pure-slow pure-fast decomposition result of the algebraic Riccati equation, as obtained in (Su *et al.*, 1992b), is presented in the next lemma.

Lemma 7.1 *Consider the closed-loop system*

$$\begin{bmatrix} \dot{x}_1 \\ \epsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 - B_1F_1 & A_2 - B_1F_2 \\ A_3 - B_2F_1 & A_4 - B_2F_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7.6}$$

then there exists a nonsingular transformation $\mathbf{T}$ *such that*

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = \mathbf{T}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} \dot{x}_s = (a_1 + a_2P_s)x_s \\ \epsilon\dot{x}_f = (b_1 + b_2P_f)x_f \end{matrix} \tag{7.7}$$

where P_s and P_f are the unique stabilizing solutions of the exact pure-slow and pure-fast algebraic Riccati equations given by

$$P_sa_1 - a_4P_s - a_3 + P_sa_2P_s = 0$$

$$P_fb_1 - b_4P_f - b_3 + P_fb_2P_f = 0 \tag{7.8}$$

where matrices a_i, b_i, $i = 1, ..., 4$, are obtained from

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_1 - T_2L, \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \epsilon LT_2 \tag{7.9}$$

with

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{7.10}$$

The solution of the original global algebraic Riccati equation (7.3) can be obtained from

$$P = \left\{\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\} \left\{\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\}^{-1} \tag{7.11}$$

where

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \mathbf{\Omega} = E_1 \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon L H \end{bmatrix} E_2 \tag{7.12}$$

with

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{n_2} \end{bmatrix}, \quad E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{7.13}$$

The matrices L and H satisfies the Chang transformation equations

$$\begin{gathered} T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \\ -H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \end{gathered} \tag{7.14}$$

The decomposition transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\Pi_1 + \Pi_2 P) \tag{7.15}$$

with

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = \Omega^{-1} \tag{7.16}$$

◇

All steps in the above lemma can be easily computed by using MATLAB. The pure-slow and pure-fast algebraic Riccati equations (7.8)-(7.9) can be solved in terms of Lyapunov iterations, which is in fact the Newton method for solving (7.8)-(7.9) as demonstrated in (Su *et al.*, 1992b). The initial conditions for the Newton method are obtained from the $O(\epsilon)$-approximate slow and fast algebraic Riccati equations derived in (Chow and Kokotovic, 1976), that is

$$\begin{gathered} A_s^T P_s^{(0)} + P_s^{(0)} A_s + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0 \\ A_4^T P_f^{(0)} + P_f^{(0)} A_4 + Q_3 - P_f^{(0)} S_2 P_f^{(0)} = 0 \end{gathered} \tag{7.17}$$

The matrices A_s, S_s, Q_s can be obtained as in (Wang and Frank, 1992) by using the fact that

$$T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} \tag{7.18}$$

The unique positive semidefinite stabilizing solutions of the above algebraic Riccati equations of the standard singularly perturbed problem ($det A_4 \neq 0$) exist under stabilizability-detectability conditions imposed on the slow and fast subsystems, (Chow and Kokotovic, 1976; Kokotovic *et al.*, 1986), which are the standard assumptions in the theory of singular perturbations. Thus, the following assumption is required.

Assumption 7.1: The triples $(A_s, \mathrm{Chol}(S_s), \mathrm{Chol}(Q_s))$ and (A_4, B_2, Q_3) are stabilizable-detectable.

It has been shown in (Su *et al.*, 1992b) that under the same assumption, the unique solutions of the pure-slow and pure-fast algebraic Riccati equations defined in (7.8) exist for sufficiently small values of the small perturbation parameter ϵ.

The Chang transformation equations (7.14) can be efficiently solved as linear algebraic equations using either fixed point iterations or the Newton method as demonstrated in (Grodt and Gajic, 1988). In addition, equations (7.14) can be solved by using the Taylor series as shown in (Derbel *et al.*, 1994) or the subspace iteration algorithm presented in (Avramovic, 1979). It has been suggested in (Lim and Gajic, 1994) that the eigenvector method can be used in the case when the above methods fail to produce the required solutions.

Solvability of equations (7.14) requires invertibility of the matrix T_4. This matrix has to be nonsingular in order to preserve the slow-fast decomposition of the corresponding state-costate variables, that is, to keep the slow variables slow and the fast variable fast. For standard singularly perturbed systems the matrix T_4 is nonsingular under Assumption 7.1 (Kokotovic *et al.*, 1986). For nonstandard singularly perturbed systems we need another assumption.

Assumption 7.2: The fast Hamiltonian matrix T_4 is nonsingular.

This assumption is satisfied under conditions stated in the lemma established in (Wang and Frank, 1992).

Lemma 7.2 *The matrix T_4 is invertible if and only if*

$$\mathrm{rank}[A_4 \;\; B_2] = n_2 \quad \text{and} \quad \mathrm{rank}\left[A_4^T \;\; C_2^T\right] = n_2 \tag{7.19}$$

◇

Lemma 7.2 produces the required conditions that assure the applicability of the results of (Su *et al.*, 1992b) to nonstandard singularly perturbed control systems.

Note that the controllability-observability condition imposed on the triple (A_4, B_2, C_2) guarantees also the invertability of the matrix T_4, according to Assumption 5.1, as established in (Fridman, 1995; 1996). The conditions stated in Assumption 5.1 are stronger than the one used in Lemma 7.2.

7.2 The Eigenvector Method for Nonsymmetric (Nonsquare) Algebraic Riccati Equation

The eigenvector method for solving the algebraic square and symmetric Riccati equation dates back to the works of (MacFarlane, 1963; Porter, 1966; Fath, 1969). The main results for numerical solution of the symmetric square algebraic Riccati equation by the eigenvector method were obtained by (Van Dooren, 1981). A nice survey of the eigenvector numerical methods for solving the algebraic Riccati equations can be found in (Bunse-Gerstner *et al.*, 1992). Analytical studies of the general nonsquare algebraic Riccati equations were reported in (Clements and Anderson, 1976; Medanic, 1982). Some results on the analytical and numerical properties of the eigenvector method for solving the algebraic Riccati equation can be found in (Bingulac and VanLandingham, 1993).

Without loss of generality, we will present results for the square nonsymmetric algebraic Riccati equation. The same approach can be used for the general nonsquare algebraic Riccati equation. The algebraic nonlinear square nonsymmetric matrix Riccati equation is defined by

$$AX + XB + C + XDX = 0 \tag{7.20}$$

where all matrices are constant and square of dimensions $n \times n$. Consider the following $2n \times 2n$ matrix

$$R = \begin{bmatrix} B & D \\ -C & -A \end{bmatrix} \tag{7.21}$$

Let the eigenvalues and eigenvectors of the matrix R be denoted, respectively, by λ_i, v_i, $i = 1, 2, ..., 2n$. Form a real $2n \times 2n$ matrix M composed of all real eigenvectors of the matrix R and for each complex-conjugate vector put in the matrix M both the real and imaginary parts of that vector and discard its complex conjugate pair. Note that there are many ways to form the matrix M. The matrix M obtained in such a manner has the following property (Bingulac and VanLandingham, 1993)

$$RM = MK = [M_1 \quad M_2]\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{7.22}$$

where the matrix M_1 contains the first n columns of M and the matrix M_2 contains the remaining n column of M. When the eigenvalues are distinct the matrices K_1 and K_2 are diagonal with the eigenvalues of R being the corresponding diagonal elements. In general, for the case of multiple eigenvalues, the matrices K_1 and K_2 represent Jordan forms. Even more

$$RM_1 = M_1 K_1, \qquad RM_2 = M_2 K_2 \tag{7.23}$$

By partitioning the matrix M_1 as

$$M_1 = \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} \tag{7.24}$$

where M_{11} and M_{12} are of dimensions $n \times n$, we obtain from (7.23)

$$\begin{aligned} BM_{11} + DM_{12} &= M_{11}K_1 \\ -CM_{11} - AM_{12} &= M_{12}K_1 \end{aligned} \tag{7.25}$$

By assuming that the matrix M_{11} is nonsingular, from the above equations we have

$$\begin{aligned} K_1 &= M_{11}^{-1}BM_{11} + M_{11}^{-1}DM_{12} \\ -C - AM_{12}M_{11}^{-1} &= M_{12}K_1M_{11}^{-1} \end{aligned} \tag{7.26}$$

By substituting K_1 into the last equation we see that

$$X = M_{12}M_{11}^{-1} \tag{7.27}$$

represents a solution to the original algebraic Riccati equation (7.20). Note that by following the same procedure another solution of (7.20) is given by $X = M_{21}M_{22}^{-1}$ under the assumption that the matrix M_{22} is nonsingular.

It is important to note that each selection of n eigenvectors comprising the matrix M_1 produces a new solution to (7.20), in other words, there are many solutions to the algebraic equation (7.20). The same statement holds for different choices of the matrix M_2 and the corresponding solutions of (7.20) obtained from $X = M_{21}M_{22}^{-1}$. In fact, if there are r $(2n \geq r \geq n)$ distinct real eigenvalues of the matrix R the total number of solutions of (7.20), denoted by s, is (Kecman, 1997)

$$s = \sum_{i=0}^{(2n-r)/2} \binom{\frac{2n-r}{2}}{i} \binom{r}{n-2i} \tag{7.28}$$

When there are less distinct eigenvalues in the matrix R than the complex-conjugate ones, two formulas have to be used for finding the number of solutions to (7.20). For n even and $n \geq r \geq 0$, we have

$$s = \sum_{i=0}^{r/2} \binom{r}{2i} \binom{\frac{2n-r}{2}}{\frac{n-2i}{2}} \tag{7.29}$$

For n odd and $n \geq r \geq 0$, the required number of solutions is given by

$$s = \sum_{i=1}^{r/2} \binom{r}{2i-1} \binom{\frac{2n-r}{2}}{\frac{n-(2i-1)}{2}} \tag{7.30}$$

In a more general setup, when the matrix X is nonsquare of dimensions $m \times n$, equation (7.20) can be solved by following the same procedure. In that case the matrix R is of dimensions $(n+m) \times (n+m)$, and the number of solution for $m > n$ and $m+n \geq r \geq m$ is given by

$$s = \sum_{i=0}^{(n+m-r)/2} \binom{\frac{n+m-r}{2}}{i} \binom{r}{n-2i} \tag{7.31}$$

It has been seen that by using the similarity transformation composed of the eigenvectors of the matrix R, this matrix can be put into diagonal form (or the Jordan form in the case of multiple eigenvalues) defined by (7.22). Another similarity transformation (Smith, 1987) that puts the matrix R into block-diagonal form is also known in the literature. Let

$$\mathrm{T}_1 = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \tag{7.32}$$

where X is a solution of (7.20) then

$$\mathbf{T}_1^{-1} R \mathbf{T}_1 = \begin{bmatrix} B + DX & D \\ 0 & -(A + XD) \end{bmatrix} \tag{7.33}$$

Even more, this upper block diagonal matrix can be put into block diagonal form by using another similarity transformation defined by

$$\mathbf{T}_2 = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \tag{7.34}$$

where Y satisfies the algebraic Lyapunov (Sylvester) equation

$$(B + DX)Y + Y(A + XD) + D = 0 \tag{7.35}$$

The unique solution to equation (7.35) is guaranteed under the assumption that matrices $B + DX$ and $-(A + XD)$ have no eigenvalues in common (Gajic and Qureshi, 1995). The second transformation produces

$$\mathbf{T}_2^{-1} \mathbf{T}_1^{-1} R \mathbf{T}_1 \mathbf{T}_2 = \mathbf{T}_D^{-1} R \mathbf{T}_D = \begin{bmatrix} B + DX & 0 \\ 0 & -(A + XD) \end{bmatrix} \tag{7.36}$$

where

$$\mathbf{T}_D = \begin{bmatrix} I & Y \\ X & I + XY \end{bmatrix}, \quad \mathbf{T}_D^{-1} = \begin{bmatrix} I + YX & -Y \\ -X & I \end{bmatrix} \tag{7.37}$$

The similarity transformation defined by (7.32), (7.35)-(7.37) is valid for both nonsquare and square algebraic Riccati equations.

For the purpose of our chapter the following lemma plays a fundamental role.

Lemma 7.3 *Let the matrix X be a solution of (7.20) obtained by using formula (7.27) with $\begin{bmatrix} M_{11}^T & M_{12}^T \end{bmatrix}^T$ consisting of $l < n$ eigenvectors spanning the stable subspace of R and $n - l$ eigenvectors from the corresponding unstable subspace. Then, the matrix $B + DX$ as defined in (7.33) and (7.36) will have l stable and $n - l$ unstable eigenvalues corresponding to the eigenvectors used in $\begin{bmatrix} M_{11}^T & M_{12}^T \end{bmatrix}^T$.*

◇

Proof: This lemma is just a special case of a more general theorem proved in (Clements and Anderson, 1976, see also Medanic, 1982, Theorem 1).

■

7.3 Exact Decomposition Algorithm for Singularly Perturbed Systems

It can be seen from Lemma 7.1 that in order to solve the linear-quadratic optimal control problem of singularly perturbed systems (7.1)-(7.5) in terms of reduced-order problems, one has in addition to solving algebraic equations (7.14) (whose solutions comprise the desired transformation) also to solve the reduced-order algebraic Riccati equations (7.8). Note that the equations for L, P_s, P_f take the form of nonsymmetric algebraic Riccati equation (7.20). The equation for H in (7.14) is a linear equation, hence its solution is straightforward.

In the following we will show that by using the eigenvector method presented in the previous section that all three equations for L, P_s, P_f can be solved at the same time.

Before we present the decomposition algorithm we first must establish some features of the matrices related with the equation for L. This equation has the form of the general nonsymmetric algebraic Riccati equation (7.20) with the corresponding matrix R_L given by

$$R_L = \begin{bmatrix} -\epsilon T_1 & \epsilon T_2 \\ T_3 & -T_4 \end{bmatrix} \tag{7.38}$$

It can be seen that the matrix R_L has n eigenvalues of $O(\epsilon)$. This implies that for very small values of ϵ the eigenvector method might lead to ill-conditioning. Thus, in the case of singularly perturbed systems the *eigenvector approach has to be used only when ϵ is not very very small.* When ϵ is very small the other methods like fixed point iterations, Newton method or even Taylor series expansions are very efficient for solving the corresponding algebraic equations.

The following lemma is valid for the matrix R_L.

Lemma 7.4 *The eigenvalues of the matrix R_L are symmetric with respect to the imaginary axis.*

◇

Proof: It is well known that the matrix R_P formed of the coefficients of the algebraic Riccati equation (7.3) as

$$R_P = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \tag{7.39}$$

under stabilizability-detectability conditions imposed on the triple $(A, \text{Chol}(S), \text{Chol}(Q))$, has the eigenvalues symmetrically distributed with respect to the imaginary axis (Kwakernaak and Sivan, 1972). By introducing a permutation matrix of the form

$$E = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & -I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & -\epsilon I_{n_2} \end{bmatrix} \tag{7.40}$$

it can be shown by matrix multiplications that the following holds

$$R_L = -\epsilon E R_P E \tag{7.41}$$

This implies that $\lambda(R_L) = -\epsilon\lambda(R_P)$, which establishes the result stated in Lemma 7.4.

●

Now we are ready to formulate the order-reduction algorithms for singularly perturbed systems based on the eigenvector approach.

Algorithm 7.1: Eigenvector Approach.

Step 1: Find the matrix L from (7.14), via the eigenvector approach applied to the matrix R_L defined in (7.38). Let this solution be obtained from a collection of n_1 eigenvectors spanning the stable subspace and n_2 eigenvectors spanning the corresponding unstable subspace.

Step 2: Use the solution obtained in Step 1 in order to solve the algebraic Sylvester equation that has the form of (7.35), that is

$$Y_L(T_4 + \epsilon L T_2) + \epsilon(-T_1 + T_2 L)Y_L + \epsilon T_2 = 0 \tag{7.42}$$

and apply the transformation defined in (7.36)-(7.37) to matrix R_L given in (7.38). This leads to

$$\mathbf{T}_D^{-1} R_L \mathbf{T}_D = \begin{bmatrix} -\epsilon R_s & 0 \\ 0 & -R_f \end{bmatrix}, \quad \mathbf{T}_D = \begin{bmatrix} I & Y_L \\ L & I + LY_L \end{bmatrix} \tag{7.43}$$

Step 3: Partition matrices R_s and R_f as

$$R_s = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad R_f = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \tag{7.44}$$

where $a_i's, i = 1,2,3,4$, are of dimensions $n_1 \times n_1$ and $b_j's, j = 1,2,3,4$, are of dimensions $n_2 \times n_2$. Define the pure-slow and pure-fast algebraic Riccati equations as given in (7.8), that is

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0$$

Form the pure-slow and pure-fast algebraic Lyapunov equations corresponding to (7.35) as

$$
\begin{aligned}
&(a_1 + a_2 P_s)Y_s - Y_s(a_4 - P_s a_2) + a_2 = 0 \\
&(b_1 + b_2 P_f)Y_f - Y_f(b_4 - P_f b_2) + b_2 = 0
\end{aligned}
\tag{7.45}
$$

Then, the similarity transformations, obtained from the solutions of (7.8) and (7.45) as

$$
\mathbf{T}_{sd} = \begin{bmatrix} I & Y_s \\ P_s & I + P_s Y_s \end{bmatrix}, \quad \mathbf{T}_{fd} = \begin{bmatrix} I & Y_f \\ P_f & I + P_f Y_f \end{bmatrix} \tag{7.46}
$$

will block diagonalize, respectively, matrices R_s and R_f along the lines of (7.35)-(7.37), that is

$$
\begin{aligned}
\mathbf{T}_{sd}^{-1} R_s \mathbf{T}_{sd} &= \begin{bmatrix} -\epsilon(a_1 + a_2 P_s) & 0 \\ 0 & -\epsilon(a_4 - P_s a_2) \end{bmatrix} \\
\mathbf{T}_{fd}^{-1} R_f \mathbf{T}_{fd} &= \begin{bmatrix} -(b_1 + b_2 P_f) & 0 \\ 0 & -(b_4 - P_f b_2) \end{bmatrix}
\end{aligned}
\tag{7.47}
$$

Thus, a successive application of transformations (7.43) and (7.47) produces in the new coordinates a four-block block-diagonal form in which the pure-slow and pure-fast state and costate variables are completely decoupled. Note that the transformations defined in (7.43) and (7.47) can be put in a compact form leading to

$$
R_{LD} = \mathbf{T}_{LD}^{-1} R_L \mathbf{T}_{LD}
$$

$$
= \begin{bmatrix} -\epsilon(a_1 + a_2 P_s) & 0 & 0 & 0 \\ 0 & -\epsilon(a_4 - P_s a_2) & 0 & 0 \\ 0 & 0 & -(b_1 + b_2 P_f) & 0 \\ 0 & 0 & 0 & -(b_4 - P_f b_2) \end{bmatrix} \tag{7.48}
$$

where

$$
\mathbf{T}_{LD} = \mathbf{T}_D \begin{bmatrix} \mathbf{T}_{sd} & 0 \\ 0 & \mathbf{T}_{fd} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{sd} & Y_L \mathbf{T}_{fd} \\ L\mathbf{T}_{sd} & \mathbf{T}_{fd} + L Y_L \mathbf{T}_{fd} \end{bmatrix} \tag{7.49}
$$

Step 4: From the obtained values for P_s and P_f calculate the solution of the global algebraic Riccati equation (7.3) by using formula (7.11).

$\triangle$

It can be observed that in Step 3 of Algorithm 7.1, we have to solve the pure-slow pure-fast nonsymmetric square algebraic Riccati equations (7.8). That can be done either by using the Newton method with appropriately chosen initial conditions (which are $O(\epsilon)$ apart from the exact solutions as demonstrated in (Su *et al.*, 1992b)) or by using the eigenvector approach presented in this chapter.

Remark: It is interesting to point out that the Newton method of (Grodt and Gajic, 1988) produces a solution of (7.14) with the minimal infinity norm. Thus, in order to keep in the feedback loop slow variables slow and fast variable fast, we have to use in Step 1 of Algorithm 7.1 the solution for L either obtained by the Newton method (when such a solution is available) or the one obtained by using the eigenvectors of R_L that correspond to its slow stable eigenvalues, which are $O(1)$ in magnitude.

In the following we present another algorithm whose important feature is that the main results of Lemma 7.1 are obtained without the need to solve independently the pure-slow and pure-fast algebraic Riccati equations as defined in (7.8). These solutions are obtained as a by-product of the decomposition algorithm.

Algorithm 7.2: Modified Eigenvector Method.

Step 1: Equal to Step 1 of Algorithm 7.1.

Step 2: Equal to Step 2 of Algorithm 7.1.

Step 3: Calculate the $2n_1 \times 2n_1$ dimensional pure-slow and $2n_2 \times 2n_2$ dimensional pure-fast eigenvector matrices according to the formula

$$\begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = \mathbf{T}_D^{-1}[M_1 \quad M_2] \tag{7.50}$$

where $[M_1 \quad M_2]$ is the eigenvector matrix of R_L obtained in Step 1.

Step 4: Calculate the $n_1 \times n_1$ dimensional solution P_s and the $n_2 \times n_2$ dimensional solution P_f from

$$P_s = M_{12s}M_{11s}^{-1}, \quad P_f = M_{12f}M_{11f}^{-1} \tag{7.51}$$

where the required matrices are obtained by appropriately partitioning matrices M_s and M_f, that is

$$M_s = \begin{bmatrix} M_{11s} & M_{21s} \\ M_{12s} & M_{22s} \end{bmatrix}, \quad M_f = \begin{bmatrix} M_{11f} & M_{21f} \\ M_{12f} & M_{22f} \end{bmatrix} \tag{7.52}$$

Step 5: Partition appropriately matrices R_s and R_f obtained in Step 2 according to formula (7.44) and form the pure-slow and pure-fast optimal feedback matrices, respectively given by

$$a_1 + a_2 P_s, \qquad \frac{1}{\epsilon}(b_1 + b_2 P_f) \tag{7.53}$$

Step 6: Use the values for P_s and P_f obtained in Step 4 to calculate the solution of the global algebraic Riccati equation (7.3) by using formula (7.11).

$\triangle$

Step 3 of Algorithm 7.2 can be justified from the following facts. We have established, in general, that

$$\mathbf{T}_D^{-1} R \mathbf{T}_D = \begin{bmatrix} B + DL & 0 \\ 0 & -(A + LD) \end{bmatrix} \tag{7.54}$$

Since the similarity transformation $\mathbf{T}_D$ preserves the eigenvalues, the pure-slow and pure-fast eigenvectors satisfy

$$\begin{bmatrix} B + DL & 0 \\ 0 & -(A + LD) \end{bmatrix} \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{7.55}$$

or

$$\begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix}^{-1} \begin{bmatrix} B + DL & 0 \\ 0 & -(A + LD) \end{bmatrix} \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{7.56}$$

Using (7.54) in (7.55) we have

$$\begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix}^{-1} \mathbf{T}_D^{-1} R \mathbf{T}_D \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{7.57}$$

Also, it is known from (7.22) that

$$[M_1 \quad M_2]^{-1} R [M_1 \quad M_2] = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{7.58}$$

It follows from the last two formulas that

$$\mathbf{T}_D \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = [M_1 \quad M_2] \tag{7.59}$$

which indicates that the pure-slow pure-fast eigenvectors can be obtained by using a very simple formula, that is

$$\begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} = \mathbf{T}_D^{-1}[M_1 \quad M_2] \tag{7.60}$$

Once the pure-slow and pure-fast eigenvectors are known, Step 4 of Algorithm 7.2 finds the solutions of the pure-slow and pure-fast algebraic Riccati equations by the eigenvector method as described at the beginning of this section. Step 6 finds the solution of the global algebraic Riccati equations that is used to find the optimal value of the performance criterion, and Step 5 produces reduced-order independent pure-slow and pure-fast subsystems as given by (7.7).

Example 7.1: In this example we only look for admissable solutions for the L–equation and discuss the number of possible solutions. Consider the nonstandard singularly perturbed system (the matrix A_4 is nonsingular) taken from (Wang and Frank, 1992). The problem matrices are given by

$$A_1 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} C_1^T C_1 & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{bmatrix}, \quad R = 1$$

A particular feature of this example is that for every ϵ in the interval $[0, 0.097175]$ all the eigenvalues of the matrix R_L (associated with the nonsymmetric algebraic Riccati equation for the matrix L) are real and symmetric with respect to the origin. This implies that there are 70 solutions to algebraic equation (7.14). According to Step 1 of Algorithm 7.1 only 36 solutions among 70 will be admissible, that is, they will produce the desired pure-slow pure-fast decomposition. For a square $n \times n$ matrix L and all real eigenvalues of R_L the number of admissible solutions for L is given by

$$s_{sp} = \binom{n}{2}\binom{n}{2}$$

For $\epsilon = 0.09$, one out of 36 admissible solutions for the matrix L, is the minimal norm solution, which is exactly the one obtained by using the Newton method for solving (7.14), (Su *et al.*, 1992b). This solution is given by

$$L = \begin{bmatrix} 0.0641 & 0.0000 & 0.0000 & 0.5511 \\ 0.0000 & -3.1859 & -0.1200 & 0.0000 \\ 0.0000 & 12.0443 & 0.4460 & 0.0000 \\ -0.0171 & 3.1859 & 0.1200 & -0.0545 \end{bmatrix}$$

By increasing ϵ above $\epsilon_0 = 0.09775$ the Newton method, starting with $L^{(0)} = T_4^{-1}T_3$ as used in (Su *et al.*, 1992b), does not converge. Now there are four real and four complex-conjugate eigenvalues of the matrix R_L. Formula (7.28) implies that there are 14 solutions to L. Out of these 14 solutions only four of them will be admissible (see Step 1 of Algorithm 7.1). For $\epsilon = 0.1$ the eigenvalues of R_L are given by

$$\lambda_{1,2} = \pm 0.9638, \ \lambda_{3,4} = \pm 0.0204$$

$$\lambda_{5,6} = 0.1335 \pm j0.0234, \ \ \lambda_{7,8} = -0.1335 \pm j0.0234$$

For ϵ between 0.1 and 1, the eigenvalues of R_L preserve the same structure, that is, there are four real and four complex conjugate eigenvalues. The eigenvector method in this case produces four admissible solutions for the matrix L and all four of them give the exact desired unique global solution for the matrix P and the desired exact reduced-order decomposition as described in Algorithm 7.1.

$\triangle$

In Section 7.5 we will consider real physical control systems and demonstrate efficiency of the proposed slow-fast decomposition algorithms 7.1 and 7.2.

7.4 Exact Decomposition Algorithm for Regular Systems

The method presented in Section 7.3 may be used for any value of the parameter ϵ, which means that for $\epsilon = 1$ *we get the reduced-order decomposition of the regular linear-quadratic optimal control problem.* Here, we present only preliminary ideas. Further research is needed to impose needed assumptions and justify all steps in the proposed decomposition technique.

For a given regular linear dynamic system of dimension n

$$\dot{x} = Ax + Bu \tag{7.61}$$

with a quadratic performance criterion to be minimized

$$J = \frac{1}{2}\int_{0}^{\infty} \left(x^T Q x + u^T R u\right) dt, \qquad Q \geq 0, \quad R > 0 \tag{7.62}$$

the triple $(A, B, \mathrm{Chol}(Q))$ must be stabilizable-detectable (Kwakernaak and Sivan, 1972), which guarantees the existence of the positive semidefinite stabilizing solution of the corresponding algebraic Riccati equation. Under this assumption the Hamiltonian matrix R_P, defined in (7.39) has n stable and n unstable eigenvalues symmetrically distributed with respect to the imaginary axis. Let us partition the problem matrices as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \tag{7.63}$$

with dimensions for A_4, Q_3 equal to $n_2 \times n_2$, $1 \leq n_2 < n$. Form matrices $T_i, i = 1, 2, 3, 4$, according to formula (7.10). The corresponding matrix R_L will also have n stable and n unstable eigenvalues symmetrically distributed with respect to the imaginary axis, see (7.41) and (7.40) for $\epsilon = 1$. By taking any collection of $n_1 = n - n_2$ eigenvectors spanning stable subspace, then Steps 1–3 of Algorithm 7.1 will produce the reduced-order subsystems, whose forms are given in (7.47) for $\epsilon = 1$ and P_s, P_f playing the roles of the solutions of the subsystem algebraic Riccati equations. Note that according to Lemma 7.3 both reduced-order subsystem matrices $(a_1 + a_2 P_s)$ and $(b_1 + b_2 P_f)$ are stable. Similarly, one can use Algorithm 7.2 to achieve this goal.

The last steps of both algorithms, in which the global solution P is found in terms of the local solutions, are carried out under the assumption that the inversion defined in formula (7.11) exists. Further research is needed to justify this step. Note that for singularly perturbed systems, it has been analytically proved in (Su *et al.*, 1992b) that the corresponding inversion exists. Example 7.3, done in the next section, demonstrates the reduced-order decomposition for a regular (standard) problem, that is, for $\epsilon = 1$.

7.5 Case Studies

In this section we solve two real-world control system examples in order to demonstrate the presented procedures. In Case Study 7.5.1, we consider a fluid catalytic cracker and solve the decomposition problem by following Algorithm 7.2 (exactly the same results are obtained by using Algorithm 7.1). Case Study 7.5.2 is done for the general case, that is, for $\epsilon = 1$. It is demonstrated on the example of an inverted pendulum.

7.5.1 Case Study: Fluid Catalytic Reactor

Consider a fluid catalytic reactor (Arkun and Ramakrishnan, 1983) that represents the standard singularly perturbed system with two slow and three fast modes. The problem matrices are given by

$$A_1 = \begin{bmatrix} -16.11 & -0.39 \\ 0.01 & -16.99 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 27.2 & 0 & 0 \\ 0 & 0 & 12.47 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1.511 & 0 \\ -5.336 & 0 \\ 0.227 & 6.91 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -5.36 & -1.657 & 7.178 \\ 0 & -10.72 & 23.211 \\ 0 & 0.2273 & -10.299 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 11.12 & -12.6 \\ -3.61 & 3.36 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2.191 & 0 \\ -5.36 & 0 \\ 6.91 & 0 \end{bmatrix}, \quad Q = I_5, \quad R = I_2$$

There are two slow $(-2.8, -7.7)$ and three fast eigenvalues $(-74, -82, -129)$ in this system. The small singular perturbation parameter $\epsilon = 0.1$ is roughly the ratio of 7.7 and 74.

The matrix R_L for all values of $\epsilon \in (0, 0.875)$ has six real and four complex conjugate eigenvalues. There are 46 choices for the eigenvector matrix M_1 which produce solutions to the corresponding algebraic Riccati equations, but only 16 of them are admissable solutions for the purpose of the reduced-order pure-slow and pure-fast decomposition. Beyond $\epsilon = 0.875$ and up to $\epsilon = 1$, the matrix R_L has all real eigenvalues (ten of them) which leads to 210 possible solutions, among them 100 solutions are admissible.

For $\epsilon = 0.1$, the admissible solution for L with the smallest norm is given by

$$L = \begin{bmatrix} -0.4966 & -0.3453 & -0.8581 & 0.3230 \\ 0.4566 & -0.8681 & 7.9021 & 1.5818 \\ -0.1048 & -0.1924 & 4.8992 & 1.4245 \\ 0.0169 & -0.0749 & -5.3177 & 0.1886 \\ 0.0258 & -0.0656 & 1.5040 & 0.1244 \\ 0.0557 & -0.1819 & 0.0238 & -0.7031 \end{bmatrix}$$

The solution of the Sylvester equation (7.42) is

$$Y_L = \begin{bmatrix} 0.4167 & -0.1132 & -0.0072 & -0.0796 & 0.5688 & 0.3550 \\ -0.0142 & -0.0123 & 0.0691 & 0.0429 & 0.1463 & 0.1431 \\ 0.0014 & 0.0020 & 0.0039 & 0.0386 & -0.0356 & 0.0076 \\ -0.0049 & -0.0074 & -0.0183 & 0.0259 & 0.0939 & 0.0195 \end{bmatrix}$$

The solutions of the pure-slow and pure-fast algebraic Riccati equations (7.8) are

$$P_s = \begin{bmatrix} 0.0699 & -0.0017 \\ -0.0111 & 0.0866 \end{bmatrix}, \quad P_f = \begin{bmatrix} 0.0716 & 0.0012 & 0.0416 \\ -0.0072 & 0.0468 & 0.0457 \\ 0.0124 & 0.0496 & 0.1533 \end{bmatrix}$$

The solution of the global algebraic Riccati equation (7.3), obtained by using (7.11) is

$$P = \begin{bmatrix} 0.0519 & -0.0050 & 0.0228 & -0.0053 & -0.0003 \\ -0.0050 & 0.0832 & 0.0005 & 0.0024 & 0.0183 \\ 0.0228 & 0.0005 & 0.0196 & -0.0028 & 0.0038 \\ -0.0053 & 0.0024 & -0.0028 & 0.0052 & 0.0046 \\ -0.0003 & 0.0183 & 0.0038 & 0.0046 & 0.0171 \end{bmatrix}$$

Note that for every choice of eigenvectors in M_1 we get different solutions for L, and thus, for P_s and P_f, but the global solution of (7.3), obtained by using (7.11) is always the same. Note that the solution for P obtained through Algorithms 7.1 or 7.2 is exact up to used computer accuracy, in our case the MATLAB package has produced the accuracy of $O\left(10^{-14}\right)$. The same accuracy holds for the optimal value of the performance criterion defined in (7.2).

7.5.2 Case Study: Inverted Pendulum

Here we solve a regular (standard) linear-quadratic optimal control problem, the classic example of an inverted pendulum mounted on a cart. This is an open-loop unstable fourth-order system with state variables representing position and speed of the cart and angular position and velocity of the inverted pendulum (Kecman, 1988). For the mass of the cart, $m_1 = 2\,\mathrm{kg}$, mass of the pendulum concentrated at its tip, $m_2 = 1\,\mathrm{kg}$, and the pendulum length, $l = 1\,\mathrm{m}$, the following matrices are obtained from (Kecman, 1988)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -4.905 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 14.715 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \end{bmatrix}$$

The eigenvalues of this system are given by $\lambda \in (0, 0, -3.836, 3.836)$. Note the open-loop unstable eigenvalue at 3.836. The performance criterion matrices are taken as $Q = I_4, R = 1$. This real system is composed of two second-order subsystems, cart and pendulum, hence the natural decomposition requires $n_1 = 2, n_2 = 2$.

An admissible solution for L, which is in fact the minimal norm solution is given by

$$L = \begin{bmatrix} 0.0004 & 0.0000 & 0.0000 & 0.0114 \\ 0.0000 & -0.0110 & -0.0114 & 0.0000 \\ 0.0000 & -0.3035 & -0.3250 & 0.0000 \\ -0.0220 & 0.0000 & 0.0000 & -0.3364 \end{bmatrix}$$

The solution of the Sylvester equation (7.42) is

$$Y_L = \begin{bmatrix} 0.3229 & 0.0000 & 0.0000 & -0.0113 \\ 0.0000 & 0.3342 & 0.0113 & 0.0000 \\ 0.0000 & -0.0218 & -0.0004 & 0.0000 \\ -0.3014 & 0.0000 & 0.0000 & 0.0109 \end{bmatrix}$$

The corresponding solutions of the reduced-order algebraic Riccati equations (7.8) have the form

$$P_s = \begin{bmatrix} 2.6631 & 3.0124 \\ 3.0797 & 8.0222 \end{bmatrix}, \quad P_f = \begin{bmatrix} 450.4339 & 116.9835 \\ 116.8961 & 30.4849 \end{bmatrix}$$

and the solution of the global algebraic Riccati equation (7.3), obtained by using (7.11) is

$$P = \begin{bmatrix} 3.1845 & 4.5707 & 20.3622 & 6.5707 \\ 4.5707 & 12.4798 & 58.2736 & 18.8488 \\ 20.3622 & 58.2736 & 728.0953 & 206.8098 \\ 6.5707 & 18.8488 & 206.8098 & 59.5733 \end{bmatrix}$$

We have experimented with this example also for $n_1 = 1$ and $n_1 = 3$ and in both cases we got the desired reduced-order decomposition.

7.6 Conclusions

In this chapter we have used the eigenvector method to solve the slow-fast order-reduction problem of linear singularly perturbed systems. The advantage of this method over the ones previously used is in the fact that once the eigenvectors of the Hamiltonian matrix are determined the solution of the reduced-order pure-slow and pure-fast algebraic Riccati equations are obtained through very simple matrix multiplications. It should be emphasized that the results obtained present the exact system decomposition and that the solution for the global algebraic Riccati equation is also exactly obtained (up to the computer's precision). The presented method is also extended to the reduced-order decomposition of regular linear-quadratic optimal control problems, which is valid under the assumption that the corresponding matrix inversion defined in (7.11) exists. Establishing under what conditions this matrix is invertible should be the subject of future research.

Using duality between the linear-quadratic optimal control and the Kalman filtering, the results reported in this chapter can be also applied for the reduced-order Kalman filtering problems of both singularly perturbed (Gajic and Lim, 1994) and regular (Hong, 1992) linear stochastic systems. An extension to the H_∞ optimal control and filtering problems is also possible. The discrete-time version of the eigenvector method for singularly perturbed linear optimal control and filtering systems is an interesting research topic. The above problems are currently under research by the authors.

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8

Additional Topics

In this chapter we discuss some additional topics related to the Hamiltonian approach to singularly perturbed linear-quadratic optimal control systems. In that direction, we consider optimal control of nonstandard continuous-time singularly perturbed linear systems, continuous-time finite horizon feedback optimization, and present the main results about the slow-fast integral manifold approach for linear-quadratic optimization problems.

In Section 8.1 we show how to exactly decompose the optimal control and filtering tasks in terms of reduced-order pure-slow and pure-fast subproblems for both the optimal control and filtering problems of nonstandard singularly perturbed linear systems. The proposed methods achieve the exact accuracy, in contrast to the decomposition methods available in the literature that produce only an $O(\epsilon)$ accuracy, where ϵ represents a small positive singular perturbation parameter. A real world control system example is solved in order to demonstrate perfect pure-slow/pure-fast decoupling of nonstandard singularly perturbed linear control and filtering tasks. The results are presented in the continuous-time domain. The discrete-time nonstandard singularly perturbed systems have not been studied in the control literature. Obtaining the pure-slow and pure-fast decomposition of nonstandard discrete-time linear control systems is a challenging research topic.

An extension of the steady state feedback optimization, presented in the previous chapters, to the continuous-time finite horizon feedback optimization problem is considered in Section 8.2. The presented results are incomplete and only guidelines are given since the research in that direction is still underway in (Coumarbatch, 2000).

In Section 8.3, we present the slow-fast integral manifold theory of Fridman, Sobolev, and Strygin in the context of linear-quadratic optimization. We also summarize the results of Fridman, who in a series of important papers has extended the slow-fast integral manifold theory to various optimal control problems of singularly perturbed linear and nonlinear systems.

8.1 Nonstandard Continuous-Time Singularly Perturbed Linear Systems

A powerful algorithm, based on results of (Su *et al.*, 1992a), for the exact slow-fast decomposition of the continuous-time algebraic Riccati equation of standard singularly perturbed systems is developed in Chapter 2 so that the optimal control and filtering tasks can be solved *exactly* and *performed independently* in slow and fast time scales. In this section, we show that the same algorithm, under the appropriate assumptions is applicable to the continuous-time algebraic Riccati equation of nonstandard singularly perturbed control systems (having singular fast subsystem matrix). Nonstandard singularly perturbed systems are the modern research trend in control theory of singular perturbations (Kokotovic *et al.*, 1986; Khalil, 1989; Wang and Frank, 1992; Wang *et al.*, 1994). The result obtained for the decomposition of the algebraic Riccati equation is used in this section to obtain the *exact* pure-slow and pure-fast decomposition of optimal control and filtering tasks of nonstandard singularly perturbed linear systems. Note that in the control literature *only approximate* results for nonstandard singularly perturbed systems are available.

Before the results of (Su *et al.*, 1992a) were available, the control engineers were able to decompose *exactly* only *linear* singularly perturbed systems by using the celebrated Chang transformation, (Chang, 1972). As a matter of fact, in (Chang, 1972) a general boundary value problem of singularly perturbed linear systems is studied. Since that time the Chang transformation has been used very often in the engineering literature of singularly perturbed linear control systems. A new version of the Chang transformation is developed in (Qureshi and Gajic, 1992).

A comprehensive overview of decoupling transformations for linear dynamical systems with emphasis on the work of (Chang, 1972) can be found in (Smith, 1987; see also (Gajic and Shen, 1993, Chapter 3, where continuous- and discrete-time versions of the Chang transformation are presented)).

In (Chow and Kokotovic, 1976) the *nonlinear* algebraic Riccati equation was decomposed into approximate slow and fast algebraic Riccati equations *with the accuracy of* $O(\epsilon)$, where ϵ is a small positive singular perturbation parameter. Note that $O(\epsilon)$ stands for $k\epsilon$, where k is a bounded constant. Hence, based on the above definition, an $O(\epsilon)$ accuracy does not necessarily mean a very high accuracy. Several real world examples done in (Gajic *et al.*, 1989; Skataric and Gajic, 1992; Mizukami and Suzumura, 1993; Gajic and Shen, 1993) indicate that often an $O(\epsilon)$ order of accuracy is not satisfactory. The results of (Su *et al.*, 1992a) are as a matter of fact the extended and improved results of (Chow and Kokotovic, 1976). It can be said that the results of (Su *et al.*, 1992a) achieve the same goal as the results of (Chow and Kokotovic, 1976), but with *perfect accuracy*.

The approach taken in (Su *et al.*, 1992a) was based on block diagonalization of the singularly perturbed Hamiltonian matrices. It has been previously used in (Grodt and Gajic, 1988) for the exact solution of the differential singularly perturbed Riccati equation and in (Su *et al.*, 1992b) for the *exact* slow-fast decomposition of the open-loop singularly perturbed linear-quadratic control problem.

The goal of this section is to show that theory developed in (Su *et al.*, 1992a) and related work, (Grodt and Gajic, 1988; Su *et al.*, 1992b; Gajic and Shen, 1993; Gajic and Lim, 1994) can be extended to the nonstandard singularly perturbed systems. It should be pointed out that mechanical control systems in the modal coordinates (Baruh and Choe, 1990) displaying slow and fast time scales are nonstandard singularly perturbed linear control systems—for example, the linearized model of a flexible space structure (Moerder and Calise, 1985).

Conditions under which the first approximation of nonstandard singularly perturbed control systems can be studied are established in (Wang and Frank, 1992). It is important to point out that the results of (Wang and Frank, 1992) produce an $O(\epsilon)$ accuracy only. In contrast, the results of this section produce the exact solution and preserve the slow-fast decomposition features of (Wang and Frank, 1992).

8.1.1 Optimal Control of Nonstandard Linear Systems

A nonstandard singularly perturbed control linear system is represented by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) \end{aligned} \tag{8.1}$$

where $x_1(t) \in \Re^{n_1}$ are slow and $x_2(t) \in \Re^{n_2}$ are fast system state space variables, $u(t) \in \Re^m$ is a vector input, and ϵ is a small positive singular perturbation parameter. Matrices A_i, $i = 1, ..., 4$, and B_j, $j = 1, 2$, are constant and of appropriate dimensions with A_4 *being singular.* Singularity of A_4 indicates the nonstandard singularly perturbed linear control system (Khalil, 1989). In the case when the matrix A_4 is nonsingular we have the so-called standard singularly perturbed control system (Kokotovic *et al.*, 1986).

With (8.1) a quadratic performance criterion to be minimized by the control action is associated

$$J = \frac{1}{2} \int_0^\infty \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt, \qquad Q \geq 0, \quad R > 0 \tag{8.2}$$

It is interesting to point out that the famous Chang transformation is not directly applicable to singularly perturbed systems having singular fast system matrix A_4. Also, the results of (Chow and Kokotovic, 1976) are not applicable for the slow-fast decomposition of the corresponding algebraic Riccati equation since they require nonsingularity of A_4. However, in the following we show that the results of (Su *et al.*, 1992a) can be applied under certain assumptions to both standard and nonstandard singularly perturbed control systems.

For the optimal control problem defined by (8.1)-(8.2) and for the *standard singularly perturbed control system* (A_4 nonsingular) the exact pure-slow pure-fast decomposition result of the algebraic Riccati equation is obtained, (Su *et al.*, 1992a). For the reason of completeness of this section, the decomposition results of (Su *et al.*, 1992a) are summarized in the lemma given below. In that lemma we have also simplified and algorithmically organized the main steps on the exact pure-slow pure-fast decomposition of singularly perturbed linear control systems originally obtained in (Su *et al.*, 1992a).

Let P be the solution of the algebraic Riccati equation corresponding to the standard singularly perturbed control system. This equation is given by

$$A^T P + PA + Q - PSP = 0, \qquad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix} \tag{8.3}$$

with

$$\begin{gathered} A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon} A_3 & \frac{1}{\epsilon} A_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix} \\ S = \begin{bmatrix} S_1 & \frac{1}{\epsilon} Z \\ \frac{1}{\epsilon} Z^T & \frac{1}{\epsilon^2} S_2 \end{bmatrix} = BR^{-1}B^T, \quad B = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{bmatrix} \end{gathered} \tag{8.4}$$

The optimal feedback control is given in terms of P as

$$\begin{gathered} u(t) = -R^{-1}B^T P x(t) = -F_1 x_1(t) - F_2 x_2(t) \\ x^T(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} \\ F_1 = R^{-1}\left(B_1^T P_1 + B_2^T P_2^T\right), \quad F_2 = R^{-1}\left(\epsilon B_1^T P_2 + B_2^T P_3\right) \end{gathered} \tag{8.5}$$

Lemma 8.1 *Consider the closed-loop system*

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 - B_1 F_1 & A_2 - B_1 F_2 \\ A_3 - B_2 F_1 & A_4 - B_2 F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{8.6}$$

There exists a nonsingular transformation $\mathbf{T}$ *such that*

$$\begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \dot{x}_s(t) &= (a_1 + a_2 P_s) x_s(t) \\ \epsilon \dot{x}_f(t) &= (b_1 + b_2 P_f) x_f(t) \end{aligned} \tag{8.7}$$

where P_s *and* P_f *are the unique solutions of the exact pure-slow and pure-fast algebraic Riccati equations given by*

$$\begin{gathered} P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0 \\ P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0 \end{gathered} \tag{8.8}$$

where matrices $a_i, b_i,\ i = 1, \ldots, 4$, *are obtained from*

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_1 - T_2 L, \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \epsilon L T_2 \tag{8.9}$$

with

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$
$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{8.10}$$

The matrix L satisfies the Chang transformation equations

$$\begin{gathered} T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \\ -H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \end{gathered} \tag{8.11}$$

The solution of the original global algebraic Riccati equation (8.3) can be obtained from

$$P = \left\{\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\}\left\{\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\}^{-1} \tag{8.12}$$

where

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \mathbf{\Omega} = E_1 \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon L H \end{bmatrix} E_2 \tag{8.13}$$

with

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{n_2} \end{bmatrix}, \quad E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{8.14}$$

The decomposition transformation $\mathbf{T}$ *is given by*

$$\mathbf{T} = (\Pi_1 + \Pi_2 P) \tag{8.15}$$

with

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = \Omega^{-1} \tag{8.16}$$

◇

For standard singularly perturbed systems, the above lemma is valid under the assumption that the slow and fast subsystems are stabilizable-detectable (Chow and Kokotovic, 1976). Let

$$A_0 = A_1 - A_2 A_4^{-1} A_3, \quad B_0 = B_1 - A_2 A_4^{-1} B_2, \quad q_0 = q_1 - q_2 A_4^{-1} A_3$$

then, the required assumption for standard singularly perturbed linear systems is given below.

Assumption 8.1: The triples (A_0, B_0, q_0) and (A_4, B_2, q_2) are stabilizable-detectable.

For nonstandard singularly perturbed systems, we are even not able to form the matrices A_0, B_0, q_0. However, in such a case, we can define the matrices A_s, S_s, Q_s as (Wang and Frank, 1992)

$$T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} \tag{8.17}$$

The assumption needed for nonstandard linear singularly perturbed control systems is as follows.

Assumption 8.2: The triples $(A_s, \mathrm{Chol}(S_s), \mathrm{Chol}(Q_s))$ and (A_4, B_2, q_2) are stabilizable-detectable.

The matrices A_s, S_s, Q_s are obtained in (Wang and Frank, 1992) as

$$T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} \tag{8.18}$$

All steps in the above lemma can be easily computed by using MATLAB. The pure-slow and pure-fast algebraic Riccati equations (8.8)-(8.9) can be solved in terms of Lyapunov iterations, which is in fact the Newton method for solving (8.8)-(8.9) as demonstrated in (Su *et al.*, 1992a). The initial conditions for the Newton method are obtained from the $O(\epsilon)$-approximate slow and fast algebraic Riccati equations derived in (Wang and Frank, 1992), that is

$$\begin{aligned} A_s^T P_s^{(0)} + P_s^{(0)} A_s + Q_s - P_s^{(0)} S_s P_s^{(0)} &= 0 \\ A_4^T P_f^{(0)} + P_f^{(0)} A_4 + Q_3 - P_f^{(0)} S_2 P_f^{(0)} &= 0 \end{aligned} \tag{8.19}$$

The unique positive semidefinite stabilizing solutions of the above algebraic Riccati equations exist under Assumption 8.2. By the Implicit Function Theorem, the unique solutions of the pure-slow and pure-fast algebraic Riccati equations (8.8) exist for sufficiently small values of the small perturbation parameter ϵ since $P_s - P_s^{(0)} = O(\epsilon)$ and $P_f - P_f^{(0)} = O(\epsilon)$.

The Chang transformation equations (8.11) can be solved as linear equations by using either the fixed point iterations or by the Newton method as demonstrated in (Grodt and Gajic, 1988). In addition, they can be solved by using the Taylor series as demonstrated in (Derbel *et al.*, 1994) and by the eigenvector method of (Kecman *et al.*, 1999).

Solvability of equations (8.11) requires invertibility of the matrix T_4. In addition, this matrix has to be nonsingular in order to preserve the slow-fast decomposition of the corresponding state-costate variables, that is, to keep slow variables slow and fast variable fast.

Note that in (Khalil, 1989), where the frequency domain technique is used to study nonstandard singularly perturbed linear systems stabilizability-detectability conditions of the slow and fast subsystems are imposed. In (Wang and Frank, 1992) the linear-quadratic control problem of nonstandard singularly perturbed systems is solved with the accuracy of $O(\epsilon)$ by requiring nonsingularity of T_4. The following lemma is established in (Wang and Frank, 1992).

Lemma 8.2 *The matrix T_4 is invertible if and only if*

$$\operatorname{rank}[A_4 \;\; B_2] = n_2 \quad \text{and} \quad \operatorname{rank}\left[A_4^T \;\; D_2^T\right] = n_2 \tag{8.20}$$

where $D_2^T D_2 = Q_3$.

◇

Lemma 8.2 produces the required conditions that assures invertibility of T_4 and applicability of results of (Su *et al.*, 1992a) for solving the algebraic Riccati equation of nonstandard singularly perturbed systems in terms of the reduced-order pure-slow and pure-fast algebraic Riccati equations. However, the stabilizability-detectability of the triple (A_4, B_2, q_2) also guarantees the invertibility of the matrix T_4. Thus, the lemma established in (Wang and Frank, 1992) states another set of conditions under which the matrix T_4 is invertible.

Example 8.1: In order to compare the results this section and that of (Wang and Frank, 1992) and to demonstrate an improvement over the results of (Wang and Frank, 1992), we have considered the following example.

$$A_1 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0.524 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; R = 1$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; Q_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x(0) = [1 \quad 0 \quad 1 \quad 0]^T$$

In Table 8.1 we compare for different values of the small singular perturbation parameter ϵ the values for the approximate optimal criterion, J_{WF92}, obtained by using the methodology of (Wang and Frank, 1992) and the optimal criterion values obtained by using the technique presented in this section.

Table 8.1: A comparison of the two methods

ϵ	J_{opt}	J_{WF92}	$J_{opt} - J_{WF92}$ (%)
0	3.1423	3.1423	0
0.01	3.2530	3.2548	0.06
0.05	3.7246	3.7780	1.43
0.1	4.3813	4.6392	5.89
0.25	6.8166	10.1217	48.5

It can be seen from Table 8.1 that for very small values of ϵ the satisfactory results are obtained by both methods. However, for relatively bigger values of ϵ the results of (Wang and Frank, 1992) are not accurate; hence the reduced-order slow-fast decomposition technique proposed in this section has to be used.

Δ

8.1.2 Kalman Filtering for Nonstandard Linear Systems

A linear stochastic nonstandard singularly perturbed system is represented by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + G_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + G_2 w(t) \end{aligned} \tag{8.21}$$

where $w(t)$ represents an r-dimensional Gaussian zero-mean stationary white noise stochastic process with intensity matrix $W \geq 0$. Matrices

G_1 and G_2 are constant and of appropriate dimensions. It should be emphasized that the matrix A_4 is singular, in contrast to the Kalman filtering problem of standard singularly perturbed systems where this matrix is nonsingular, (Gajic and Lim, 1994). With system (8.21) a measurement equation is associated in the form

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + v(t) \tag{8.22}$$

where $y(t)$ is a p–dimensional measurement vector, $v(t)$ is a p–dimensional measurement zero-mean stationary Gaussian white noise stochastic process with intensity matrix $V > 0$, and C_1, C_2 are constant matrices of appropriate dimensions.

The Kalman filtering problem can be studied by using duality with the optimal linear-quadratic control problem. In this case, the duality is achieved by replacing matrices T_1, T_2, T_3, T_4, defined in (8.10), respectively by the following matrices

$$\begin{aligned}
T_{1_F} &= \begin{bmatrix} A_1^T & -C_1^T V^{-1} C_1 \\ -G_1 W G_1^T & -A_1 \end{bmatrix} \\
T_{2_F} &= \begin{bmatrix} A_3^T & -C_1^T V^{-1} C_2 \\ -G_1 W G_2^T & -A_2 \end{bmatrix} \\
T_{3_F} &= \begin{bmatrix} A_2^T & -C_2^T V^{-1} C_1 \\ -G_2 W G_1^T & -A_3 \end{bmatrix} \\
T_{4_F} &= \begin{bmatrix} A_4^T & -C_2^T V^{-1} C_2 \\ -G_2 W G_2^T & -A_4 \end{bmatrix}
\end{aligned} \tag{8.23}$$

Using (8.23), we can form the matrix dual to the matrix defined in (8.17), that is

$$T_{1_F} - T_{2_F} T_{4_F}^{-1} T_{3_F} = \begin{bmatrix} A_{s_F}^T & -V_s \\ -W_s & -A_{s_F} \end{bmatrix} \tag{8.24}$$

The following assumption, dual to Assumption 8.2, will be needed in the remaining part of this section.

Assumption 8.3: The triples $(A_{s_F}, \mathrm{Chol}(V_s), \mathrm{Chol}(W_s))$ and (A_4, C_2, G_2) are stabilizable-detectable.

Lemma 8.2 of (Wang and Frank, 1992) in the case of Kalman filtering of nonstandard singularly perturbed linear systems should be replaced by a dual lemma stated below.

Lemma 8.3 *The matrix T_{4_F}, defined by (8.23), is invertible if and only if*

$$\operatorname{rank}\left[A_4^T \;\; C_2^T\right] = n_2 \quad \text{and} \quad \operatorname{rank}[A_4 \; G_2] = n_2 \tag{8.25}$$

However, as pointed out for the regulator problem, the stabilizability-detectability conditions imposed on the fast subsystem in Assumption 8.3 also guarantee nonsingularity of the matrix T_{4F}.

According to the results of (Gajic and Lim, 1994), the exact reduced-order, independent, pure-slow and pure-fast, Kalman filters driven by the system measurements are now given by

$$\begin{aligned} \dot{\hat{x}}_s(t) &= \left(a_{1_F} + a_{2_F} P_{s_F}\right)^T \hat{x}_s(t) + K_s y(t) \\ \epsilon \dot{\hat{x}}_f(t) &= \left(b_{1_F} + b_{2_F} P_{f_F}\right)^T \hat{x}_f(t) + K_f y(t) \end{aligned} \tag{8.26}$$

where the newly defined matrices are

$$\begin{bmatrix} a_{1_F} & a_{2_F} \\ a_{3_F} & a_{4_F} \end{bmatrix} = \left(T_{1_F} - T_{2_F} L_F\right), \quad \begin{bmatrix} b_{1_F} & b_{2_F} \\ b_{3_F} & b_{4_F} \end{bmatrix} = \left(T_{4_F} + \epsilon L_F T_{2_F}\right) \tag{8.27}$$

$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon} K_f \end{bmatrix} = \left(\Pi_{1_F} + \Pi_{2_F} P_F\right)^{-T} P_F \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} V^{-1} \tag{8.28}$$

The matrix P_F can be obtained by using formula (8.12) with the solutions of the corresponding pure-slow and pure-fast algebraic filter Riccati equations obtained from

$$\begin{aligned} P_{s_F} a_{1_F} - a_{4_F} P_{s_F} - a_{3_F} + P_{s_F} a_{2_F} P_{s_F} &= 0 \\ P_{f_F} b_{1_F} - b_{4_F} P_{f_F} - b_{3_F} + P_{f_F} b_{2_F} P_{f_F} &= 0 \end{aligned} \tag{8.29}$$

The remaining matrices in (8.12) are

$$\begin{aligned} \Omega_F^{-1} &= \begin{bmatrix} \Omega_{1_F} & \Omega_{2_F} \\ \Omega_{3_F} & \Omega_{4_F} \end{bmatrix}^{-1} = \begin{bmatrix} \Pi_{1_F} & \Pi_{2_F} \\ \Pi_{3_F} & \Pi_{4_F} \end{bmatrix} \\ &= E_2^T \begin{bmatrix} I - \epsilon H_F L_F & -\epsilon H_F \\ L_F & I \end{bmatrix} E_3 \end{aligned} \tag{8.30}$$

where

$$E_3 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & \frac{1}{\epsilon}I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{8.31}$$

The Chang transformation Kalman filter decoupling algebraic equations are given by

$$\begin{gathered} T_{4F}L_F - T_{3F} - \epsilon L_F(T_{1F} - T_{2F}L_F) = 0 \\ -H_F(T_{4F} + \epsilon L_F T_{2F}) + T_{2F} + \epsilon(T_{1F} - T_{2F}L_F)H_F = 0 \end{gathered} \tag{8.32}$$

The initial conditions for the Newton method for solving the pure-slow and pure-fast algebraic Riccati equations are obtained respectively from the approximate, reduced-order, pure-slow and pure-fast, symmetric, algebraic filter Riccati equations

$$\begin{gathered} P_{s_F}^{(0)}A_{s_F}^T + A_{s_F}P_{s_F}^{(0)} + W_s - P_{s_F}^{(0)}V_s P_{s_F}^{(0)} = 0 \\ P_{f_F}^{(0)}A_4^T + A_4 P_{f_F}^{(0)} + G_2 W G_2^T - P_{f_F}^{(0)} C_2^T V^{-1} C_2 P_{f_F}^{(0)} = 0 \end{gathered} \tag{8.33}$$

The unique positive definite stabilizing solutions of (8.33) exist under Assumption 8.3. It can be easily shown that these initial conditions are $O(\epsilon)$ approximations of the exact solutions.

The optimal estimates obtained from (8.26) are related at steady state to the optimal estimates of the state variables given in (8.21) by the following nonsingular transformation

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = (\Pi_{1_F} + \Pi_{2_F} P_F)^T \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_f(t) \end{bmatrix} \tag{8.34}$$

8.1.3 Linear Quadratic Optimal Stochastic Controller

In this section we combine control and filtering results from Sections 8.1.2 and 8.1.3 in order to solve the optimal control singularly perturbed stochastic problem. Consider the linear-quadratic optimal stochastic control problem defined by the system state space equation

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \qquad G = \begin{bmatrix} G_1 \\ \frac{1}{\epsilon}G_2 \end{bmatrix} \tag{8.35}$$

system measurements

$$y(t) = Cx(t) + v(t), \quad C = [C_1 \quad C_2] \tag{8.36}$$

and the performance criterion to be minimized by the control action

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} E\left\{ \int_0^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt \right\} \tag{8.37}$$

The optimal feedback solution to this problem is given by the well-known separation principle (Kwakernaak and Sivan, 1972)

$$u_{opt}(t) = -\mathbf{F}\hat{x}(t) \tag{8.38}$$

where the optimal regulator gain $\mathbf{F} \in \Re^{m \times n}$ is defined in (8.5) and the optimal estimate $\hat{x}(t)$ is obtained from (8.34).

By using (8.34) in (8.38) we have

$$u_{opt}(t) = -\mathbf{F}\hat{x}(t) = -\mathbf{F}(\Pi_{1_F} + \Pi_{2_F} P_F)^T \begin{bmatrix} \hat{x}_s \\ \hat{x}_f \end{bmatrix} = -\mathbf{F_1}\hat{x}_s(t) - \mathbf{F_2}\hat{x}_f(t) \tag{8.39}$$

where $\mathbf{F} \in \Re^{m \times n}$ and $\mathbf{F}_i \in \Re^{m \times n_i}, i = 1, 2$. The new optimal gain $\mathbf{F}$ can be expressed in terms of the quantities defined in the previous sections

$$\mathbf{F} = [\mathbf{F}_1 \quad \mathbf{F}_2] = R^{-1}B^T P(\Pi_{1F} + \Pi_{2F} P_F)^T \tag{8.40}$$

where P and P_F are the positive semidefinite stabilizing solutions of the regulator and filter algebraic Riccati equations, obtained respectively from the reduced-order independent subsystem algebraic Riccati equations (8.8) and (8.29) and formula (8.12).

Based on the results presented in the previous sections, it can be concluded that completely decoupled optimal local filters driven by system measurements and control inputs are given by

$$\begin{aligned} \dot{\hat{x}}_s(t) &= (a_{1F} + a_{2F} P_{sF})^T \hat{x}_s(t) + B_s u(t) + K_s y(t) \\ \epsilon\dot{\hat{x}}_f(t) &= (b_{1F} + b_{2F} P_{fF})^T \hat{x}_f(t) + B_f u(t) + K_f y(t) \end{aligned} \tag{8.41}$$

with K_s and K_f defined in (8.28). It can be seen that the pure-slow and pure-fast Kalman filters are of the reduced-order, well-conditioned, and completely decoupled. Hence, they process input and measurement signals in parallel, the slow ones with the slow sampling rate and the fast ones with the fast sampling rate. Note that the original global full-order Kalman filter is ill-conditioned since it has to process both the slow and fast signals with the fast sampling rate. Note that the results obtained are valid at steady state.

Similarly to (8.28), the matrices B_s and B_f are obtained from (note that we have to use the state transformation (8.34))

$$\begin{bmatrix} B_s \\ \frac{1}{\epsilon}B_f \end{bmatrix} = \left(\Pi_{1_F} + \Pi_{2_F} P_F\right)^{-T} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} \tag{8.42}$$

The optimal value for the performance criterion can be calculated by using the formula

$$J_{opt} = \operatorname{tr}\left\{PGWG^T + P_F F^T RF\right\} = \operatorname{tr}\left\{PKVK^T + P_F Q\right\} \tag{8.43}$$

The slow-fast decomposition results presented in this section will be demonstrated in the next section on a real physical control system example, a flexible space structure.

8.1.4 Case Study: A Flexible Space Structure

For the modeling issues of a singularly perturbed flexible space structure the reader is referred to (Moerder and Calise, 1985). The corresponding problem matrices are given by

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -0.176 & 0 & 0 & 0 \\ 0 & -0.176 & 0 & 0 \\ 0 & 0 & -4.41 & 0 \\ 0 & 0 & 0 & -4.41 \end{bmatrix}$$

$$A_1 = 0^{4\times4}, \; A_4 = 0^{4\times4}, \; B_1 = 0^{4\times4}, \; Q_1 = I_4, \; Q_{12} = 0^{4\times4}, \; R = I_4$$

$$B_2 = \begin{bmatrix} -9.20 & -1.40 & 0.92 & -1.40 \\ 0.65 & 1.60 & 0.65 & -1.60 \\ 1.40 & -1.00 & 1.40 & 1.00 \\ 2.05 & -0.80 & -2.00 & -0.80 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 21.06 & 0 & 0 & -6.12 \\ 0 & 23.86 & -5.90 & 0 \\ 0 & -5.90 & 38.74 & 0 \\ -6.12 & 0 & 0 & 38.74 \end{bmatrix}$$

The small singular perturbation parameter is $\epsilon = 0.2$. Note that conditions of Assumption 8.2 and Lemma 8.2 are satisfied, that is, the matrix T_4 is invertible.

By following the procedure presented in the previous sections, we have obtained the following values for the solutions of the pure-slow and pure-fast algebraic regulator Riccati equations defined in (8.8)

$$P_s = \begin{bmatrix} 4.5712 & 0.0020 & 0.0019 & -0.7627 \\ 0.0020 & 4.8830 & -0.6624 & 0.0336 \\ 0.0251 & -0.6235 & 18.3212 & 4.7280 \\ -0.7415 & 0.0590 & 4.7283 & 15.1487 \end{bmatrix}$$

$$P_f = \begin{bmatrix} 0.6711 & 0.3010 & 0.5923 & 0.4962 \\ 0.3001 & 2.3274 & 0.5990 & 0.4965 \\ 0.5943 & 0.6049 & 3.5727 & 0.9076 \\ 0.4975 & 0.5008 & 0.9071 & 2.8346 \end{bmatrix}$$

By using these solutions in formula (8.12), the solution of the global algebraic regulator Riccati equation (8.3) is obtained as

$$P = \begin{bmatrix} 4.5782 & 0.0050 & 0.0206 & -0.7452 & 0.0296 & 0.0125 & 0.0255 & 0.0276 \\ 0.0050 & 4.9022 & -0.6361 & 0.0498 & 0.0120 & 0.0892 & 0.0292 & 0.0196 \\ 0.0206 & -0.6361 & 18.3139 & 4.7178 & -0.0187 & -0.0604 & 0.0034 & -0.0169 \\ -0.7452 & 0.0498 & 4.7178 & 15.1468 & -0.0167 & -0.0432 & -0.0131 & 0.0071 \\ 0.0296 & 0.0120 & -0.0187 & -0.0167 & 0.1344 & 0.0602 & 0.1187 & 0.0995 \\ 0.0125 & 0.0892 & -0.0604 & -0.0432 & 0.0602 & 0.4669 & 0.1205 & 0.0997 \\ 0.0255 & 0.0292 & 0.0034 & -0.0131 & 0.1187 & 0.1205 & 0.7150 & 0.1816 \\ 0.0276 & 0.0196 & -0.0169 & 0.0071 & 0.0995 & 0.0997 & 0.1816 & 0.5673 \end{bmatrix}$$

This solution is $O\left(10^{-14}\right)$ close to the global solution obtained by using MATLAB for solving full order linear-quadratic optimal control problem. This is as a matter of fact the standard accuracy of MATLAB, hence we can consider that these two solutions are identical. The solution of the regulator algebraic equation is used to find the optimal feedback gains.

Consider now the filtering part of the above flexible space structure optimal control problem with

$$G_1 = B_1, \quad G_2 = B_2, \quad W = I_4, \quad V = I_4$$

$$C_1 = I, \quad C_2 = \mathrm{Chol}(Q_3)$$

Similarly to the regulator part, we can find first the solutions of the pure-slow and pure-fast algebraic filter Riccati equations (8.29), that is P_{sF}

and P_{fF}. Then, the following independent closed-loop, pure-slow and pure-fast, independent, Kalman filters are obtained according to (8.41)

$$\dot{\hat{x}}_s(t) = \begin{bmatrix} -0.2229 & -0.0009 & -0.0286 & -0.0641 \\ -0.0007 & -0.2098 & -0.0739 & -0.0258 \\ -0.0037 & -0.0247 & -0.4927 & -0.1309 \\ -0.0238 & -0.0044 & -0.1315 & -0.4067 \end{bmatrix} \hat{x}_s(t)$$

$$+ \begin{bmatrix} 0.0011 & -0.0119 & -0.0047 & -0.0133 \\ 0.0044 & -0.0074 & 0.0151 & -0.0114 \\ 0.0231 & -0.0696 & 0.0833 & -0.0274 \\ 0.0163 & -0.0625 & -0.0394 & -0.0718 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0.2223 & 0.0002 & 0.0002 & 0.0206 \\ 0.0001 & 0.2082 & 0.0182 & 0.0002 \\ 0.0006 & 0.0188 & 0.1642 & 0.0008 \\ 0.0209 & 0.0006 & 0.0008 & 0.1641 \end{bmatrix} y(t)$$

and

$$\epsilon\dot{\hat{x}}_f(t) = \begin{bmatrix} -42.3548 & 1.8066 & 8.2920 & 21.0576 \\ 2.6185 & -11.4535 & 4.4927 & 0.8508 \\ 4.6697 & 3.1381 & -13.4006 & 0.9154 \\ 7.6494 & 0.7283 & 2.1213 & -17.7029 \end{bmatrix} \hat{x}_f(t)$$

$$+ \begin{bmatrix} -1.8399 & -0.2806 & 0.1839 & -0.2806 \\ 0.1302 & 0.3194 & 0.1306 & -0.3204 \\ 0.2807 & -0.2022 & 0.2820 & 0.1989 \\ 0.4105 & -0.1618 & -0.4005 & -0.1618 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 1.7959 & -0.1080 & -0.2799 & -0.5163 \\ -0.1201 & 0.4618 & -0.1043 & -0.0382 \\ -0.2095 & -0.0792 & 0.4333 & -0.0463 \\ -0.2652 & -0.0359 & -0.0687 & 0.5534 \end{bmatrix} y(t)$$

The optimal control in terms of pure-slow and pure-fast estimates is given by the following expression

$$u_{opt}(t) = \begin{bmatrix} 0.0111 & -0.0348 & -0.4321 & -0.2261 \\ 0.0739 & 0.0016 & 2.3759 & 2.0235 \\ -0.0046 & -0.1157 & -2.6683 & 0.6214 \\ 0.0748 & 0.1651 & 1.6479 & 2.5491 \end{bmatrix} \hat{x}_s(t) +$$

$$
+\begin{bmatrix} -20.7134 & 3.0129 & 8.8839 & 14.0832 \\ -7.2369 & 11.5865 & -20.7909 & -15.3395 \\ 3.2618 & 8.2193 & 20.6592 & -18.0590 \\ -6.1646 & -19.8110 & 5.1647 & -14.3441 \end{bmatrix} \hat{x}_f(t)
$$

where the optimal gains are obtained using (8.40). Note that the knowledge of P_F is needed to find the optimal gains for pure-slow and pure-fast filters. P_F can be also found by appropriately using formula (8.12).

Finally, we can evaluate the optimal performance criterion from (8.43). It is given by $J_{opt} = 853.0123$.

8.2 On the Finite Horizon Feedback Optimization Problem

Finite horizon continuous-time, linear-quadratic, open-loop optimization problem for singularly perturbed systems, in which the optimal control is a time function, is solved in Section 2.2. The corresponding discrete-time open-loop optimal control problem is considered in Section 3.4. In those sections, we have indicated how to obtain analytically the exact decomposition of the original optimal open-loop control problems into pure-slow and pure-fast optimal subproblems. In many applications, the closed-loop solution, in which the optimal control is a function of state variables, is desirable for finite horizon optimization. Solving the feedback equivalents of Sections 2.2 and 3.4 seems to be computationally and analytically much more involved. The study in that direction is underway in (Coumarbatch, 2000). Here, we only indicate the main ideas for such a decomposition. In that direction, we first formulate the feedback optimization problem for linear singularly perturbed systems over a finite horizon, whose solution requires solving the singularly perturbed nonlinear differential Riccati equation. We will be faced with the problem of studying the transient behavior of systems of singularly perturbed differential equations. That behavior is also known as the boundary layer behavior (it is present in the neighborhood of the boundary conditions, initial and terminal ones).

It should be pointed that the recursive approach to slow-fast decomposition of the differential Riccati equation obtained in (Grodt and Gajic, 1988) is very efficient for achieving a very high accuracy. It has been demonstrated in (Grodt and Gajic, 1988) on a real power system example that the accuracy of $O(\epsilon^{12})$ can be easily obtained, see Appendix 8.1.

Singularly perturbed differential equations have been studied in mathematics (Tikhonov, 1948; Vasileva and Butuzov, 1973; O'Malley, 1974a,b; 1991) and control systems engineering (Kokotovic *et al.*, 1986; Kokotovic and Khalil, 1986; Gajic and Shen, 1993) for quite some time. These differential equations are characterized by simultaneous presence of slow and fast variables with a small positive singular perturbation parameter ϵ multiplying derivatives of the fast variables. Simultaneous presence of slow and fast phenomena causes numerical ill-conditioning (Kreiss and Kreiss, 1981) so that an important goal in studying singularly perturbed systems of differential equations is to separate slow and fast variables. Since the initial work of (Tikhonov, 1948), the singularly perturbed differential equations have been studied analytically by using different series expansion techniques and matching corresponding terms (Vasileva and Butuzov, 1973; O'Malley, 1974a,b, 1991) so that the approximate solutions with accuracy of $O(\epsilon^k)$, $k = 1, 2, 3, ..$ have been obtained for many important problems.

Consider the singularly perturbed continuous-time linear system (Kokotovic and Yackel, 1972; O'Malley, 1974a,b, 1991; Kokotovic *et al.*, 1986)

$$\begin{aligned} \dot{x}(t,\epsilon) &= A_1 x(t,\epsilon) + A_2 z(t,\epsilon) + B_1 u(t,\epsilon), \qquad & x(0,\epsilon) = x_0 \\ \epsilon \dot{z}(t,\epsilon) &= A_3 x(t,\epsilon) + A_4 z(t,\epsilon) + B_2 u(t,\epsilon), \qquad & z(0,\epsilon) = z_0 \end{aligned} \tag{8.44}$$

where $x(t,\epsilon)$ and $z(t,\epsilon)$ are n_1 and n_2–dimensional slow and fast state vectors respectively, $u(t,\epsilon)$ is an m-dimensional vector control input, and ϵ is a small positive singular perturbation parameter. The matrices $A_i, i = 1, 2, 3, 4$, and $B_j, j = 1, 2$, are constant and of appropriate dimensions.

With (8.44) consider the finite horizon performance criterion to be minimized by the control action

$$\begin{aligned} J = \frac{1}{2}\int_0^{t_f} \left\{ \begin{bmatrix} x(t,\epsilon) \\ z(t,\epsilon) \end{bmatrix}^T Q \begin{bmatrix} x(t,\epsilon) \\ z(t,\epsilon) \end{bmatrix} + u^T(t,\epsilon) R u(t,\epsilon) \right\} dt \\ + \frac{1}{2} \begin{bmatrix} x(t_f,\epsilon) \\ z(t_f,\epsilon) \end{bmatrix}^T Q_{t_f}(\epsilon) \begin{bmatrix} x(t_f,\epsilon) \\ z(t_f,) \end{bmatrix} \end{aligned} \tag{8.45}$$

which has to be minimized along trajectories of (8.44). The following assumption is commonly used for the criterion penalty matrices.

Assumption 8.4: The penalty matrices in the performance criterion (8.45) are symmetric. In addition, the matrix R is positive definite, and the matrices Q and $F(\epsilon)$ are positive semidefinite.

The very well-known feedback solution to the optimal control problem defined by (8.44)-(8.45) is given by

$$u_{opt}(x, z, t, \epsilon) = F_1(t, \epsilon)x(t, \epsilon) + F_2(t, \epsilon)z(t, \epsilon) \tag{8.46}$$

with

$$\begin{aligned} F_1(t, \epsilon) &= R^{-1}\left(B_1^T P_1(t, \epsilon) + B_2^T P_2^T(t, \epsilon)\right) \\ F_2(t, \epsilon) &= R^{-1}\left(\varepsilon B_1^T P_2(t, \epsilon) + B_2^T P_3(t, \epsilon)\right) \end{aligned} \tag{8.47}$$

where $P_i(t, \epsilon), i = 1, 2, 3$, are the corresponding partitions of the positive semidefinite solution of the regulator matrix Riccati differential equation (Kokotovic and Yackel, 1972; O'Malley, 1974a)

$$\begin{gathered} -\dot{P}(t, \epsilon) = A^T(\epsilon)P(t, \epsilon) + P(t, \epsilon)A(\epsilon) - P(t, \epsilon)S(\epsilon)P(t, \epsilon) + Q \\ P(t_f, \epsilon) = Q_{t_f}(\epsilon) \end{gathered} \tag{8.48}$$

with newly defined matrices given by

$$\begin{gathered} A(\epsilon) = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 \end{bmatrix}, \quad B(\varepsilon) = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}, \quad S(\epsilon) = B(\epsilon)R^{-1}B^T(\epsilon) \\ Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \qquad Q_{tf}(\epsilon) = \begin{bmatrix} Q_{1t_f} & \epsilon Q_{2tf} \\ \epsilon Q_{2t_f}^T & \epsilon Q_{3t_f} \end{bmatrix} \end{gathered} \tag{8.49}$$

Note that the fact that the problem matrices are constant and Assumption 8.4 imposed on the penalty matrices, that is, $R = R^T > 0$, $Q = Q^T \geq 0$, $Q_f(\epsilon) = Q_f^T(\epsilon) \geq 0$, guarantee the existence of the unique positive semidefinite symmetric matrix $P(t, \epsilon)$ for all $t \in [0, t_f]$ (see for example, Wilde and Kokotovic, 1972).

The required solution of (8.48), $P(t, \epsilon)$, is scaled compatible to nature of the problem matrices as follows (Kokotovic and Yackel, 1972; O'Malley, 1974a)

$$P(t, \epsilon) = \begin{bmatrix} P_1(t, \epsilon) & \epsilon P_2(t, \epsilon) \\ \epsilon P_2^T(t, \epsilon) & \epsilon P_3(t, \epsilon) \end{bmatrix} \tag{8.50}$$

The partitioned form of (8.48) produces the following system of singularly perturbed matrix differential equations

$$-\dot{P}_1(t,\epsilon) = P_1(t,\epsilon)A_1 + A_1^T P\ (t,\epsilon) + P_2(t,\epsilon)A_3 + A_3^T P_2^T(t,\epsilon) -$$
$$P_1(t,\epsilon)S_1P_1(t,\epsilon) - P_1(t,\epsilon)SP_2^T(t,\epsilon) - P_2(t,\epsilon)S^TP_1(t,\epsilon)$$
$$-P_2(t,\epsilon)S_2P_2^T(t,\epsilon) + Q_1, \qquad P_1(t_f,\epsilon) = Q_{1t_f} \tag{8.51}$$

$$-\epsilon\dot{P}_2(t,\epsilon) = P_1(t,\epsilon)A_2 + P_2(t,\epsilon)A_4 + \epsilon A_1^T P_2(t,\epsilon) + A_3^T P_3(t,\epsilon)$$
$$-\epsilon P_1(t,\epsilon)S_1P_2(t,\epsilon) - P_1(t,\epsilon)SP_3(t,\epsilon) - \epsilon P_2(t,\epsilon)S^TP_2(t,\epsilon)$$
$$-P_2(t,\epsilon)S_2P_3(t,\epsilon) + Q_2, \qquad P_2(t_f,\epsilon) = Q_{2t_f} \tag{8.52}$$

$$-\epsilon\dot{P}_3(t,\epsilon) = P_3(t,\epsilon)A_4 + A_4^T P_3(t,\epsilon) + \varepsilon P_2^T(t,\varepsilon)A_2 + \epsilon A_2^T P_2(t,\epsilon)$$
$$-\epsilon^2 P_2^T(t,\epsilon)S_1P_2(t,\epsilon) - \epsilon P_2^T(t,\epsilon)SP_3(t,\epsilon) - \epsilon P_3(t,\epsilon)S^TP_2(t,\epsilon)$$
$$-P_3(t,\epsilon)S_2P_3(t,\epsilon) + Q_3, \qquad P_3(t_f,\epsilon) = Q_{3t_f} \tag{8.53}$$

with

$$S_i = B_iR^{-1}B_i^T, \qquad i = 1,2, \qquad S = B_1R^{-1}B_2^T \tag{8.54}$$

The singularly perturbed system of matrix differential equations (8.51)-(8.53) has been studied analytically in (Kokotovic and Yackel, 1972; O'Malley 1974a) using asymptotic series expansions, and numerically in (Grodt and Gajic, 1988) by employing block diagonalization of the corresponding Hamiltonian matrix. In (Yackel and Kokotovic, 1973) the results are obtained using control oriented assumptions (boundary layer controllability and boundary layer observability assumptions). These assumptions are relaxed in (O'Malley, 1974a) into a set of linear algebra assumptions. One of the main assumptions used in (O'Malley, 1974a) is that the Hamiltonian matrix of the fast subsystem is nonsingular.

Assumption 8.5: The fast subsystem Hamiltonian matrix is nonsingular, that is

$$det\left\{\begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix}\right\} \neq 0$$

We have seen from the previous chapters of this book that Assumption 8.5 is a fundamental assumption of the Hamiltonian approach to singularly perturbed linear-quadratic optimal control problems.

The optimal open-loop optimization problem (8.44)-(8.45) has the solution given by

$$u_{opt}(t,\epsilon) = -R^{-1}B^T \begin{bmatrix} p(t,\epsilon) \\ q(t,\epsilon) \end{bmatrix} \tag{8.55}$$

where the costate variables $p(t,\epsilon)$ and $q(t,\epsilon)$ satisfy the Hamiltonian system of linear differential equations defined by

$$\begin{bmatrix} \dot{x}(t,\epsilon) \\ \dot{z}(t,\epsilon) \\ \dot{p}(t,\epsilon) \\ \dot{q}(t,\epsilon) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t,\epsilon) \\ z(t,\epsilon) \\ p(t,\epsilon) \\ q(t,\epsilon) \end{bmatrix} \tag{8.56}$$

with boundary conditions

$$\begin{bmatrix} x(t_0) \\ z(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \quad \begin{bmatrix} p(t_f) \\ q(t_f) \end{bmatrix} = Q_{tf} \begin{bmatrix} x(t_f) \\ z(t_f) \end{bmatrix} \tag{8.57}$$

The complete solution to the exact pure-slow and pure-fast decomposition for the optimal open-loop control problem has been presented in Section 2.2. In the following we indicate an idea for achieving such a decomposition for the corresponding closed-loop optimal control problem, whose solution is given by (8.46)-(8.47).

The Hamiltonian form (8.56)-(8.57), in view of (8.49), can be written in more detailed form as

$$\begin{bmatrix} \dot{x}(t,\epsilon) \\ \dot{z}(t,\epsilon) \\ \dot{p}(t,\epsilon) \\ \dot{q}(t,\epsilon) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & -S_1 & -\frac{1}{\epsilon}S \\ \frac{1}{\epsilon}A_3 & \frac{1}{\epsilon}A_4 & -\frac{1}{\epsilon}S^T & -\frac{1}{\epsilon^2}S_2 \\ -Q_1 & -Q_2 & -A_1^T & -\frac{1}{\epsilon}A_3^T \\ -Q_2^T & -Q_3 & -A_2^T & -\frac{1}{\epsilon}A_4^T \end{bmatrix} \begin{bmatrix} x(t,\epsilon) \\ z(t,\epsilon) \\ p(t,\epsilon) \\ q(t,\epsilon) \end{bmatrix} \tag{8.58}$$

with boundary conditions given by

$$\begin{bmatrix} x(0,\epsilon) \\ z(0,\epsilon) \\ p(t_f,\epsilon) \\ q(t_f,\epsilon) \end{bmatrix} = \begin{bmatrix} x_0 \\ z_0 \\ Q_{1t_f}x(t_f,\epsilon) + \epsilon Q_{2t_f}z(t_f,\epsilon) \\ \epsilon Q_{2t_f}^T x(t_f,\epsilon) + \epsilon Q_{3t_f}z(t_f,\epsilon) \end{bmatrix} \tag{8.59}$$

Introducing the change of variables as $q(t,\epsilon) = \epsilon r(t,\epsilon)$ and interchanging the order of equations for $z(t,\epsilon)$ and $p(t,\epsilon)$, we get the standard linear singularly perturbed system

$$\begin{bmatrix} \dot{x}(t,\epsilon) \\ \dot{p}(t,\epsilon) \\ \epsilon\dot{z}(t,\epsilon) \\ \epsilon\dot{r}(t,\epsilon) \end{bmatrix} = \begin{bmatrix} \mathbf{A_1} & \mathbf{A_2} \\ \mathbf{A_3} & \mathbf{A_4} \end{bmatrix} \begin{bmatrix} x(t,\epsilon) \\ p(t,\epsilon) \\ z(t,\epsilon) \\ r(t,\epsilon) \end{bmatrix} \tag{8.60}$$

where

$$\mathbf{A_1} = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad \mathbf{A_2} = \begin{bmatrix} A_2 & -S \\ -Q_2 & -A_3^T \end{bmatrix}$$
$$\mathbf{A_3} = \begin{bmatrix} A_3 & -S^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad \mathbf{A_4} = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{8.61}$$

The boundary conditions for (8.60) are

$$V \begin{bmatrix} x(0,\epsilon) \\ p(0,\epsilon) \\ z(0,\epsilon) \\ r(0,\epsilon) \end{bmatrix} + N(\epsilon) \begin{bmatrix} x(t_f,\epsilon) \\ p(t_f,\epsilon) \\ z(t_f,\epsilon) \\ r(t_f,\epsilon) \end{bmatrix} = c \tag{8.62}$$

with

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$N(\epsilon) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ Q_{1t_f} & -I_{n_1} & \epsilon Q_{2t_f} & 0 \\ 0 & 0 & 0 & 0 \\ Q_{2t_f}^T & 0 & Q_{3t_f} & -I_{n_2} \end{bmatrix}, \quad c = \begin{bmatrix} x_0 \\ 0 \\ z_0 \\ 0 \end{bmatrix} \tag{8.63}$$

The Chang decoupling transformation (Chang, 1972) will be used to decouple (8.60) into pure-slow and pure-fast subsystems. The transformation is defined by

$$\begin{bmatrix} x(t,\epsilon) \\ p(t,\epsilon) \\ z(t,\epsilon) \\ r(t,\epsilon) \end{bmatrix} = \mathbf{T}(\epsilon) \begin{bmatrix} x_s(t,\epsilon) \\ p_s(t,\epsilon) \\ z_f(t,\epsilon) \\ r_f(t,\epsilon) \end{bmatrix}$$

$$\mathbf{T}(\epsilon) = \begin{bmatrix} I_{2n_1} & \epsilon M(\epsilon) \\ -L(\epsilon) & I_{2n_2} - \epsilon L(\epsilon)M(\epsilon) \end{bmatrix} \tag{8.64}$$

$$\mathbf{T}^{-1}(\varepsilon) = \begin{bmatrix} I_{2n_1} - \epsilon M(\epsilon)L(\epsilon) & -\epsilon M(\epsilon) \\ L(\epsilon) & I_{2n_2} \end{bmatrix}$$

where the $2n_2 \times 2n_1$ dimensional matrix $L(\epsilon)$ satisfies

$$\mathbf{A_4}L(\epsilon) - \mathbf{A_3} - \epsilon L(\epsilon)(\mathbf{A_1} - \mathbf{A_2}L(\epsilon)) = 0 \tag{8.65}$$

and $2n_1 \times 2n_2$ dimensional matrix $M(\epsilon)$ is obtained from

$$M(\epsilon)\mathbf{A_4} - \mathbf{A_2} + \epsilon M(\epsilon)L(\epsilon)\mathbf{A_2} - \epsilon(\mathbf{A_1} - \mathbf{A_2}L(\epsilon))M(\epsilon) = 0 \tag{8.66}$$

The unique solution to equation (8.64) exists under Assumption 8.5 (matrix $\mathbf{A_4}$ is invertible). It can be easily found numerically as a solution of a sequence of linear equations. This can be accomplished by using either the fixed-point iterations or the Newton method (Kokotovic *et al.*, 1980; Grodt and Gajic, 1988). Also, the eigenvector method is available for solving (8.64), see (Kecman *et al.*, 1999). The fixed point algorithm for solving (8.64) is given by

$$L^{(i+1)}(\epsilon) = \mathbf{A_4^{-1}}\left[\mathbf{A_3} - \epsilon L^{(i)}(\epsilon)\Big(\mathbf{A_1} - \mathbf{A_2}L^{(i)}(\epsilon)\Big)\right]$$

$$L^{(0)}(\epsilon) = L^{(0)}(0) = \mathbf{A_4^{-1}}\mathbf{A_3}, \qquad i = 0,1,2,... \tag{8.67}$$

It can be easily shown that

$$L(\epsilon) = L^{(0)}(0) + O(\epsilon), \qquad L^{(0)}(0) = O(1) \quad \Rightarrow \quad L(\epsilon) = O(1) \tag{8.68}$$

Having obtained a solution for $L(\epsilon)$, one can obtain a solution for $M(\epsilon)$ by using the following iterative scheme

$$M^{(i+1)}(\epsilon) = \left[\mathbf{A_2} - \epsilon M^{(i)}(\epsilon)L(\epsilon)\mathbf{A_2} + \epsilon(\mathbf{A_1} - \mathbf{A_2}L(\epsilon))M^{(i)}(\epsilon)\right]\mathbf{A_4^{-1}}$$

$$M^{(0)}(\epsilon) = M^{(0)} = \mathbf{A_2}\mathbf{A_4^{-1}}, \qquad i = 0,1,2,... \tag{8.69}$$

It is easy to establish that

$$M(\epsilon) = M^{(0)}(0) + O(\epsilon), \qquad M^{(0)}(0) = O(1) \quad \Rightarrow \quad M(\epsilon) = O(1) \tag{8.70}$$

In the following, we will use the partitioned matrices $L(\epsilon)$ and $M(\epsilon)$ and the partitioned product $L(\epsilon)M(\epsilon)$, that is, we need to define

$$\begin{aligned} L^{2n_2 \times 2n_1}(\epsilon) &= \begin{bmatrix} L_1^{n_2 \times n_1}(\epsilon) & L_2^{n_2 \times n_1}(\epsilon) \\ L_3^{n_2 \times n_1}(\epsilon) & L_4^{n_2 \times n_1}(\epsilon) \end{bmatrix} \\ M^{2n_1 \times 2n_2}(\epsilon) &= \begin{bmatrix} M_1^{n_1 \times n_2}(\epsilon) & M_2^{n_1 \times n_2}(\epsilon) \\ M_3^{n_1 \times n_2}(\epsilon) & M_4^{n_1 \times n_2}(\epsilon) \end{bmatrix} \\ (L(\epsilon)M(\epsilon))^{2n_2 \times 2n_2} &= H(\epsilon) = \begin{bmatrix} H_1^{n_2 \times n_2}(\epsilon) & H_2^{n_2 \times n_2}(\epsilon) \\ H_3^{n_2 \times n_2}(\epsilon) & H_4^{n_2 \times n_2}(\epsilon) \end{bmatrix} \end{aligned} \tag{8.71}$$

Applying the Chang transformation (Chang, 1972) to two-point boundary value problem defined by (8.60)-(8.63), the system is transformed into the new coordinates where the slow and fast state variables are dynamically decoupled, that is,

$$\begin{bmatrix} \dot{x}_s(t,\epsilon) \\ \dot{p}_s(t,\epsilon) \end{bmatrix} = (\mathbf{A}_1 - \mathbf{A}_2 L(\epsilon)) \begin{bmatrix} x_s(t,\epsilon) \\ p_s(t,\epsilon) \end{bmatrix} \tag{8.72}$$

$$\epsilon \begin{bmatrix} \dot{z}_f(t,\epsilon) \\ \dot{r}_f(t,\epsilon) \end{bmatrix} = (\mathbf{A}_4 + \epsilon L(\epsilon)\mathbf{A}_2) \begin{bmatrix} z_f(t,\epsilon) \\ r_f(t,\epsilon) \end{bmatrix} \tag{8.73}$$

with the boundary conditions satisfying

$$V_1(\epsilon) \begin{bmatrix} x_s(0,\epsilon) \\ p_s(0,\epsilon) \\ z_f(0,\epsilon) \\ r_f(0,\epsilon) \end{bmatrix} + N_1(\epsilon) \begin{bmatrix} x_s(t_f,\epsilon) \\ p_s(t_f,\epsilon) \\ z_f(t_f,\epsilon) \\ r_f(t_f,\epsilon) \end{bmatrix} = c \tag{8.74}$$

It is easy to show that

$$N_1(\epsilon) = N(\epsilon)\mathbf{T}(\epsilon) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_1(\epsilon) & \alpha_2(\epsilon) & \alpha_3(\epsilon) & \alpha_4(\epsilon) \\ 0 & 0 & 0 & 0 \\ \beta_1(\epsilon) & \beta_2(\epsilon) & \beta_3(\epsilon) & \beta_4(\epsilon) \end{bmatrix}$$

with

$$\alpha_1(\epsilon) = Q_{1t_f} - \epsilon Q_{2t_f} L_1(\epsilon), \qquad \alpha_2(\epsilon) = -I_{n_1} - \epsilon Q_{2t_f} L_2(\epsilon)$$

$$\alpha_3(\epsilon) = \epsilon\big[Q_{1t_f} M_1(\epsilon) - M_3(\epsilon) + Q_{2t_f}(I - \epsilon H_1(\epsilon))\big]$$

$$\alpha_4(\epsilon) = \epsilon\big(Q_{1t_f} M_2(\epsilon) - M_4(\epsilon) - \epsilon Q_{2t_f} H_2(\epsilon)\big)$$

$$\beta_1(\epsilon) = Q_{2t_f}^T - Q_{3t_f} L_1(\epsilon) + L_3(\epsilon), \qquad \beta_2(\epsilon) = Q_{3t_f} L_2(\epsilon) + L_4(\epsilon)$$

$$\beta_3(\epsilon) = Q_{3t_f}(I_{n_2} - \epsilon H_1(\epsilon)) + \epsilon\Big(Q_{2t_f}^T M_1(\epsilon) + H_3(\epsilon)\Big)$$

$$\beta_4(\epsilon) = -I_{n_2} + \epsilon\Big(Q_{2t_f}^T M_2(\epsilon) - Q_{3t_f} H_2(\epsilon) + H_4(\epsilon)\Big) \tag{8.75}$$

and

$$V_1(\epsilon) = V\mathbf{T}(\epsilon) = \begin{bmatrix} I_{n_1} & 0 & \epsilon M_1(\epsilon) & \epsilon M_2(\epsilon) \\ 0 & 0 & 0 & 0 \\ -L_1(\epsilon) & -L_2(\epsilon) & I_{n_2} - \epsilon H_1(\epsilon) & -\epsilon H_2(\epsilon) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Equations (8.72) and (8.73) can be represented in the form

$$\begin{bmatrix} \dot{x}_s(t,\epsilon) \\ \dot{p}_s(t,\epsilon) \end{bmatrix} = \begin{bmatrix} a_1(\epsilon) & a_2(\epsilon) \\ a_3(\epsilon) & a_4(\epsilon) \end{bmatrix} \begin{bmatrix} x_s(t,\epsilon) \\ p_s(t,\epsilon) \end{bmatrix} \tag{8.76}$$

$$\epsilon \begin{bmatrix} \dot{z}_f(t,\epsilon) \\ \dot{r}_f(t,\epsilon) \end{bmatrix} = \begin{bmatrix} b_1(\epsilon) & b_2(\epsilon) \\ b_3(\epsilon) & b_4(\epsilon) \end{bmatrix} \begin{bmatrix} z_f(t,\epsilon) \\ r_f(t,\epsilon) \end{bmatrix} \tag{8.77}$$

The expressions for the newly defined constant matrices in (8.76)-(8.77) are

$$\begin{aligned} a_1(\epsilon) &= A_1 - A_2 L_1(\epsilon) + S L_3(\epsilon) \\ a_2(\epsilon) &= -S_1 - A_2 L_2(\epsilon) + S L_4(\epsilon) \\ a_3(\epsilon) &= -Q_1 + Q_2 L_1(\epsilon) + A_3^T L_3(\epsilon) \\ a_4(\epsilon) &= -A_1^T + Q_2 L_2(\epsilon) + A_3^T L_4(\epsilon) \end{aligned} \tag{8.78}$$

and

$$\begin{aligned} b_1(\epsilon) &= A_4 + \epsilon(L_1(\epsilon) A_2 - L_2(\epsilon) Q_2) \\ b_2(\epsilon) &= -S_2 - \epsilon\big(L_1(\epsilon) S + L_2(\epsilon) A_3^T\big) \\ b_3(\epsilon) &= -Q_3 + \epsilon(L_3(\epsilon) A_2 - L_4(\epsilon) Q_2) \\ b_4(\epsilon) &= -A_4^T - \epsilon\big(L_3(\epsilon) S + L_4(\epsilon) A_3^T\big) \end{aligned} \tag{8.79}$$

The boundary value problem defined by (8.72)-(8.74) can be solved by using the invariant imbedding method (see for example Meyer, 1973; Bryson, 1999). The solution may be sought using the following relations

$$
\begin{aligned}
p_s(t,\epsilon) &= P_s(t,\epsilon)x_s(t,\epsilon) + f_{s1}(t,\epsilon)z_f(t,\epsilon) + f_{s2}(t,\epsilon)r_f(t,\epsilon) \\
r_f(t,\epsilon) &= P_f(t,\epsilon)z_f(t,\epsilon) + f_{f1}(t,\epsilon)x_s(t,\epsilon) + f_{f2}(t,\epsilon)p_s(t,\epsilon)
\end{aligned}
\tag{8.80}
$$

where $P_s(t,\epsilon)$ and $P_f(t,\epsilon)$ are the solutions of the pure-fast and pure-slow differential Riccati equations, respectively given by

$$
\begin{aligned}
&-\dot{P}_s(t,\epsilon) = P_s(t,\epsilon)a_1(\epsilon) - a_4(\epsilon)P_s(t,\epsilon) - a_3(\epsilon) \\
&+P_s(t,\epsilon)a_2(\epsilon)P_s(t,\epsilon), \qquad P_s(t_f,\epsilon) = -\alpha_2^{-1}(\epsilon)\alpha_1(\epsilon)
\end{aligned}
\tag{8.81}
$$

$$
\begin{aligned}
&-\epsilon\dot{P}_f(t,\epsilon) = P_f(t,\epsilon)b_1(\epsilon) - b_4(\epsilon)P_f(t,\epsilon) - b_3(\epsilon) \\
&+P_f(t,\epsilon)b_2(\epsilon)P_f(t,\epsilon), \qquad P_f(t_f,\epsilon) = -\beta_4^{-1}(\epsilon)\beta_3(\epsilon)
\end{aligned}
\tag{8.82}
$$

The search for the functions $f_{s1}(t,\epsilon)$, $f_{s2}(t,\epsilon)$ and $f_{f1}(t,\epsilon)$, $f_{f2}(t,\epsilon)$ that under appropriate assumptions satisfy (8.79) and the imposed boundary conditions is underway in (Coumarbatch, 2000).

8.3 Slow-Fast Decomposition of Fridman, Sobolev, and Strygin

The slow-fast decomposition of singularly perturbed systems via integral manifold theory (Mitropolsky and Lykova, 1973; Henry, 1981) originated in (Fridman and Strygin, 1984; Sobolev 1984; Strygin *et al.*, 1985; Fridman, 1986; see also Kokotovic *et al.*, 1986). The results of (Sobolev, 1984) are extended in several papers by E. Fridman to various problems of linear and nonlinear optimal control theory (Fridman, 1990a,b, 1995, 1996a,b, 1999, 2000; Fridman and Shaked, 2000).

The work of (Fridman and Strygin, 1984; Sobolev 1984; Strygin *et al.*, 1985; Fridman, 1990a,b) based on slow-fast integral manifold theory resulted also in the exact pure-slow and pure-fast decomposition of the linear-quadratic optimal control problems (Fridman, 1990a, 1995, 1996a) like the Hamiltonian approach of (Su *et al.*, 1992b). It should be pointed out that the results of (Fridman, 1995, 1996a) hold for both finite-time (horizon) and steady state optimization problems.

The slow-fast integral manifold theory is extended by E. Fridman to linear singularly perturbed systems with delays (Fridman, 1990a, 1996b), to H_∞ optimization with time delays and for the static output feedback

(Fridman and Shaked, 1999, 2000), and to some classes of nonlinear optimal control problems in (Sobolev, 1984; Fridman, 1999, 2000). In the following, we first present a brief introduction to slow-fast integral manifolds and then discuss their applications to linear and nonlinear singularly perturbed control systems.

8.3.1 Integral Manifolds for Singularly Perturbed Systems

In this section we follow the work of (Sobolev, 1984) with some minor modifications and simplification on the expense of a rigorous mathematical presentation. Consider a nonlinear differential equation

$$\dot{x}(t) = f(x(t), t) \tag{8.83}$$

of dimension n, that is $x(t) \in \Re^n$ and $t \in \Re$. An integral manifold of the above differential equation can be defined as follows.

Definition 8.1: A set $M \subset \Re \times \Re^n$ is an integral manifold of $\dot{x}(t) = f(x(t), t)$ if for some $(x(t_0), t_0) \in M$ the solution $(x(t), t)$ belongs to M for all $t \geq t_0$

Consider now a general nonlinear singularly perturbed time varying differential equation represented by

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t), t, \epsilon) \\ \epsilon\dot{x}_2(t) &= f_2(x_1(t), x_2(t), t, \epsilon) \end{aligned} \tag{8.84}$$

where $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$, $n = n_1 + n_2$, are respectively slow and fast variables, and $\epsilon \in [0, \epsilon^*]$ is a small singular perturbation parameter. Let $x_1^\epsilon(t), x_2^\epsilon(t)$ denote solutions of (8.84) for the given initial conditions.

An integral manifold of the singularly perturbed nonlinear differential equation exists under assumptions defined below (existence of the isolated root, closeness to the isolated root, smoothness of the functions involved, and asymptotic stability of the fast subsystem).

(a) The algebraic equation $0 = f_2(x_1(t), x_2(t), t, 0)$ has the *isolated root* $x_2^0(t) = h_0(x_1(t), t)$.

(b) The solution $x_2^\epsilon(t)$ of (8.84) is sufficiently *close* to $x_2^0(t)$, that is

$$\left|x_2^\epsilon(t) - x_2^0(t)\right| = \left|x_2^\epsilon(t) - h_0(x_1(t), t)\right| < \rho$$

(c) The functions f_1, f_2, h_0 are sufficiently *smooth* (at least twice continuously differentiable).

(d) The fast subsystem is asymptotically stable.

Under conditions (a)-(d), there exists an integral manifold of (8.84) defined by

$$M_\epsilon : \qquad x_2(t) = h(x_1(t), t, \epsilon) \tag{8.85}$$

The dynamics on the manifold (8.85) is dictated by the following differential equation

$$\dot{x}_1(t) = f_1(x_1, h(x_1, t, \epsilon), t, \epsilon) \tag{8.86}$$

Note that (8.86) represents the exact slow subsystem and that (8.85) is the slow invariant manifold of (8.84). Using the second equation in (8.84) together with equation (8.85), and assuming that $h(x_1(t), t, \epsilon)$ is continuously differentiable, the fast differential equation can be written as

$$\epsilon \frac{\partial h}{\partial t} + \epsilon \frac{\partial h}{\partial x_1} f_1(x_1, h(x_1, t, \epsilon), t, \epsilon) = f_2(x_1, h(x_1, t, \epsilon), t, \epsilon) \tag{8.87}$$

which gives the manifold condition that the function $h(x_1(t), t, \epsilon)$ has to satisfy. If the functions f_1 and f_2 are smooth then the function h can be represented by

$$h(x_1, t, \epsilon) = h_0(x_1, t) + \epsilon h_1(x_1, t) + \epsilon^2 h_2(x_1, t) + \cdots \tag{8.88}$$

The functions $h_i, i = 0, 1, 2, ...$ can be obtained using only algebraic operations (see Sobolev, 1984, and references therein).

Example 8.2: For a linear singularly perturbed time invariant system defined by

$$\dot{x}_1(t) = A_1 x_1(t) + A_2 x_2(t)$$

$$\epsilon \dot{x}_2(t) = A_3 x_1(t) + A_4 x_2(t)$$

with asymptotically stable fast dynamics ($\mathrm{Re}(\lambda(A_4)) < 0$), the manifold condition (8.87) becomes (Kokotovic *et al.*, 1986)

$$\epsilon \frac{\partial h(x_1, t, \epsilon)}{\partial x_1} (A_1 x_1(t) + A_2 h(x_1, t, \epsilon)) = A_3 x_1(t) + A_4 h(x_1, t, \epsilon)$$

This partial differential equation has a simple solution

$$h(x_1(t), t, \epsilon) = -L(\epsilon)x_1(t)$$

so that the integral manifold is given by

$$x_2(t) = -L(\epsilon)x_1(t)$$

The matrix $L(\epsilon)$ can be easily determined as a solution of the following algebraic equation

$$-\epsilon L(\epsilon)(A_1 - A_2 L(\epsilon)) = A_3 - A_4 L(\epsilon)$$

The unique solution of the algebraic equation obtained exists under the assumption that the matrix A_4 is nonsingular. Even more, we can recognize that this algebraic equation is identical to the corresponding Chang transformation decoupling algebraic equation, hence it can be solved by using any of the methods previously discussed.

The dynamics on the slow integral manifold is governed by

$$\dot{x}_1(t) = (A_1 - A_2 L(\epsilon))x_1(t)$$

Δ

Define new variables $e_1(t)$ and $e_2(t)$ that in fact represent, respectively, the deviations from the exact slow subsystem and from the slow invariant manifold

$$\begin{aligned} e_1(t) &= x_1^\epsilon(t) - \eta(t) \\ e_2(t) &= x_2^\epsilon(t) - h(\eta + e_1, t, \epsilon) \end{aligned} \tag{8.89}$$

where

$$\dot{\eta}(t) = \mathcal{F}_1(\eta(t), t, \epsilon) = f_1(\eta, h(\eta, t, \epsilon), t, \epsilon) \tag{8.90}$$

Taking the derivatives in (8.89) we obtain

$$\begin{aligned} \dot{e}_1(t) &= f_1(\eta + e_1, e_2 + h(\eta + e_1, t, \epsilon), t, \epsilon) - f_1(\eta, h(\eta, t, \epsilon), \epsilon) \\ &= E_1(\eta, e_1, e_2, t, \epsilon) \end{aligned} \tag{8.91}$$

and

$$
\begin{aligned}
\epsilon\dot{e}_2(t) = E(\eta, e_1, e_2, t, \epsilon) &= f_2(\eta + e_1, e_2 + h(\eta + e_1, t, \epsilon), t, \epsilon) \\
&- f_2(\eta + e_1, h(\eta + e_1, t, \epsilon), \epsilon) \\
&- \epsilon \frac{\partial h(\eta + e_1, t, \epsilon)}{\partial(\eta + e_1)} f_1(\eta + e_1, e_2 + h(\eta + e_1, t, \epsilon), t, \epsilon) \\
&\epsilon \frac{\partial h(\eta + e_1, t, \epsilon)}{\partial(\eta + e_1)} f_1(\eta + e_1, h(\eta + e_1, t, \epsilon), t, \epsilon)
\end{aligned}
\tag{8.92}
$$

Note that the manifold condition (8.87) has been used in the derivation of this differential equation. It has been shown in (Sobolev, 1984) that the system of differential equations (8.90)-(8.92) has the integral manifold $e_1 = \epsilon L(\eta, e_2, t, \epsilon)$. The flow on this manifold is governed by (8.90) and (8.92).

In summary, under the above stated invariant manifold conditions, there exists a transformation defined by

$$
\begin{aligned}
x_1^\epsilon(t) &= \eta(t) + \epsilon L(\eta, v, t, \epsilon) \\
x_2^\epsilon(t) &= v(t) + h(\eta + \epsilon L(\eta, v, t, \epsilon), t, \epsilon)
\end{aligned}
\tag{8.93}
$$

such that in the new coordinates we have two decoupled subsystems

$$
\begin{aligned}
\dot{\eta}(t) &= \mathcal{F}_1(\eta(t), t, \epsilon) \\
\epsilon\dot{v}(t) = \mathcal{F}_2(\eta(t), v(t), t, \epsilon) &= E_2(\eta, \epsilon L(\eta, v, t, \epsilon), v, t, \epsilon)
\end{aligned}
\tag{8.94}
$$

The unknown quantity L can be obtained using the results from (Sobolev, 1984) by solving the following partial differential equation

$$
\begin{aligned}
&\epsilon \frac{\partial L}{\partial t} + \frac{\partial L}{\partial \eta} \mathcal{F}_1(\eta, t, \epsilon) + \frac{\partial L}{\partial v} E_2(\eta, \epsilon L, v, t, \epsilon) \\
&= f_1(\eta + e_1, e_2 + h(\eta + e_1, t, \epsilon), t, \epsilon) - f_1(\eta, h(\eta, t, \epsilon), \epsilon)
\end{aligned}
\tag{8.95}
$$

In the next section we review the integral manifold theory results for linear-quadratic optimal control of time invariant singularly perturbed systems.

8.3.2 Linear Optimal Control via Slow and Fast Integral Manifolds

The linear-quadratic optimal finite horizon control problem is defined in (8.44)-(8.45). Its state-costate form (Hamiltonian form) is given by (8.60)-(8.63). Based in the integral manifold theory presented in the previous section, it has been recommended in (Sobolev, 1984) to apply the following transformation to (8.60)

$$\begin{bmatrix} x(t) \\ p(t) \\ z(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} I & 0 & \epsilon G_1 & \epsilon G_2 \\ 0 & I & \epsilon G_3 & \epsilon G_4 \\ L_1 & L_2 & D_1 & D_2 \\ L_3 & L_4 & D_3 & D_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} \tag{8.96}$$

with

$$\begin{gathered} L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \quad \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = V \begin{bmatrix} I & H \\ M & I + MH \end{bmatrix} \\ \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} = (I + \epsilon LV) \begin{bmatrix} I & H \\ M & I + MH \end{bmatrix} \end{gathered} \tag{8.97}$$

where the transformation matrices L, P, H, and M satisfy

$$\epsilon \dot{L} = -\epsilon L(\mathbf{A}_1 + \mathbf{A}_2 L) + \mathbf{A}_3 + \mathbf{A}_4 L \tag{8.98}$$

$$\epsilon \dot{V} = -V(\mathbf{A}_4 - \epsilon L \mathbf{A}_2) + \epsilon(\mathbf{A}_1 + \mathbf{A}_2 L)V + \mathbf{A}_2 \tag{8.99}$$

$$\begin{gathered} \epsilon \dot{M} = -M[A_4 + \epsilon K_1 + (\epsilon K_2 - S_2)M] - Q_3 + \\ \epsilon K_3 + \left(-A_4^T + \epsilon K_4\right) M \end{gathered} \tag{8.100}$$

$$\begin{gathered} \epsilon \dot{H} = H\left[A_4^T - \epsilon K_4 + M(\epsilon K_2 - S_2)\right] - S_2 + \epsilon K_2 \\ +[A_4 + \epsilon K_1 + (\epsilon K_2 - S_2)M]H \end{gathered} \tag{8.101}$$

and

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = -L\mathbf{A}_2 = -\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \begin{bmatrix} A_2 & -S \\ -Q_2 & -A_3^T \end{bmatrix} \tag{8.102}$$

Matrices $\mathbf{A}_\mathrm{i}$ are defined in (8.61). It should be pointed out that the transformation defined in (8.96) is time varying due to the fact that the corresponding matrices are obtained as the solutions of the differential equations. Hence, the methodology presented is applicable to time varying linear singularly perturbed systems. Even more, the same technique under appropriate assumptions is applicable to the corresponding H_∞ optimization problem as demonstrated in (Fridman, 1995, 1996a). Note that equations (8.98)-(8.99) are time varying equivalents of the Chang transformation decoupling equations. Equation (8.100) corresponds to the reduced-order pure-fast differential Riccati equation introduced in (8.82).

The application of the transformation (8.96) to (8.60) produces in the new coordinates the following subsystems, a pure slow subsystem

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = (\mathbf{A}_1 + \mathbf{A}_2 L) \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \tag{8.103}$$

and two pure-fast subsystems

$$\begin{aligned} \epsilon\dot{\zeta}_1(t) &= (A_4 - S_2 M + \epsilon(K_1 + K_2 M))\zeta_1(t) \\ \epsilon\dot{\zeta}_2(t) &= \left(-A_4^T + M S_2 + \epsilon(K_4 - M K_2)\right)\zeta_2(t) \end{aligned} \tag{8.104}$$

It has been shown in (Sobolev, 1984) that the proposed mapping of the original boundary conditions (8.57) produces in the new coordinates independent boundary conditions for (8.103) and initial conditions for (8.104). Hence, the original two-point boundary value problem is decoupled into the independent pure-slow boundary value problem and two pure-fast initial value problems. The solution obtained in (8.103)-(8.104) can be considered as the open-loop optimal solution.

In (Fridman, 1995, 1996a) the closed-loop optimal solution of the corresponding linear-quadratic control problem defined by (8.44)-(8.45) is obtained. In these papers, the solution of the differential Riccati equation (8.48) is obtained in terms of solutions reduced-order pure-slow differential Riccati equation and three pure-fast linear differential equations using the following formula

$$\begin{aligned} &P(t)\begin{bmatrix} I + \epsilon G_2 P_{21}(t) & \epsilon(G_1 + G_2 P_{22}(t)) \\ L_1 + L_2 P_s(t) + D_2 P_{21}(t) & D_1 + D_2 P_{22}(t) + \epsilon L_2 P_{12}(t) \end{bmatrix} \\ &= \begin{bmatrix} P_s(t) + \epsilon G_4 P_{21}(t) & \epsilon(P_{12}(t) + G_3 + G_4 P_{22}(t)) \\ \epsilon(L_3 + L_4 P_s(t) + D_4 P_{21}(t)) & \epsilon(D_3 + D_4 P_{22}(t) + \epsilon L_4 P_{12}(t)) \end{bmatrix} \end{aligned} \tag{8.105}$$

where the pure-slow differential Riccati equation is given by

$$
\begin{gathered}
-\dot{P}_s(t) = P_s(t)a_1 - a_4P_s(t) - a_3 + P_s(t)a_2P_s(t) \\
P_s(t_f) = U_{11}
\end{gathered}
\tag{8.106}
$$

with the coefficients $a_i, i = 1, \ldots, 4$, defined in (8.78). $P_{12}(t), P_{21}(t)$, $P_{22}(t)$ are the solutions of the following pure-fast linear differential equations

$$
\begin{gathered}
\epsilon\dot{P}_{12}(t) = -P_{12}(t)(A_4 - S_2M + \epsilon(K_1 + K_2M + a_2)) + \epsilon a_4P_{12}(t) \\
P_{12}(t_f) = U_{12}
\end{gathered}
\tag{8.107}
$$

$$
\begin{gathered}
\epsilon\dot{P}_{21}(t) = -\Big((A_4 - S_2M)^T - \epsilon(K_4 - MK_2)\Big)P_{21}(t) \\
-\epsilon P_{21}(t)(a_1 + a_2P_s(t)), \qquad P_{21}(t_f) = U_{21}
\end{gathered}
\tag{8.108}
$$

$$
\begin{gathered}
\epsilon\dot{P}_{22}(t) = -P_{22}(t)(A_4 - S_2M + \epsilon(K_1 + K_2M)) \\
-\epsilon\Big((A_4 - S_2M)^T - \epsilon(K_4 - MK_2)\Big)P_{22}(t), \qquad P_{22}(t_f) = U_{22}
\end{gathered}
\tag{8.109}
$$

Note that (8.106) is identical in form to (8.81). They might differ only in the terminal conditions. The terminal conditions for the above differential equations are obtained from

$$
\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}^{-1}_{|t=t_f}
\tag{8.110}
$$

where $Y_i, i = 1, \ldots, 4$, matrices are given by

$$
\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} I + \epsilon VL & -\epsilon V \\ -\Psi L & \Psi \end{bmatrix} \begin{bmatrix} I & 0 \\ Q_{1t_f} & \epsilon Q_{2t_f} \\ 0 & I \\ Q_{2t_f}^T & Q_{3t_f} \end{bmatrix}
\tag{8.111}
$$

with

$$
\Psi = \begin{bmatrix} I + HM & -H \\ -M & I \end{bmatrix}
\tag{8.112}
$$

The matrix whose inversion is needed in (8.110) is

$$\begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ * & I + H^{(0)}\left(M^{(0)} - Q_{3t_f}\right) \end{bmatrix} + O(\epsilon) \tag{8.113}$$

The following assumption is required (Fridman, 1996a) to assure invertibility of the matrix defined in (8.113).

Assumption 8.6: The matrix $I + H^{(0)}\left(M^{(0)} - Q_{3t_f}\right)$ is invertible at $t = t_f$.

It has been shown in (Fridman, 1996a), under some standard assumptions used in the literature on singularly perturbed linear-quadratic optimal control problem, that the solution for the singularly perturbed differential Riccati equation (8.48) can be obtained from formula (8.105) since the corresponding coefficient matrix is nonsingular for sufficiently small values of the singular perturbation parameter ϵ. Having obtained the solution for $P(t)$, the closed-loop optimal control is found.

The linear-quadratic finite horizon optimal control problem of singularly perturbed linear continuous-time systems with time delays defined by

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + A_2 z(t) + B_1 u(t), \qquad & x(0,\epsilon) = x_0 \\ \epsilon \dot{z}(t) &= A_3 x(t) + A_4 z(t) + A_5 z(t-\epsilon) + B_2 u(t), \qquad & z(0,\epsilon) = z_0 \end{aligned} \tag{8.114}$$

and the performance criterion given by (8.45) via integral manifold theory have been considered in (Fridman, 1990a). Generalization of the Chang decoupling transformation to singularly perturbed systems with time delays can be found in (Fridman, 1996b).

Exact slow-fast decompositions of some classes of nonlinear singularly perturbed optimal control problems (nonlinear only with respect to the slow variable as considered in (Chow and Kokotovic, 1981), and nonlinear with respect to both slow and fast variables, but linear with respect to control) via invariant manifold theory have been obtained in (Fridman, 1999, 2000).

H_∞ linear-quadratic optimal control problems of singularly perturbed systems have been considered in (Fridman, 1995, 1996a). In a very recent paper, (Fridman and Shaked, 2000), the solution to the static output feedback control of singularly perturbed systems in the H_∞ setup has been presented.

Dynamic systems described by neutral type singularly perturbed differential equations have been studied via asymptotic expansions using integral slow and fast manifolds in (Fridman, 1990b).

8.4 Conclusions

We have shown that the exact slow-fast decomposition of (Su *et al.*, 1992a) is applicable under certain assumptions to *nonstandard* singularly perturbed linear systems. In contrast to (Wang and Frank, 1992), where the $O(\epsilon)$ accuracy is obtained, the presented methodology keeps all good features of (Wang and Frank, 1992) (reduced-order slow-fast decomposition and numerical well-conditioning—the original problem is numerically ill-conditioned) and *produces the exact solution* to the linear-quadratic optimal control problem of nonstandard singularly perturbed systems. In addition, we have shown how to decompose exactly the full-order, ill-conditioned Kalman filter of nonstandard singularly perturbed linear systems into reduced-order, well-conditioned, pure-slow and pure-fast Kalman filters and how to solve the corresponding linear-quadratic optimal stochastic control problem. The presentation about nonstandard singularly perturbed systems mostly follows the work of (Kecman and Gajic, 1999).

In the second of this chapter we have indicated a strategy for solving the finite horizon linear-quadratic optimal control problem in terms of slow and fast subproblems (pure-slow and pure-fast differential Riccati equations). The presented results via the Hamiltonian approach represent only ideas. On the other hand, the complete solution to this problem via integral manifold theory has been presented in Section 8.3.2. It seems that integral slow-fast manifold theory is a very promising tool for slow-fast decomposition of singularly perturbed time varying linear control systems and finite horizon optimization problems.

Appendix 8.1

A differential Riccati equation of a singularly perturbed system is given by (Grodt and Gajic, 1988)

$$-\dot{P}(t) = P(t)A + A^T P(t) + Q - P(t)SP(t), \quad P(T) = F \tag{8.115}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad Q \geq 0 \tag{8.116}$$

$$S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{Z}{\epsilon} \\ \frac{Z^T}{\epsilon} & \frac{S_2}{\epsilon^2} \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & \epsilon F_3 \end{bmatrix}, \quad R > 0$$

are $n \times n$ constant matrices and ϵ is a small positive parameter. The presence of the small parameter ϵ makes this problem numerically ill-defined, producing the so-called stiff numerical problem (huge slope at terminal time), (Miranker, 1981). In order to overcome this difficulty, the Taylor series expansion approach, with respect to a small parameter ϵ has been taken in (Yackel and Kokotovic, 1973) leading to a family of well-defined reduced-order problems. However, the Taylor series expansion method is not recursive in its application. When one is interested in a high degree of accuracy, or when ϵ is not very small, the size of computations required can be considerable. In such cases, the advantage of using the series expansion method (the important theoretical tool) is questionable from the numerical point of view, and sometimes that method is almost not applicable.

We will exploit the Hamiltonian form of the solution of the Riccati equation, and a nonsingular transformation (Chang, 1972) in order to obtain an efficient recursive numerical method for solving (8.115). The Chang transformation is used to block diagonalize the Hamiltonian, so that the required solution is obtained in terms of reduced-order problems.

The solution of (8.115) can be sought in the form

$$P(t) = M(t)N^{-1}(t) \tag{8.117}$$

where matrices $M(t)$ and $N(t)$ satisfy a system of linear equations (Kwakernaak and Sivan, 1972)

$$\begin{aligned} \dot{M}(t) &= -A^T M(t) - QN(t), \quad & M(T) = F \\ \dot{N}(t) &= -SM(t) + AN(t), \quad & N(T) = I \end{aligned} \tag{8.118}$$

and $N(t)$ is assumed to be nonsingular for $\forall t,\ t < T$. This approach is considered as the most efficient numerical method for the solution of the differential Riccati equation (Kenney and Leipnik, 1985), where the invertibility problem of $N(t)$ is solved by performing a reinitialization along the path $t_0 < t < T$ whenever $N(t)$ is close to being singular.

Knowing the nature of the solution of (8.115), which is properly scaled as (Kokotovic and Khalil, 1986; Yackel and Kokotovic, 1973)

$$P(t) = \begin{bmatrix} P_1(t) & \epsilon P_2(t) \\ \epsilon P_2^T(t) & \epsilon P_3(t) \end{bmatrix}, \quad P(T) = F = \begin{bmatrix} F_1 & \epsilon F_2 \\ \epsilon F_2^T & \epsilon F_3 \end{bmatrix} \tag{8.119}$$

where $\dim P_1 = n_1 \times n_1$, $\dim P_3 = n_2 \times n_2$, $n_1 + n_2 = n$ (n_1-slow variables, n_2-fast variables), we introduce compatible partitions of $M(t)$ and $N(t)$ matrices

$$M(t) = \begin{bmatrix} M_1(t) & M_2(t) \\ M_3(t) & M_4(t) \end{bmatrix}, \quad N(t) = \begin{bmatrix} N_1(t) & N_2(t) \\ N_3(t) & N_4(t) \end{bmatrix} \tag{8.120}$$

The invertibility of the matrix $N(t)$ for every t, $t_0 \leq t < T$, plays an important role in the proposed method. The condition under which $N(t)$ is an invertible matrix is stated in the following lemma proved in (Grodt and Gajic, 1988).

Lemma 8.4 *If the triple* $(A, B, \sqrt{Q})$ *is stabilizable-observable, then the matrix* $N(t)$, *with* $N(T) = I$ *is invertible for any* $t \in (t_0, T)$.

◇

Partitioning (8.118), according to (8.120), will reveal a decoupled structure, that is, equations for M_1, M_3, N_1, and N_3 are independent of equations for M_2, M_4, N_2, and N_4 and vice versa. Introducing the notation

$$U = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}, \quad \epsilon V = \begin{bmatrix} M_3 \\ \epsilon N_3 \end{bmatrix}, \quad X = \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}, \quad \epsilon Y = \begin{bmatrix} M_4 \\ \epsilon N_4 \end{bmatrix} \tag{8.121}$$

$$\begin{gathered} T_1 = \begin{bmatrix} -A_1^T & -Q_1 \\ -S_1 & A_1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -A_3^T & -Q_2 \\ -Z & A_2 \end{bmatrix} \\ T_3 = \begin{bmatrix} -A_2^T & -Q_2^T \\ -Z^T & A_3 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -A_4^T & -Q_3 \\ -S_2 & A_4 \end{bmatrix} \end{gathered} \tag{8.122}$$

and after doing some algebra, we get two systems of singularly perturbed matrix differential equations

$$\begin{aligned} \dot{U} &= T_1 U + T_2 V, \quad U(T) = \begin{bmatrix} F_1 \\ I \end{bmatrix} \\ \epsilon \dot{V} &= T_3 U + T_4 V, \quad V(T) = \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \end{aligned} \tag{8.123}$$

$$\begin{aligned} \dot{X} &= T_1 X + T_2 Y, \quad X(T) = \begin{bmatrix} \epsilon F_2 \\ 0 \end{bmatrix} \\ \epsilon \dot{Y} &= T_3 X + T_4 Y, \quad Y(T) = \begin{bmatrix} F_3 \\ I \end{bmatrix} \end{aligned} \tag{8.124}$$

Note that these two systems have exactly the same form and they differ in terminal conditions only. From this point we will proceed by applying the Chang transformation to (8.123) and (8.124). This transformation is defined by (Chang, 1972)

$$\mathbf{T}_1 = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix} \tag{8.125}$$

and

$$\mathbf{T}_1^{-1} = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon LH \end{bmatrix} \tag{8.126}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{8.127}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0 \tag{8.128}$$

Applying this transformation to (8.123) and (8.124) we obtain

$$\dot{\widehat{U}}(t) = (T_1 - T_2 L)\widehat{U}(t), \quad \widehat{U}(T) = (I - \epsilon HL)U(T) - \epsilon HV(T) \tag{8.129}$$

$$\epsilon\dot{\widehat{V}}(t) = (T_4 + \epsilon L T_2)\widehat{V}(t), \quad \widehat{V}(T) = LU(T) + V(T) \tag{8.130}$$

$$\dot{\widehat{X}}(t) = (T_1 - T_2 L)\widehat{X}(t), \quad \widehat{X}(T) = (I - \epsilon HL)X(T) - \epsilon HY(T) \tag{8.131}$$

$$\epsilon\dot{\widehat{Y}}(t) = (T_4 + \epsilon L T_2)\widehat{Y}(t), \quad \widehat{Y}(T) = LX(T) + Y(T) \tag{8.132}$$

Solutions of (8.129)-(8.132) are given by

$$\widehat{U}(t) = e^{(T_1 - T_2 L)(t-T)}\widehat{U}(T) \tag{8.133}$$

$$\widehat{V}(t) = e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)}\widehat{V}(T) \tag{8.134}$$

$$\widehat{X}(t) = e^{(T_1 - T_2 L)(t-T)}\widehat{X}(T) \tag{8.135}$$

$$\widehat{Y}(t) = e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)}\widehat{Y}(T) \tag{8.136}$$

so that in the original coordinates we have

$$U(t) = e^{(T_1 - T_2 L)(t-T)}\widehat{U}(T) + \epsilon H e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)}\widehat{V}(T) \tag{8.137}$$

$$V(t) = -Le^{(T_1 - T_2 L)(t-T)}\widehat{U}(T) + (I - \epsilon LH)e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)}\widehat{V}(T) \tag{8.138}$$

$$X(t) = e^{(T_1 - T_2 L)(t-T)} \widehat{X}(T) + \epsilon H e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)} \widehat{Y}(T) \tag{8.139}$$

$$Y(t) = -L e^{(T_1 - T_2 L)(t-T)} \widehat{X}(T) + (I - \epsilon L H) e^{\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t-T)} \widehat{Y}(T) \tag{8.140}$$

Partitioning (8.137)-(8.140) according to (8.121) will produce all components of matrices $M(t)$ and $N(t)$, that is,

$$\begin{bmatrix} M_1(t) \\ N_1(t) \end{bmatrix} = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = U(t), \quad \begin{bmatrix} M_2(t) \\ N_2(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = X(t)$$

$$\begin{bmatrix} \frac{1}{\epsilon} M_3(t) \\ N_3(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = V(t), \quad \begin{bmatrix} \frac{1}{\epsilon} M_4(t) \\ N_4(t) \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = Y(t)$$

so that the required solution of (8.115) is given by

$$P(t) = \begin{bmatrix} U_1(t) & X_1(t) \\ \epsilon V_1(t) & \epsilon Y_1(t) \end{bmatrix} \begin{bmatrix} U_2(t) & X_2(t) \\ V_2(t) & Y_2(t) \end{bmatrix}^{-1} \tag{8.141}$$

Thus, in order to get the numerical solution of (8.115), that is $P(t)$, which has dimensions $n \times n = (n_1 + n_2) \times (n_1 + n_2)$, we have to solve two simple algebraic equations (8.127) and (8.128) of dimensions of $(2n_2 \times 2n_1)$ and $(2n_1 \times 2n_2)$, respectively. The existing numerical algorithms for solving (8.127) and (8.128) can be found in (Kokotovic *et al.*, 1980; Gajic, 1986; Grodt and Gajic, 1988). Then, two exponential forms $\exp[(T_1 - T_2 L)(t - T)]$ and $\exp\left[\frac{1}{\epsilon}(T_4 + \epsilon L T_2)(t - T)\right]$ have to be transformed in the matrix forms by using some of the well-known approaches (Molen and Van Loan, 1978). Finally, the inversion of the matrix $N(t)$ has to be performed. Since the matrices $M(t)$ and $N(t)$ contain unstable modes of the Hamiltonian also (Kwakernaak and Sivan, 1972), then even though a product $M(t)$ and $N^{-1}(t)$ tends to be a constant as $t \to \infty$, the inversion of the nonsingular matrix $N(t)$, which contains huge elements, will hurt the accuracy.

The reinitialization version of the Hamiltonian approach, which leads to the known Kalman-Englar method (Kwakernaak and Sivan, 1972), is considered as the most efficient numerical method for the solution of the general matrix differential Riccati equation. The reinitialization technique applied to the previously obtained formulas will modify (8.118), (8.123)-(8.124), respectively, in

$$M(k\triangle t) = P(k\triangle t) \tag{8.142}$$

$$U(k\Delta t) = \begin{bmatrix} P_1(k\Delta t) \\ I \end{bmatrix}, \quad V(k\Delta t) = \begin{bmatrix} P_2^T(k\Delta t) \\ 0 \end{bmatrix} \tag{8.143}$$

$$X(k\Delta t) = \begin{bmatrix} \epsilon P_2(k\Delta t) \\ 0 \end{bmatrix}, \quad Y(k\Delta t) = \begin{bmatrix} P_3(k\Delta t) \\ I \end{bmatrix} \tag{8.144}$$

where k represents the number of steps and Δt is an integration step. This will introduce slight modifications in formulas (8.129)-(8.132), namely, instead of the final time T a discrete time $k\Delta t$ has to be used. These changes can be implemented very easily from the programming point of view.

The efficiency of the above method finding the solution of the differential matrix Riccati equation of singularly perturbed systems has been demonstrated in (Grodt and Gajic, 1988) on a seventh-order model of a synchronous machine connected to an infinite bus (Kokotovic *et al.*, 1980). It has been shown in (Grodt and Gajic, 1988) that in order to get the accuracy of four decimal digits, it takes 12 iterations (the fixed point iterations method has been used for solving the algebraic equations composing the Chang transformation—in order to be able to compare the proposed recursive scheme to the power-series expansion method, since both methods are producing the same order of accuracy). That means if the power-series expansion method had been used, in order to get the same accuracy, it would have required 12 terms, that is (Yackel and Kokotovic, 1973)

$$P(t, \epsilon) = \sum_{m=0}^{11} \frac{\epsilon^m}{m!} \left\{ P_s^{(m)}(t) + P_f^{(m)}(\tau) \right\} + O(\epsilon^{12}), \quad \tau = \frac{t - T}{\epsilon}$$

where

$$P_s^{(m)}(t) = \begin{bmatrix} P_{1s}^{(m)}(t) & \epsilon P_{2s}^{(m)}(t) \\ \epsilon P_{2s}^{(m)^T}(t) & \epsilon P_{3s}^{(m)}(t) \end{bmatrix}$$

$$P_f^{(m)}(\tau) = \begin{bmatrix} P_{1f}^{(m)}(\tau) & \epsilon P_{2f}^{(m)}(\tau) \\ \epsilon P_{2f}^{(m)^T}(\tau) & \epsilon P_{3f}^{(m)}(\tau) \end{bmatrix}$$

It is shown in (Yackel and Kokotovic, 1973), (p. 21, formula 32) that the right-hand sides of differential equations for $P_{1f}^{(1)}(\tau)$, $P_{2f}^{(1)}(\tau)$, and $P_{3f}^{(1)}(\tau)$ contain respectively 7, 23, and 22 terms, each consisting of a product of two or three matrices. Thus, the size of the computations required for only an $O(\epsilon^2)$ accuracy is already enormous. The complexity of the right-hand side of differential equations for $P_f^{(m)}(\tau)$ grows extremely quickly with the increase of m so that this nice theoretical method is not convenient for practical computations. For an $O(\epsilon^{12})$ accuracy, the right-hand sides of the differential equations for the power-series expansion method will contain hundreds or even thousands of terms, and this example can not be efficiently solved by using the power-series expansion method.

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9

Concluding Remarks

We have presented, more or less, the complete story about the Hamiltonian approach (Grodt and Gajic, 1988; Su *et al.*, 1992a,b; Gajic and Lim, 1994; Aganovic *et al.*, 1995) for time scale decomposition order-reduction and parallel signal processing of continuous-time linear-quadratic optimal steady state control and filtering problems. The corresponding steady state *discrete-time* results are either obtained or can be obtained by extending dual continuous-time results, except for the *discrete-time high gain and cheap linear control problems and small measurement noise linear filtering problems*, where meaningful problem formulations are still missing in the control literature.

Recently developed, the *eigenvector approach* (Kecman *et al.*, 1999) for pure-slow and pure-fast decomposition of linear-quadratic continuous-time singularly perturbed systems is also awaiting its extension to the discrete-time domain.

Areas of *time varying* linear-quadratic optimal control and filtering problems and *finite horizon optimization* problems in both continuous- and discrete-time domains are widely open for future research. It is known (Aganovic and Gajic, 1995) that the solution of the above problems via pure-slow and pure-fast decomposition techniques will open a door for reduced-order linear optimal control of singularly perturbed *bilinear systems*.

Integral manifold theory has been already established as a promising tool for studying time varying and finite horizon optimization problems of linear singularly perturbed systems (Sobolev, 1984; Fridman, 1996) in terms of slow and fast subproblems. Even more, that theory has successfully extended the corresponding slow-fast decomposition of some classes of nonlinear singularly perturbed optimal control problems (Fridman, 1999, 2000).

In conclusion, we want to point out that the Hamiltonian approach presented is this book is a continuation of the recursive methods and parallel algorithms (Gajic *et al.*, 1990; Gajic and Shen, 1993) that still remain powerful tools for some classes of singularly perturbed control problems such as Nash and Stackelberg differential games, jump parameter linear stochastic systems, output feedback control, and optimal control of bilinear systems, and some other more complex classes of singularly perturbed control systems (Kokotovic *et al.*, 1986).

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