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Qing-Guo Wang

Decoupling Control

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1. Introduction

Prior to the 1940s, most industrial systems were run essentially manually or using on-off control. Many operators were needed to keep watch on the many variables in the plant. With the continuous industrialization over the last several decades, manufacturing and production have taken off at an incredibly high speed in almost every part of the world. As a consequence of the expanding scale and volume of production, there could be hundreds or even thousands of variables to be controlled in a plant. The manual effort thus needed in operation is tremendous. With increasing labor and equipment costs, and with eager demand for high precision, quality and efficiency, the idea of employing operators for the control of physical systems rapidly became uneconomical and infeasible. Automatic control thus becomes a solution much sought after. The fundamental component in an automatic control system is the so called “controller”. It could either be a piece of hardware or software code in a computer. Its job is to receive information about the system from a variety of sensors, process it, and automatically generate commands for corrective action to bring the variable of interest to its desired value or trajectory. Wide applications of automatic control have driven great attention to its theoretical development. Since the age of the familiar classical control theory, many new and sophisticated theories have evolved and striking developments have taken place especially since 1950s. Optimal and stochastic control, state-space theory and adaptive control are only a few to name. The advances in microelectronics which give cheap microprocessors whose computational power is continuously increasing have also acted as a supportive catalyst in the development of new concepts by providing convenient test grounds and platforms for extensive simulations to be performed in the various stages of design and verification, and enabling effective implementation of advanced control algorithms. Today, automatic controllers have found a much wider scope of usage than only in industrial processes. It may range from missile tracking in military applications, down to water and temperature control in washing machines. Automatic control systems have permeated life in all advanced societies.

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1.1 Multivariable Systems and Decoupling Control

A typical system (natural or man-made) will have several variables to be controlled and is called multivariable systems if so. Multivariable systems can be found almost everywhere. In the office, the temperature and humidity are crucial to comfort. Water flow speed and rate are two key measures of a river. A robot needs six degree-of-freedoms in order to have a full range of positioning, and the same can be said to airplanes and missiles. Some phenomena are unique to multivariable systems only, and could not occur in single variable systems. For instance, a multivariable system can have a pole and zero which coincide with each other and yet do not cancel each other. A zero of some element in a multivariable system play no role in the system properties. The most important feature with a multivariable system is possible cross-couplings or interactions between its variables, i.e., one input variable may affect all the output variables. They prevent the control engineer from designing each loop independently as adjusting controller parameters of one loop affects the performance of another, sometimes to the extent of destabilizing the entire system. It is the multivariable interactions that accounts for essential difference for design methodologies between single-variable and multivariable control systems. In general, multivariable control is much more difficult than single variable control, and it is the topic of interest in the present book.

A multivariable system will be simplified to a number of single variable systems if it has no cross-couplings between variables, and is called decoupled in this case. As mentioned above, a given multivariable plant usually has couplings. A design strategy is then to design a multivariable controller which can decouple the plant, that is, the resulting control system has no more couplings between the desired reference variables and the plant output variables. It should be stressed that decoulping control is popular NOT mainly because it can simplify multivariable control system design, BUT rather because it is a desired feature in many practical applications. Firstly, decoupling is required for ease of system operations, at least in process and chemical industry, as otherwise, technicians operating a multivariable control system can hardly decide the values of multiple set-points to meet their target. Secondly, poor decoupling could be the principal common control problem in industry, see the recent survey (Kong, 1995) from the leading controller manufactures such as Fish-Rosemount, Yokogawa and Foxboro. According to our own industrial experience, poor decoupling in closed-loop could, in many cases, lead to poor diagonal loop performance. Conversely, good decoupling is helpful for good loop performance. An evidence is that some optimal control without any decoupling requirement could eventually result in a decoupled closed-loop (Dickman and Sivan, 1985; Linnemann and Wang, 1993). Thirdly, it should be noted that even if the decoupling requirement is relaxed to the limited coupling, it will still lead to almost decoupling if the number of inputs/outputs is large. Roughly, if the number is 10, total couplings from all other loops are limited to 30%, then each off-diagonal element will have relative gain less than 3%, so the system is almost decoupled.

It should be, however, pointed out that there are cases where decoupling should not be used, instead, couplings are deliberately employed to boost performance such as airplane control systems. The extreme case is that decoupling cannot be used at all since it may destabilize the system. In short, decoupling is a common practice in industry, but needs to be justified from the real situation and requirements before use.

The study of decoupling linear time-invariant multivariable systems has received considerable attentions in both control theory and industrial practice for several decades, dating back at least to (Voznesenskii, 1938). During that early times, this problem was treated with transfer matrices, see (Kavanagh, 1966; Tsien, 1954) for reviews. The state space approach to decoupling was initiated by Morgan (Morgan, 1964). A seminal paper by Falb and Wolovich (Falb and Wolovich, 1967) presented a necessary and sufficient condition on the solvability of decoupling problem for square systems described by state space models. The equivalent condition for systems described by transfer matrices was given by Gilbert (Gilbert, 1969) later. In these investigations, the problem is confined to the case of scalar input and output blocks and hence with equal number of inputs and outputs. The more general block decoupling problem was first defined and solved by Wonham and Morse (Morse and Wonham, 1970; Wonham and Morse, 1970) using geometric approach, see also the monograph (Wonham, 1986). For this kind of general block decoupling problem, an alternative approach was developed by Silverman and Payne (Silverman and Payne, 1971) based on structure algorithm proposed by Silverman (Silverman, 1969). In these works, it is still required that the transformation matrix between the new control inputs and old control inputs is nonsingular. The complete solution to general Morgan problem, without any assumptions on either system matrices or feedback matrices, is due to Descusse *et al.* (Descusse, 1991; Descusse *et al.*, 1988). Other developments can be found in (Commault *et al.*, 1991; Desoer and Gundes, 1986; Hautus and Heymann, 1983; Koussiouris, 1979; Koussiouris, 1980; Pernebo, 1981; Williams and Antsaklis, 1986), to name a few.

A parallel avenue for the study of decoupling problem is for unity output feedback systems based on input-output models. The existence condition for decoupling by unity output feedback compensation is that the plant has full row rank, which is very simple and well known. But the problem of decoupling in such a configuration while maintaining internal stability of the decoupled system appears to be much more difficult. Under the assumption that the plant has no unstable poles coinciding with its zeros, the diagonal decoupling problem is solvable and a parameterization of controllers solving the problem could be obtained (Safonov and Chen, 1982; Vardulakis, 1987; Peng, 1990; Desoer, 1990; Lin and Hsieh, 1991). The assumption is however not necessary and is relaxed later by Linnemann and Maier (1990) for 2×2 plants, and by Wang (1992) for square plants, along with a controller parameterization. Alternative necessary and sufficient conditions for this problem was also derived based on transfer matrices and residue by Lin (1997) and on the new notion of the minimal C^+-decoupler by Wang and Yang (2002). The results of Wang (1992) are generalized to the **block** decoupling problem for possibly **non-square** plants and the performance limitations in the decoupled system are compared with those of non-decoupled systems in Linnemann and Wang (1993).

The works cited above consider delay-free systems only. Time delay is however present and popular in process and chemical industries, and causes a serious obstacle to good process operation and control. It prevents high gain of a conventional controller from being used, leading to offset and sluggish system response. The problem is usually further complicated by multivariable nature of most plants in operation. Input-output loops in a multivariable plant usually have different time delays, and for a particular loop its output could be affected by all the inputs through likely different time delays. As a result, such a plant may then be represented by a multivariable transfer function matrix having multiple (or different) time delays around a operating point. Wang *et al.* (1997*a*) presented a transfer function matrix approach to decoupling design. It is shown there that the controller diagonal elements can be first designed with respect to the "equivalent processes" only and the controller off-diagonal elements then determined to achieve decoupling. A systematic method was proposed in Zhang (1999) for the decoupling design of general multivariable controller for processes with multi-delays. A controller of lowest complexity is found so as to achieve fastest loops with acceptable overshoot and minimum loop interactions.

For single variable processes with delay, Smith (1957) suggested a compensation scheme which can remove the delay from the closed-loop characteristic equation and thus eases feedback control design and improves set-point response greatly. Like the SISO case, multivariable delay compensation may be devised in hope of performance enhancement. This was done for multivariable systems with single delay (Alevisakis and Seborg, 1973; Alevisakis and Seborg, 1974) and for multi-delays (Ogunnaike and Ray, 1979; Ogunnaike *et al.*, 1983). In the perfect model-process match case, they can, like SISO Smith's, eliminate all the time delays from the system characteristic equation but, unlike SISO Smith's, fails to achieve the desired output performance expected from the delay-free design because the delays in the process could mix up the outputs of delay-free part to generate messy actual output response. Thus, the performance of this MIMO Smith might sometimes be poor (Garcia and Morari, 1985). Jerome and Ray (1986) proposed a different version of multivariable Smith predictor control to improve overall performance. However, their design of the primary controller is based on a transfer matrix with time delays and is difficult to carry out. This also contradicts the Smith's philosophy of delay removal from closed-loop characteristic equation for ease of stable design. The problem was solved in Wang *et al.* (2000) by incorporating decoupling into the MIMO Smith scheme.

It is well known that a Smith predictor controller can be put into an equivalent internal model control (IMC) structure. The IMC was introduced by Garcia and Morari (1982) and studied thoroughly in Morari and Zafiriou (1989) as a powerful control design strategy for linear systems. In principle, if a multivariable delay model, $\hat{G}(s)$, is factorized as $\hat{G}(s) = \hat{G}^{+}(s)\hat{G}^{-}(s)$ such that $\hat{G}^{+}(s)$ is diagonal and contains all the time delays and non-minimum phase zeros of $\hat{G}(s)$, then the IMC controller can be designed, like the scalar case, as $K(s) = \{\hat{G}^{-}(s)\}^{-1}$ with a possible filter appended to it, and the performance improvement of the resultant IMC controller over various versions of the multivariable Smith predictor schemes can be expected (Garcia and Morari, 1985). For a scalar transfer function, it is quite easy to obtain $\hat{G}^{+}(s)$. However, it becomes much more difficult for a transfer matrix with multi-delays. The factorization is affected not only by the time delays in individual elements but also by their distributions within the transfer function matrix, and a non-minimum phase zero is not related to that of elements of the transfer matrix at all (Holt and Morari, 1985*b*; Holt and Morari, 1985*a*). In Zhang (1999), an effective approach to the IMC analysis and design is proposed for decoupling and stabilizing linear stable square multivariable processes with multi-delays. There, the characteristics of all the controllers which solve this decoupling problem with stability and the resulting closed-loop systems will be given in terms of their unavoidable time delays and non-minimum phase zeros which actually determine performance limitations for IMC systems.

It can be concluded that decoupling control is now well developed, at least for nominal design, and it is timely and desirable to write a book which is devoted to such an important topic and gives a comprehensive and in-depth coverage of the topic. In the author's opinion, the emphasis of future research should be on robust decoupling and in particular, practical and stable design techniques which can achieve the specified coupling constraints under the given plant uncertainty.

1.2 Disturbance Decoupling

It is well known (Astrom and Hagglund, 1995) that the attenuation of load disturbance is of a primary concern for any control system design, and is even the ultimate objective for process control, where the set-point may be kept unchanged for years (Luyben, 1990). In fact, countermeasure of disturbance is one of the top factors for successful and failed applications (Takatsu and Itoh, 1999). If the disturbance is measurable, feedforward control is a useful tool for cancelling its effect on the system output (Ogata, 1997), while feedback control may still be used as usual for stability and robustness. Formally, the problem of (dynamic) disturbance decoupling is to find a compensator such that the resulting closed-loop transfer function from the disturbance to the controlled output is equal to zero. This problem is well-known and has been investigated extensively. It was actually the starting point for the development of a geometric approach to systems theory. Numerous investigators, employing a variety of mathematical formulations and techniques, have studied this problem.

In a geometric setting the problem using a state feedback was considered by Basile and Marro (1969) and Wonham (1986). Equivalent frequency domain solvability conditions were given by Bhattacharyya (1980), Bhattacharyya (1982). It is more realistic to assume that only output is available for feedback. In this context, the problem was solved by Akashi and Imai (1979) and Schumacher (1980). However, they did not consider the issue of stability. The disturbance decoupling by output feedback with stability or pole placement was solved for the first time by Willems and Commault (1981). This problem was also considered from an input-output viewpoint by Kucera (1983) for single variable system and by Pernebo (1981) for multivariable systems. The structure of the control system which decouples the disturbances is now well understood in both geometric terms and frequency domain terms.

To reject unmeasurable disturbances which are more often encountered in industry, one possible way is to rely on the single controller in the feedback system in addition to normal performance requirements. Then, there will be inevitably a design trade-off between the set-point response and disturbance rejection performance (Astrom and Hagglund, 1995). To alleviate this problem, a control scheme which introduces some add-on mechanism to the conventional feedback system was introduced, first in the filed of the servo control in mechatronics, and is called the disturbance observer (Ohnishi, 1987). It was further refined by Umeno and Hori (1991). The disturbance observer estimates the equivalent disturbance as the difference between the actual process output and the output of the nominal model. The estimate is then fed to a process inverse model to cancel the disturbance effect on the output. The disturbance observer makes use of the plant inverse, and its application needs some approximation as the plant inverse is usually not physically un-realizable due to improperness and/or time delay.

The complete disturbance decoupling is usually difficult to achieve. A more realistic requirement will be static or asymptotic disturbance decoupling. The latter problem falls in a general framework of asymptotic tracking and regulation problem, a fundamental control system problem. Wonham (1986) gives a complete exposition, where solutions together with (necessary and sufficient) solvability conditions, and a procedure for the construction of a compensator are obtained. This problem had been alternatively been considered by Davison (1975) in an algebraic setting. Both works are carried out in a state-space viewpoint. This problem, which can also be formulated in a frequency domain setting, was also approached from an input-output viewpoint such as Francis (1977) who considered the case where feedback signals coincide with the outputs; Chen and Pearson (1978) who considered the case where feedback signals are certain functions of outputs; and finally

Pernebo (1981) who considered the general case where feedback signals may be different from outputs and not related to each other. The crucial condition for the solvability in frequency domain setting is the existence condition for the solution of Diophantine equation (Kucera, 1979), which, in a special case, is simplified to skew-primeness (Wolovich, 1978). Now, the problem has been well solved in both state space and frequency domains. The relevant solutions in terms of the internal model principle are popular in control society.

In practice, we often encounter the situation where the reference commands to be tracked and/or disturbance to be rejected are periodic signals, e.g., repetitive commands or operations for mechanical systems such as industrial robots, and disturbances depending on the frequency of the power supply (Hara *et al.*, 1988). Disturbances acting on the track-following servo system of an optical disk drive inherently contain significant periodic components that cause tracking errors of a periodic nature. For such disturbances, the controller designed for step type reference tracking and/or disturbance rejection will inevitably give an uncompensated error. Though the internal model principle seems applicable, the unstable modes of a periodic disturbance is infinite and no longer rational, and actually it contains a time delay term in its denominator. For such delay and unstable systems, no stabilizing scheme could be found in the literature. Therefore, one has to abandon the internal model principle and seek other solutions. Hara *et al.* (1988) pioneered a method for periodic disturbance rejection, which is now called repetitive control. However, this method potentially makes the closed-loop system prone to instability, because the internal positive feedback loop that generates a periodic signal reduces the stability margin. Consequently, the trade-off between system stability and disturbance rejection is an important yet difficult aspect of the repetitive control system design. Moon *et al.* (1998) proposed a repetitive controller design method for periodic disturbance rejection with uncertain plant coefficients. The design is performed by analyzing the frequency domain properties, and Nyquist plots play a central role throughout the design phase. It is noted that such a method is robust at the expense of performance deterioration compared with the nominal case, and further, the disturbance cannot be fully compensated for even under the ideal case. Another way to solve the periodic disturbance problem is to use the double controller scheme (Tian and Gao, 1998). However, the complexity and the lack of tuning rules hinder its application. A plug-in adaptive controller (Hu and Tomizuka, 1993; Miyamoto *et al.*, 1999) is proposed to reject periodic disturbances. An appealing feature is that turning on or off the plug module will not affect the original structure. However, the shortcoming of this method is that the analysis and implementations are somewhat more complex than the conventional model based algorithm. An alternative scheme, called the virtual feedforward control, is proposed in Zhang (2000) for asymptotic rejection of periodic disturbance. The periodic disturbance is estimated when a steady state periodic error is detected, and the virtual feedforward control is then activated to compensate for such a disturbance.

1.3 Organization of the Book

The book assumes the pre-requisite of basic linear system theory from readers. It is organized as follows.

Chapter 2 reviews some notions and results on linear systems. Multivariable linear dynamic systems and their representations are introduced. Polynomial and rational function matrices are studied in details. Matrix fraction descriptions and multivariable pole and zeros are covered. General methods for model reduction are highlighted and two special algorithms taking care of stability of reduced models are presented. Popular formulas for conversions between continuous time and discrete time systems are discussed and a technique with high accuracy and stability is described.

Chapter 3 considers stability and robustness of linear feedback systems. Internal stability of general interconnected systems is addressed and a powerful stability condition is derived which is applicable to systems with any feedback and/or feedforward combinations. A special attention is paid to the conventional unity output feedback configuration and the simplifying conditions and Nyquist-like criteria are given. The plant uncertainties and feedback system robustness are introduced.

Chapter 4 considers decoupling problem by state feedback. Necessary and sufficient solvability conditions are derived and formulas for calculating feedback and feedforward gain matrices given. Pole and zeros of decoupled systems are discussed. Geometric methods are not covered.

Chapter 5 addresses decoupling problem by unity output feedback compensation. The polynomial matrix approach is adopted. The diagonal decoupling problem for square plants is first solved and then extended to the general block decoupling case for non-square plants. A unified and independent solution is also presented. In all the cases, stability is included in the discussion, the necessary and sufficient condition for solvability is given, and the set of all the compensators solving the problem is characterized.

Chapter 6 discusses decoupling problem for plants with time delay. Conventional unity output feedback configuration is used exclusively. Our emphasis is to develop decoupling methodologies with necessary theoretical supports as well as the controller design details for possible practical applications. The new decoupling equations are derived in a transfer function matrix setting, and achievable performance after decoupling is analyzed where characterization of the unavoidable time delays and non-minimum phase zeros that are inherent in a feedback loop is given. An effective decoupling control design method is then presented with stability and robustness analysis.

Chapter 7 considers decoupling problem in connection with time delay compensation. The presence of time delay in a feedback control loop could impose serious control performance limitations, and they can be released only by time delay compensation. The Smith control scheme and the internal model control (IMC) structure are popular for SISO time delay compensation, and are extended to MIMO delay systems in this chapter.

Chapter 8 addresses near-decoupling problem. In most applications, exact decoupling is not necessary or impossible. The concept of near-decoupling control is then proposed. Design methods for near-decoupling controllers are presented for four general (block-decoupling) cases: (i) exact models and state feedback, (ii) uncertain models and state feedback, (iii) exact models and dynamic output feedback, and (iv) uncertain models and dynamic output feedback. All theses results are given in the form of linear matrix inequalities, which makes the numerical solutions easily tractable.

Chapter 9 deals with decoupling a disturbance from the plant output in dynamic sense. For measurable disturbances, the feedforward compensation scheme is employed, and necessary and sufficient conditions as well as controller parameterization are given. For unmeasurable disturbances, the disturbance observer scheme is introduced, and its theory and design are presented. Stability is a crucial issue and is covered in both schemes.

Chapter 10 is concerned with static or asymptotic disturbance decoupling. For general disturbances, the internal model principle is developed and the relevant design and computations are discussed. For a specific yet common case of periodic disturbances, the virtual feedforward control is proposed. It is an add-on function on top of normal feedback control. It is shown that the closed-loop stability is not affected by the proposed control and thus there is no design trade-off between disturbance rejection and stability.

The relations among the chapters are simple: Chapters 4-10 are almost independent of each other, and they may require some knowledge of linear systems and robust control covered in Chapters 2 and 3. For instance, Chapter 5 makes use of polynomial and rational matrix theory (Sections 2 and 3 of Chapter 2) as well as stability conditions on unity output feedback systems Section 2 of Chapter 3); Chapters 6 and 7 need model reduction (Section 4 of Chapter 2) and stability and robustness analysis (Sections 2 and 3 of Chapter 3).

2. Representations of Linear Dynamic Systems

In this chapter, we will review some preliminary notions and results on linear systems that will be needed subsequently for the solution of various decoupling problems in later chapters. Multivariable linear dynamic systems and their representations are introduced. Polynomial and rational function matrices are studied in details. Matrix fraction descriptions and multivariable poles and zeros are covered. State-space realizations from matrix fraction descriptions are discussed. Model reduction is important in many ways, existing methods are highlighted and two algorithms taking care of stability of reduced models are presented. Conversions between continuous time and discrete time systems are reviewed and a technique with high accuracy and stability preservation is described.

2.1 Linear Dynamical Systems

Let a finite dimensional linear time-invariant dynamical system be described by the following linear constant coefficient equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \tag{2.1}$$

$$y(t) = Cx(t) + Du(t), \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is called the system *state*, $x(t_0)$ the *initial condition* of the system, $u(t) \in \mathbb{R}^m$ the system *input*, and $y(t) \in \mathbb{R}^p$ the system *output*. The A, B, C and D are real constant matrices with appropriate dimensions. A dynamical system with $m = 1$ and $p = 1$ is known as a single-input and single-output (SISO) system, otherwise as an multiple-input and multiple-output (MIMO) system. In this book, we shall be concerned mainly with MIMO systems.

For the given initial condition $x(t_0)$ and the input $u(t)$, the dynamical system will have its solution, or response $x(t)$ and $y(t)$ for $t \geq t_0$, which can be obtained from the following formulas:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \tag{2.3}$$

$$y(t) = Cx(t) + Du(t). \tag{2.4}$$

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The matrix exponential e^{At} is defined by

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \tag{2.5}$$

and it can also be calculated by

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}, \tag{2.6}$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ stand for the Laplace and inverse Laplace transforms, respectively.

In the case of $u(t) = 0$, $\forall t \geq t_0$, it is easy to see from the solution (2.3) that for any $t_1 \geq t_0$ and $t \geq t_0$, we have

$$x(t) = e^{A(t-t_1)}x(t_1).$$

Therefore, the matrix function $\Phi(t, t_1) = e^{A(t-t_1)}$ acts as a transformation from one state to another, and thus $\Phi(t, t_1)$ is usually called the *state transition matrix*. Since the state of a linear system at one time can be obtained from the state at another time through the transition matrix, we can assume without loss of generality that $t_0 = 0$. This will be assumed in the sequel.

Definition 2.1.1. (Controllability) *The dynamical system described by (2.1) or the pair (A, B) is said to be controllable if there exists an input $u(\tau)$, $0 \leq \tau \leq t_1$, which transfers the initial state $x(0) = x_0$ to any desired state $x(t_1) = x_1$.*

Theorem 2.1.1. *The system (2.1) is controllable if and only if one of the following conditions is true:*

i) *The controllability matrix:*

$$Q_c := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix},$$

has full row rank (over the real number field $\mathbb{R}$);

ii) *For $\alpha \in \mathbb{R}^n$, $\alpha^T e^{At}B = 0$ implies $\alpha = 0$ for any $t > 0$, i.e, the n rows of $e^{At}B$ are linearly independent (over the real number field $\mathbb{R}$);*

iii) *The matrix $[sI - A, \ B]$ has full row rank for any $s \in \mathbb{C}$, the complex number field;*

iv) *The eigenvalues of $A - BK$ can be freely assigned (with the restriction that complex eigenvalues are in conjugate pairs) by a suitable choice of K.*

The observability is a dual concept to controllability, and it can be tested by Theorem 2.1.1 with the substitution of $(A, B) = (A^T, C^T)$.

Definition 2.1.2. *The dynamical system (2.1) is said to be (internally) stable if all the eigenvalues $\lambda(A)$ of A are in the open left half plane, i.e, $Re\ \lambda(A) < 0$. A matrix A with such a property is said to be stable or Hurwitz.*

Note that the unforced solution $x(t)$ to the stable system (2.1), i.e., under $u(t) \equiv 0$, meets $x(t) = e^{At}x(0) \to 0$ when $t \to \infty$.

Definition 2.1.3. *The dynamical system (2.1), or the pair (A, B), is said to be state-feedback stabilizable if there exists a state feedback $u = -Kx$ such that the feedback system,*

$$\dot{x} = (A - Bk)x,$$

is stable.

Theorem 2.1.2. *The system in (2.1) is state-feedback stabilize if and only if the matrix $[sI - A,\ B]$ has full row rank for all Re $s \geq 0$.*

Taking the Laplace transform of the system in (2.1) and (2.2) under the zero initial condition $x(0) = 0$ yields the transfer function matrix of the system:

$$G(s) = C(sI - A)^{-1}B + D. \tag{2.7}$$

On the other hand, given the transfer function matrix $G(s)$, the system in (2.1) and (2.2) is said to be a (state-space) realization of $G(s)$ if (2.7) holds. The realization is called minimal if (A, B) is controllable and (A, C) is observable.

The computation of the transfer matrix $G(s)$ from the state-space model can be carried out with the Faddeev algorithm as follows. Let the characteristic polynomial of the $n \times n$ matrix A be denoted by

$$\det(sI - A) := \phi(s) = s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \cdots + \alpha_0. \tag{2.8}$$

Then, write

$$(sI - A)^{-1} = \frac{1}{\phi(s)}(\Gamma_{n-1}s^{n-1} + \Gamma_{n-2}s^{n-2} + \cdots + \Gamma_0), \tag{2.9}$$

where $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_0, \Gamma_{n-1}, \Gamma_{n-2}, \cdots, \Gamma_0$ can be calculated in a recursive manner from

$$\begin{array}{ll}
\Gamma_{n-1} = I, & \alpha_{n-1} = -\mathrm{tr}\,(A\Gamma_{n-1}), \\
\Gamma_{n-2} = A\Gamma_{n-1} + \alpha_{n-1}I, & \alpha_{n-2} = -\mathrm{tr}\,(A\Gamma_{n-2})/2, \\
\Gamma_{n-3} = A\Gamma_{n-2} + \alpha_{n-2}I, & \alpha_{n-3} = -\mathrm{tr}\,(A\Gamma_{n-3})/3, \\
\cdots & \cdots \\
\Gamma_i = A\Gamma_{i+1} + \alpha_{i+1}I, & \alpha_i = -\mathrm{tr}\,(A\Gamma_i)/(n-i), \\
\cdots & \cdots \\
\Gamma_0 = A\Gamma_1 + \alpha_1 I, & \alpha_0 = -\mathrm{tr}\,(A\Gamma_0)/n, \\
0 = A\Gamma_0 + \alpha_0 I, &
\end{array}$$

where tr (X), the trace of X, is the sum of all the diagonal elements of the matrix X.

Example 2.1.1. Consider the following multivariable state-space system:

$$\dot{x}(t) = \begin{bmatrix} -1 & -2 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t).$$

Using the algorithm shown above, we compute

$$(sI - A)^{-1} = \frac{\Gamma_2 s^2 + \Gamma_1 s + \Gamma_0}{\phi(s)},$$

where

$$\Gamma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 9 & -2 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 20 & -10 & 0 \\ -15 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

and

$$\phi(s) = s^3 + 10s^2 + 23s - 10,$$

which in turn lead to

$$\begin{aligned} C(sI - A)^{-1}B &= \frac{1}{\phi(s)}\{C\Gamma_2 B s^2 + C\Gamma_1 B s + C\Gamma_0 B\} \\ &= \frac{1}{\phi(s)} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -2 & 9 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} -10 & 20 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \frac{1}{\phi(s)} \begin{bmatrix} -2s - 10 & s^2 + 9s + 20 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} \frac{-2s-10}{\phi(s)} + 1 & \frac{s^2+9s+20}{\phi(s)} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

◊

The transfer function matrices for MIMO systems without time delay are in fact *rational function matrices* of complex variable, namely, matrices whose generic element is a rational function, a ratio of polynomials with real coefficients. A transfer function matrix is said to be *proper* when each element is a proper rational function, i.e.,

$$\lim_{s \to \infty} G(s) = K < \infty,$$

where the notation $K < \infty$ means that each element of matrix K is finite. Analogously, $G(s)$ is strictly proper if

$$\lim_{s\to\infty} G(s) = 0.$$

A rational matrix G(s) is said to be stable or system (2.1) is said to be *input-output* stable if $G(s)$ is proper and all elements of $G(s)$, whenever expressed as ratio of polynomials without common roots, have their poles in the open left half plane only.

2.2 Polynomial Matrices

As one has seen in the proceeding section, a multivariable state-space model is a collection of coupled first-order differential equations. But more naturally, the model one gets from first-principle modelling and/or system identification is usually a set of coupled differential equations whose order may be greater than one. A elective circuit described by

$$\begin{cases} \ddot{y}_1(t) + 3\dot{y}_1(t) + 2y_2(t) = 2\ddot{u}_1(t) + u_2(t) + 3\dot{u}_2(t), \\ \dot{y}_2(t) + 3y_1(t) = u_1(t) + 2u_2(t), \end{cases}$$

is such an example. Taking the Laplace transform under zero initial condition yields

$$D(s)Y(s) = N(s)U(s),$$

where $Y(s) = [Y_1(s)\ Y_2(s)]^T$, $U(s) = [U_1(s)\ U_2(s)]^T$;

$$D(s) = \begin{bmatrix} s^2 + 3s & 2 \\ s & 3 \end{bmatrix} \quad \text{and} \quad N(s) = \begin{bmatrix} 2s^2 & 3s+1 \\ 1 & 2 \end{bmatrix}$$

are polynomial matrices. The transfer function matrix of the system is

$$G(s) = D^{-1}(s)N(s),$$

which is a rational function matrix expressed as a fraction of two polynomial matrices.

Therefore, a study of polynomial and rational function matrices is important as it is a basic building block in the analysis and synthesis of multivariable systems. We will discuss polynomial matrices in this section while rational matrices are the topic of the next section. We begin with the introduction of some standard notations.

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}[s]$ the ring of polynomials with coefficients in $\mathbb{R}$ and $\mathbb{R}(s)$ the field of fractions over $\mathbb{R}[s]$:

$$\mathbb{R}(s) := \left\{ t(s) | t(s) = \frac{n(s)}{d(s)}, n(s), d(s) \in R[s], d(s) \not\equiv 0 \right\}.$$

$\mathbb{R}(s)$ is called the field of (real) rational functions. A matrix $P(s)$ whose elements are polynomials is called a *polynomial matrix*. The set of $p \times m$

polynomial matrices is denoted by $\mathbb{R}^{p\times m}[s]$. Similarly, Let $\mathbb{R}^{p\times m}(s)$ denote the set of $p \times m$ matrices with elements in $\mathbb{R}(s)$. A matrix $T(s) \in \mathbb{R}^{p\times m}(s)$ is called a (real) *rational function matrix*. The rank of a polynomial or rational function matrix is the maximum number of its linearly independent row vectors or its column vectors over the field $\mathbb{R}(s)$ and it is denoted by rank $T(s)$.

2.2.1 Unimodular Matrices and Elementary Operations

Let $P(s) \in \mathbb{R}^{m\times m}[s]$ be a square polynomial matrix. In general, its determinant, denoted by $\det P(s)$, is a polynomial. The following two special cases deserve attention.

Definition 2.2.1. *$P(s)$ is called singular if $detP(s) = 0$; otherwise, it is called nonsingular; $P(s)$ is called unimodular if $detP(s) = \alpha$, where α is a non-zero real number.*

Note that if P(s) is nonsingular, then its inverse $P^{-1}(s)$ exists and is in general a rational function matrix. In the case of unimodular $P(s)$, however, the inversion formula, $P^{-1}(s) = adj(P(s))/\det P(s)$, tells us that $P^{-1}(s)$ becomes a polynomial matrix, and is also unimodular due to $\det P(s) \det P^{-1}(s) = 1$. In fact, $P(s)$ is unimodular *if and only if* $P(s)$ and $P^{-1}(s)$ are both polynomial matrices.

Unimodular matrices are closely related to elementary operations. To be precise, elementary row and column operations on any polynomial matrix $P(s) \in \mathbb{R}^{p\times m}[s]$ are defined as follows:

i) interchange any two rows or columns of P(s);
ii) multiply row or column i of P(s) by a non-zero real number;
iii) add to row or column i of P(s) a polynomial multiple of row or column j, $j \neq i$.

These elementary operations can be accomplished by multiplying the given $P(s)$ on the left or on the right by elementary unimodular matrices, namely matrices obtained by performing the above elementary operations on the identity matrix I. It can also be shown that every unimodular matrix may be represented as the product of a finite number of elementary matrices.

2.2.2 Degree and Division

For a polynomial $p(s)$, we denote its degree by $\partial(p(s))$ or $\partial p(s)$ in case of no confusion. Recall that

(i) for $a(s), b(s) \in \mathbb{R}[s]$ such that $a(s)b(s) \neq 0$, there holds $\partial(a(s)b(s)) \geq \partial(a(s))$;

(ii) the polynomial division theorem: for every $a(s), b(s) \in \mathbb{R}[s], b(s) \neq 0$, there exists two elements $q(s), r(s) \in \mathbb{R}[s]$ such that $a(s) = q(s)b(s)+r(s)$ and either $r(s) = 0$ or $\partial r(s) < \partial b(s)$.

We now extend these facts to the matrix case.

Consider a $p \times m$ polynomial matrix P(s). The $i - th$ column (resp. row) degree of P(s) is defined as the highest degree of all m polynomials in the $i - th$ column (resp. row) of $P(s)$, and denoted by $\partial_{ci}(P(s))$ (resp. $\partial_{ri}(P(s))$). Let $\partial_{ci}P(s) = \mu_i$, $i = 1, 2, \ldots, m$; $\partial_{ri}P(s) = \nu_i$, $i = 1, 2, \ldots, p$; $S_c(s) = diag\{s^{\mu_1}, s^{\mu_2}, \cdots, s^{\mu_m}\}$ and $S_r(s) = diag\{s^{\nu_1}, s^{\nu_2}, \cdots, s^{\nu_p}\}$. $P(s)$ can be expressed by

$$\begin{aligned} P(s) &= P_{hc}S_c(s) + L_c(s) \\ &= S_r(s)P_{hr} + L_r(s), \end{aligned}$$

where P_{hc} (resp. P_{hr}) is the highest column (resp. row) degree coefficient matrix of $P(s)$, and $L_c(s)$ (resp. $L_r(s)$) has strictly lower column (resp. row) degrees than $\partial_{ci}P(s)$ (resp. $\partial_{ri}P(s)$).

In the case of square matrices, $p = m$, it readily follows from linear algebra that

$$\det P(s) = \det P_{hc} \cdot s^{\mu} + \ lower - degree\ terms\ in\ s$$

and

$$\det P(s) = s^{\nu} \det P_{hr} + \ lower - degree\ terms\ in\ s$$

with $\mu = \sum_{i=1}^{m} \mu_i$ and $\nu = \sum_{i=1}^{m} \nu_i$, which imply, respectively

$$\partial\ detP(s) \leq \sum_{i=1}^{m} \mu_i\ ,$$

and

$$\partial\ detP(s) \leq \sum_{i=1}^{m} \nu_i\ .$$

The equality holds if P_{hc} (resp. P_{hr}) is nonsingular.

Definition 2.2.2. *A polynomial matrix is called column (resp. row)-reduced if P_{hc} (resp. P_{hr}) is nonsingular.*

Example 2.2.1. Let

$$P(s) = \begin{bmatrix} s^2 + 1 & s^2 \\ 2s & s \end{bmatrix}.$$

One writes

$$P(s) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{P_{hc}} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}}_{S_c(s)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 2s & s \end{bmatrix}}_{L_c(s)}$$

$$= \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}}_{S_r(s)} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{P_{hr}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{L_r(s)} .$$

$P(s)$ is row-reduced since $\det P_{hr} \neq 0$ but not column-reduced since $\det P_{hc} = 0$. ◊

By post- (resp. pre-) multiplication by an appropriate unimodular matrix, it is possible to reduce nonsingular $P(s)$ to a column (resp. row)-reduced form.

Proposition 2.2.1. *For a nonsingular polynomial matrix $P(s)$, there are unimodular matrices $U_R(s)$ and $U_L(s)$ such that*

$$\overline{P}(s) = P(s)U_R(s)$$

is column-reduced, and

$$\widetilde{P}(s) = U_L(s)P(s)$$

is row-reduced.

Continue Example 2.2.1, where $P(s)$ is not column-reduced. To reduce column degrees, we examine just the highest-degree terms in each column:

$$\begin{bmatrix} s^2 & s^2 \\ 0 & 0 \end{bmatrix},$$

which can be transformed to

$$\begin{bmatrix} 0 & s^2 \\ 0 & 0 \end{bmatrix}$$

by a post-multiplication with

$$U_R(s) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Thus, we perform

$$P(s)U_R(s) = \begin{bmatrix} s^2+1 & s^2 \\ 2s & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s^2 \\ s & s \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\bar{P}_{hc}} \underbrace{\begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix}}_{\bar{S}_c(s)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}}_{\bar{L}_c(s)},$$

which is already column-reduced.

Consider now two polynomial matrices $N(s) \in \mathbb{R}^{p\times m}[s]$ and $D(s) \in \mathbb{R}^{m\times m}[s]$ with $D(s)$ nonsingular. In general, $N(s)D^{-1}(s)$ will be a rational function matrix. By carrying out polynomial divisions in each element of $N(s)D^{-1}(s)$, we have

$$N(s)D^{-1}(s) = Q(s) + W(s),$$

where $Q(s) \in \mathbb{R}^{p\times m}[s]$ is polynomial and $W(s) \in \mathbb{R}^{p\times m}(s)$ is strictly proper. It can be re-written as

$$N(s) = Q(s)D(s) + R(s), \tag{2.10}$$

where $R(s) := W(s)D(s) = N(s) - Q(s)D(s)$ is polynomial and $R(s)D^{-1}(s) = W(s)$ is strictly proper as mentioned before. Such $Q(s)$ and $R(s)$ are unique since if $\bar{Q}(s)$ and $\bar{R}(s)$ is another pair such that (2.10) holds and $\bar{R}(s)D(s)^{-1}$ is strictly proper, then from $N(s) = \bar{Q}(s)D(s) + \bar{R}(s)$ and (2.10), it follows that

$$Q(s) - \bar{Q}(s) = \left[\bar{R}(s) - R(s)\right]D^{-1}(s), \tag{2.11}$$

where the left-hand side is polynomial and the right-hand side is strictly proper. For (2.11) to hold true, both sides must be zero. Hence $Q(s) = \bar{Q}(s)$ and $R(s) = \bar{R}(s)$.

Proposition 2.2.2. *Let $D(s) \in \mathbb{R}^{m\times m}[s]$ be nonsingular. Then, for any $N(s) \in \mathbb{R}^{p\times m}[s]$ there exist unique $Q(s), R(s) \in \mathbb{R}^{p\times m}[s]$ such that (2.10) holds true with $R(s)D(s)^{-1} \in \mathbb{R}^{p\times m}(s)$ strictly proper.*

Analogously, given a nonsingular $A(s) \in \mathbb{R}^{p\times p}[s]$, then for any $B(s) \in \mathbb{R}^{p\times m}[s]$ there exist unique $Q(s), R(s) \in \mathbb{R}^{p\times m}[s]$ such that

$$B(s) = A(s)Q(s) + R(s), \tag{2.12}$$

with $A(s)^{-1}R(s) \in \mathbb{R}^{p\times m}(s)$ being strictly proper.

Example 2.2.2. Let

$$D(s) = \begin{bmatrix} s & 1 \\ -1 & 1 \end{bmatrix}, \; N(s) = \begin{bmatrix} s & 1 \\ 0 & s^2 \\ -1 & 0 \end{bmatrix}.$$

Given that

$$D^{-1}(s) = \frac{1}{s+1}\begin{bmatrix} 1 & -1 \\ 1 & s \end{bmatrix},$$

we obtain

$$N(s)D^{-1}(s) = \frac{1}{s+1}\begin{bmatrix} s+1 & s \\ s^2 & s^3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s-1 & s^2-s+1 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+1}\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Hence,

$$Q(s) = \begin{bmatrix} 1 & 0 \\ s-1 & s^2-s+1 \\ 0 & 0 \end{bmatrix}, \quad R(s) = \frac{1}{s+1}\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} s & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}. \quad \diamondsuit$$

We now present an algorithm for direct determination of Q(s) and R(s). This method is the most natural generalization of the scalar polynomial division algorithm. To begin, let $D(s)$ be column-reduced with column degrees $\partial_{ci}D(s) = \mu_i$, $i = 1, 2, \cdots, m$. Let $\partial_{ci}N(s) = \nu_i$, $i = 1, 2, \cdots, m$, $\mu = max\{\mu_i, i = 1, 2, \cdots, m\}$, $l = \mu + max\{\nu_i - \mu_i, i = 1, 2, \cdots, m\}$, and $S_c(s) = diag\{s^{\mu_1}, s^{\mu_2}, \cdots, s^{\mu_m}\}$. Then $D(s)$, $N(s)$ and $R(s)$ can be written as

$$D(s) = [D_\mu + D_{\mu-1}s^{-1} + \cdots + D_1 s^{-\mu+1} + D_0 s^{-\mu}]S_c(s), \tag{2.13}$$

$$N(s) = [N_l s^{l-\mu} + N_{l-1}s^{l-\mu-1} + \cdots + N_1 s^{-\mu+1} + N_0 s^{-\mu}]S_c(s), \tag{2.14}$$

$$R(s) = [R_{\mu-1}s^{-1} + R_{\mu-2}s^{-2} + \cdots + R_1 s^{-\mu+1} + R_0 s^{-\mu}]S_c(s), \tag{2.15}$$

where D_μ is nonsingular, and the columns of D_i, N_i and R_i corresponding to the negative powers of s are zero. Finally, let $Q(s)$ be

$$Q(s) = Q_\gamma s^\gamma + Q_{\gamma-1}s^{\gamma-1} + \cdots + Q_1 s + Q_0, \tag{2.16}$$

where γ, Q_i as well as R_i are to be determined.

Substituting (2.13)-(2.16) into (2.10) and equating the terms of the same degree on its both sides yield

$$N_i = \sum_{j=0}^{\mu} Q_{i-j}D_j + R_i\ , \qquad i = \gamma+\mu, \gamma+\mu-1, \cdots, 2, 1, 0,$$

$$Q_{i-\mu} = \left[N_i - \sum_{j=0}^{\mu-1} Q_{i-j}D_j - R_i \right] D_\mu^{-1}\ , \qquad i = \gamma+\mu, \gamma+\mu-1, \cdots, 2, 1, 0,$$

where

$$N_i = 0,\ i > l,$$

$$Q_i = 0,\ i > \gamma \ or \ i < 0,$$

$$R_i = 0,\ i > \mu - 1.$$

Then we have

$$Q_i = 0, \quad i > l - \mu; \tag{2.17}$$

$$Q_i = \left[N_{i+\mu} - \sum_{j=0}^{\mu-1} Q_{i+\mu-j} D_j \right] D_\mu^{-1}, \ i = l-\mu, l-\mu-1, \ldots, 1, 0; \tag{2.18}$$

$$R_i = N_i - \sum_{j=0}^{i} Q_{i-j} D_j, \quad i = \mu-1, \mu-2, \cdots, 1, 0. \tag{2.19}$$

It is interesting to note that the degree of $Q(s)$ is $l-\mu$, i.e., $\max\{v_i - \mu_i : i = 1, 2, \cdots, m\}$. If the column degrees of N(s) are strictly less than that of D(s), i.e., $\nu_i < \mu_i$ for all i, then the quotient $Q(s)$ is zero and $R(s) = N(s)$. This is because $N(s)D(s)^{-1}$ is itself strictly proper in this case.

Example 2.2.3. To illustrate the algorithm, consider

$$N(s) = \begin{bmatrix} s^2+3 & 1 \\ 2 & -4s^2+s \\ s+1 & s^2-4 \end{bmatrix}, \qquad D(s) = \begin{bmatrix} s+1 & 4s-5 \\ 2 & s^2-3s+2 \end{bmatrix}.$$

The matrix $D(s)$ is column-reduced with column degrees $\mu_1 = 1, \mu_2 = 2$, so $\mu = max\{\mu_1, \mu_2\} = 2$. Write $D(s)$ as

$$\begin{aligned} D(s) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix} s^{-1} + \begin{bmatrix} 0 & -5 \\ 0 & 2 \end{bmatrix} s^{-2} \right\} \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix} \\ &:= [D_2 + D_1 s^{-1} + D_0 s^{-2}] S_c(s). \end{aligned}$$

For $N(s), \nu_1 = \nu_2 = 2$, then $l = \mu + max\{\nu_1 - \mu_1, \nu_2 - \mu_2\} = 2 + 1 = 3$, and $N(s)$ is expressed as

$$\begin{aligned} N(s) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} s^{-1} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -4 \end{bmatrix} s^{-2} \right\} \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix} \\ &:= [N_3 s + N_2 + N_1 s^{-1} + N_0 s^{-2}] S_c(s). \end{aligned}$$

Since $l - \mu = 1, Q(s)$ can be expressed as

$$Q(s) = Q_1 s + Q_0.$$

From (2.18), we have

$$Q_1 = N_3 D_2^{-1} = N_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Q_0 = \left[N_2 - Q_1 D_1 \right] D_2^{-1} = \begin{bmatrix} -1 & -4 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}.$$

According to (2.15), $R(s)$ should be

$$R(s) = [R_1 s^{-1} + R_0 s^{-2}]S_c(s).$$

It follows from (2.19) that

$$R_1 = N_1 - (Q_1 D_0 + Q_0 D_1) = \begin{bmatrix} 12 & -3 \\ 10 & -11 \\ -2 & -1 \end{bmatrix},$$

$$R_0 = N_0 - Q_0 D_0 = \begin{bmatrix} 0 & 4 \\ 0 & -8 \\ 0 & -1 \end{bmatrix}.$$

Finally, we obtain

$$Q(s) = Q_1 s + Q_0 = \begin{bmatrix} s-1 & -4 \\ 0 & -4 \\ 1 & 1 \end{bmatrix},$$

$$R(s) = (R_1 s^{-1} + R_0 s^{-2})S_c(s) = \begin{bmatrix} 12 & -3s+4 \\ 10 & -11s+8 \\ -2 & -s-1 \end{bmatrix}. \qquad \diamond$$

2.2.3 Stability of Polynomial Matrices

Consider an $m \times m$ nonsingular polynomial matrix $P(s)$. $P(s)$ is said to be *stable* if $\det(P(s))$ has all roots in $\mathbb{C}^-$, the open left-half of the complex plane. For a nonsingular polynomial matrix $P(s)$, it has been shown before that there is a unimodular polynomial matrix $U(s)$ such that $U(s)P(s)$ is row-reduced, i.e., there are integers $\nu_1, \nu_2, \cdots, \nu_m$ such that for some $S_r(s) = diag\{s^{\nu_1}, s^{\nu_2}, \cdots, s^{\nu_m}\}$, the limit $\lim_{s\to\infty} S_r^{-1}(s)U(s)P(s) = P_{hr}$ exists and is nonsingular. Without loss of generality, P_{hr} can be assumed to be an identity matrix. Therefore, we let the nonsingular polynomial matrix $P(s)$ have the form

$$P(s) = S_r(s)\{I_m + P_{\nu-1}s^{-1} + P_{\nu-2}s^{-2} + \cdots + P_1 s^{-\nu+1} + P_0 s^{-\nu}\}, \qquad (2.20)$$

where $\nu = max\{\nu_1, \nu_2, \cdots, \nu_m\}$. Also, let

$$D_r(s) = \mathrm{diag}\,(s^{\nu-\nu_1}, s^{\nu-\nu_2}, \cdots, s^{\nu-\nu_m}).$$

Then, we have

$$\bar{P} := D_r(s)P(s) = I_m s^{\nu} + P_{\nu-1}s^{\nu-1} + P_{\nu-2}s^{\nu-2} + \cdots + P_1 s + P_0, \qquad (2.21)$$

$$\det(\bar{P}(s)) = det(D_r(s)) \cdot \det(P(s)) = s^{\Delta\nu} \cdot \det(P(s)), \qquad (2.22)$$

where $\Delta\nu = \sum_{i=1}^{m}(\nu - \nu_i)$. Introduce the constant matrix A as follows:

$$A = \begin{bmatrix} 0 & & 0 & -P_0 \\ I_m & & & -P_1 \\ & \ddots & & \vdots \\ 0 & & I_m & -P_{\nu-1} \end{bmatrix}. \tag{2.23}$$

It is well known that

$$\det(\bar{P}(s)) = \det(sI - A). \tag{2.24}$$

If $P(0)$ is singular, $\det P(0)$ is zero, and $P(s)$ has a root at the origin so that $P(s)$ is unstable. On the other hand, if $P(0)$ is nonsingular, $\det P(0)$ is non-zero so that $\det P(s)$ has no roots at the origin. Then, by (2.22), the non-zero roots of $\det P(s)$ will be the same as those of $\det \overline{P}(s)$. We have actually established the following theorem.

Theorem 2.2.1. *Suppose nonsingularity of $P(0)$. The polynomial matrix $P(s)$ in (2.20) is stable if and only if all the eigenvalues of A in (2.23) are in $\mathbb{C}^-$.*

It is believed that the theorem is useful from the computation point of view because there is a function available in Matlab to compute all eigenvalues of a matrix.

Example 2.2.4. Let

$$P(s) = \begin{bmatrix} s^2+4s+4 & s+2 \\ 1 & s+2 \end{bmatrix},$$

which has nonsingular $P(0)$. It is rewritten as

$$P(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \left\{ I_2 + \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} s^{-1} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} s^{-2} \right\}.$$

Then, we form

$$A = \begin{bmatrix} 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix},$$

and obtain its non-zero eigenvalues as -1, -2, -3, which are all in $\mathbb{C}^-$. By Theorem 2.2.1, $P(s)$ is thus stable. ♢

2.2.4 Hermite and Smith Forms

Two polynomial matrices $P_1(s), P_2(s) \in \mathbb{R}^{p\times m}[s]$ are called equivalent (in $\mathbb{C}$) if there exist unimodular matrices $U_L(s) \in \mathbb{R}^{p\times p}[s], U_R(s) \in \mathbb{R}^{m\times m}[s]$ such that

$$U_L(s)P_1(s)U_R(s) = P_2(s).$$

With postmultiplying and premultiplying polynomial matrices by appropriate unimodular matrices, it is possible to reduce them to the so called standard forms. The advantage of a standard form is that it reveals the basic structure of the matrix and serves as a typical representation of a whole class of equivalent matrices. Two commonly used standard forms are the Hermite form (similar to a triangular matrix) if only row or column operations are allowed; and the Smith form (similar to a diagonal matrix) if both row and column operations are permitted.

Theorem 2.2.2. *(**Hermite row form**) Let $P(s) \in \mathbb{R}^{p\times m}[s]$ have rank r. Then, there exists unimodular matrix $U(s) \in \mathbb{R}^{p\times p}[s]$ (obtained from elementary row operations) such that $U(s)P(s) = H(s)$ is in the Hermite row form which is upper quasi-triangular and defined as follows:*

(i) If $p > r$, the last $p - r$ rows are identically zero;
(ii) In column j, $1 \le j \le r$, the diagonal element is monic (i.e., having the leading coefficient 1), any element below it is zero, and any element above it has lower degree than it; and
(iii) If $m > r$, no particular statements can be made about the elements in the last $m - r$ columns and first r rows.

Proof. Choose among nonzero elements of the first column of $P(s)$ a polynomial of the lowest degree and bring it to position (1,1) by suitable interchange of the rows. We denote it by $\bar{p}_{11}(s)$. Then divide other elements $\bar{p}_{i1}(s)$ in the column by $\bar{p}_{11}(s)$, i.e.

$$\bar{p}_{i1}(s) = \bar{p}_{11}(s)q_{i1}(s) + r_{i1}(s), \quad i = 2, \dots, m. \tag{2.25}$$

Subtracting the $i-th$ row by q_{i1} times $1st$ row, we actually replace $\bar{p}_{i1}(s)$ by $r_{i1}(s)$. Among nonzero $r_{i1}(s)$, choose one with the lowest degree and bring it to position (1,1) by suitable interchange of the rows. Continuing this procedure until all elements in the first column are zero except one at position (1,1). Now consider the second column of the resulting matrix and, temporarily ignoring the first row, repeat the above procedure until all the elements below the (2,2) element are zero. If the (1,2) element does not have lower degree than the (2,2) element, the division algorithm and an elementary row operation can be used to replace the (1,2) element by a polynomial of lower degree than the diagonal or (2,2) element. Continuing this procedure with the third column, fourth column, and so on finally gives the desired Hermite row form. □

Remark 2.2.1. By interchanging the roles of rows and columns, one can obtain a similar *Hermite column form*, the details of which we shall not spell out.

Example 2.2.5. We proceed as follows:

$$\begin{bmatrix} s & 3s+1 \\ -1 & s^2+s-2 \\ -1 & s^2+2s-1 \end{bmatrix} \xrightarrow{-r_3 \leftrightarrow r_1} \begin{bmatrix} 1 & -(s^2+2s-1) \\ -1 & s^2+s-2 \\ s & 3s+1 \end{bmatrix}$$

$$\xrightarrow[-sr_1 \to r_3]{r_1 \to r_2} \begin{bmatrix} 1 & -(s^2+2s-1) \\ 0 & -(s+1) \\ 0 & s^3+2s^2+2s+1 \end{bmatrix} \xrightarrow{-r_2 \to r_2} \begin{bmatrix} 1 & -(s^2+2s-1) \\ 0 & (s+1) \\ 0 & (s+1)(s^2+s+1) \end{bmatrix}$$

$$\xrightarrow[-(s^2+s+1)r_2 \to r_3]{(s+1)r_2 \to r_1} \begin{bmatrix} 1 & 2 \\ 0 & s+1 \\ 0 & 0 \end{bmatrix} := H(s)$$

which is in the Hermite row form. The corresponding unimodular matrix is obtained from the above operations (take note of the sequence) as

$$\begin{bmatrix} 1 & (s+1) & 0 \\ 0 & 1 & 0 \\ 0 & -(s^2+s+1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -s & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(s+1) & s \\ 0 & -1 & 1 \\ 1 & (s^2+s+1) & -(s^2+1) \end{bmatrix} := U(s). \qquad \diamondsuit$$

Theorem 2.2.3. (**Smith form**) *Let $P(s)$ be a polynomial matrix of rank r, then there are unimodular polynomial matrices $U_L(s)$ and $U_R(s)$ such that*

$$U_L(s)P(s)U_R(s) = \begin{bmatrix} \lambda_1(s) & & & & 0 \\ & \lambda_2(s) & & & \\ & & \ddots & & \\ & & & \lambda_r(s) & \\ & 0 & & & 0 \end{bmatrix} := \Lambda(s)$$

is the so-called Smith form of P(s) in which each $\lambda_i(s), i = 1, 2, \cdots, r$, is a monic polynomial satisfying the divisibility property:

$$\lambda_i(s) | \lambda_{i+1}(s), \quad i = 1, 2, \cdots, r-1,$$

i.e., $\lambda_i(s)$ divides $\lambda_{i+1}(s)$ without remainder. $\lambda_i(s)$ are called invariant polynomials of P(s). Moreover, define the determinant divisors of $P(s)$ by $\Delta_i(s)=$

the monic greatest common divisor (GCD) of all $i \times i$ minors of $P(s)$, $i = 1, 2, \cdots, r$, with $\Delta_0(s) = 1$ by convention. Then, one can identify the invariant polynomials $\lambda_i(s)$ by

$$\lambda_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}}(s), \quad i = 1, 2, \cdots, r.$$

Proof. Bring the least degree element of $P(s)$ to the (1,1) position. As in construction of the Hermite forms, by elementary row and column operations, make all other elements in the first row and column zero. If this (1,1) element does not divide every other element in the matrix, using the polynomial division algorithm and row and column interchanges, we can bring a lower-degree element to the (1,1) position, and then repeat the above steps to zero all other elements in the first row and column. Then we reach

$$U_1 P V_1 = \begin{bmatrix} \lambda_1(s) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P_1(s) & \\ 0 & & & \end{bmatrix},$$

where λ_1 divides every element of P_1. Now repeat the above procedure on P_1. Proceeding in this way gives the Smith form $\Lambda(s)$. □

Example 2.2.6. Continue Example 2.2.5. With the additional elementary column operation corresponding to

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} := U_R(s),$$

we get the Smith form:

$$H(s)U_R(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \\ 0 & 0 \end{bmatrix} := \Lambda(s). \qquad \diamondsuit$$

Example 2.2.7. Consider a polynomial matrix,

$$P(s) = \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{bmatrix}.$$

One proceeds,

$$\begin{aligned} \Delta_0(s) &= 1, \\ \Delta_1(s) &= GCD\{1, -1, s^2-s-4, 2s^2-s-8, s^2-4, 2s^2-8\} = 1, \\ \Delta_2(s) &= GCD\{\begin{vmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ s^2-4 & 2s^2-8 \end{vmatrix}, \\ &\qquad \begin{vmatrix} s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{vmatrix}\} \\ &= (s+2)(s-2). \end{aligned}$$

It follows that

$$\lambda_1(s) = \frac{\Delta_1}{\Delta_0} = 1, \quad \lambda_2(s) = \frac{\Delta_2}{\Delta_1} = (s+2)(s-2).$$

Thus, one obtains the Smith form:

$$\Lambda(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+2)(s-2) \\ 0 & 0 \end{bmatrix}. \qquad \Diamond$$

2.2.5 Common Divisor and Coprimeness

Definition 2.2.3. *Let $P(s), L(s)$, and $R(s)$ be polynomial matrices such that*

$$P(s) = L(s)R(s).$$

Then $R(s)$ (resp. $L(s)$) is called a right (resp. left) divisor of $P(s)$, and $P(s)$ a left (resp. right) multiple of $R(s)$ (resp. $L(s)$).

For instance, the polynomial matrix:

$$P(s) = \begin{bmatrix} s(s+1)(s+2) \\ \\ s(s+1)(s+3) \end{bmatrix},$$

has $R_1(s) = 1,\ R_2(s) = s,\ R_3(s) = s+1$, and $R_4(s) = s(s+1)$ as its right divisors. One notes that $R_4(s) = s(s+1)$ is the monic greatest common factor of two scalar polynomials in $P(s)$.

If $P(s)$ is nonsingular, then all its square divisions (right and left) are nonsingular, too. For example, the matrix:

$$P(s) = \begin{bmatrix} 1 & s(s+1)(s+2) \\ \\ 0 & s(s+1)(s+3) \end{bmatrix},$$

is obviously nonsingular. Its right divisors include

$$R_1(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix},$$

$$R_3(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 1 & 0 \\ 0 & s(s+1) \end{bmatrix},$$

which are all square and nonsingular.

Definition 2.2.4. *Let two polynomial matrices $N(s)$ and $D(s)$ have the same number of columns.*

(i) *A square polynomial matrix* $R(s)$ *is called a common right divisor of* $N(s)$ *and* $D(s)$ *if* $R(s)$ *is a right divisor of both, i.e., there are polynomial matrices* $\tilde{N}(s)$ *and* $\tilde{D}(s)$ *such that*

$$N(s) = \tilde{N}(s)R(s),$$

$$D(s) = \tilde{D}(s)R(s);$$

(ii) $R(s)$ *is called a greatest common right divisor (GCRD) of* $N(s)$ *and* $D(s)$ *if* $R(s)$ *is a left multiple of any other common right divisor* $\bar{R}(s)$ *of* $N(s)$ *and* $D(s)$*, i.e. there is a polynomial matrix* $L(s)$ *such that*

$$R(s) = L(s)\bar{R}(s);$$

(iii) $N(s)$ *and* $D(s)$ *are called right coprime if all common right divisions of* $N(s)$ *and* $D(s)$ *are unimodular.*

Analogously, we can define greatest common left divisors (GCLD) and left coprimeness of two polynomial matrices having the same number of rows. Greatest common divisors can be found by elementary operations on a composite matrix formed by $N(s)$ and $D(s)$, as indicated by the following theorem.

Theorem 2.2.4. *Let* $N(s) \in \mathbb{R}^{p\times m}[s]$ *and* $D(s) \in \mathbb{R}^{m\times m}[s]$*.*

(i) *If a unimodular matrix* $U(s) \in \mathbb{R}^{(p+m)\times(p\times m)}[s]$ *and a square polynomial matrices* $R(s) \in \mathbb{R}^{m\times m[s]}$ *are such that*

$$U(s)\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}. \tag{2.26}$$

Then, $R(s)$ *is a GCRD of* $N(s)$ *and* $D(s)$*.*

(ii) *Further, if there holds*

$$rank\,[D^T(s) \quad N^T(s)]^T = m, \tag{2.27}$$

then all GCRDs of $N(s)$ *and* $D(s)$ *must be nonsingular and can differ only by unimodular (left) divisors. In particular, if a GCRD is unimodular, then all GCRDs must be unimodular.*

Proof. (i) Partition $U(s)$ in the form:

$$U(s) = \begin{bmatrix} U_{11}(s) \; U_{12}(s) \\ U_{21}(s) \; U_{22}(s) \end{bmatrix}, \tag{2.28}$$

where $U_{11}(s) \in \mathbb{R}^{m\times m}[s]$ and $U_{22}(s) \in \mathbb{R}^{p\times p}[s]$. Then the polynomial matrix $U^{-1}(s)$ can be partitioned similarly as

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}^{-1} = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} \begin{bmatrix} R(s) \\ 0 \end{bmatrix},$$

or equivalently

$$D(s) = V_{11}(s)R(s), \quad N(s) = V_{21}(s)R(s),$$

indicating that $R(s)$ is a common right divisor of $D(s)$ and $N(s)$. Notice next from (2.26) that

$$R(s) = U_{11}(s)D(s) + U_{12}(s)N(s).$$

Now, if $R_1(s)$ is another common right divisor,

$$D(s) = D_1(s)R_1(s), \qquad N(s) = N_1(s)R_1(s),$$

then one has

$$R(s) = [U_{11}(s)D_1(s) + U_{12}(s)N_1(s)]R_1(s),$$

so that $R_1(s)$ is a right divisor of $R(s)$, and thus $R(s)$ is a GCRD.

(ii) Since elementary operations do not change rank, $[R^T, 0]^T$ has full rank or R is nonsingular if (2.27) holds true. If $\bar{R}$ is any other GCRD, then $R(s)$ and $\bar{R}(s)$ are related by definition as

$$R(s) = L_1(s)\bar{R}(s), \quad \bar{R}(s) = L_2(s)R(s),$$

for polynomial L_i. Thus, $\bar{R}(s)$ is nonsingular. And we can write

$$R(s) = L_1(s)L_2(s)R(s),$$

which implies $\det(L_1)\det(L_2) = 1$ and both L_1 and L_2 are unimodular. □

Notice that the reduction of $[D^T(s) \quad N^T(s)]^T$ to $[R^T(s) \quad 0]^T$ can be achieved by elementary row operations such that the last p rows from the bottom on the right-hand side become zero. In particular, we could use the procedure for getting the Hermite row form, but this is more than necessary.

Example 2.2.8. Find a GCRD of

$$D(s) = \begin{bmatrix} s & 3s+1 \\ -1 & s^2+s-2 \end{bmatrix}, \qquad N(s) = [-1 \quad s^2+2s-1].$$

It follows from Example 2.2.5 that there is a unimodular matrix $U(s)$ such that

$$U(s)\begin{bmatrix} D(s) \\ \cdots \\ N(s) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & s+1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix}.$$

Then, one knows from Theorem 2.2.4 that

$$R(s) = \begin{bmatrix} 1 & 2 \\ 0 & s+1 \end{bmatrix}$$

is a GCRD of the given $D(s)$ and $N(s)$. $\Diamond$

Theorem 2.2.5. *Suppose full rank of* $[D^T(s) \quad N^T(s)]^T$. *$N(s)$ and $D(s)$ are right coprime if and only if one of the following holds true:*

(i) There exist polynomial matrices $X(s)$ and $Y(s)$ such that

$$X(s)N(s) + Y(s)D(s) = I; \tag{2.29}$$

(ii) $[D^T(s) \quad N^T(s)]^T$ *has full rank for every $s \in \mathbb{C}$.*

Proof. (i) Based on the Theorem 2.2.4, it is always possible to write a GCRD $R(s)$ of $N(s)$ and $D(s)$ as $R(s) = \hat{X}(s)N(s) + \hat{Y}(s)D(s)$ for polynomial $\hat{X}$ and $\hat{Y}$. Moreover, if $N(s)$ and $D(s)$ are coprime, $R(s)$ must be unimodular so that

$$\begin{aligned} I &= R^{-1}(s)[\hat{X}(s)N(s) + \hat{Y}(s)D(s)] \\ &= X(s)N(s) + Y(s)D(s), \end{aligned}$$

where $X(s) := R^{-1}(s)\hat{X}(s)$ and $Y(s) := R^{-1}(s)\hat{Y}(s)$ are both polynomial. Conversely, suppose that there exist two matrices $X(s)$ and $Y(s)$ satisfying (2.29). Let $R(s)$ be a GCRD of $N(s)$ and $D(s)$, i.e.

$$N(s) = \hat{N}(s)R(s), \quad D(s) = \hat{D}(s)R(s).$$

Equation (2.29) then becomes

$$I = \left[X(s)\hat{N}(s) + Y(s)\hat{D}(s)\right]R(s),$$

yielding

$$R^{-1}(s) = X(s)\hat{N}(s) + Y(s)\hat{D}(s),$$

a polynomial matrix. This entails that $R(s)$ is a unimodular matrix so that $N(s)$ and $D(s)$ are right coprime.

(ii) From (2.26) we see that $[D^T(s) \quad N^T(s)]^T$ has full rank for all s if and only if $R(s)$ is unimodular, i.e., if and only if all GCRDs of $N(s)$ and $D(s)$ are unimodular, which by definition means that $N(s)$ and $D(s)$ are right coprime. $\square$

Example 2.2.9. Check right coprimeness of

$$D(s) = \begin{bmatrix} s^2+2s & s+3 \\ 2s^2-s & 3s-2 \end{bmatrix}, \quad N(s) = \begin{bmatrix} s & 1 \end{bmatrix}.$$

One readily sees that

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}_{s=0} = \begin{bmatrix} 0 & 3 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

is of rank defect, and then $D(s)$ and $N(s)$ are not coprime. One may write

$$\begin{aligned} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} &= \begin{bmatrix} s^2+2s & s+3 \\ 2s^2-s & 3s-2 \\ s & 1 \end{bmatrix} = \begin{bmatrix} s+2 & s+3 \\ 2s-1 & 3s-2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \\ &:= \begin{bmatrix} \tilde{D}(s) \\ \tilde{N}(s) \end{bmatrix} R(s). \end{aligned}$$

and find that the right divisor $R(s)$ is not unimodular as $detR(s) = s \neq const$, which is in agreement with non-coprimeness of $D(s)$ and $N(s)$. One notes that the first and third rows of $\begin{bmatrix} \tilde{D}^T(s) & \tilde{N}^T(s) \end{bmatrix}^T$ form a nonsingular matrix for any $s \in \mathbb{C}$. Thus, $\tilde{D}(s)$ and $\tilde{N}(s)$ are right coprime and $R(s)$ is actually a GCRD of them. ◊

2.3 Rational Matrices and Polynomial Matrix Fractions

A rational function matrix $G(s)$ is a matrix whose elements are rational functions. We are interested to generalize polynomial fractions of scalar rational functions to the matrix case.

Definition 2.3.1. *A right polynomial matrix fraction (PMF) for the $p\times\ m$ rational function matrix $G(s)$ is an expression of the form*

$$G(s) = N(s)D^{-1}(s),$$

where $N(s)$ is a $p\times\ m$ polynomial matrix, and $D(s)$ is an $m\times\ m$ nonsingular polynomial matrix. Such a PMF is further called a right coprime PMF if $N(s)$ and $D(s)$ are right coprime. The degree of a right PMF is the degree of the polynomial $detD(s)$. A left polynomial matrix fraction for $G(s)$ is an expression of the form

$$G(s) = D_L^{-1}(s)N_L(s),$$

where $N_L(s)$ is a $p\times\ m$ polynomial matrix, and $D_L(s)$ is a $p\times\ p$ nonsingular polynomial matrix. Similar definitions apply for the coprimeness and degree of a left PMF.

Of course, this definition is familiar if $m = p = 1$. In the multi-input and multi-output case, a simple device can be used to exhibit PMFs for a given $G(s)$. Suppose that $d(s)$ is a least common denominator of all elements of $G(s)$. (In fact, any common denominators can be used). Then

$$N_d(s) = d(s)G(s)$$

is a $p \times\ m$ polynomial matrix, and we can write either a right or left PMF:

$$G(s) = N_d(s)[d(s)I_m]^{-1} = [d(s)I_p]^{-1}N_d(s). \tag{2.30}$$

The degrees of these two PMFs are different in general, and it should not be surprising that lower-degree PMFs can be found if some effort is invested. Let $d_{ci}(s)$ (resp. $d_{ri}(s)$) be a least common denominator of all elements in the ith column (resp. ith row) of $G(s)$, then

$$N(s) := G(s) diag\{d_{c1}, d_{c2}, \cdots, d_{cm}\}$$

and

$$N_L(s) := diag\{d_{r1}, d_{r2}, \cdots, d_{rp}\}G(s)$$

are both polynomial. We obtain PMFs as

$$\begin{aligned} G(s) &= N(s) \cdot diag^{-1}\{d_{c1}, d_{c2}, \cdots, d_{cm}\} \\ &= diag^{-1}\{d_{r1}, d_{r2}, \cdots, d_{rp}\} \cdot N_L(s). \end{aligned}$$

Example 2.3.1. Consider a rational matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

A least common denominator of it is $d(s) = (s+1)(s+2)(s+3)$ so that

$$\begin{aligned} G(s) &= \begin{bmatrix} (s+2)(s+3) & (s+1)(s+3) \\ 0 & (s+1)(s+2) \end{bmatrix} [d(s)I]^{-1} \\ &= [d(s)I]^{-1} \begin{bmatrix} (s+2)(s+3) & (s+1)(s+3) \\ 0 & (s+1)(s+2) \end{bmatrix} \end{aligned}$$

where two PMFs are both of degree 6 but neither of them is coprime. On the other hand, one can write

$$\begin{aligned} G(s) &= \begin{bmatrix} 1 & (s+3) \\ 0 & (s+2) \end{bmatrix} \begin{bmatrix} (s+1) & 0 \\ 0 & (s+2)(s+3) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+3) \end{bmatrix}^{-1} \begin{bmatrix} (s+2) & (s+1) \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where the degree of two fractions is now 3 and both of them are actually coprime. ◊

2.3.1 Coprime Polynomial Matrix Fractions

In the single-input and single output case, the issue of common factors in the numerator and denominator polynomials of $G(s)$ draws attention. The utility of PMFs begins to emerge from the corresponding concept in the matrix case. For PMFs, one of the polynomial matrices is always nonsingular, so only nonsingular common divisors occur. Suppose that $G(s)$ is given by a right PMF:

$$G(s) = N(s)D^{-1}(s),$$

and that $R(s)$ is a common right divisor of $N(s)$ and $D(s)$. Then

$$\widetilde{N}(s) = N(s)R^{-1}(s) \quad \text{and} \quad \widetilde{D}(s) = D(s)R^{-1}(s) \tag{2.31}$$

are polynomial matrices, and they provide another right PMF for $G(s)$ since

$$\widetilde{N}(s)\widetilde{D}^{-1}(s) = N(s)R^{-1}(s)R(s)D^{-1}(s) = G(s).$$

The degree of this new PMF is no greater than the degree of the original, since

$$\deg\,[\det D(s)] = \deg\,[\det \widetilde{D}(s)] + \deg\,[\det R(s)].$$

Of course, the largest degree reduction occurs if $R(s)$ is a greatest common right divisor, and no reduction occurs if $N(s)$ and $D(s)$ are right coprime. This discussion indicates that extracting common right divisors of a right PMF is a generalization of the process of cancelling common factions in a scalar transfer function.

Theorem 2.3.1. *For any $p \times m$ rational function matrix $G(s)$, there is a right coprime PMF; Further, for any two right coprime PMFs, $G(s) = N(s)D^{-1}(s) = N_1(s)D_1^{-1}(s)$, there exists a unimodular polynomial matrix $U(s)$ such that $N(s) = N_1(s)U(s)$ and $D(s) = D_1(s)U(s)$.*

Proof. Given a $G(s)$, it is always possible, say by (2.30), to get a right PMF, $G(s) = N(s)D^{-1}(s)$. Using Theorem 2.2.4, we can extract a GCRD, $R(s)$, of $N(s)$ and $D(s)$ from

$$U(s)\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}. \tag{2.32}$$

A right coprime PMF is then obtained as $N_R(s)D_R^{-1}(s)$ where

$$N_R(s) = N(s)R^{-1}(s), \quad D_R(s) = D(s)R^{-1}(s).$$

Next, for any two right coprime PMFs, $G(s) = N(s)D^{-1}(s) = N_1(s)D_1^{-1}(s)$, by Theorem 2.2.5, there exist polynomial matrices $X(s)$, $Y(s)$, $A(s)$, and $B(s)$ such that

$$X(s)N_1(s) + Y(s)D_1(s) = I_m, \tag{2.33}$$

and

$$A(s)N(s) + B(s)D(s) = I_m. \tag{2.34}$$

Since $N(s)D^{-1}(s) = N_1(s)D_1^{-1}(s)$, we have $N_1(s) = N(s)D^{-1}(s)D_1(s)$. Substituting this into (2.33) gives

$$X(s)N(s)D^{-1}(s)D_1(s) + Y(s)D_1(s) = I_m,$$

or

$$X(s)N(s) + Y(s)D(s) = D_1^{-1}(s)D(s).$$

A similar calculation using $N(s) = N_1(s)D_1^{-1}(s)D(s)$ in (2.34) gives

$$A(s)N_1(s) + B(s)D_1(s) = D^{-1}(s)D_1(s).$$

One sees that both $D_1^{-1}D(s)$ and $D^{-1}(s)D_1(s)$ are polynomial matrices, but they are inverses of each other. Thus, both must be unimodular. Let

$$U(s) = D_1^{-1}(s)D(s).$$

Then, we have

$$N(s) = N_1(s)U(s), \quad D(s) = D_1(s)U(s).$$

The proof is complete. □

Surprisingly, the extraction procedure in (2.32) gives us more than a right coprime PMF. Partition $U(s)$ as

$$U(s) = \begin{bmatrix} Y_R & X_R \\ -N_L & D_L \end{bmatrix},$$

and $U^{-1}(s)$ as

$$U^{-1}(s) = \begin{bmatrix} D_R & -X_L \\ N_R & Y_L \end{bmatrix}.$$

Then, there holds

$$UU^{-1} = \begin{bmatrix} Y_R & X_R \\ -N_L & D_L \end{bmatrix} \begin{bmatrix} D_R & -X_L \\ N_R & Y_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{2.35}$$

Theorem 2.3.2. *Let $N(s)D^{-1}(s)$ is a right PMF of $G(s)$ with the operations leading to (2.32) performed. Then $N_R D_R^{-1}$ is a right coprime PMF of $G(s)$, and $D_L^{-1}N_L$ is a left coprime PMF of $G(s)$.*

Proof. Premultiplying (2.32) by U^{-1} yields

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D_R & -X_L \\ N_R & Y_L \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} D_R R \\ N_R R \end{bmatrix},$$

so that $D = D_R R$ with D_R nonsingular since so is D. Furthermore, one sees

$$N_R D_R^{-1} = N_R R R^{-1} D_R^{-1} = (N_R R)(D_R R)^{-1} = N D^{-1} = G.$$

The (1, 1) block of (2.35) gives

$$Y_R D_R + X_R N_R = I,$$

where Y_R and X_R are both polynomial. Thus $N_R D_R^{-1}$ is a right coprime MFD of $G(s)$.

The (2,1) block of (2.35) gives

$$D_L N_R = N_L D_R. \tag{2.36}$$

We now show nonsingularity of D_L by contradiction. Assume that there is a non-zero polynomial vector $\alpha(s)$ such that $\alpha^T D_L = 0$. Then by (2.36), one has $0 = \alpha^T D_L N_R = \alpha^T N_L D_R$. This implies $\alpha^T N_L = 0$ since D_R is nonsingular. It then follows that

$$\alpha^T [N_L X_L + D_L Y_L] = \alpha^T N_L X_L + \alpha^T D_L Y_L = 0.$$

But the (2,2) block of (2.35):

$$N_L X_L + D_L Y_L = I, \tag{2.37}$$

shows

$$\alpha^T [N_L X_L + D_L Y_L] = \alpha^T I = \alpha^T \neq 0.$$

From (2.36), $D_L^{-1} N_L = N_R D_R^{-1} = G(s)$, so that $D_L^{-1} N_L$ is a left PMF of $G(s)$, and it is coprime in view of (2.37). □

Example 2.3.2. Consider $G(s) = N(s) D^{-1}(s)$ where

$$D(s) = \begin{bmatrix} s & (3s+1) \\ -1 & (s^2+s-2) \end{bmatrix}, \qquad N(s) = \begin{bmatrix} -1 & (s^2+2s-1) \end{bmatrix}.$$

It follows from Example 2.2.5 that U which makes $U[D^T \quad N^T]^T = [R^T \quad 0]^T$ is

$$U = \begin{bmatrix} 0 & -(s+1) & s \\ 0 & -1 & 1 \\ 1 & (s^2+s+1) & -(s^2+1) \end{bmatrix},$$

from which we get a left coprime PMF, $D_L^{-1}(s) N_L(s)$, of $G(s)$ where

$$D_L(s) = -(s^2+1), \quad N_L(s) = -[1 \quad (s^2+s+1)].$$

Further, one calculates

$$U^{-1} = \begin{bmatrix} s & 1 & * \\ -1 & s & * \\ -1 & s+1 & * \end{bmatrix},$$

so that a right coprime PMF is obtained as

$$N_R D_R^{-1} = \begin{bmatrix} -1 & (s+1) \end{bmatrix} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}^{-1}. \qquad \Diamond$$

Theorem 2.3.3. (**Generalized Bozout Identity**) *Let $N_R D_R^{-1}$ be a right coprime PMF. Then there are six polynomial matrices X_R, Y_R, N_L, D_L, X_L and Y_L such that*

$$\begin{bmatrix} Y_R & X_R \\ -N_L & D_L \end{bmatrix} \begin{bmatrix} D_R & -X_L \\ N_R & Y_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{2.38}$$

Proof. Since N_R and D_R are right coprime, there are polynomials X_R and Y_R such that $Y_R D_R + X_R N_R = I$. By Theorem 2.3.2, we can have a left coprime polynomial matrices N_L and D_L such that $D_L^{-1} N_L = N_R D_R^{-1}$. The left coprimeness of N_L and D_L implies the existence of polynomials $\widetilde{X}_L$ and $\widetilde{Y}_L$ such that $D_L \widetilde{Y}_L + N_L \widetilde{X}_L = I$. Writing these relations in matrix form gives

$$\begin{bmatrix} Y_R & X_R \\ -N_L & D_L \end{bmatrix} \begin{bmatrix} D_R & -\widetilde{X}_L \\ N_R & \widetilde{Y}_L \end{bmatrix} = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix},$$

where $Q = -Y_R \widetilde{X}_L + X_R \widetilde{Y}_L$. Post-multiplying both sides by

$$\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}$$

produces (2.38), where $X_L = D_R Q + \tilde{X}_L$ and $Y_L = \widetilde{Y}_L - N_R Q$. □

2.3.2 Smith-McMillan Form

Write a rational matrix $G(s)$ as $G(s) = \frac{N(s)}{d(s)}$, where $d(s)$ is the monic least common denominator of all elements of $G(s)$. Apply Theorem 2.2.3 to the polynomial matrix, $N(s) = G(s)d(s)$, to get its unique Smith form $\Lambda(s) = \mathrm{diag}[\lambda_1(s), \lambda_2, \cdots, \lambda_r, 0]$ with $N(s) = U_L(s)\Lambda(s)U_R(s)$ holding for some unimodular matrices $U_L(s)$ and $U_R(s)$. Using $G(s) = U_L(s)\frac{\Lambda(s)}{d(s)}U_R(s)$ and cancelling all common factors for each pair of λ_i and d, we will get the Smith-McMillan form.

Theorem 2.3.4. (**Smith-McMillan form**) *Let a rational $G \in \mathbb{R}^{p\times m}(s)$ have rank r. Then there exist unimodular matrices $U_L \in \mathbb{R}^{p\times p}[s]$ and $U_R \in \mathbb{R}^{m\times m}[s]$ such that*

$$G = U_L M U_R,$$

and

$$M = \begin{bmatrix} \epsilon_1/\psi_1 & & & & 0 \\ & \epsilon_2/\psi_2 & & & \\ & & \ddots & & \\ & & & \epsilon_r/\psi_r & \\ & 0 & & & 0 \end{bmatrix} \in \mathbb{R}^{p\times m}(s) \tag{2.39}$$

is the so called Smith-McMillan form of $G(s)$, where (ϵ_i, ψ_i) are pairs of monic polynomials copime to each other, satisfying the following divisibility properties

$$\psi_{i+1}|\psi_i \ , \quad i = 1, 2, \cdots, r-1,$$

$$\epsilon_i|\epsilon_{i+1} \ , \quad i = 1, 2, \cdots, r-1,$$

where $\psi_1 = d$ is the monic least common divisor of all the elements of G.

Example 2.3.3. Let

$$G(s) = \begin{bmatrix} \frac{1}{s^2+3s+2} & \frac{-1}{s^2+3s+2} \\ \frac{s^2+s-4}{s^2+3s+2} & \frac{2s^2-s-8}{s^2+3s+2} \\ \frac{s-2}{s+1} & \frac{2s-4}{s+1} \end{bmatrix}.$$

It is written as

$$G(s) = \frac{1}{(s+1)(s+2)} P(s),$$

where $P(s)$ is given in Example 2.2.7. So

$$\frac{\Lambda(s)}{d(s)} = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s-2}{s+1} \\ 0 & 0 \end{bmatrix} := M(s)$$

is the Smith-McMillan form of $G(s)$. ◊

Given the Smith-McMillan form of $G(s)$, it is straightforward to arrive at coprime factorization of G. Define

$$W = \begin{bmatrix} \mathrm{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_r) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{p\times m}[s],$$
$$\Psi_R = \begin{bmatrix} \mathrm{diag}(\psi_1, \psi_2, \cdots, \psi_r) & 0 \\ 0 & I_{m-r} \end{bmatrix},$$
$$\Psi_L = \begin{bmatrix} \mathrm{diag}(\psi_1, \psi_2, \cdots, \psi_r) & 0 \\ 0 & I_{p-r} \end{bmatrix}.$$

G could be expressed in two different ways as follows

$$G = U_L M U_R = (U_L W)(U_R^{-1}\Psi_R)^{-1} := N_R D_R^{-1}, \tag{2.40}$$
$$= (\Psi_L U_L^{-1})^{-1}(W U_R) := D_L^{-1} N_L. \tag{2.41}$$

Note that even though the non-zero polynomial elements of Ψ_r and Ψ_l are the same, Ψ_L is a $p \times p$ matrix and Ψ_R is a $m \times m$ matrix.

We now show that the polynomial matrix pair (N_R, D_R) is right coprime. Write

$$\begin{bmatrix} D_R \\ N_R \end{bmatrix} = \begin{bmatrix} U_R^{-1}\Psi_R \\ U_L W \end{bmatrix} = \begin{bmatrix} U_R^{-1} & 0 \\ 0 & U_L \end{bmatrix} \begin{bmatrix} \Psi_R \\ W \end{bmatrix}.$$

$\begin{bmatrix} D_R \\ N_R \end{bmatrix}$ and $\begin{bmatrix} \Psi_R \\ W \end{bmatrix}$ have the same rank because the transformation matrix relating them is unimodular. Consider the rank of $\begin{bmatrix} \Psi_R \\ W \end{bmatrix}$. The first r columns of $\begin{bmatrix} \Psi_R \\ W \end{bmatrix}$ are linearly independent and the individual columns do not vanish for any $s \in \mathbb{C}$. This follows from the fact that the only non-zero elements of the ith column, namely ψ_i and ϵ_i, do not vanish simultaneously for any $s \in \mathbb{C}$ as they are coprime to each other. The remaining $(m - r)$ columns have only one non-zero element, namely unity for each column and they are linearly independent. Hence the rank of $\begin{bmatrix} \Psi_R \\ W \end{bmatrix}$ is equal to m for all $s \in \mathbb{C}$. The matrix pair $(U_L W, U_R^{-1}\Psi_R)$ is therefore right coprime. Similarly it can be shown that $(\Psi_L U_L^{-1}, W U_R)$ is left coprime.

Using the above fact and Theorem 2.3.4, we establish the following theorem.

Theorem 2.3.5. *Let* $G \in \mathbb{R}^{p\times m}(s)$.

(i) $N_R D_R^{-1}$ *and* $D_L^{-1} N_L$ *in (2.40) and (2.41) are right and left coprime PMFs of* $G(s)$*, respectively;*

(ii) All the numerator matrices, whether they belong to the right or left coprime PMFs of G*, are equivalent and have the same Smith form as* W*;*

(iii) The denominator matrices of any (right or left) coprime PMFs of G have the same non-unity invariant polynomials. Hence, in particular, their determinants have the same degree and the same roots. Moreover, if $G(s)$ is square, then these denominators are equivalent.

2.3.3 Poles and Zeros

Definition 2.3.2. (**poles and zeroes**) *Let $G(s)$ be a rational matrix with the Smith-McMillan form $M(s)$ in (2.39), define the pole (or characteristic) polynomial and zero polynomial of $G(s)$, respectively, as*

$$p(s) = \psi_1(s)\psi_2(s)\cdots\psi_r(s),$$
$$z(s) = \varepsilon_1(s)\varepsilon_2(s)\cdots\varepsilon_r(s).$$

The degree of $p(s)$ is called the McMillan degree of $G(s)$.

Example 2.3.4. The pole and zero polynomials of $G(s)$ in Example 2.3.3 are

$$p(s) = (s+1)^2(s+2),$$
$$z(s) = (s-2).$$

Hence $G(s)$ has poles $\{-1, -1, -2\}$ and a zero $\{2\}$. ♢

The following is a straightforward corollary of Theorem 2.3.5.

Corollary 2.3.1. *Let $G(s) \in \mathbb{R}^{p\times m}(s)$ have rank r, and $N(s)$ and $D(s)$ be the numerator and denominator of any coprime (right or left) PMF of $G(s)$, respectively. Then,*

(i) ρ is a pole of $G(s)$ if and only if $\det D(\rho) = 0$;
(ii) ζ is a zero of $G(s)$ if and only if rank $N(\zeta) < r$.

In view of the above corollary, one can find poles and zeros of $G(s)$ from any coprime PMF of it (one may not necessarily get its Smith-McMillan form). It is easily shown that ρ is a pole of $G(s)$ if and only if it is a pole of some element of $G(s)$. But one can hardly tell its multiplicity in $p(s)$ from individual elements. The case is worse for zeros: a zero of elements in $G(s)$ has no relationship with zeros of $G(s)$. Another interesting property of multivariable poles and zeros is that a multivariable system can have a pole and a zero at the same place, but they do not cancel each other. For instance,

$$G = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{s+1}{s} \end{bmatrix},$$

has a pole and a zero both at $s = -1$.

Definition 2.3.3. *For a rational matrix $G(s)$, if there are polynomial matrices N_L, D and N_R such that $N_R D^{-1} N_L = G$, then $N_L D^{-1} N_R$ is called a right-left PMF of G. Furthermore, if N_R and D are right coprime and D and N_L are left coprime, $N_R D^{-1} N_L$ is called a right-left coprime PMF. Obviously, for $N_L = I$ (resp. $N_R = I$), $N_R D^{-1} N_L$ reduces to normal right (resp. left) PMF $N_R D^{-1}$ (resp.$N_L D^{-1}$).*

Theorem 2.3.6. *For a right-left coprime PMF, $G = N_R D^{-1} N_L$, $\det D$ is the pole polynomial of $G(s)$ modulo a nonzero constant.*

Proof. Let $\overline{N}_L \overline{D}^{-1}$ be a right coprime PMF of $D^{-1}N_L$, so that $D^{-1}N_L = \overline{N}_L \overline{D}^{-1}$ and $detD = det\overline{D}$ modulo a nonzero real number. Noting that $(N_R \overline{N}_L)(\overline{D})^{-1} = G$ and Theorem 2.3.5, the proof would be completed if we show that $(N_R \overline{N}_L, \overline{D})$ is a right coprime.

Since both pairs, (N_R, D) and $(\overline{N}_L, \overline{D})$, are coprime, respectively, there are polynomials X, Y, $\overline{X}$ and $\overline{Y}$ such that

$$XD + YN_R = I, \tag{2.42}$$

$$\overline{X}\,\overline{D} + \overline{Y}\,\overline{N}_L = I. \tag{2.43}$$

By postmultiplying (2.42) by $D^{-1}N_L$, we get successively

$$XN_L + YN_R D^{-1} N_L = D^{-1} N_L,$$

$$XN_L + YN_R \overline{N}_L \overline{D}^{-1} = \overline{N}_L \overline{D}^{-1},$$

$$XN_L \overline{D} + YN_R \overline{N}_L = \overline{N}_L.$$

Substituting this into (2.43) yields

$$(\overline{X} + \overline{Y} X N_L)\overline{D} + (\overline{Y} Y) N_R \overline{N}_L = I,$$

indicating coprimeness of $N_R \overline{N}_L$ and $\overline{D}$. □

Example 2.3.5. Consider a general state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t).$$

Its transfer function matrix is given by

$$G(s) = C(sI - A)^{-1}B.$$

So a state-space realization of the system naturally gives rise to a right-left PMF of the transfer matrix. Furthermore, if the model is both observable and controllable or minimal, then by Theorem Theorem 2.1a, $(C, sI - A)$ is a right coprime, and $(sI - A, B)$ is left coprime, and $C(sI - A)^{-1}B$ is actually a right-left coprime fraction of $G(s)$. It follows form Theorem Theorem 2.13a that $\det(sI - A)$ coincides with the pole polynomial of $G(s)$. ♢

2.3.4 Proper Rational Matrices and Realization

Recall from Section 2.1 that a rational transfer function matrix $G(s)$ is said to be proper if $\lim_{s\to\infty} G(s) < \infty$ and strictly proper if $\lim_{s\to\infty} G(s) = 0$. In the scalar case, a transfer function is proper if the degree of the numerator polynomial is less than or equal to the degree of the denominator polynomial. The situation is not so simple in the matrix case, as we shall now explore.

Lemma 2.3.1. *Let $G(s) \in \mathbb{R}^{p\times m}(s)$ be proper (resp. strictly proper) and have a right PMF, $N(s)D^{-1}(s)$. Then, every column of $N(s)$ has degree less than or equal to (resp. strictly less than) that of the corresponding column of $D(s)$.*

Proof. We have $N(s) = G(s)D(s)$, or equivalently, elementwise,

$$n_{ij}(s) = \sum_{k=1}^{m} g_{ik}d_{kj}, \quad i = 1, 2, \cdots, p.$$

Let now $\mu_j = \partial_{cj}D(s)$, the j-th column degree of $D(s)$, and evaluate

$$\lim_{s\to\infty} n_{ij}(s)s^{-\mu_j} = \sum_{k=1}^{m}(\lim_{s\to\infty} g_{ik}(s))(\lim_{s\to\infty} d_{kj}(s)s^{-\mu_j}),$$

which is finite (resp. zero) if g_{ik} is proper (resp. strictly proper). Hence we have $\partial_{cj}N(s) \le \partial_{cj}D(s)$ (resp. $\partial_{cj}N(s) < \partial_{cj}D(s)$). □

However, somewhat surprisingly, the converse of the result is not always true. For example, let

$$N(s) = [s \quad 1], \quad D(s) = \begin{bmatrix} s^2 & s^2 \\ s^2 & s^2+1 \end{bmatrix}.$$

The degrees of the columns of $N(s)$ are less than those of the corresponding columns of $D(s)$. But

$$N(s)D^{-1}(s) = \begin{bmatrix} \frac{s^2-s+1}{s} & (1-s) \end{bmatrix}$$

is not proper.

To obtain necessary and sufficient conditions for the properness of $N(s)\times D^{-1}(s)$, suppose that $D(s)$ is column-reduced with its highest column degree coefficient matrix D_{hc} being nonsingular. Let $S_c(s) = diag[s^{\mu_1}, s^{\mu_2}, ..., s^{\mu_m}]$. Then one can write

$$N(s)D^{-1}(s) = [N(s)S_c^{-1}(s)][D(s)S_c^{-1}(s)]^{-1},$$

and take the limit

$$\begin{aligned}\lim_{s\to\infty} N(s)D^{-1}(s) &= \lim_{s\to\infty}[N(s)S_c^{-1}(s)]\lim_{s\to\infty}[D(s)S_c(s)^{-1}]\\ &= N_{hc}D_{hc}^{-1},\end{aligned}$$

where N_{hc} is finite (resp. zero) if $\partial_{cj}N(s) \le \partial_{cj}D(s)$ (resp. $\partial_{cj}N(s) < \partial_{cj}D(s)$). Hence, we have the following theorem.

Theorem 2.3.7. *If $D(s)$ is column-reduced, then $G(s) = N(s)D^{-1}(s)$ is strictly proper (resp. proper) if and only if each column of $N(s)$ has degree less than (resp. less than or equal to) the degree of the corresponding column of $D(s)$.*

If $\overline{G}(s) \in \mathbb{R}^{p\times m}(s)$ is proper, we can have a state space realization (A, B, C, D) in (2.1-2.2). Obviously, one sees $D = G(\infty)$ and $\overline{G}(s) - G(\infty) = G(s)$ will be strictly proper. We thus aims to get A, B and C from a right PMF $\bar{N}(s)\bar{D}^{-1}(s)$ of a strictly proper $G(s)$. It follows from Proposition 2.2.1 that there is a unimodular matrix $U(s)$ such that $D(s) = \bar{D}(s)U(s)$ is column-reduced. So, without loss of generality, we assume that $N(s)D^{-1}(s)$ is a right PMF of a $p \times m$ strictly proper $G(s)$ with $D(s)$ column-reduced and $\partial_{cj}N(s) < \partial_{cj}D(s) = \mu_j$, $j = 1, 2, \cdots, m$. Let $n = \Sigma\mu_j$ and write $D(s)$ as

$$D(s) = D_{hc}S_c(s) + D_{lc}\Phi_c(s)$$

for suitable constant matrices D_{hc} and D_{lc}, where $S_c(s) = diag\{s^{\mu_1}, s^{\mu_2}, \cdots, s^{\mu_m}\}$ as before,

$$\Phi_c(s) = block\,diag\{\phi_{c1}(s), \phi_{c2}(s), \cdots, \phi_{cm}(s)\},$$

$$\phi_{ci}(s) = [s^{\mu_i-1}, s^{\mu_i-2}, \cdots, s, 1]^T, \quad i = 1, 2, \ldots, m.$$

Similarly, we write

$$N(s) = N_c\Phi_c(s)$$

for a suitable constant matrix N_c. Define

$$\begin{aligned}A_o &= block\,diag\,\{A_{o1}, A_{o2}, \cdots, A_{om}\},\\ A_{oj} &= \begin{bmatrix} 0 & & & 0\\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0\end{bmatrix} \in \mathbb{R}^{\mu_j\times\mu_j}, \qquad j = 1, 2, \cdots, m,\\ B_o &= block\,diag\,\{b_{o1}, b_{o2}, \cdots, b_{om}\},\\ b_{0j} &= [1, 0, \cdots, 0]^T \in \mathbb{R}^{\mu_j\times 1}, \quad j = 1, 2, \ldots, m.\end{aligned}$$

We now claim that a controllable realization of $G(s)$ is given by

$$A = A_o - B_o D_{hc}^{-1} D_{lc}, \tag{2.44}$$
$$B = B_o D_{hc}^{-1}, \tag{2.45}$$
$$C = N_c. \tag{2.46}$$

We write $D(s)$ as

$$\begin{aligned} D(s) &= D_{hc}(S_c(s) + D_{hc}^{-1} D_{lc} \Phi_c(s)) \\ &:= D_{hc} \Gamma(s), \end{aligned}$$

and look at

$$\begin{aligned} (sI - A)\Phi_c(s) &= (sI - A_o + B_o D_{hc}^{-1} D_{lc})\Phi_c(s) \\ &= (sI - A_o)\Phi_c(s) + B_o D_{hc}^{-1} D_{lc} \Phi_c(s) \\ &= B_o S_c(s) + B_o D_{hc}^{-1} D_{lc} \Phi_c(s) \\ &= B_o \Gamma(s), \end{aligned}$$

which can be written as

$$(sI - A)^{-1} B_o = \Phi_c(s) \Gamma^{-1}(s).$$

Post-multiplying the above by D_{hc}^{-1} gives

$$(sI - A)^{-1} B_o D_{hc}^{-1} = \Phi_c(s)[D_{hc}\Gamma(s)]^{-1},$$

or

$$(sI - A)^{-1} B = \Phi_c(s) D^{-1}(s).$$

Pre-multiplying by $C = N_c$ yields

$$C(sI - A)^{-1} B = N(s) D^{-1}(s),$$

and hence (A, B, C) is a realization of $N(s)D^{-1}(s)$. Furthermore, (A_o, B_o) is controllable, since by inspection, $Rank[sI - A_o \quad B_o]$ has full row rank for all s. Write

$$[sI - A \quad B] = [sI - A_o \quad B_o] \begin{bmatrix} I & 0 \\ D_{hc}^{-1} D_{lc} & D_{hc}^{-1} \end{bmatrix},$$

which is also of full row rank due to nonsingularity of the second matrix on the right hand side. Thus, (A, B) is controllable.

Theorem 2.3.8. *Let $N(s)D^{-1}(s)$ be a right PMF of a strictly proper rational function matrix $G(s)$ with $D(s)$ column-reduced. Then*

(i) *(A, B, C) in (2.44)-(2.46) is a controllable realization of $G(s)$;*
(ii) *(A, B, C) is a controllable and observable (or minimal) realization of $G(s)$ if $N(s)D^{-1}(s)$ is coprime; and*

(iii) $det(sI - A) = \alpha\, det D(s)$ for a nonzero real number α if $N(s)D^{-1}(s)$ is coprime. Hence, $det(sI - A)$ and the determinants of the denominators in any coprime fractions of $G(s)$ have the same set of roots precisely, and $\dim A$ is equal to the McMillan degree of $G(s)$.

The proof of (ii) and (iii) can be found, say, in Chen (1984). With obvious changes, dual results to Theorems 2.3.7 and 2.3.8 can be established for properness and realization of left PMFs.

Example 2.3.6. Consider a transfer matrix

$$G(s) = \begin{bmatrix} \frac{2s+3}{(s+1)(s+2)} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}.$$

It is easy to get a right PMF,

$$G(s) = \begin{bmatrix} (2s+3) & 1 \\ (s+1) & 1 \end{bmatrix} \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & (s+2) \end{bmatrix}^{-1} := \tilde{N}(s)\tilde{D}^{-1}(s).$$

After extracting its GCRD,

$$R(s) = \begin{bmatrix} 1 & -1 \\ 0 & s+2 \end{bmatrix},$$

One gets

$$\begin{aligned} \bar{D}(s) := \tilde{D}(s)R^{-1}(s) &= \begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\ &= \begin{bmatrix} (s+1)(s+2) & s+1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \bar{N}(s) := \tilde{N}(s)R^{-1}(s) &= \begin{bmatrix} 2s+3 & 1 \\ s+1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\ &= \begin{bmatrix} 2s+3 & 2 \\ s+1 & 1 \end{bmatrix} \end{aligned}$$

for a right coprime PMF $\bar{N}(s)\bar{D}^{-1}(s)$ of $G(s)$.

But $\bar{D}(s)$ is not column reduced, since its highest column degree coefficient matrix,

$$\bar{D}_{hc} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

is singular. We see that (1,1) element will become zero if the 2nd column is multiplied by $-(s+2)$ and the result is added to the first column. This means

$$\bar{D}(s)U(s) = \begin{bmatrix} (s+1)(s+2) & s+1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(s+2) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & s+1 \\ -(s+2) & 1 \end{bmatrix} := D(s),$$

which is now column reduced as desired. The corresponding numerator in the right coprime PMD $N(s)D^{-1}(s) = G(s)$ is

$$N(s) := \bar{N}U(s) = \begin{bmatrix} 2s+3 & 2 \\ s+1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(s+2) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Using the realization formulas, we have

$$S_c(s) = \begin{bmatrix} s^{\mu_1} & 0 \\ 0 & s^{\mu_2} \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix},$$

$$\Phi_c(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

$$D(s) = \begin{bmatrix} 0 & s+1 \\ -(s+2) & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} I_2,$$

$$N(s) = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} I_2 = N_c\Phi_c,$$

$$A_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = A_o - B_oD_{hc}^{-1}D_{lc} = -D_{hc}^{-1}D_{lc} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix},$$

$$B = B_oD_{hc}^{-1} = D_{hc}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$C = N_c = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Besides, (A, B, C) is minimal. $\Diamond$

2.4 Model Reduction

Reduced-order models are often required for simplifying the design and implementation of control systems. A reduced-order model is usually adequate, provided it has dynamic characteristics close to that of the original high-order system in the frequency range which is most relevant for control system design. Therefore, model reduction has been an active research area in engineering, especially in model-based prediction, control and optimization, and is a key step in many designs to be presented in later chapters. For plants with rational transfer functions, balanced truncation and optimal Hankel norm approximation are popular. However, the transfer functions which will appear in the subsequent chapters usually contain time delay and may even not fit into the form of a rational function plus time delay. In this section, we present two approaches to model reduction, based on the recursive Least Squares and step response construction, respectively. A great attention is paid to the preservation of stability for the reduced-order models. These two approaches are generally applicable provided that plant transfer function or frequency response is available.

2.4.1 Recursive Least Square Reduction

Consider a scalar system of possibly complicated dynamics with its transfer function $G(s)$ (probably non-rational) or frequency response $G(j\omega)$ available. The problem at hand is to find a nth-order rational function plus dead time model:

$$\hat{G}(s) = \hat{g}_0(s)e^{-Ls} = \frac{\beta_n s^n + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} e^{-Ls}, \tag{2.47}$$

such that the approximation error e defined by

$$e = \sum_{i=1}^{M} \left| W(j\omega_i)(G(j\omega_i) - \hat{G}(j\omega_i)) \right|^2, \tag{2.48}$$

is minimized, where the interval $[\omega_1,\ \omega_M]$ defines the frequency range of interest and $W(j\omega_k)$ serves as the weighting. Note that (2.47) contains the unknown time delay L to be estimated and it makes the problem nonlinear. The solution to our original model reduction is obtained by minimizing the approximation error over the possible range of L. This is a one-dimensional search problem and can be easily solved if an estimation of the range of L is given. A reasonable search range of L is $0.5 \sim 2.0$ times of L_0 which is the initial estimate of the time delay of $G(s)$, and evaluate $15 \sim 20$ points in that range to find the optimal estimate $\hat{L}$. An alternative is to use a Newton-Raphson type of algorithms to search for the optimal L (Lilja, 1989), which requires no prior range for L.

If the time delay L is known, then the problem becomes to approximate a modified plant $g_0(s) = G(s)e^{Ls}$ with a rational transfer function

$$\hat{g}_0(s) = \frac{\beta_n s^n + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0},$$

such that

$$e_0 \triangleq \sum_{i=1}^{M} |W(j\omega_i)(g_0(j\omega_i) - \hat{g}_0(j\omega_i))|^2, \tag{2.49}$$

is minimized. Equation (2.49) falls into the framework of transfer function identification in frequency domain. For this identification, a number of methods are available (Pintelon *et al.*, 1994), and the recursive least square (RLS) algorithm is simple and effective and is briefly described as follows.

$$\begin{aligned} e_0^{(k)} \triangleq \sum_{i=1}^{M} |\bar{W}_i^{(k)} \{ & g_0(j\omega_i)[(j\omega_i)^n + \alpha_{n-1}^{(k)}(j\omega_i)^{n-1} + ... + \alpha_1^{(k)}(j\omega_i) + \alpha_0^{(k)}] \\ & -[\beta_n^{(k)}(j\omega_i)^n + ... + \beta_1^{(k)}(j\omega_i) + \beta_0^{(k)}]\}|^2, \end{aligned} \tag{2.50}$$

where

$$\bar{W}_i^{(k)} \triangleq \frac{W(j\omega_i)}{(j\omega_i)^n + \alpha_{n-1}^{(k-1)}(j\omega_i)^{n-1} + ... + \alpha_1^{(k-1)}(j\omega_i) + \alpha_0^{(k-1)}} \tag{2.51}$$

acts as a weighting function in the standard least squares problem in braces. Equation (2.50) is re-arranged to yield

$$e_0^{(k)} \triangleq \sum_{i=1}^{M} |\eta_i^{(k)} - \phi_i^{(k)T}\theta^{(k)}|^2,$$

where

$$\eta_i^{(k)} = -g_0(j\omega_i)(j\omega_i)^n \bar{W}_i^{(k)}, \tag{2.52}$$

$$\theta^{(k)} = [\, \alpha_{n-1}^{(k)} \; \cdots \; \alpha_0^{(k)} \; \beta_n^{(k)} \; \cdots \; \beta_1^{(k)} \; \beta_0^{(k)} \,]^T, \tag{2.53}$$

$$\phi_i^{(k)} = [\, g_0(j\omega_i)(j\omega_i)^{n-1} \; \cdots \; g_0(j\omega_i) \; -(j\omega_i)^n \; \cdots \; -(j\omega_i) \; -1 \,]^T \bar{W}_i^{(k)}. \tag{2.54}$$

Then, we have the recursive equation for $\theta^{(i)}$ as

$$\theta^{(k,i)} = \theta^{(k,i-1)} + K^{(k,i)}\varepsilon^{(k,i)}, \quad i = 1, 2, \cdots, M, \tag{2.55}$$

where

$$\varepsilon^{(k,i)} = \eta_i^{(k)} - \phi_i^{(k)T}\theta^{(k,i-1)}. \tag{2.56}$$

$$K^{(k,i)} = P^{(k,i-1)}\phi_i^{(k)}(I + \phi_i^{(k)T}P^{(k,i-1)}\phi_i^{(k)})^{-1}, \tag{2.57}$$

$$P^{(k,i)} = (I - K^{(k,i)}\phi_i^{(k)T})P^{(k,i-1)}, \tag{2.58}$$

Once the above RLS in (2.55)-(2.56) has been completed, the resultant parameter vector $\theta^{(k)} = \theta^{(k,M)}$ is used to update $\bar{W}_i^{(k)}$ to

$$\bar{W}_i^{(k+1)} = \frac{1}{(j\omega_i)^n + \alpha_{n-1}^{(k)}(j\omega_i)^{n-1} + ... + \alpha_1^{(k)}(j\omega_i) + \alpha_0^{(k)}}, \tag{2.59}$$

and (2.55)-(2.56) are repeated to calculate $\theta^{(k+1)}$. On convergence, the resultant parameter vector will form one solution to (2.49).

Different weighting functions are employed in the various methods (Pintelon *et al.*, 1994). For simplicity, it is recommended that the $\bar{W}_i^{(k)}$ is chosen as

$$\frac{1}{(j\omega_i)^n + \alpha_{n-1}^{(k-1)}(j\omega_i)^{n-1} + ... + \alpha_1^{(k-1)}(j\omega_i) + \alpha_0^{(k)}}.$$

Simulation shows that the most important frequency range for the model reduction is a decade above and below ω_c, where ω_c is the unity gain crossover frequency of the transfer function $G_0(s)$. Therefore, the frequency range $[\omega_1,\ \omega_M]$ in the optimal fitting problem (2.49) is chosen to span M logarithmically equally spaced points between $0.1\omega_c$ and $10\omega_c$.

Once a reduced-order model $\hat{G}(s)$ is found, the following frequency response maximum relative estimation error can be evaluated

$$E := \max_{\omega \in (0,\ \omega_M)} \left|\frac{\hat{G}(j\omega) - G(j\omega)}{G(j\omega)}\right|. \tag{2.60}$$

And the following criterion will be used to validate the solution

$$E \leq \epsilon \tag{2.61}$$

where ϵ is a user-specified fitting error threshold. ϵ is specified according to the desired accuracy of the RSL approximation to the original dynamics $G(s)$. Usually ϵ may be set between $1\% \sim 10\%$. If (2.61) is met, the procedure stops. Otherwise, one increases n by 1 until the smallest integer n such that $E \leq \epsilon$.

Algorithm 2.4.1 *Seek a reduced-order model $\hat{G}(s)$ of order n in (2.47) given $G(s)$ or $G(j\omega)$, approximation threshold ϵ and the initial time delay L_0 and parameter vector θ^0.*

step 1. Choose N between $15 \sim 20$, set $\Delta L = \frac{1.5L_0}{N}$, and obtain $L_k = 0.5L_0 + k\Delta L,\quad k = 0, 1, \cdots, N$.
step 2. Start from $\hat{G}(s)$ with $n = 1$.
step 3. For each L_k, find the nth order rational approximation solution $\hat{g}_0(s)$ to the modified process $g_0(s) = G(s)e^{L_k s}$ with the RLS method in (2.55) - (2.56) and evaluate the corresponding approximation error E in (2.61) for $\hat{G}(s) = \hat{g}_0(s)e^{-L_k s}$.

step 4. Take $\hat{G}(s)$ as the solution if it yields the minimum error E and $E \leq \epsilon$. Otherwise, $n+1 \to n$ and go to Step 3.

The preservation of stability is a crucial issue in frequency domain based model reduction. Let $\hat{G}(s) = \hat{g}_0(s)e^{-Ls}$ be the one that yields the smallest approximation error e in (2.48). It is noted that Algorithm 2.4.1 with zero initial parameter vector might result in unstable $\hat{G}(s)$, especially for high-order models, even though $G(s)$ is stable. One can use the stability tests and projection algorithm (Ljung and Soderstrom, 1983) to constrain the poles of the model to be in the stable region. However, simulation shows that this method can slow down the convergence of the recursive least square considerably and may result in large modelling error, if the dynamics of the transfer function to be modelled is complicated. Here, we notice that since (2.48) is a nonlinear problem, Algorithm 2.4.1 may yield different local optimal solutions, if it starts with different initial settings. Among those solutions, only stable $\hat{G}(s)$ models are required given a stable $G(s)$. If we initiate the algorithm with a stable model, the algorithm is likely to reach a stable approximate upon convergence.

When $n=1$, we set the initial model for $\hat{G}(s)$ as

$$\hat{G}_0 = \frac{\beta_0}{s+\alpha_0}e^{-L_0 s}. \tag{2.62}$$

Matching $\hat{G}_0(j\omega)$ to $G(j\omega)$ at $\omega = 0$ and $\omega = \omega_c$, where ω_c is the phase cross over frequency of $G(s)$, i.e., $\angle G(j\omega_c) = -\pi$, we get

$$\begin{cases} \alpha_0 = \omega_c\sqrt{\frac{|G(j\omega_c)|^2}{G^2(0)-|G(j\omega_c)|^2}}, \\ \beta_0/\alpha_0 = G(0), \\ L_0 = \frac{1}{\omega_c}\{-\arg[G(j\omega_c)] - \tan^{-1}(\frac{\omega_c}{\alpha_0})\}, \end{cases} \tag{2.63}$$

When $n=2$, one may adopt the following structure,

$$\hat{G}_0 = \frac{\beta_0}{s^2+\alpha_1 s+\alpha_0}e^{-L_0 s}. \tag{2.64}$$

Similarly, match $\hat{G}_0(j\omega)$ to $G(j\omega)$ at the two points $\omega = \omega_b$ and $\omega = \omega_c$, where $\angle G(j\omega_b) = -\frac{\pi}{2}$ and $\angle G(j\omega_c) = -\pi$. It then follows (Wang *et al.*, 1999*a*) that the parameters β_0, α_1, α_0 and L_0 can be determined as

$$\begin{cases} \frac{\sin(\omega_c L_0)}{\cos(\omega_b L_0)} = \frac{\omega_c|G(j\omega_c)|}{\omega_b|G(j\omega_b)|}, \\ \beta_0 = (\omega_c^2-\omega_b^2)\left[\frac{\sin(\omega_b L_0)}{|G(j\omega_b)|} + \frac{\cos(\omega_c L_0)}{|G(j\omega_c)|}\right]^{-1}, \\ \alpha_1/\beta_0 = \frac{\sin(\omega_c L_0)}{\omega_c|G(j\omega_c)|}, \\ \alpha_0/\beta_0 = (\omega_c^2-\omega_b^2)\left[\frac{\omega_c^2\sin(\omega_b L_0)}{|G(j\omega_b)|} + \frac{\omega_b^2\cos(\omega_c L_0)}{|G(j\omega_c)|}\right]^{-1}. \end{cases} \tag{2.65}$$

For the cases of $n > 2$, we can set the initial model $\hat{G}_0(s)$of the respective orders with β_0, α_1, α_0 and L_0 determined as in (2.65), while all the remaining high-degree coefficients are set to 0. Our extensive simulation shows that this technique works very well.

Example 2.4.1. Consider a high-order plant (Maffezzoni and Rocco, 1997):

$$G(s) = 2.15\frac{(-2.7s+1)(158.5s^2+6s+1)}{(17.5s+1)^4(20s+1)}e^{-14s}.$$

With zero initial parameter vector, Algorithm 2.4.1 gives a reduced order model

$$\hat{G}(s) = \frac{-0.0275s^2 - 0.0010s - 0.0001}{s^3 - 0.0129s^2 - 0.0029s - 0.0001}e^{-63.20s}, \tag{2.66}$$

which is unstable. When the parameters obtained from (2.65) are adopted for the initial model, we get

$$\hat{G}(s) = \frac{0.4687s^2 + 0.0038s + 0.0024}{s^3 + 1.2160s^2 + 0.0665s + 0.0011}e^{-41.90s}, \tag{2.67}$$

which is stable with $E = 2.71\%$. The frequency responses of the actual and estimated models are shown in Figure 2.1. $\Diamond$

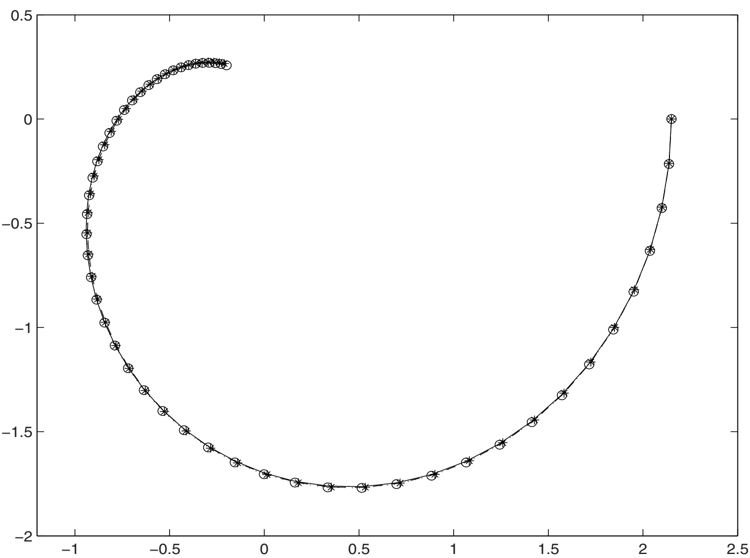


Fig. 2.1. Frequency Domain Model Reduction
(—— actual process; $***$ unstable model; $-\circ-\circ-$ stable model)

2.4.2 Stable Model Reduction

The method described in the preceding subsection cannot guarantee the preservation of stability of the reduced-order models. This is a typical problem associated with frequency domain methods. On the other hand, time domain model reduction is easier to deal with stability preservation. One sees that the step response of a stable $G(s)$ will remain finite while that of an unstable $\hat{G}(s)$ will tend to infinity. The squared error between them will diverge, and unstable $\hat{G}(s)$ will be excluded from the solutions to the problem of minimizing such an error, in other words, the solution must be stable. Based on this idea, we present a time domain model reduction algorithm. This algorithm not only preserves stability but also enables non-iterative estimation of all model parameters including time delay.

Construction of Step Response from $G(s)$ or $G(j\omega)$. Suppose that plant is stable with $G(s)$ or $G(j\omega)$ given. Let the plant input $u(t)$ be of step type with size h. Then the plant output step response in Laplace domain is

$$Y(s) = G(s)\frac{h}{s}. \tag{2.68}$$

It seems very easy to obtain the corresponding time response $y(t)$ by simply applying the inverse Fourier transform ($\mathcal{F}^{-1}$). However, since the steady state part of $y(t)$ is not absolutely integrable, such a calculation is inapplicable and meaningless (Wang *et al.*, 1997*b*). To solve this obstacle, $y(t)$ is decomposed into

$$y(t) = y(\infty) + \Delta y(t) = G(0)h + \Delta y(t).$$

Applying the Laplace transform to both sides gives

$$Y(s) = \frac{G(0)}{s} + \mathcal{L}\{\Delta y(t)\}. \tag{2.69}$$

Bring (2.68) in, we have

$$\mathcal{L}\{\Delta y(t)\} = h\frac{G(s) - G(0)}{s}.$$

Applying the inverse Laplace transform yields

$$\Delta y(t) = h\mathcal{F}^{-1}\{\frac{G(s) - G(0)}{s}\}.$$

Thus, the plant step response is constructed as

$$y(t) = h[G(0) + \mathcal{F}^{-1}\{\frac{G(j\omega) - G(0)}{j\omega}\}], \tag{2.70}$$

where $\mathcal{F}^{-1}$ may easily be implemented by the inverse fast Fourier transform (IFFT). Alternatively, the step response $y(t)$ may be obtained from a real time step test on the given stable plant.

Identification from Step Response. Suppose that the given stable step response is to be fitted into the following model:

$$Y(s) = \hat{G}(s)U(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} e^{-Ls} U(s), \tag{2.71}$$

or equivalently by

$$\begin{aligned} &y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_{n-1}y^{(1)}(t) + a_n y(t) = b_1 u^{(n-1)}(t-L) \\ &+b_2 u^{(n-2)}(t-L) + \ldots + b_{n-1}u^{(1)}(t-L) + b_n u(t-L), \end{aligned} \tag{2.72}$$

where $L > 0$. For an integer $m \geqslant 1$, define

$$\int_{[0,t]}^{(m)} f = \int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} f(\tau_1) d\tau_1 \cdots d\tau_m.$$

Under zero initial conditions, for $u(t) = h\mathbf{1}(t)$, it follows that

$$\int_{[0,t]}^{(m)} u(t-L) = \frac{1}{m!}(t-L)^m h.$$

Integrating (2.72) n times gives

$$\begin{aligned} y(t) &= -a_1 \int_{[0,t]}^{(1)} y - a_2 \int_{[0,t]}^{(2)} y \cdots - a_{n-1} \int_{[0,t]}^{(n-1)} y - a_n \int_{[0,t]}^{(n)} y \\ &\quad +hb_1(t-L) + \frac{1}{2}hb_2(t-L)^2 + \cdots + \frac{1}{(n-1)!}hb_{n-1}(t-L)^{n-1} \\ &\quad +\frac{1}{n!}hb_n(t-L)^n, \\ &= -a_1 \int_{[0,t]}^{(1)} y - \cdots - a_n \int_{[0,t]}^{(n)} y + ht^0 \sum_{j=1}^{n} \frac{b_j(-L)^j}{j!} \\ &\quad +ht^1 \sum_{j=1}^{n} \frac{b_j(-L)^{j-1}}{(j-1)!} + \frac{ht^2}{2!} \sum_{j=2}^{n} \frac{b_j(-L)^{j-2}}{(j-2)!} + \cdots \\ &\quad +\frac{ht^{n-1}}{(n-1)!} \sum_{j=n-1}^{n} \frac{b_j(-L)^{j-(n-1)}}{1!} + \frac{ht^n}{n!} b_n. \end{aligned} \tag{2.73}$$

Define

$$\begin{cases} \gamma(t) = y(t), \\ \phi^T(t) = \left[-\int_{[0,t]}^{(1)} y, \ \cdots, \ -\int_{[0,t]}^{(n)} y, \ h, \ ht, \ \frac{ht^2}{2!}, \ \cdots, \ \frac{t^n h}{n!} \right], \\ \theta^T = \Big[a_1, \ \cdots, \ a_n, \ \sum_{j=1}^{n} \frac{b_j(-L)^j}{j!}, \ \sum_{j=1}^{n} \frac{b_j(-L)^{j-1}}{(j-1)!}, \ \cdots, \\ \qquad \sum_{j=n-1}^{n} b_j(-L)^{j-(n-1)}, \ b_n \Big]. \end{cases} \tag{2.74}$$

Then (2.73) can be expressed as

$$\gamma(t) = \phi^T(t)\theta, \qquad if \quad t \geqslant L.$$

Choose $t = t_i$ and $L \leqslant t_1 < t_2 < \cdots < t_N$. Then

$$\Gamma = \Phi\theta, \tag{2.75}$$

where $\Gamma = [\gamma(t_1), \ \gamma(t_2), \ ..., \ \gamma(t_N)]^T$, and $\Phi = [\phi(t_1), \ \phi(t_2), \ ...., \ \phi(t_N)]^T$.

In (2.75), the parameter vector θ which minimizes the following error,

$$\min_{\theta}(\Gamma - \Phi\theta)^T(\Gamma - \Phi\theta),$$

is given by the least square solution:

$$\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T\Gamma. \tag{2.76}$$

In practice, the true plant output $\hat{y}$ may be corrupted by computational errors or measurement noise $v(t)$, and

$$y = \hat{y} + v, \tag{2.77}$$

where v is supposed to be a strictly stationary stochastic process with zero mean. In this case, (2.75) is again modified to be

$$\Gamma = \Phi\theta + \Delta, \tag{2.78}$$

where $\Delta = [\delta_1, \ \delta_2, \ \cdots, \ \delta_N]^T$, and

$$\delta_i = [-v(t_i) \quad - a_1 \int_{[0,t_i]}^{(1)} v \quad - a_2 \int_{[0,t_i]}^{(2)} v \quad \cdots \quad - a_n \int_{[0,t_i]}^{(n)} v].$$

To get an unbiased estimate, the instrumental variable matrix Z is constructed as

$$Z = \begin{bmatrix} \frac{1}{t_1^{n+1}} & \frac{1}{t_1^{n+2}} & \cdots & \frac{1}{t_1^{2n}} & 1 & t_1 & \cdots & t_1^n \\ \frac{1}{t_2^{n+1}} & \frac{1}{t_2^{n+2}} & \cdots & \frac{1}{t_2^{2n}} & 1 & t_2 & \cdots & t_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t_N^{n+1}} & \frac{1}{t_N^{n+2}} & \cdots & \frac{1}{t_N^{2n}} & 1 & t_N & \cdots & t_N^n \end{bmatrix}, \tag{2.79}$$

and the estimate is then given by

$$\hat{\theta} = (Z^T\Phi)^{-1}Z^T\Gamma. \tag{2.80}$$

Theorem 2.4.1. *For a linear time-invariant process of n-th or higher-order, if the noise in the output measurement is a zero-mean strictly stationary stochastic process and the number of output samples N satisfies $N \geqslant 2n+1$, then the estimate given by*

$$\hat{\theta} = (Z^T \Phi)^{-1} Z^T \Gamma \tag{2.81}$$

is consistent, where Z is given in (2.79). If there is no noise, Z may be replaced by Φ.

Once θ is estimated from (2.80), one has to recover the process model coefficients: L, a_i and b_i, $i = 1, 2, \cdots, n$. It follows from (2.74) that

$$a_k = \theta_k \quad k = 1, 2, \cdots, n;$$

and

$$\begin{bmatrix} -L & \frac{(-L)^2}{2!} & \frac{(-L)^3}{3!} & \cdots & \frac{(-L)^{n-1}}{(n-1)!} & \frac{(-L)^n}{n!} \\ 1 & -L & \frac{(-L)^2}{2!} & \cdots & \frac{(-L)^{n-2}}{(n-2)!} & \frac{(-L)^{n-1}}{(n-1)!} \\ 0 & 1 & -L & \cdots & \frac{(-L)^{n-3}}{(n-3)!} & \frac{(-L)^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{(-L)^{n-4}}{(n-4)!} & \frac{(-L)^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -L \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} \theta_{n+1} \\ \theta_{n+2} \\ \theta_{n+3} \\ \theta_{n+4} \\ \vdots \\ \theta_{2n} \\ \theta_{2n+1} \end{bmatrix}. \tag{2.82}$$

The last n rows of (2.82) give b_i in terms of L:

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & L & \frac{L^2}{2!} & \cdots & \frac{(L)^{n-2}}{(n-2)!} & \frac{(L)^{n-1}}{(n-1)!} \\ 0 & 1 & L & \cdots & \frac{(L)^{n-3}}{(n-3)!} & \frac{(L)^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & L \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \theta_{n+2} \\ \theta_{n+3} \\ \theta_{n+4} \\ \vdots \\ \theta_{2n} \\ \theta_{2n+1} \end{bmatrix}. \tag{2.83}$$

Substituting (2.83) into the first row of (2.82) yields the following n-degree polynomial equation in L:

$$\sum_{i=0}^{n} \frac{\theta_{n+i+1}}{i!} L^i = 0. \tag{2.84}$$

Equation (2.84) has n roots for L. In selecting a suitable solution, a rule of thumb is to choose one that leads to the minimal output error between the step response of the estimated model and the given one. Once L is determined, b_i can be easily computed from (2.83).

Algorithm 2.4.2 *Seek a reduced-order model $\hat{G}(s)$ of order n in (2.71) given $G(s)$ or $G(j\omega)$.*

step 1. Construct the output step response $y(t)$ from $G(s)$ or $G(j\omega)$ with the IFFT by (2.70);
step 2. Compute the regression vector $\hat{\theta}$ by (2.76) or (2.80);
step 3. Recover the model parameters from $\hat{\theta}$ by (2.76) and (2.83) and (2.84).

Some special cases can simplify the formulas. For first-order plus dead time (FOPDT) modelling:

$$G(s) = \frac{b}{s+a} e^{-Ls},$$

(2.74) becomes

$$\begin{cases} \gamma(t) = y(t), \\ \phi^T(t) = \left[-\int_0^t y(\tau)d\tau,\, h,\, ht \right], \\ \theta^T = \left[a,\, -bL,\, b \right], \end{cases}$$

and the model parameters are recovered from

$$\left[a,\, b,\, L \right]^T = \left[\theta_1,\, \theta_3,\, -\tfrac{\theta_1}{\theta_3} \right]^T.$$

For second-order plus dead time (SOPDT) modelling:

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} e^{-Ls},$$

(2.74) becomes

$$\begin{cases} \gamma(t) = y(t), \\ \phi^T(t) = \left[-\int_0^t y(\tau)d\tau,\, -\int_0^t \int_0^\tau y(\tau_1)d\tau_1 d\tau,\, h,\, th,\, \tfrac{1}{2}t^2 h, \right] \\ \theta^T = \left[a_1,\, a_2,\, -b_1 L + \tfrac{1}{2}b_2 L^2,\, b_1 - b_2 L,\, b_2, \right] \end{cases}$$

and the model parameters are recovered from

$$\left[a_1,\, a_2,\, b_1,\, b_2,\, L \right]^T = \left[\theta_1,\, \theta_2,\, \beta,\, \theta_5,\, -\tfrac{-\theta_4+\beta}{\theta_5} \right]^T$$

with

$$\beta = \begin{cases} -\sqrt{\theta_4^2 - 2\theta_5\theta_3}, & \text{if a inverse response is detected;} \\ \sqrt{\theta_4^2 - 2\theta_5\theta_3}, & \text{otherwise.} \end{cases}$$

A few parameters need to be specified by users before Algorithm 2.4.2 is applied. The guidelines for their selection are given now, which can lead to effective implementation of this algorithm.

Choice of t_1**.** It is noted from the above development that the first sample $y(t_1)$ should not be taken into the algorithm until $t_1 \geqslant L$, when the output deviates from the previous steady state. In practice, the selection of the logged $y(t)$ after $t_i \geqslant L$ can be made as follows. Before the step input is applied, the output will be monitored for a period called the 'listening period', during which the noise band B_n can be found. After a step change in the input is applied, the time t at which $y(t)$ satisfies

$$\text{abs}(\text{mean}(y(t-\tau, t))) > 2B_n$$

is considered to meet $t > L$, where τ is user-specified time interval and is used for averaging.

Choice of t_N**.** The initial part of the step response contains more high frequency information, while the part of the response after the steady state contains only zero frequency information, i.e., at $\omega = 0$. Extensive simulation suggests that t_N be set at $1.2 \sim 1.5T_{set}$, where T_{set} is the settling time, defined as the time required for the output to settle within $\pm 2\%$ of its steady state.

Choice of N**.** The computational effort becomes heavy if too many samples are taken into consideration, which leads to the large size of Φ. Moreover, $\Phi^T\Phi$ or $Z^T\Phi$ may tend to become ill-conditioned for very large N and this may cause the computational difficulty in estimating $\hat{\theta}$. Therefore, N has to be limited. For the case with a large number of recorded data, the default value of N is recommended to be 200, and t_i may be set as

$$t_i = t_1 + \frac{i-1}{N}(t_N - t_1), \qquad i = 1, 2, \cdots, N.$$

For a better assessment of model accuracy, identification errors in both time domain and frequency domain are used. The time domain identification error is measured over the transient period by standard deviation:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}[y(t_i) - \hat{y}(t_i)]^2, \tag{2.85}$$

where $y(t_i)$ and $\hat{y}(t_i)$ are the step responses of the plant $G(s)$ and its model $\hat{G}(s)$, respectively, under the same step input. Once the identification is carried out, the model $\hat{G}(s)$ is available and the frequency error E in (2.60) is measured as the *worst-case error*. Here, the frequency range $[0, \omega_c]$ is considered, where $\angle G(j\omega_c) = -\pi$, since this range is the most significant for control design. To show the robustness of the proposed method, noise is introduced into the output response. In the context of system identification, noise-to-signal ratio defined by

$$NSR = \frac{\text{mean}(\text{abs}(\text{noise}))}{\text{mean}(\text{abs}(\text{signal}))}.$$

is used to represent noise level.

Example 2.4.2. Reconsider the high-order plant in Example 2.4.1. The output step response is first constructed using the IFFT. The area method (Rake, 1980) gives the model:

$$\hat{G}(s) = \frac{2.15}{46.69s + 1} e^{-53.90s},$$

with $\epsilon = 1.15 \times 10^{-2}$ and $E = 60.87\%$. The FOPDT model estimated by Algorithm 2.4.2 is

$$\hat{G}(s) = \frac{0.0396}{s + 0.0184} e^{-49.9839s},$$

with $\epsilon = 8.6 \times 10^{-3}$ and $E = 48.12\%$. Our SOPDT model is

$$\hat{G}(s) = \frac{0.0011}{s^2 + 0.0343s + 0.0005} e^{-28.8861s},$$

with $\epsilon = 4.0631 \times 10^{-4}$ and $E = 5.81\%$. For third-order plus dead time model (TOPDT), we have

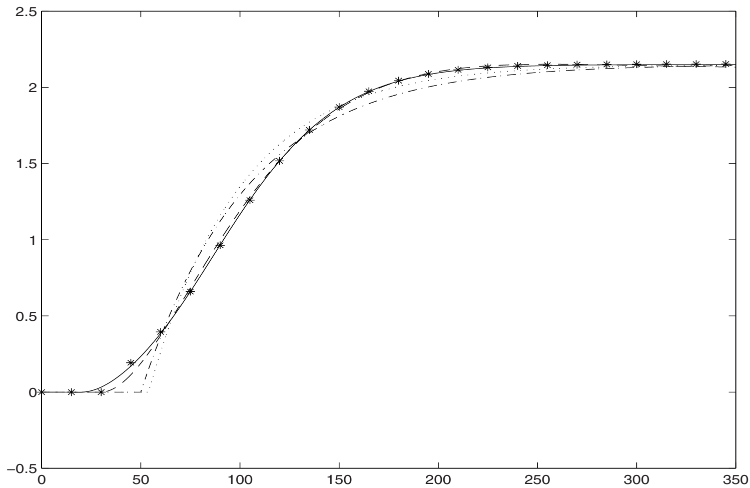
$$\hat{G}(s) = \frac{0.2059s^2 + 0.0005s + 0.0001}{s^3 + 0.9969s^2 + 0.0382s + 0.0001} e^{-31.9265s},$$

with $\epsilon = 2.0009 \times 10^{-4}$ and $E = 1.27\%$. It can be seen that the results from Algorithm 2.4.2 are consistently better than that of the area method. The estimation errors decrease with model order n. Especially from FOPDT to SOPDT, errors decrease dramatically, but the error decrease slows down from SOPDT to TOPDT. In this case, an SOPDT model is good enough. Step and frequency responses of the actual and estimated models are shown in Figure 2.2. ◊

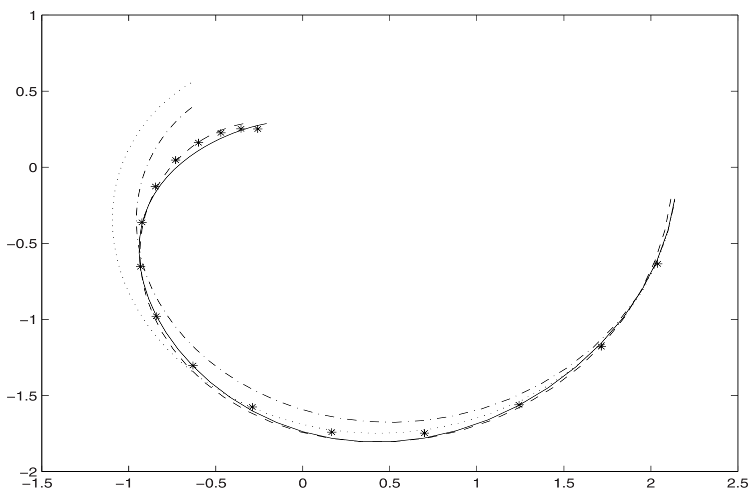
In this section, we have discussed two model reduction approaches. Recursive Lease Square (RLS) is simple to employ, and stability can be taken care of but cannot be guaranteed. The time domain method involves several steps but can preserve stability.

2.5 Conversions between Continuous and Discrete Systems

This book will address control system design for continuous-time systems only. For computer implementation, the designed continuous controllers have to be converted to discrete forms. In general, signals from computer related applications are often manipulated in digital form. Conversion between continuous and discrete time systems is thus needed in these applications. The conversion accuracy is important and directly affects the final performance of



(a) *Time responses*



(b) *Frequency responses*

Fig. 2.2. Time Domain Model Reduction
(—— actual process; $\cdots$ area method; $-\cdot-\cdot-\cdot$ proposed FOPDT;$---$ proposed SOPDT; * * * proposed TOPDT)

the system. In this section, popular discrete equivalents to continuous transfer functions are first reviewed. Then a frequency response fitting method based on Recursive Least Square (RLS) to solve the model conversion problem is presented. The objective of this method is to obtain the discrete transfer function (DTF) so that its frequency response fits that of the continuous transfer function (CTF) as well as possible. A DTF model can be obtained either from a CTF or directly from the system frequency response such as Bode plot data.

2.5.1 Popular Rules

We consider this problem: Given a continuous time transfer function $G(s)$, what discrete time transfer function $G(z)$ will approximately have the same characteristics?

Discrete equivalents by numerical integration. We recast a dynamic system $G(s)$ to the corresponding differential equation, numerically integrate with the step size T the equation to get the discrete solution. Different numerical integration rules will yield different solutions. Most popular are forward, backward and Trapezoid (Bilinear) rules. Applying z-transform to discretized systems will get respective $G(z)$. The results are summarized in Table. 2.1. Note that the conversions can be in two ways, i.e, either from $G(s)$ to $G(z)$, or from $G(z)$ to $G(s)$. To know stability preservation properties of these rules, we let $s = j\omega$ in these equations and obtain the boundaries of the regions in the z-plane. The shaded areas in Figure 2.3 corresponds to the stability region in s-plane (Re $s < 0$).

Table 2.1. Integration Based Rules

Method	$G(s) \to G(z)$	$G(z) \to G(s)$
Forward Rule	$s = \frac{z-1}{T}$	$z = 1 + Ts$
Backward Rule	$s = \frac{z-1}{Tz}$	$z = \frac{1}{1-Ts}$
Bilinear Rule	$s = \frac{2z-1}{Tz+1}$	$z = \frac{1+Ts/2}{1-Ts/2}$

Zero-pole Mapping Equivalents. A very simple but effective method of obtaining a discrete equivalent to a continuous $G(s)$ is to relate poles of $G(s)$ to the poles of $G(z)$ according to $z = e^{sT}$. The same mapping is applied to zeros, too. The following heuristic rules are used to locate poles and zeros of $G(z)$.

(i) if $G(s)$ has a pole at $s = -a$, then $G(z)$ has a pole at $z = e^{-aT}$. If $G(s)$ has a pole at $-a + jb$, then $G(z)$ has a pole at $re^{j\theta}$, where $r = e^{-aT}$ and $\theta = bT$.

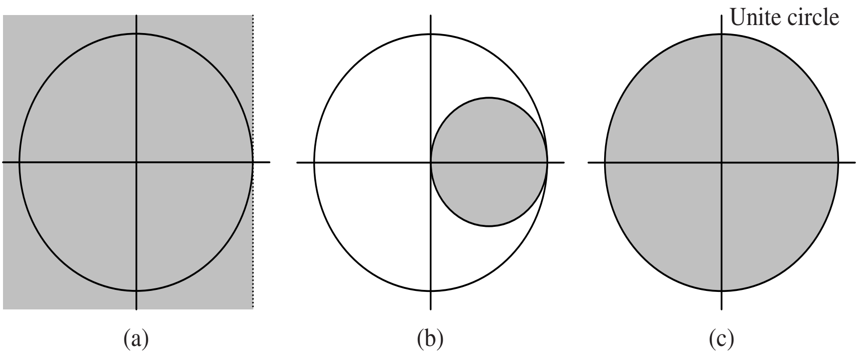


Fig. 2.3. Maps of the left-half s-plane to the z-plane by the integration rules in Table .2.1. Stable s-plane poles map into the shaded regions in the z-plane. The unite circle is shown for reference. (a) Forward rectangular rule. (b) Backward rectangular rule.(c) Trapezoid or bilinear rule.

(ii) All finite zeros are mapped in the same way.
(iii) The zeros of $G(s)$ at $s = \infty$ are mapped in $G(z)$ to the point $z = -1$. or
 (iiia) One zero of $G(s)$ at $s = \infty$ is mapped into $z = \infty$.
(iv) There holds gain matching, i.e.

$$G(s)|_{s=0} = G(z)|_{z=1}.$$

Application of these rules to $G(s) = a/(s + a)$ gives

$$G(z) = \frac{(z + 1)(1 - e^{-aT})}{2(z - e^{-aT})}.$$

or using (iiia), we get

$$G(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}.$$

Hold Equivalents. In this case, we want to design a discrete system $G(z)$ that, with an input consisting of samples of a continuous signal $u(t)$, has an output which approximates the output of the continuous system $G(z)$ whose input is the continuous signal $u(t)$. We can do so through hold operation. If the hold is of the zero-order, the conversion is given by

$$G(z) = (1 - z^{-1})\mathcal{Z}\{\frac{G(s)}{s}\}.$$

Suppose again

$$G(s) = \frac{a}{s + a}.$$

It follows that

$$\mathcal{Z}\{\frac{G(s)}{s}\} = \mathcal{Z}\{\frac{a}{s(s+a)}\} = \mathcal{Z}\{\frac{1}{s}\} - \mathcal{Z}\{\frac{1}{s+a}\}$$
$$= \frac{(1-e^{-aT}z^{-1}) - (1-z^{-1})}{(1-z^{-1})(1-e^{-aT}z^{-1})},$$

so that

$$G(z) = \frac{1-e^{-aT}}{z-e^{-aT}}.$$

2.5.2 A Frequency Response Fitting Approach

The rules discussed above are applicable mainly to rational $G(s)$ only. We now develop a conversion technique which is suitable for non-rational $G(s)$, too. Let a continuous transfer function $G_c(s)$ or frequency response $G_c(j\omega)$ be given. An equivalent discrete transfer function

$$\hat{G}_d(z) = \frac{\beta_m z^m + \beta_{m-1}z^{m-1} + ... + \beta_1 z + \beta_0}{z^m + \alpha_{m-1}z^{m-1} + ... + \alpha_1 z + \alpha_0}, \tag{2.86}$$

is to be found to match the frequency response of $\hat{G}_d(z)$ to $G_c(s)$ with respect to the map,

$$z = e^{Ts}. \tag{2.87}$$

This problem is formulated as

$$\min_{\hat{G}_d} J \triangleq \min_{\hat{G}_d} \sum_{i=1}^{M} |G_c(j\omega_i) - \hat{G}_d(e^{jT\omega_i})|^2, \tag{2.88}$$

where $(\omega_1,\ \omega_M)$ defines the frequency range of interest. Equation (2.88) falls into the framework of transfer function identification in frequency domain, in which G_c is the known frequency response while $\hat{G}_d$ is a parametric transfer function to be identified. Obviously, the RLS based method (*Algorithm 2.4.1*) in the preceding section is immediately applicable. To make the best use of it, two minor remarks are given as follows.

The frequency range in the optimal fitting is a useful tuning parameter. In general, our studies suggest that the frequency range $(\omega_1 \ \cdots \ \omega_M)$ be chosen as $(\frac{1}{100}\omega_b,\ \omega_b)$ with the step of $(\frac{1}{100} \sim \frac{1}{10})\omega_b$, where ω_b is the bandwidth of $G_c(j\omega)$. In this range, RLS yields satisfactory fitting results in frequency domain. Next, the preservation of stability is again a crucial issue. We adopt the same strategy, i.e., make the initial model stable. But, the formulas in (2.62) and (2.64) are no longer useful as they only ensure stability of continuous systems. To get a stable initial $G_{d0}(z)$ from the given stable $G_c(s)$, note from Figure 2.3(a) that the Bilinear rule preserves stability under continuous and discrete conversions. Thus, for a rational $G_c(s)$, we set the initial model $G_{d0}(z)$ according to the Bilinear rule. For other $G_c(s)$, the idea behind (2.62) and (2.64) can still be used, that is, match $G_{d0}(z)$ to $G_c(s)$ at two selected frequencies to get a stable $G_{d0}(z)$. Our extensive simulation shows that this technique works very well.

The method described above is also applicable to inverse conversion, i.e., from DTF to CTF, with obvious changes of the rules of continuous and discrete frequency responses in (2.88).

We now illustrate the above technique with some typical plants. In simulation, the sampling time T is chosen to meet the standard rule:

$$6 \lesssim \frac{\omega_s}{\omega_b} \lesssim 40, \tag{2.89}$$

where $\omega_s = \frac{2\pi}{T}$ is the sampling frequency. Once a discrete equivalent is found, the frequency response maximum relative error similar to (2.60) is used to assess the conversion accuracy.

Example 2.5.1. Consider the continuous filter transfer function (Franklin *et al.*, 1990, pp. 141):

$$G_c(s) = \frac{1}{s^3 + 2s^2 + 2s + 1},$$

with unity pass bandwidth. For $T = 1$ sec, i.e., $\frac{\omega_s}{\omega_b} = 2\pi > 6$, the Bilinear rule gives

$$\hat{G}_d(z) = \frac{0.0476z^3 + 0.1429z^2 + 0.1429z + 0.0476}{z^3 - 1.1900z^2 + 0.7143z - 0.1429},$$

with the error $E = 24.67\,\%$, while the zero-pole mapping transformation does

$$\hat{G}_d(z) = \frac{0.0920(z+1)^2}{(z - 0.3679)(z^2 - 0.7859z + 0.3679)},$$

with $E = 62.84\,\%$. For $m = 3$, the fitting method yields

$$\hat{G}_d(z) = \frac{0.0080z^3 + 0.2257z^2 + 0.1381z - 0.0072}{z^3 - 1.1583z^2 + 0.6597z - 0.1367},$$

with $E = 0.26\,\%$. In Figure 2.4, the responses of three models are compared, and the fitting method yields much better performance than others. Moreover, Figure 2.5 shows that in the reasonable range of ω_s given in (2.89), the fitting method exhibits a consistent superior approximation performance than Bilinear and zero-pole mapping schemes. ◊

Example 2.5.2. Consider the system with time delay

$$G_c(s) = \frac{s-1}{s^2 + 4s + 5} e^{-0.35s}, \tag{2.90}$$

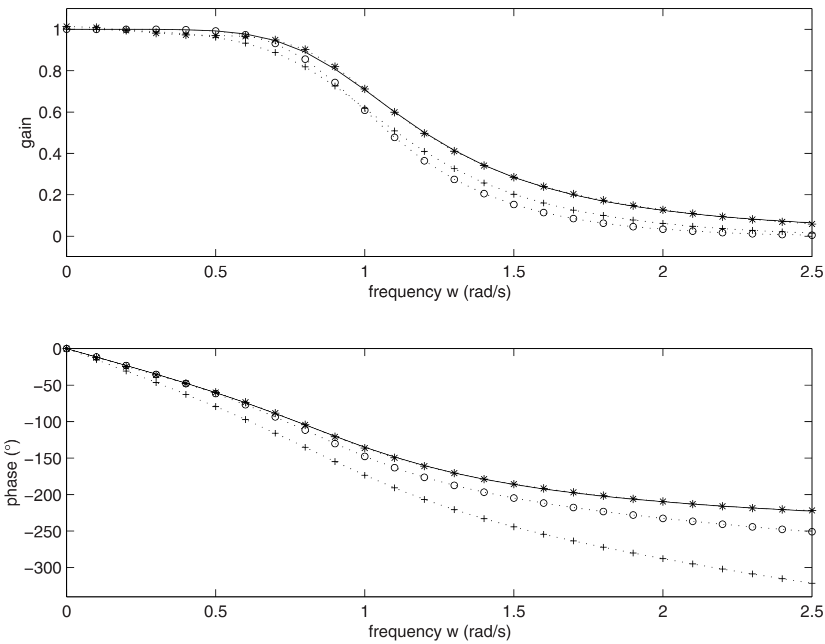


Fig. 2.4. Digital equivalents to 3rd-order lowpass filter
(—— CTF, $\cdots\ *\ \cdots$ Fitting, $\cdots\ o\ \cdots$ Bilinear, $\cdots\ +\ \cdots$ P-Z mapping)

with $T = 0.1$ sec (Control System Toolbox User's Guide, MathWorks, ver. 4.2, 1998, *c2d* function). The Toolbox, via Triangle approximation, gives

$$\hat{G}_d(z) = z^{-3}\frac{0.0115z^3 + 0.0456z^2 - 0.0562z - 0.0091}{z^3 - 1.6290z^2 + 0.6703z},$$

with $E = 5.12\%$. The fitting method generates

$$\hat{G}_d(z) = z^{-3}\frac{0.0042z^2 + 0.0684z - 0.0807}{z^2 - 1.6333z + 0.6740},$$

with $E = 0.50\%$, a much improved accuracy. $\Diamond$

To end this section, it is worth mentioning that the fitting method is suitable for general systems with either parametric or non-parametric source models, and can achieve high performance with good chance of stability preservation. In fact, it can be also applied to many other cases such as determining optimal sampling time, optimal model order, and re-sampling a discrete systems. Readers are referred to Wang and Yang (2001) for details.

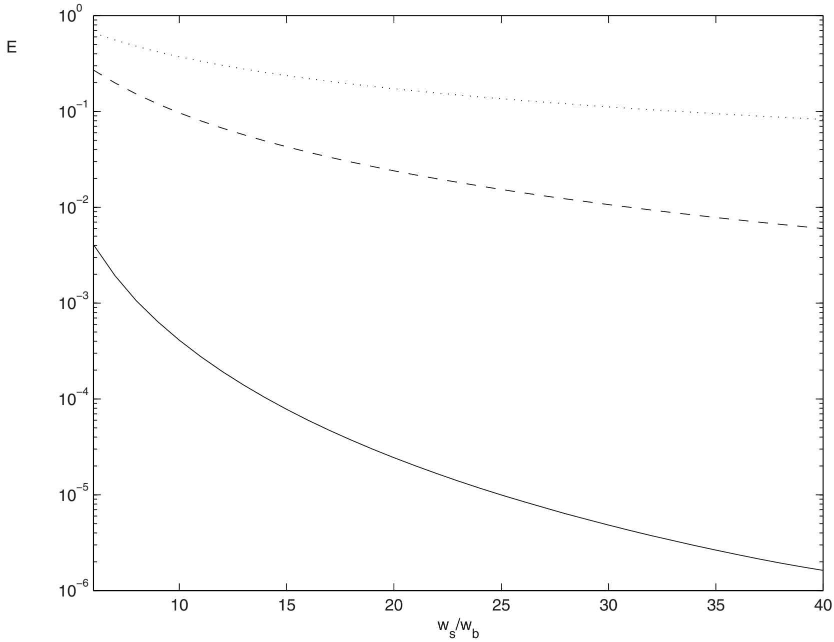


Fig. 2.5. Approximation error E versus sample rate ω_s
(—— Fitting, $- \cdot - \cdot -$ Bilinear, $\cdots\cdots$ P-Z mapping)

2.6 Notes and References

The standard linear system theory is well covered in Kailath (1980) and Chen (1984). The Polynomial matrix approach to linear multivariable system analysis and design became popular since 1970's (Rosenbrock, 1974; Callier and Desoer, 1982; Wolovich, 1978; Kucera, 1979). For model reduction, the most popular are the balanced truncation method (Moore, 1981; Pernebo and Silverman, 1982; Enns, 1984), and the Hankel norm approximation method (Glover, 1984). Optimization-based approaches to model reduction are discussed in Meier and Luenberger (1967), Hyland and Bernstein (1985), Baratchart *et al.* (1991), Spanos *et al.* (1992), Zhou (1995). But the problem for delay systems is less developed and stability of reduced-order models usually cannot be guaranteed. The algorithms given in Section 4 are from Yang (2001). Standard formulas for converting a continuous time model to a discrete-time one are well known (Kuo, 1980; Rattan, 1984; Franklin *et al.*, 1990). And optimization based discretization has been widely studied by Chen and Francis (1991), Chen and Francis (1995), Keller and Anderson (1992), Shieh *et al.* (1992) and Rafee *et al.* (1997), which are only applicable for delay-free cases. However, the presence of time delay makes a lot of difference and stability is a crucial issue. The method in Section 5 is based on Wang and Yang (2001).

3. Stability and Robustness of Feedback Systems

Feedback is the main tool for control systems. Stability and robustness are two key issues associated with any feedback design, and are the topics of this chapter. Section 1 addresses internal stability of general interconnected systems and derives a powerful stability condition which is applicable to systems with any feedback and/or feedforward combinations. Section 2 focuses on the conventional unity output feedback configuration and gives the simplifying conditions and Nyquest-like criteria. The plant uncertainties and feedback system robustness to them are discussed in Section 3, both structured and unstructured perturbations are introduced. A special attention is paid to the case where the phase perturbation is limited, a realistic situation.

3.1 Internal Stability

Stability is a fundamental requirement for any feedback systems. In the state-space setting, all signals inside the system can be expressed as a linear combination of its state variables and they are bounded for any bounded inputs if and only if all state variables of the system are bounded for any bounded inputs. However, state-space descriptions of systems may not be available in many cases. In this section, internal stability of interconnected systems is considered from an input-output setting. It is shown that a system consisting only of single-input and single-output (SISO) plants is internally stable if and only if $\Delta \prod_i p_i(s)$ has all its roots in the open left half of the complex plane, where $p_i(s)$ are the denominators of the plant transfer functions and Δ is the system determinant as defined in the Mason's formula. A general theorem is also presented which is applicable to the case where the system may have multi-input and/or multi-output plants. Several typical control schemes are employed as illustrative examples to demonstrate the simplicity and usefulness of these results in internal stability analysis and stabilization synthesis.

3.1.1 Stability Criterions

Let a general interconnected system have n plants with proper transfer function matrices $G_i(s)$, $i = 1, 2, \cdots, n$. Assume that for each i, $G_i(s) =$

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 65-114, 2003.

$A_i^{-1}(s)B_i(s)$ is a left coprime polynomial matrix fraction (PMF). Then $p_i(s) = \det\big(A_i(s)\big)$ is the characteristic polynomial of $G_i(s)$. $G_i(s)$ can be considered (Rosenbrock 1974) to arise from the equations:

$$A_i(s)\xi_i(s) = B_i(s)u_i(s), \tag{3.1a}$$
$$y_i(s) = \xi_i(s), \tag{3.1b}$$

where u_i and y_i are the input and output vectors of plant $G_i(s)$ respectively. n plants in form of (3.1) may be combined to

$$A(s)\xi(s) = B(s)U(s), \tag{3.2a}$$
$$Y(s) = \xi(s), \tag{3.2b}$$

where

$$\begin{aligned}
A(s) &= block\ diag\{A_1(s), A_2(s), \cdots, A_n(s)\},\\
B(s) &= block\ diag\{B_1(s), B_2(s), \cdots, B_n(s)\},\\
\xi &= \begin{bmatrix}\xi_1^T & \xi_2^T & \cdots & \xi_n^T\end{bmatrix}^T,\\
U &= \begin{bmatrix}u_1^T & u_2^T & \cdots & u_n^T\end{bmatrix}^T.
\end{aligned}$$

In an interconnected system, the outputs of some plants may be injected into other plants as inputs. Besides, various exogenous signals (references and disturbances) r_j, $j = 1, 2, \cdots, m$, may also come into the system. Thus, U has the general form:

$$U(s) = CY(s) + DR(s), \tag{3.3}$$

where $R = [r_1 \quad r_2 \quad \cdots \quad r_m]^T$, and C represents the system internal input-output connection between plants, and C and D can be easily read out by inspection of the plants' interconnections.

Combining (3.2) and (3.3) yields

$$(A(s) - B(s)C)\xi(s) = B(s)DR(s). \tag{3.4}$$

Definition 3.1.1. *The interconnected system described in (3.2) and (3.3) is called internally stable if and only if the system characteristic polynomial* $p_c(s)$ *defined by*

$$p_c(s) = det(A(s) - B(s)C) \tag{3.5}$$

has all its roots in the open left half of the complex plane.

Let $G(s) = A^{-1}(s)B(s) = block\ diag\{G_1, G_2, \cdots, G_n\}$. To proceed with stability analysis, we need a well-posedness assumption.

Assumption 3.1.1 $\det(I - G(\infty)C) \neq 0$.

This assumption ensures the properness of all elements of the transfer function matrix from the exogenous signal vector $R(s)$ to the internal signal vector $\xi(s)$. To see this, one notes that $A(s) - B(s)C = A(s)(I - G(s)C)$ is nonsingular under Assumption 3.1.1, and has from (3.4)

$$\begin{aligned}\xi(s) &= (A(s) - B(s)C)^{-1}B(s)DR(s)\\ &= (I - G(s)C)^{-1}(s)G(s)DR(s).\end{aligned}$$

The transfer matrix between $\xi(s)$ and $R(s)$ meets

$$\lim_{s\to\infty}\left[(I - G(s)C)^{-1}G(s)D\right] = (I - G(\infty)C)^{-1}G(\infty)D < \infty$$

for proper $G(s)$. Obviously, Assumption 3.1.1 is true if $G(s)$ is strictly proper, i.e., all plants $G_i(s)$ are strictly proper.

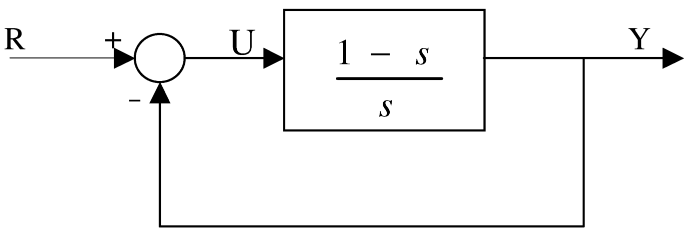


Fig. 3.1. System well-posedness

Assumption 3.1.1 is necessary to rule out any possibility of improper closed-loop transfer functions resulting from proper G. For example, the system in Figure 3.1 yields an improper transfer function,

$$H_{YR} = \frac{\frac{1-s}{s}}{1 + \frac{1-s}{s}} = 1 - s,$$

though $G(s) = \frac{1-s}{s}$ is proper. For this case, one sees that $U = R - Y$ and $C = -1$, which leads to $1 - G(s)C = 1 + G = \frac{1}{s}$ and $\det(1 - G(\infty)C) = 0$, violating Assumption 3.1.1.

Recall that a rational function is stable if it is proper and all its poles have negative real parts. The properness has to be included, otherwise, a bounded input may generate an unbounded output even though all the poles of the rational function have negative real parts. Take the above example again for instance, one has $p_c(s) = det(A(s) - B(s)C) = (s+1-s) = 1$ for which there is no pole and thus no unstable pole. But $H_{YR} = 1 - s$, for the bounded input, say, $r(t) = \sin(t^2)$, the output response under zero initial condition is $y(t) = \sin(t^2) - 2t\cos(t^2)$ and unbounded.

Refer back to the interconnected system described in (3.2) and (3.3), for ease of future reference, we define the system connection matrix as

$$W(s) = I - G(s)C, \tag{3.6}$$

and $p_c(s)$ in (3.5) can be rewritten as

$$\begin{aligned} p_c(s) &= det(A(s))det(I - A^{-1}(s)B(s)C) \\ &= det(I - G(s)C)\prod_{i=1}^{n} p_i(s) \\ &= \det(W(s))\prod_{i=1}^{n} p_i(s). \end{aligned} \tag{3.7}$$

Hence, we have the following result.

Theorem 3.1.1. *Under Assumption 3.1.1, a linear time-invariant interconnected system is internally stable if and only if $p_c(s)$ in (3.7) has all its roots in the open left half of the complex plane.*

The system which Theorem 3.1.1 can be applied to can contain multi-input plants. In such a case, it is here called the system with vector signals since its input and/or output are vector signals. On the other hand, if a linear time-invariant interconnected system consists only of SISO plants $G_i(s)$, then such a system is here called the system with scalar signals because all the signals in the system is scalar. For such systems, we are going to develop an elegant stability criterion which makes use of plants' characteristic polynomials and the system determinant only without evaluating $\det W(s)$ (which may be difficult to calculate for large-scale systems). This criterion may also be applied to systems with vector signals if all multi-input and/or multi-output plants involved in the system are viewed to consist of several SISO subsystems and are represented by their coprime polynomial matrix fractions (PMFs) or minimal state-space realizations in the signal flow graph of the system.

For a system with scalar signals, we first draw its signal flow graph. Recall that a signal flow graph consists of nodes and directed branches. A node performs the functions of addition of all incoming signals and then transmission of it to all outgoing branches. A branch connected between two nodes acts as a one-way signal multiplier and the multiplication factor is the transfer function of the corresponding plant. A loop is a closed path that starts at a node and ends at the same node. We say that a signal goes through a loop if and only if it passes through all nodes and branches in the loop once. If two loops have neither common nodes nor branches, then they are said to be nontouched. The loop gain for a loop is the product of all transfer functions of the plants in the loop. The system determinant Δ is defined (Mason 1956) by

$$\Delta = 1 - \sum_i F_{1i} + \sum_j F_{2j} - \sum_k F_{3k} + \cdots , \tag{3.8}$$

where F_{1i} are the loop gains, F_{2j} are the products of 2 non-touching loop gains, F_{3k} are products of 3 non-touching loop gains, $\cdots$. The following is the major result of this section.

Theorem 3.1.2. *Let an interconnected system consist of n SISO plants $G_i(s)$, $i = 1, 2, \cdots, n$, and the characteristic polynomial of $G_i(s)$ be $p_i(s)$ (simply the transfer function denominator) for each i. Suppose $\Delta(\infty) \neq 0$. Then, the system is internally stable if and only if*

$$p_c(s) = \Delta(s) \prod_{i=1}^{n} p_i(s) \tag{3.9}$$

has all its roots in the open left half of the complex plane.

Some insights may be drawn from the above theorem. Firstly, any pole-zero cancellations will be revealed by $p_c(s)$ in (3.9). To see this, let $p(s)$ be a least common denominator of all the loop gains after pole-zero cancellations and $p_o(s) = \prod_i p_i(s)$. Then $p(s)$ divides $p_o(s)$ and the quotient contains the poles of some plants that have been cancelled by other plant zeros but still appear in $p_c(s)$ as the closed-loop poles. Secondly, one may see that any unstable poles in the plants outside of the loop will be retained as the roots of $p_c(s)$.

It should be pointed out that Theorem 3.1.1 can actually be applied to systems with scalar signals too, since the scalar signal is a special case of a vector signal. However, Theorem 3.1.2 is recommended in the case of scalar signals as it is much simpler to use than Theorem 3.1.1. The internal stability of several typical control schemes is now investigated with the help of these results.

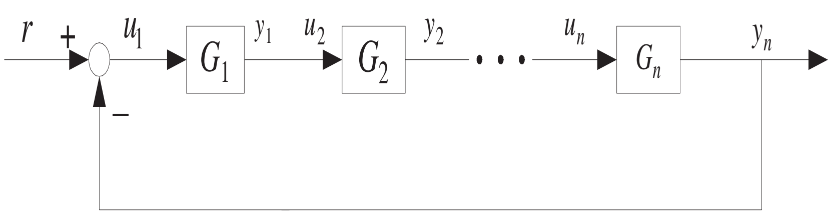


Fig. 3.2. Single-loop Feedback System

Example 3.1.1. Consider the single loop feedback system shown in Figure 3.2. It follows that

$$W = I - GC = I - diag\{G_i\} \begin{bmatrix} 0 & 0 & 0 & \cdots & -I \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 & 0 & \cdots & G_1 \\ -G_2 & I & 0 & \cdots & 0 \\ 0 & -G_3 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}.$$

Post-multiplying the i-th column by $G_iG_{i-1}\cdots G_1$ and subtracting it from the n-th column, $i = 1, 2, \cdots, n-1$, yields

$$\det(W) = \det \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -G_2 & I & 0 & \cdots & G_2G_1 \\ 0 & -G_3 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}$$
$$= \cdots$$
$$= det[I + G_nG_{n-1}\cdots G_1].$$

Thus, it follows from Theorem 3.1.1 that the system is internally stable if and only if $p_c(s) = \det(I + G_nG_{n-1}\cdots G_1)\prod_{i=1}^n p_i(s)$ has all its roots in the open left half of the complex plane, where $p_i(s)$ are the characteristic polynomials of $G_i(s)$ respectively. ◊

By using a different indexing on G_i or the matrix identity:

$$\det \begin{bmatrix} I & Y \\ -X & I \end{bmatrix} = \det(I + XY) = \det[I + YX]$$

for compatible X and Y, we obtain

$$\begin{aligned} p_c(s) &= \det[I + G_nG_{n-1}\cdots G_1]\textstyle\prod_{i=1}^n p_i(s) \\ &= \det[I + G_{n-1}\cdots G_1G_n]\textstyle\prod_{i=1}^n p_i(s) \\ &\cdots \\ &= \det[I + G_1G_n\cdots G_2]\textstyle\prod_{i=1}^n p_i(s). \end{aligned}$$

Example 3.1.2. Consider the Smith predictor control system depicted in Figure 3.3. Assume that all the plants in the system are of SISO. Let $g(s) = g_0(s)e^{-Ls}$, $g_0(s) = b(s)/a(s)$ and $k(s) = \beta(s)/\alpha(s)$ and be coprime polynomial fractions. The system consists of 4 plants, $g(s)$, $g_0(s)$, e^{-Ls} and $k(s)$. Their characteristic polynomials are $a(s)$, $a(s)$, 1 and $\alpha(s)$ respectively, so that $\prod_{i=1}^4 p_i(s) = a^2(s)\alpha(s)$. To find the system determinant $\Delta(s)$, one easily

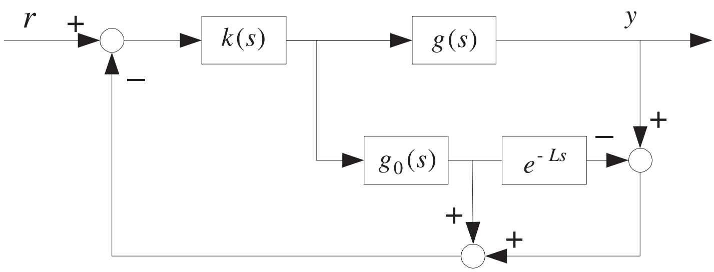


Fig. 3.3. Smith Predictor Control System

sees that the system has three loops which pass through (k, g_0), (k, g_0, e^{-Ls}) and (k, g) respectively. Note that all these loops have the common branch $k(s)$ and thus they are touched. It follows that $\Delta = 1+kg_0+kg-kg_0e^{-Ls} = 1+kg_0$. According to (3.9), $p_c(s)$ for this Smith system is

$$p_c(s) = \Delta(s)\prod_{i=1}^{4} p_i(s) = \big(b(s)\beta(s) + a(s)\alpha(s)\big)a(s).$$

It is well known that $\big(b(s)\beta(s) + a(s)\alpha(s)\big)$ can be assigned to an arbitrary stable polynomial by a proper choice of $\alpha(s)$ and $\beta(s)$ in $k(s)$ for coprime $a(s)$ and $b(s)$. Hence, the system is internally stabilizable if and only if $a(s)$ is a stable polynomial, or if and only if the process $g(s)$ is stable. $\Diamond$

Example 3.1.3. Consider the input-output feedback system shown in Figure 3.4. Let $g(s) = b(s)/a(s)$ be a coprime polynomial fraction, $p_f(s)$ and $q(s)$ be two given polynomials, and $f(s) = a(s) - p_f(s)$. Polynomials $k(s)$ and $h(s)$ are determined by solving the polynomial equation:

$$k(s)a(s) + h(s)b(s) = q(s)f(s).$$

To analyze the system with our method, note that there are two touched loops in Figure 3.4 whose gains are $k(s)/q(s)$ and $g(s)h(s)/q(s)$. It follows that

$$\begin{aligned} p_c(s) &= \left[1 - \frac{k(s)}{q(s)} - g(s)\frac{h(s)}{q(s)}\right] q(a)a(s) \\ &= q(s)a(s) - k(s)a(s) - h(s)b(s) = q(s)p_f(s), \end{aligned}$$

where both $q(s)$ and $p_f(s)$ are specified by the user. Thus, all the poles of the system can be assigned arbitrarily. $\Diamond$

3.1.2 Proof of Theorem 3.1.2

For a system with scalar signals, all plants in the system have SISO transfer functions. To emphasize this SISO property, we replace $G_i(s)$ with $g_i(s)$.

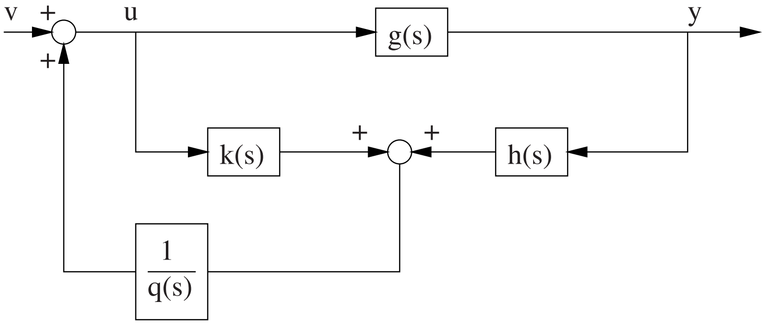


Fig. 3.4. Pole-assignment Control System

Consider the matrix C which represents the input-output connection between plants. If the output of some plant is directly fed back as one of its inputs, we view this feedback path as an additional plant with the unity gain. With this convention, no output of a plant is an input to itself and thus $c_{ii} = 0$ for the matrix C . It follows that C now reads as

$$c_{ij} = \begin{cases} 0, \; i = j; \\ 1, \; i \neq j, \text{ and } g_j' \text{s output is an input to } g_i; \\ 0, \; i \neq j, \text{ and } g_j' \text{s output is not an input to } g_i. \end{cases} \tag{3.10}$$

As a result, the system interconnection matrix $W(s) = I - diag\{G_i(s)\}C$ is determined as

$$w_{ij} = \begin{cases} 1, & i = j; \\ -g_i(s), & i \neq j, \text{ and } g_j' \text{s output is an input to } g_i; \\ 0, & i \neq j, \text{ and } g_j' \text{s output is not an input to } g_i. \end{cases} \tag{3.11}$$

which, in fact, can be obtained directly by inspection of the system without referring to C at all.

What remains for proof of Theorem 3.1.2 is to show that for W given in (3.11) there holds $\det(W(s)) = \alpha\Delta(s)$ for a constant α. Obviously, $\det(W)$

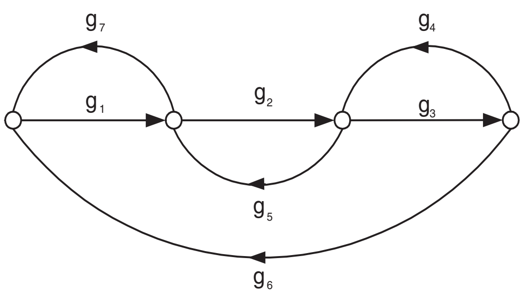


Fig. 3.5. Signal Flow Graph of a Feedback System

involves matrix calculations while Δ deals with the system signal flow graph. To link them together, we will perform a series of expansions on $\det(W)$ and express $\det(W)$ in terms of those items which have clear meanings on the signal flow graph. It should be pointed out that the method we will use to prove Theorem 3.1.2 below is general and applicable to any interconnected system. But for ease of understanding, we shall frequently refer to an example for demonstration when we introduce notations, concepts, derivations and results. The example is the signal flow graph in Figure 3.5, and according to (3.11) its system connection matrix W easily reads as

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ -g_2 & 1 & 0 & 0 & -g_2 & 0 & 0 \\ 0 & -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & 0 & -g_4 & 1 & 0 & 0 & 0 \\ 0 & -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & 0 & -g_6 & 0 & 0 & 1 & 0 \\ -g_7 & 0 & 0 & 0 & -g_7 & 0 & 1 \end{bmatrix}. \tag{3.12}$$

Let $W_{i_1,i_2,\cdots,i_v}$ denote the system connection matrix of the resultant system obtained by removing plants $g_{i_1}, g_{i_2}, \cdots, g_{i_v}$ from the system. For each $g_{i_k}, k = 1, 2, \cdots, v$, the removal of g_{i_k} from the system is equivalent to the system where there is no signal out of g_{i_k} and no signal injected into it. Then it is obvious that matrix $W_{i_1,i_2,\cdots,i_v}$ is identical to the matrix formed by deleting columns $(i_1, i_2, \cdots, i_v)$ and rows $(i_1, i_2, \cdots, i_v)$ from W. For example, if plants g_3 and g_5 are removed from the system in Figure 3.5, then its signal flow graph becomes Figure 3.6. It follows from (3.11) that the system connection matrix is

$$W_{3,5} = \begin{bmatrix} 1 & 0 & 0 & -g_1 & -g_1 \\ -g_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -g_7 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.13}$$

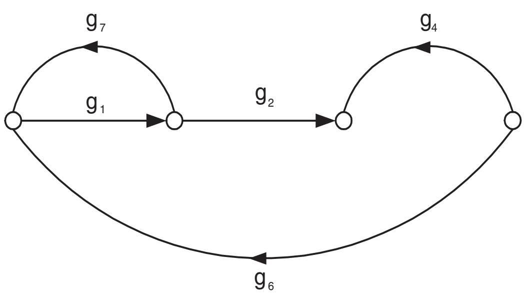


Fig. 3.6. Signal Flow Graph by deleting Plants g_3 and g_5 in Figure 3.5

Indeed, (3.13) is identical to the matrix in (3.12) after deleting the third and fifth columns and the third and fifth rows. Recall (Griffel, 1989) that a minor of a matrix is the determinant of its submatrix. Let $M^{i_1,i_2,\cdots,i_v}_{j_1,j_2,\cdots,j_v}$ be the minor of W formed by deleting rows $(i_1,i_2,\cdots,i_v)$ and columns $(j_1,j_2,\cdots,j_v)$ from W. We have just shown that matrix $W_{i_1,i_2,\cdots,i_v}$ is identical to the matrix formed by deleting columns $(i_1,i_2,\cdots,i_v)$ and rows $(i_1,i_2,\cdots,i_v)$ from W. Thus, the following fact is true.

Fact 1 $\det(W_{i_1,i_2,\cdots,i_v}) = M^{i_1,i_2,\cdots,i_v}_{i_1,i_2,\cdots i_v}$.

The key idea to prove Theorem 3.1.2 is to expand $\det(W)$ step by step and express it in terms of smaller sizes of minors until those minors are reached which correspond to some closed-paths in the system signal graph. For later reference, we define a closed-path as a signal flow path which starts from a plant, may pass through other plants and nodes, and comes back to the same plant. It should be noted that a closed-path may pass through a node many times but a loop (Mason, 1956) is a special closed-path which can pass through a node only once. For example, g_1, g_2, g_3 and g_6 form a loop, and g_1, g_2, g_5 and g_7 form a closed-path but not a loop. Turn back to the expansion of $\det(W)$, and use W in (3.12) as an example. At the first stage, we expand $\det(W)$ by the first column as

$$\begin{aligned}
\det(W) &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ -g_2 & 1 & 0 & 0 & -g_2 & 0 & 0 \\ 0 & -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & 0 & -g_4 & 1 & 0 & 0 & 0 \\ 0 & -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & 0 & -g_6 & 0 & 0 & 1 & 0 \\ -g_7 & 0 & 0 & 0 & -g_7 & 0 & 1 \end{vmatrix} \\
&= \det(W_1) + (-1)^4 \times g_2 \begin{vmatrix} 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & -g_4 & 1 & 0 & 0 & 0 \\ -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -g_7 & 0 & 1 \end{vmatrix} \\
&\quad + (-1)^9 \times g_7 \begin{vmatrix} 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ 1 & 0 & 0 & -g_2 & 0 & 0 \\ -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & -g_4 & 1 & 0 & 0 & 0 \\ -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & -g_6 & 0 & 0 & 1 & 0 \end{vmatrix} \\
&= \det(W_1) + (-1)^4 g_2 M^2_1 + (-1)^9 g_7 M^7_1.
\end{aligned} \tag{3.14}$$

One may note from (3.14) that $\det(W)$ has two items left in addition to $\det(W_1)$. This is because the output of g_1 is injected only into g_2 and g_7. The second stage of the expansion then deals with those nonzero minors (M_1^2 and M_1^7) left from the first stage expansion. The columns on which we expand these minors further are specific and are chosen as follows. For the minor M_1^i, we expand it on the i-th column of W(note that it is the i-th column of W but not of M_1^i). For M_1^2, this rule leads to

$$M_1^2 = \begin{vmatrix} 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & -g_4 & 1 & 0 & 0 & 0 \\ -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -g_7 & 0 & 1 \end{vmatrix}$$

$$= (-1)^4 g_3 \begin{vmatrix} 0 & 0 & 0 & -g_1 & -g_1 \\ -g_4 & 1 & 0 & 0 & 0 \\ 0 & -g_5 & 1 & 0 & 0 \\ -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -g_7 & 0 & 1 \end{vmatrix} + (-1)^6 g_5 \begin{vmatrix} 0 & 0 & 0 & -g_1 & -g_1 \\ 1 & -g_3 & 0 & 0 & 0 \\ -g_4 & 1 & 0 & 0 & 0 \\ -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -g_7 & 0 & 1 \end{vmatrix}$$

$$= (-1)^4 g_3 M_{1,2}^{2,3} + (-1)^6 g_5 M_{1,2}^{2,5}, \tag{3.15}$$

from which one may note once again that $M_{1,2}^{2,3}$ and $M_{1,2}^{2,5}$ are left because the output of g_2 is injected into g_3 and g_5. Similarly, we expand M_1^7 on the 7-th column of W as

$$M_1^7 = \begin{vmatrix} 0 & 0 & 0 & 0 & -g_1 & -g_1 \\ 1 & 0 & 0 & -g_2 & 0 & 0 \\ -g_3 & 1 & -g_3 & 0 & 0 & 0 \\ 0 & -g_4 & 1 & 0 & 0 & 0 \\ -g_5 & 0 & -g_5 & 1 & 0 & 0 \\ 0 & -g_6 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= (-1)^8 g_1 \begin{vmatrix} 1 & 0 & 0 & -g_2 & 0 \\ -g_3 & 1 & -g_3 & 0 & 0 \\ 0 & -g_4 & 1 & 0 & 0 \\ -g_5 & 0 & -g_5 & 1 & 0 \\ 0 & -g_6 & 0 & 0 & 1 \end{vmatrix} = (-1)^8 g_1 M_{1,7}^{7,1}. \tag{3.16}$$

Note from Figure 3.5 that the output of g_7 comes back to the input of g_1 and form a closed-path. The corresponding minor $M_{1,7}^{7,1}$, by Fact 1, is equal to $\det(W_{1,7})$, the determinant of the system connection matrix formed by removing all plants in this closed-path. In general, if the plants involved in a particular minor $M_{j_1,j_2,\cdots,j_v}^{i_1,i_2,\cdots,i_v}$ form a closed-path, i.e., $\{i_1, i_2, \cdots, i_v\} = \{j_1, j_2, \cdots, j_v\}$ (namely, two sets are equal), we stop expanding this minor and express it as $\det(W_{i_1,i_2,\cdots,i_v})$; Otherwise, continue to expand the minor. With this rule, we proceed from (3.15) as

$$M_{1,2}^{2,3} = \begin{vmatrix} 0 & 0 & 0 & -g_1 & -g_1 \\ -g_4 & 1 & 0 & 0 & 0 \\ 0 & -g_5 & 1 & 0 & 0 \\ -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -g_7 & 0 & 1 \end{vmatrix}$$

$$= (-1)^4 g_4 \begin{vmatrix} 0 & 0 & -g_1 & -g_1 \\ -g_5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -g_7 & 0 & 1 \end{vmatrix} + (-1)^6 g_6 \begin{vmatrix} 0 & 0 & -g_1 & -g_1 \\ 1 & 0 & 0 & 0 \\ -g_5 & 1 & 0 & 0 \\ 0 & -g_7 & 0 & 1 \end{vmatrix}$$

$$= (-1)^4 g_4 M_{1,2,3}^{2,3,4} + (-1)^6 g_6 M_{1,2,3}^{2,3,6}, \tag{3.17}$$

and

$$M_{1,2}^{2,5} = \begin{vmatrix} 0 & 0 & 0 & -g_1 & -g_1 \\ 1 & -g_3 & 0 & 0 & 0 \\ -g_4 & 1 & 0 & 0 & 0 \\ -g_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -g_7 & 0 & 1 \end{vmatrix} = (-1)^9 g_7 \begin{vmatrix} 0 & 0 & -g_1 & -g_1 \\ 1 & -g_3 & 0 & 0 \\ -g_4 & 1 & 0 & 0 \\ -g_6 & 0 & 1 & 0 \end{vmatrix}$$

$$= (-1)^9 g_7 M_{1,2,5}^{2,5,7}. \tag{3.18}$$

Since the minors $M_{1,2,5}^{2,5,7}$, $M_{1,2,3}^{2,3,4}$ and $M_{1,2,3}^{2,3,6}$ do not satisfy the condition on stopping expansions, we continue to expand them as

$$M_{1,2,5}^{2,5,7} = \begin{vmatrix} 0 & 0 & -g_1 & -g_1 \\ 1 & -g_3 & 0 & 0 \\ -g_4 & 1 & 0 & 0 \\ -g_6 & 0 & 1 & 0 \end{vmatrix} = (-1)^6 g_1 \begin{vmatrix} 1 & -g_3 & 0 \\ -g_4 & 1 & 0 \\ -g_6 & 0 & 1 \end{vmatrix}$$

$$= (-1)^6 g_1 M_{1,2,5,7}^{2,5,7,1} = (-1)^6 g_1 \det(W_{1,2,5,7}), \tag{3.19}$$

where it has been noticed that plant g_1, g_2, g_5 and g_7 form a closed-path, which is clear from Figure 3.5 and also from the fact $\{1,2,5,7\} = \{2,5,7,1\}$, and no further expansion on $W_{1,2,5,7}$ will be carried out. Similarly,

$$M_{1,2,3}^{2,3,4} = (-1)^4 g_5 \begin{vmatrix} 0 & -g_1 & -g_1 \\ 0 & 1 & 0 \\ -g_7 & 0 & 1 \end{vmatrix} = (-1)^4 g_5 M_{1,2,3,4}^{2,3,4,5},$$

$$= (-1)^4 g_5 (-1)^5 g_7 \begin{vmatrix} -g_1 & -g_1 \\ 1 & 0 \end{vmatrix} = (-1)^9 g_5 g_7 M_{1,2,3,4,5}^{2,3,4,5,7}$$

$$= (-1)^9 g_5 g_7 (-1)^4 g_1 M_{1,2,3,4,5,7}^{2,3,4,5,7,1} = (-1)^{13} g_5 g_7 g_1 \det(W_{1,2,3,4,5,7}); \tag{3.20}$$

$$M_{1,2,3}^{2,3,6} = (-1)^5 g_1 \begin{vmatrix} 1 & 0 & 0 \\ -g_5 & 1 & 0 \\ 0 & -g_7 & 1 \end{vmatrix} = (-1)^5 g_1 M_{1,2,3,6}^{2,3,6,1} = (-1)^5 g_1 \det(W_{1,2,3,6}), \tag{3.21}$$

where all the minors left have reached the corresponding closed-paths. To be more clear, we substitute expressions (3.21), (3.20), (3.19), (3.18), (3.17), (3.16), (3.15) into (3.14), and the expanding procedure looks as

$$
\begin{aligned}
&\det(W) \\
&= \det(W_1) + (-1)^4 g_2 M_1^2 + (-1)^9 M_1^7 \\
&= \det(W_1) + (-1)^4 g_2 [(-1)^4 g_3 M_{1,2}^{2,3} + (-1)^6 g_5 M_{1,2}^{2,5}] + (-1)^9 g_7 (-1)^8 g_1 M_{1,7}^{7,1} \\
&= \det(W_1) + (-1)^4 g_2 \{(-1)^4 g_3 [(-1)^4 g_4 M_{1,2,3}^{2,3,4} + (-1)^6 g_6 M_{1,2,3}^{2,3,6}] \\
&\quad +(-1)^6 g_5 (-1)^9 g_7 M_{1,2,5}^{2,5,7}\} + (-1) g_1 g_7 \det(W_{1,7}) \\
&= \det(W_1) + (-1)^4 g_2 \{(-1)^4 g_3 [(-1)^4 g_4 (-1)^4 g_5 M_{1,2,3,4}^{2,3,4,5} \\
&\quad +(-1)^6 g_6 (-1)^5 g_1 M_{1,2,3,6}^{2,3,6,1}] + (-1)^6 g_5 (-1)^9 g_7 (-1)^6 g_1 M_{1,2,5,7}^{2,5,7,1}\} \\
&\quad +(-1) g_1 g_7 \det(W_{1,7}) \\
&= \det(W_1) + (-1)^4 g_2 \{(-1)^4 g_3 [(-1)^4 g_4 (-1)^4 g_5 (-1)^5 g_7 M_{1,2,3,4,5}^{2,3,4,5,7} \\
&\quad +(-1)^6 g_6 (-1)^5 g_1 \det(W_{1,2,3,6})] + (-1)^6 g_5 (-1)^9 g_7 (-1)^6 g_1 \det(W_{1,2,5,7})\} \\
&\quad +(-1) g_1 g_7 \det(W_{1,7}) \\
&= \det(W_1) \\
&\quad +(-1)^4 g_2 \{(-1)^4 g_3 [(-1)^4 g_4 (-1)^4 g_5 (-1)^5 g_7 (-1)^4 g_1 \det(W_{1,2,3,4,5,7}) \\
&\quad +(-1)^6 g_6 (-1)^5 g_1 \det(W_{1,2,3,6})] + (-1)^6 g_5 (-1)^9 g_7 (-1)^6 g_1 \det(W_{1,2,5,7})\} \\
&\quad +(-1) g_1 g_7 \det(W_{1,7}) \\
&= \det(W_1) \\
&\quad +(-1)^5 g_2 g_3 g_4 g_5 g_7 g_1 \det(W_{1,2,3,4,5,7}) + (-1)^3 g_2 g_3 g_6 g_1 \det(W_{1,2,3,6}) \\
&\quad +(-1)^3 g_2 g_5 g_7 g_1 \det(W_{1,2,5,7}) + (-1) g_7 g_1 \det(W_{1,7}).
\end{aligned}
$$

At a result, $\det(W)$ has been expressed by the summation of $\det(W_1)$ and $\det(W_{i_1,i_2,\cdots,i_v})$, where plants $(i_1, i_2, \cdots, i_v)$ form closed-paths which pass through plant g_1.
Let $\delta_i(1, \cdots)$, $i = 1, 2, \cdots, n_1$, be all the distinct closed-paths each of which passes through g_1, $F_{\delta_i(1,\cdots)}$ the gain of the closed-path $\delta_i(1, \cdots)$, v_{δ_i} the total number of plants involved in the closed-path δ_i, and W_{δ_i} the system connection matrix formed by removing all the plants involved in the closed-path δ_i from the signal graph. With the above expanding procedure, we can establish the following lemma for a general connection matrix W.

Lemma 3.1.1. *Any system connection matrix W can be expanded as*

$$\det(W) = \det(W_1) - \sum_{i=1}^{n_1} F_{\delta_i(1,\cdots)} \det(W_{\delta_i(1,\cdots)}). \tag{3.22}$$

Proof. A system connection matrix W has the form of

$$W = [w_{ij}]_{n\times n} = \begin{bmatrix} 1 & * \cdots * \\ * & \\ \vdots & W_1 \\ * & \end{bmatrix},$$

where, for $i \neq 1$, $w_{i1} = g_i$ if the output of plant g_1 is one of the inputs of plant g_i; otherwise, $w_{i1} = 0$. Expanding $\det(W)$ by its first column gives

$$\det(W) = \det(W_1) + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} M_1^{i_{\alpha_1}}, \tag{3.23}$$

where the output of plant g_1 is an input to plant $g_{i_{\alpha_1}}$. Next, we expand $M_1^{i_{\alpha_1}}$ by the i_{α_1}-th column in W as

$$M_1^{i_{\alpha_1}} = \sum_{i_{\alpha_2}} (-1)^{p_{\alpha_2}+1} g_{i_{\alpha_2}} M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}},$$

where the output of plant $g_{i_{\alpha_1}}$ is an input to plant $g_{i_{\alpha_2}}$. If $i_{\alpha_2} = 1$, i.e., plant $g_{i_{\alpha_1}}$ and plant g_1 form a closed-path, then stop expanding $M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}}$ and we have

$$M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}} = M_{1,i_{\alpha_1}}^{1,i_{\alpha_1}} = \det(W_{1,i_{\alpha_1}}).$$

Compared with the Expansion Theorem (Griffel, 1989) of a determinant carefully, the term $(-1)^{p_{\alpha_1}+p_{\alpha_2}}$ is equal to $(-1)^{\gamma}$, where $\gamma = 1$ is the minimal number of permutations of $\{i_{\alpha_1}, 1\}$ defined by Griffel (1989). Otherwise, for those $i_{\alpha_2} \neq 1$, expanding $M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}}$ by the i_{α_2}-th column in W yields

$$M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}} = \sum_{i_{\alpha_3}} (-1)^{p_{\alpha_3}+1} g_{i_{\alpha_3}} M_{1,i_{\alpha_1},i_{\alpha_2}}^{i_{\alpha_1},i_{\alpha_2},i_{\alpha_3}},$$

where the output of plant $g_{i_{\alpha_2}}$ is one of the inputs of plant $g_{i_{\alpha_3}}$. Again, if $i_{\alpha_3} = 1$, i.e., plants $g_{i_{\alpha_1}}, g_{i_{\alpha_2}}$ andg_1 form a closed-path, then stop expanding and we have

$$M_{1,i_{\alpha_1},i_{\alpha_2}}^{i_{\alpha_1},i_{\alpha_2},i_{\alpha_3}} = M_{1,i_{\alpha_1},i_{\alpha_2}}^{1,i_{\alpha_1},i_{\alpha_2}} = \det(W_{1,i_{\alpha_1},i_{\alpha_2}}),$$

where $(-1)^{p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}} = (-1)^{\gamma}$, $\gamma = 2$ is the minimal number of permutations of $\{i_{\alpha_1}, i_{\alpha_2}, 1\}$ defined by Griffel (1989). Otherwise, for those $i_{\alpha_3} \neq 1$, continue to expand $M_{1,i_{\alpha_1},i_{\alpha_2}}^{i_{\alpha_1},i_{\alpha_2},i_{\alpha_3}}$ by the i_{α_3}-th column in W. It follows from this procedure that

$$
\begin{aligned}
&\det(W)\\
&= \det(W_1) + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} M_1^{i_{\alpha_1}}\\
&= \det(W_1) + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} \sum_{i_{\alpha_2}} (-1)^{p_{\alpha_2}+1} g_{i_{\alpha 2}} M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}}\\
&= \det(W_1) + \sum_{i_{\alpha_2}=1, v_{\delta_i}=2} (-1)^{p_{\alpha_1}+p_{\alpha_2}+2} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} \sum_{i_{\alpha_2}\neq 1} (-1)^{p_{\alpha_2}+1} g_{i_{\alpha_2}} M_{1,i_{\alpha_1}}^{i_{\alpha_1},i_{\alpha_2}}\\
&= \det(W_1) + \sum_{i_{\alpha_2}=1, v_{\delta_i}=2} (-1)^{p_{\alpha_1}+p_{\alpha_2}+2} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} \sum_{i_{\alpha_2}\neq 1} (-1)^{p_{\alpha_2}+1} g_{i_{\alpha_2}} \sum_{i_{\alpha_3}} (-1)^{p_{\alpha 3}+1} g_{i_{\alpha_3}} M_{1,i_{\alpha_1},i_{\alpha_2}}^{i_{\alpha_1},i_{\alpha_2},i_{\alpha_3}}\\
&= \det(W_1) + \sum_{i_{\alpha_2}=1, v_{\delta_i}=2} (-1)^{p_{\alpha_1}+p_{\alpha_2}+2} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \sum_{i_{\alpha_3}=1, v_{\delta_i}=3} (-1)^{p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+3} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \sum_{i_{\alpha_1}} (-1)^{p_{\alpha_1}+1} g_{i_{\alpha_1}} \sum_{i_{\alpha_2}\neq 1} (-1)^{p_{\alpha_2}+1} g_{i_{\alpha_2}} \sum_{i_{\alpha_3}\neq 1} (-1)^{p_{\alpha_3}+1} g_{i_{\alpha_3}} M_{1,i_{\alpha_1},i_{\alpha_2}}^{i_{\alpha_1},i_{\alpha_2},i_{\alpha_3}}\\
&\quad \cdots\\
&= \det(W_1) + \sum_{i_{\alpha_2}=1, v_{\delta_i}=2} (-1)^{p_{\alpha_1}+p_{\alpha_2}+2} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \sum_{i_{\alpha_3}=1, v_{\delta_i}=3} (-1)^{p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+3} F_{\delta_i} \det(W_{\delta_i})\\
&\quad + \cdots + \sum_{i_{\alpha_{\bar v}}=1, v_{\delta_i}=\bar v} (-1)^{p_{\alpha_1}+p_{\alpha_2}+\cdots+p_{\alpha_x}+x} F_{\delta_i} \det(W_{\delta_i})\\
&= \det(W_1) + \sum_{v_{\delta_i}} (-1)^{p_{\alpha_1}+p_{\alpha_2}+\cdots+p_{\alpha_v}+v} F_{\delta_i} \det(W_{1,i_{\alpha_1},\cdots,i_{\alpha_{v-1}}}),
\end{aligned}
$$

where $\bar v = \max\{v_{\delta_i}, i = 1, 2, \cdots, n_1\}$ and plants $(i_{\alpha_1}, i_{\alpha_2}, \cdots, i_{\alpha_{v-1}}, 1)$ form a closed-path which passes through plant g_1. $(-1)^{p_{\alpha_1}+p_{\alpha_2}+\cdots+p_{\alpha_v}} = (-1)^\gamma$, where $\gamma = v - 1$ is the minimal number of that permutations of $\{i_{\alpha_1}, i_{\alpha_2}, \cdots, i_{\alpha_{v-1}}, 1\}$ (Griffel, 1989). Hence (3.22) is true. □

We now turn to consider the system determinant Δ and attempt to expand it into a similar form to (3.22). Let $\rho(k_1, k_2, \cdots, k_{i_v})$ denote a loop which contains plants $g_i, i = k_1, k_2, \cdots, k_{i_v}$, and then $F_{(k_1,k_2,\cdots,k_{i_v})} \triangleq g_{k_1} g_{k_2} \cdots g_{k_{i_v}}$ is the gain of this loop. For ease of reference, we may also denote a loop by

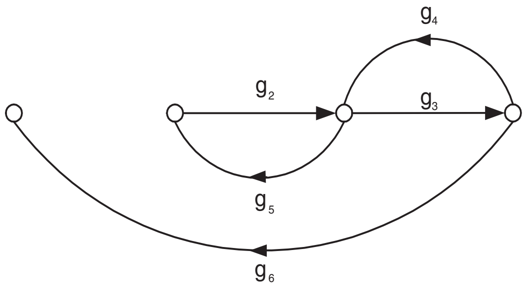


Fig. 3.7. Signal Flow Graph for $\bar{\Delta}_{\rho_1}$

ρ_i in case that the involved individual plants in the loop are clear from the context. For a loop ρ_i, let $\bar{\Delta}_{\rho_i}$ denote the system determinant for the system formed by removing loop ρ_i. Here, it should be pointed out that removing a loop from the system means that not only all the plants but also all the nodes in the loop are removed from the system. Let Δ_{ρ_i} denote the system determinant for the system formed by removing all the plants in loop ρ_i but retaining all the relevant nodes. For example, $\rho_1 = \rho(1, 7)$ is a loop of the system shown in Figure 3.5, then the graph formed by removing ρ_1 is shown in Figure 3.7 and the graph by removing all the plants in ρ_1 only is shown in Figure 3.8. Furthermore, $\bar{\Delta}_{\rho_1} = 1 - g_3g_4$ and $\Delta_{\rho_1} = 1 - g_2g_5 - g_3g_4$. In general, $\bar{\Delta}_{\rho_i} \neq \Delta_{\rho_i}$.

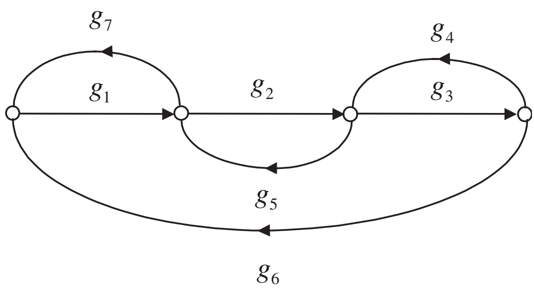


Fig. 3.8. Signal Flow Graph for Δ_{ρ_1}

If plant g_1 is removed from a given system with its system determinant Δ, the system determinant of the resultant system is denoted by Δ_1. Then, it follows from (3.8) that Δ may be decomposed into two parts: one is independent of g_1 and the system determinant Δ_1 of the system formed by removing plant g_1; and other is related to plant g_1, i.e. $\Delta - \Delta_1 = -g_1 Z$ for some Z. For example, consider the system determinant shown in Figure 3.5:

$$\begin{aligned}\Delta &= 1 - \sum_i F_{1i} + \sum_j F_{2j} = 1 - g_1 g_7 - g_2 g_5 - g_3 g_4 - g_1 g_2 g_3 g_6 + g_1 g_7 g_3 g_4 \\ &= (1 - g_2 g_5 - g_3 g_4) - g_1 g_2 g_3 g_6 - g_1 g_7 + g_1 g_7 g_3 g_4. \end{aligned} \tag{3.24}$$

The signal flow graph after removing g_1 is shown in Figure 3.9 and its system determinant is $\Delta_1 = 1 - g_3 g_4 - g_2 g_5$, which is, of course, independent of g_1. The remaining part which is related to plant g_1 is

$$\Delta - \Delta_1 = -g_1 g_2 g_3 g_6 - g_1 g_7 + g_1 g_7 g_3 g_4 = -g_1 Z$$

for $Z = g_2 g_3 g_6 + g_7 - g_7 g_3 g_4$. Look into each item in $g_1 Z$ and one sees that it is related to one and only one loop containing g_1, which is $F_{\rho_1} = g_1 g_7$ or $F_{\rho_2} = g_1 g_2 g_3 g_6$. More precisely, $g_1 Z$ can be written as $g_1 Z = g_1 g_2 g_3 g_6(1) + g_1 g_7 (1 - g_3 g_4) = F_{\rho(1,2,3,6)} \bar{\Delta}_{\rho(1,2,3,6)} + F_{\rho(1,7)} \bar{\Delta}_{\rho(1,7)}$. For a general system, $g_1 Z$ is the sum of loop gains and the products of those nontouching loop gains for those loops containing g_1, and in all these items the plant g_1 appears once and only once. This fact is due to Mason (1956).

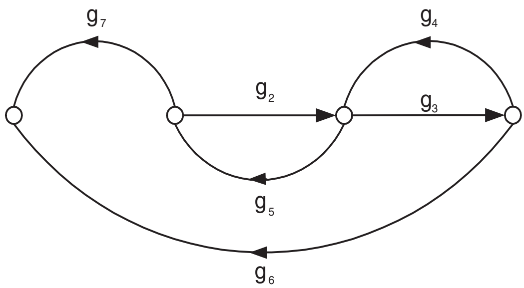


Fig. 3.9. Signal Flow Graph corresponding to Δ_1

Lemma 3.1.2. *$\Delta = \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \bar{\Delta}_{\rho_i(1,\cdots)}$, where $\rho_i(1,\cdots)$ are the loops each of which contains plant g_1.*

Now, we consider the relationship between Δ_{ρ_i} and $\bar{\Delta}_{\rho_i}$. Assume that loop ρ_j has a common node with ρ_i, then F_{ρ_j} will not appear in $\bar{\Delta}_{\rho_i}$. It is because this node has been removed from the graph when we calculate $\bar{\Delta}_{\rho_i}$ and the relevant plants involved in the original ρ_j does not form a loop in the new system anymore. However, F_{ρ_j} will appear in Δ_{ρ_i} because only the plants involved in ρ_i are removed from the original graph when we calculate Δ_{ρ_i}, but the relevant nodes are retained so that ρ_j is still a loop in the resultant graph. For instance, consider the system in Figure 3.5. For $\rho_i = \rho(1,7)$, we have $\Delta_{\rho_i} = 1 - g_2 g_5 - g_3 g_4$ and $\bar{\Delta}_{\rho_i} = 1 - g_3 g_4$. The difference between them is $g_2 g_5$, the gain of the $\rho_j = \rho(2,5)$ which is touched with loop ρ_i only by one node. It follows that $\bar{\Delta}_{\rho_i} - \Delta_{\rho_i} = g_2 g_5 = F_{\rho(2,5)}$. One further sees that $F_{\rho_i}(\bar{\Delta}_{\rho_i} - \Delta_{\rho_i}) = F_{\rho_i} F_{\rho_j}$ is the gain of the closed-path which is formed by $\rho(1,7)$ and $\rho(2,5)$ that have a common node. For a general system, let ρ_j^i ($\rho_k^{i,j}, \cdots$, respectively) be the loops each of which has a common node with $\rho_{i(1,\cdots)}$ (ρ_i or $\rho_j, \cdots$, respectively) but does not contain the plant g_1. It follows from Lemma 3.1.2 that

$$\begin{aligned}
\Delta &= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \bar{\Delta}_{\rho_i(1,\cdots)} \\
&= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} (\Delta_{\rho_i(1,\cdots)} + \bar{\Delta}_{\rho_i(1,\cdots)} - \Delta_{\rho_i(1,\cdots)}) \\
&= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \Delta_{\rho_i(1,\cdots)} - \sum_i F_{\rho_i(1,\cdots)} (\bar{\Delta}_{\rho_i(1,\cdots)} - \Delta_{\rho_i(1,\cdots)}) \\
&= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \Delta_{\rho_i(1,\cdots)} - \sum_i F_{\rho_i} [\sum_j F_{\rho_j^i} \bar{\Delta}_{\rho_i \rho_j^i} \\
&\quad - \sum_{j_1, j_2} F_{\rho_{j_1}^i} F_{\rho_{j_2}^i} \bar{\Delta}_{\rho_i \rho_{j_1}^i \rho_{j_2}^i} + \cdots] \\
&= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \Delta_{\rho_i(1,\cdots)} - \sum_i F_{\rho_i} \sum_j F_{\rho_j^i} \Delta_{\rho_i \rho_j^i} \\
&\quad - \sum_i F_{\rho_i} [\sum_j F_{\rho_j^i} (\bar{\Delta}_{\rho_i \rho_j^i} - \Delta_{\rho_i \rho_j^i}) - \sum_{j_1, j_2} F_{\rho_{j_1}^i} F_{\rho_{j_2}^i} \bar{\Delta}_{\rho_i \rho_{j_1}^i \rho_{j_2}^i} + \cdots] \\
&= \Delta_1 - \sum_i F_{\rho_i(1,\cdots)} \Delta_{\rho_i(1,\cdots)} - \sum_i F_{\rho_i} \sum_j F_{\rho_j^i} \Delta_{\rho_i \rho_j^i} \\
&\quad - \sum_i F_{\rho_i} (\sum_{j,k} F_{\rho_j^i} F_{\rho_k^{i,j}} \bar{\Delta}_{\rho_i \rho_j^i \rho_k^{i,j}} + \cdots) \\
&= \cdots,
\end{aligned}$$

where loops $\rho_{j_1}^i$ and$\rho_{j_2}^i$ are nontouched to each other but both touched with loop ρ_i, $\rho_k^{i,j}$ the loop touched with loops ρ_i or ρ_j^i, $\Delta_{\rho_i \rho_j^i \rho_k^{i,j} \cdots}$ the system determinant by removing all the plants in loops $\rho_i, \rho_j^i, \rho_k^{i,j}, \cdots$ from the system. It is noted that the set of the closed-paths $\delta_i(1,\cdots)$ each of which passes through g_1 consists of the loops $\rho_i(1,\cdots)$ and the non-loop closed-paths formed by $\rho_i(1,\cdots)$ plus ρ_j^i , $\rho_i(1,\cdots)$ plus ρ_j^i and $\rho_k^{i,j}$, $\cdots$. Thus, we have the following lemma.

Lemma 3.1.3. $\Delta = \Delta_1 - \sum_i F_{\delta_i(1,\cdots)} \Delta_{\delta_i(1,\cdots)}$.

For example, there exist four closed-paths through plant g_1 in Figure 3.5 and they are $\delta_1(1,7)$, $\delta_2(1,2,5,7)$, $\delta_3(1,2,3,4,5,7)$ and $\delta_4(1,2,3,6)$. The corresponding gains are $F_{\delta_1} = g_1 g_7$, $F_{\delta_2} = g_1 g_2 g_5 g_7$, $F_{\delta_3} = g_1 g_2 g_3 g_4 g_5 g_7$ and $F_{\delta_4} = g_1 g_2 g_3 g_6$. Obviously, $\delta_1 = \rho_1$ and $\delta_4 = \rho_2$ are loops, but δ_2 and δ_3 are not. We have $\Delta_1 = 1 - g_2 g_5 - g_3 g_4$, $\Delta_{\delta_1} = 1 - g_3 g_4 - g_2 g_5$, $\Delta_{\delta_2} = 1 - g_3 g_4$, $\Delta_{\delta_3} = 1$ and $\Delta_{\delta_4} = 1$. It then follows that

$$\begin{aligned}
&\varDelta_1 - F_{\delta_1}\varDelta_{\delta_1} - F_{\delta_2}\varDelta_{\delta_2} - F_{\delta_3}\varDelta_{\delta_3} - F_{\delta_4}\varDelta_{\delta_4} \\
&= (1 - g_3g_4 - g_2g_5) - g_1g_7(1 - g_3g_4 - g_2g_5) - g_1g_2g_5g_7(1 - g_3g_4) \\
&\quad - g_1g_2g_3g_4g_5g_7(1) - g_1g_2g_3g_6(1) \\
&= 1 - g_3g_4 - g_2g_5 - g_1g_7 - g_1g_2g_3g_6 + g_1g_7g_3g_4,
\end{aligned}$$

which, by (3.24), is equal to $\varDelta$, indeed. It may also be verified that

$$\begin{aligned}
F_{\rho_1}(\bar{\varDelta}_{\rho_1} - \varDelta_{\rho_1}) &= (g_1g_7)(g_2g_5) = g_1g_7g_2g_5(1 - g_3g_4 + g_3g_4) \\
&= g_1g_7g_2g_5(1 - g_3g_4) + g_1g_7g_2g_5g_3g_4(1) \\
&= F_{\delta_2}\varDelta_{\delta_2} + F_{\delta_3}\varDelta_{\delta_3},
\end{aligned}$$

and

$$F_{\rho_2}(\bar{\varDelta}_{\rho_2} - \varDelta_{\rho_2}) = g_1g_2g_3g_6(1 - 1) = 0,$$

so that

$$\begin{aligned}
&\varDelta_1 - F_{\rho_1}\bar{\varDelta}_{\rho_1} - F_{\rho_2}\bar{\varDelta}_{\rho_2} \\
&= \varDelta_1 - F_{\rho_1}\varDelta_{\rho_1} - F_{\rho_2}\varDelta_{\rho_2} - F_{\rho_1}(\bar{\varDelta}_{\rho_1} - \varDelta_{\rho_1}) - F_{\rho_2}(\bar{\varDelta}_{\rho_2} - \varDelta_{\rho_2}) \\
&= \varDelta_1 - F_{\delta_1}\varDelta_{\delta_1} - F_{\delta_2}\varDelta_{\delta_2} - F_{\delta_3}\varDelta_{\delta_3} - F_{\delta_4}\varDelta_{\delta_4}.
\end{aligned}$$

With Lemmas 3.1.1 and 3.1.3, we are now in the position to prove Theorem 3.1.2.

Proof of Theorem 3.1.2 By Theorem 3.1.1, the proof will be completed if there holds

$$\varDelta = \det(W). \tag{3.25}$$

We will prove (3.25) by induction. If the system has only one plant, then $\det(W) = 1$ and $\varDelta = 1$, (3.25) is thus true. When the system contains two plants g_1 and g_2, then $n = 2$ and there are only two cases to be considered. Case 1 is that these two plants form a loop, and then its connection matrix is

$$W = \begin{bmatrix} 1 & -g_1 \\ -g_2 & 1 \end{bmatrix},$$

and $\det(W) = 1 - g_1g_2$. From (3.8), $\varDelta = 1 - g_1g_2$. (3.25) thus holds in this case. Case 2 is that these two plants do not form a loop, and then we have

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and $\det(W) = \varDelta = 1$. (3.25) is thus proven for the case of two plants. Assume now that (3.25) holds for systems with $(n-1)$ plants, we need to show that it is also true for the system with n plants. The above assumption implies that $\det(W_1) = \varDelta_1$ for $(n-1)$ plants and $\det(W_{\delta_i}) = \varDelta_{\delta_i}$ for $(n-1)$ plants. Substituting them into (3.22) yields

$$\begin{aligned}\det(W) &= \Delta_1 - \sum_i F_{\delta_{i(1,\cdots)}} \Delta_{\delta_{i(1,\cdots)}} \\ &= \Delta,\end{aligned}$$

where the last equality follows from Lemma 3.1.3. Hence (3.25) holds true for any integer n. □

In this section, two theorems on internal stability of interconnected systems, one for scalar signals and other for vector signals, have been established. They are stated in terms of a single polynomial only. This polynomial is readily found from a given system and can be read off by inspection in many cases. The application to a few typical control systems has shown simplicity and effectiveness of the results.

3.2 Unity Feedback Systems

A typical feedback control system has a plant, a sensor, a controller $K(s)$, and an actuator as its principal components. The control objective is to make the plant output to track the reference (or set-point) $R(s)$ as closely as possible in the presence of possible disturbances $V(s)$. For system analysis and design, the plant, sensor, and actuator are usually combined and viewed as a single object (still called the plant $G(s)$), and this gives rise to the most popular control configuration, the unity (output) feedback system, as depicted in Figure 3.10. We assume throughout the rest of this chapter that $G(s)$ and $K(s)$ are proper and there holds $det(I + G(\infty)K(\infty)) \neq 0$. The system can be described by the input-output relationship:

$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = H(s) \begin{bmatrix} R(s) \\ V(s) \end{bmatrix}.$$

To get $H(s)$, it follows from inspection of Figure 3.10 that

$$E(s) + G(s)U(s) = R(s), \tag{3.26}$$
$$-K(s)E(s) + U(s) = V(s), \tag{3.27}$$

leading via the matrix inversion to

$$H = \begin{bmatrix} I & G(s) \\ -K(s) & I \end{bmatrix}^{-1}, \tag{3.28}$$

and via direct variable elimination to

$$\begin{aligned} E(s) + G(s)(V(s) + K(s)E(s)) &= R(s), \\ -K(s)(R(s) - G(s)U(s)) + U(s) &= V(s), \end{aligned}$$

or

$$H = \begin{bmatrix} (I + GK)^{-1} & -(I + GK)^{-1}G \\ (I + KG)^{-1}K & (I + KG)^{-1} \end{bmatrix}. \tag{3.29}$$

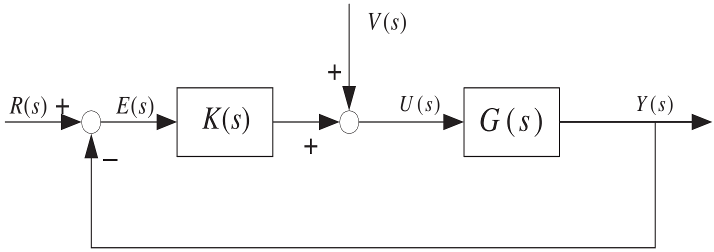


Fig. 3.10. Unity Feedback System

Let $G = D^{-1}N = N_R D_R^{-1}$ be coprime PMFs of G and $K = YX^{-1} = X_L^{-1}Y_L$ be coprime PMFs of K, p_G and p_K be the characteristic polynomials of G and K, respectively.

Theorem 3.2.1. *The following are equivalent:*
(i) The system in Figure 3.10 is internally stable;
(ii) $p_c := p_G p_K \det[I + GK] = p_G p_K \det[I + KG]$ *is a stable polynomial;*
(iii) $\det[DX + NY]$ *is a stable polynomial, or* $\det[X_L D_R + Y_L N_R]$ *is a stable polynomial;*
(iv) $H(s)$ *is stable.*

Proof. $(i) \Longleftrightarrow (ii)$ It follows from Example 3.1.1 for the case of $n = 2$.

$(ii) \Longleftrightarrow (iii)$ Notice $\det[DX + NY] = \det(D)\det[I + GK]\det(X)$, and it is shown in Chapter 2 that $\det(D)$ and $\det(X)$ are equal to p_G and p_K modulo a nonzero constant factor, respectively. Hence $\det[DX+NY]$ is equal to $p_c(s)$ modulo a nonzero constant, and so is $\det[X_L D_R+Y_L N_R]$ by a similar reasoning.

$(iii) \Longleftrightarrow (iv)$ Rewrite H in (3.28):

$$\begin{aligned} H &= \begin{bmatrix} I & D^{-1}N \\ -YX^{-1} & I \end{bmatrix}^{-1} \\ &= \left\{ \begin{bmatrix} D^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix}^{-1} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}, \end{aligned} \tag{3.30}$$

which is a left and right PMF of $H(s)$. For the composite matrix,

$$\begin{bmatrix} DX & N \\ -Y & I \\ X & 0 \\ 0 & I \end{bmatrix},$$

subtract the last block row from the second block row to get

$$\begin{bmatrix} DX & N \\ -Y & 0 \\ X & 0 \\ 0 & I \end{bmatrix},$$

which is ensured to have full column rank due to the assumed coprimeness of X and Y. Thus,

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix}^{-1}$$

is right coprime. Similarly, one can easily show that

$$\begin{bmatrix} DX & N \\ -Y & I \end{bmatrix}^{-1} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$$

is left coprime. Then, the pole polynomial for H is (see Chapter 2)

$$p_H(s) = \det \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix}.$$

Notice

$$\begin{aligned} \det \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix} &= \det \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix} \det \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \\ &= \det \left\{ \begin{bmatrix} DX & N \\ -Y & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \right\} \\ &= \det \begin{bmatrix} DX + NY & N \\ 0 & I \end{bmatrix} \\ &= \det[DX + NY]. \end{aligned}$$

Hence stability of H is equivalent to that of the polynomial $\det[DX + NY]$, and the proof is completed. □

The fact that stable $\det[DX+NY]$ implies stability of H can also be seen in an explicit way as follows. By the following well-known matrix identities,

$$(I + KG)^{-1}K = K(I + GK)^{-1},$$

and

$$\begin{aligned} (I + KG)^{-1} &= I - (I + GK)^{-1}KG \\ &= I - K(I + GK)^{-1}G, \end{aligned}$$

(3.29) may now be rewritten as

$$H(s) = \begin{bmatrix} (I+GK)^{-1} & -(I+GK)^{-1}G \\ K(I+GK)^{-1} & I-K(I+GK)^{-1}G \end{bmatrix}, \tag{3.31}$$

which involves $(I+GK)^{-1}$ only but not $(I+KG)^{-1}$. Substituting $G = D^{-1}N$ and $K = YX^{-1}$ into (3.31) yields

$$H(s) = \begin{bmatrix} X(DX+NY)^{-1}D & -X(DX+NY)^{-1}N \\ Y(DX+NY)^{-1}D & I-Y(DX+NY)^{-1}N \end{bmatrix}.$$

Obviously, stability of the polynomial $\det[DX+NY]$ implies stability of the rational matrix $H(s)$.

The important message conveyed by Theorem 3.3 (iv) is that internal stability of the system is equivalent to (input-output) stability of $H(s)$. But note that all four blocks, $H_{ij}, i, j = 1, 2$, in H, must be stable to ensure internal stability of the feedback system, and stability of all H_{ij} can rule out the possibility of unstable pole-zero cancellation between G and K. Otherwise, unstable pole-zero cancellation may occur in the loop, and the system is not internally stable. Yet, some (but not all) of H_{ij} is stable. This is precisely the problem associated with input-output stability, but can be revealed by internal stability analysis.

Example 3.2.1. Let

$$G(s) = \begin{bmatrix} 0 & \frac{1}{s+1} \\ -\frac{s}{s+2} & \frac{2}{s+2} \end{bmatrix},$$

$$K(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 1 & \frac{1}{s} \end{bmatrix}.$$

We find coprime fractions for $G(s)$ and $K(s)$:

$$G(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -s & 2 \end{bmatrix},$$

$$K(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ o & s \end{bmatrix}^{-1}.$$

Then, $p_G(s) = (s+1)(s+2)$, $p_K(s) = s^2$, and

$$\begin{aligned} \det[I+GK] &= \det \begin{bmatrix} (1+\frac{1}{s+1}) & \frac{1}{s(s+1)} \\ \frac{1}{s+2} & (1+\frac{2}{s(s+2)}) \end{bmatrix} \\ &= (\tfrac{s+2}{s+1})(\tfrac{s^2+2s+2}{s(s+2)}) - \tfrac{1}{s(s+1)(s+2)} \\ &= \tfrac{s^3+4s^2+6s+3}{s(s+1)(s+2)}. \end{aligned}$$

We have

$$p_c(s) = s(s^3 + 4s^2 + 6s + 3),$$

a unstable polynomial, so the system is unstable. One can easily verify that $\det[DX + NY] = p_c(s)$. One also find

$$\begin{aligned} GK &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s(s+1)} \\ \frac{1}{s+2} & \frac{2}{s(s+2)} \end{bmatrix}, \\ H_{21} &= (I + KG)^{-1}K \\ &= \begin{bmatrix} \frac{s^3+3s^2+4s+3}{s(s^3+4s^2+6s+3)} & -\frac{s+2}{s(s^3+4s^2+6s+3)} \\ \frac{s^3+4s^2+7s+4}{s^3+4s^2+6s+3} & \frac{1}{s} \end{bmatrix}, \end{aligned}$$

which is unstable. ◊

Instability of this example is caused by some closed right half plane (RHP) pole-zero cancellation between G and K. Indeed,

$$\begin{aligned} GK &= \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} s & 1 \\ s & 2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}^{-1} \\ &= \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1} \end{aligned}$$

indicates a RHP pole-zero cancellation at $s = 0$ between G and K.

Another cause for instability is that some RHP pole is lost when forming $\det[I + GK]$, as can be seen in the following example.

Example 3.2.2. Let

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+1} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

GK has no pole-zero cancellation as K has neither zero nor pole. Further, one sees that

$$\det[I + GK] = \left(1 + \frac{1}{s+1}\right)^2 = \left(\frac{s+2}{s+1}\right)^2$$

has stable roots only. However, simple calculations give

$$p_c(s) = p_G p_K \det[I + GK] = (s-1)(s+1)^2 \cdot 1 \cdot (\frac{s+2}{s+1})^2 = (s-1)(s+2)^2,$$

a unstable polynomial. Thus, the system is not internally stable. Instability of the system can be clearly exhibited with Figure 3.11, where the unstable element of the plant, $\frac{1}{s-1}$, stays outside any loop and is impossible to be stabilized by any feedback. ◊

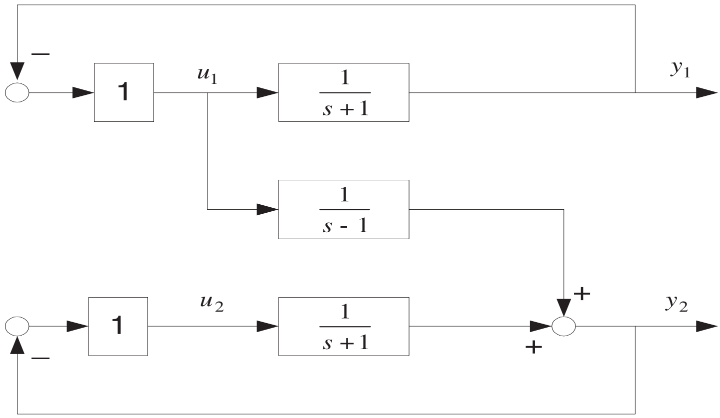


Fig. 3.11. An unstable system

If $K(s)$ and $G(s)$ are both unstable, then it is necessary to check all four blocks of the H-matrix to ensure internal stability. However, if at least one of them is stable, then there is less work to do.

Theorem 3.2.2. *If $K(s)$ is stable, then the feedback system shown in Figure 3.10 is internally stable if and only if $H_{12}(s) = -[I + G(s)K(s)]^{-1}G(s)$ is stable.*

Proof. "only if": Since all H_{ij} are stable, H_{12} is stable, too.

"if": It follows from (3.29) that

$$\begin{aligned} H_{11} &= (I + GK)^{-1} \\ &= (I + GK)^{-1}(I + GK - GK) \\ &= I + H_{12}K \end{aligned}$$

is stable since I, H_{12} and K are so. By (3.31), $H_{21} = K(I + GK)^{-1} = KH_{11}$ is stable, and $H_{22} = I - K(I + GK)^{-1}G = I - KH_{12}$ is stable, too. □

Theorem 3.2.1 leads us to examine the following polynomial matrix equation,

$$D(s)X(s) + N(s)Y(s) = P(s), \tag{3.32}$$

where $D(s) \in R^{p\times p}[s]$, $N(s) \in R^{p\times m}[s]$, and $P(s) \in R^{p\times p}[s]$ are given (with $P(s)$ being stable in contents of control engineering), and $X(s) \in R^{p\times p}[s]$ and $Y(s) \in R^{m\times p}[s]$ are sought to satisfy the equation. Such an equation is called Diophantine equation. Equation (3.32) is linear in X and Y and this enables us to establish the following results.

Theorem 3.2.3. *Consider (3.32).*

(i) There is a solution pair of $X(s)$ and $Y(s)$ if and only if a greatest common left divisor (GCLD) is a left divisor of $P(s)$;

(ii) Let $[D \quad N]$ have full row rank, $\widehat{N}\widehat{D}^{-1}$ be a right coprime PMF of $D^{-1}N$, and (X_0, Y_0) be a particular solution of (3.32), then the general solution of (3.32) is given by

$$X(s) = X_0(s) + \widehat{N}(s)T(s), \tag{3.33a}$$

$$Y(s) = Y_0(s) - \widehat{D}(s)T(s), \tag{3.33b}$$

for an arbitrary polynomial matrix $T(s)$; and

(iii) If $D(s)$ and $N(s)$ are left coprime, then (3.32) always admits a solution and has the general solution in the form of (3.33).

Proof. (i) Let $L(s)$ be a GCLD of $D(s)$ and $N(s)$. Then, it follows that there is a unimodular $U(s)$ such that

$$[D \quad N]U = [L \quad 0]$$

whose first block column gives

$$DU_1 + NU_2 = L \tag{3.34}$$

for some polynomial matrices U_1 and U_2. If $L(s)$ is also a left divisor of $P(s)$, i.e. $P(s) = L(s)\overline{P}(s)$ for some polynomial matrix $\overline{P}$, then post-multiplying (3.34) by $\overline{P}$ yields

$$D(U_1\overline{P}) + N(U_2\overline{P}) = L\overline{P} = P(s),$$

showing that $X(s) = U_1\overline{P}$ and $Y(s) = U_2\overline{P}$ meets (3.32), and thus form a solution pair. Conversely, if (3.32) has a solution (X, Y), it can be written as

$$L\overline{D}X + L\overline{N}Y = P,$$

or

$$L(\overline{D}X + \overline{N}Y) = P,$$

so that L is a left divisor of P.

(ii) Since (X_0, Y_0) is a solution to (3.32), there holds

$$DX_0 + NY_0 = P. \tag{3.35}$$

Subtracting (3.35) from (3.32) gives

$$D(X - X_0) + N(Y - Y_0) = 0,$$

or

$$\begin{bmatrix} D \; N \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \end{bmatrix} = 0. \tag{3.36}$$

Since [D N] is of size $p \times (p+m)$ and rank p, the right null space of [D N] is spanned by a $(p+m) \times m$ matrix $B(s)$ such that [D N]$B = 0$ and B has rank m. The general solution of the homogeneous equation (3.36) can then be expressed as

$$\begin{bmatrix} X - X_0 \\ Y - Y_0 \end{bmatrix} = BT \tag{3.37}$$

for an arbitrary polynomial matrix T. From $D^{-1}N = \widehat{N}\widehat{D}^{-1}$, we have

$$\begin{bmatrix} D \;\; N \end{bmatrix} \begin{bmatrix} \widehat{N} \\ -\widehat{D} \end{bmatrix} = 0.$$

And, due to the assumed coprimeness of $\widehat{N}$ and $\widehat{D}$, $[\widehat{N}^T \; -\widehat{D}^T]^T$ has full rank m. Thus, we can take

$$B = \begin{bmatrix} \widehat{N} \\ -\widehat{D} \end{bmatrix},$$

and substituting it into (3.37) produces the required form (3.33).

(iii) The identity matrix I is a GCLD of coprime D and N, and it is a left divisor of any matrix like $P(s)$. Hence, the result follows from (i) and (ii). □

Among all solutions of (3.32), there must be at least one, denoted by (X_{min}, Y_{min}) such that it has minimum column degrees, i.e.

$$\partial_i \begin{bmatrix} X_{min} \\ Y_{min} \end{bmatrix} \leq \partial_{ci} \begin{bmatrix} X \\ Y \end{bmatrix}$$

for any other solution (X, Y). In view of (3.33), we can obtain it as follows. Take column i of any solution

$$\begin{bmatrix} X \\ Y \end{bmatrix},$$

and subtract from it a polynomial combination of the columns of

$$\begin{bmatrix} \widehat{N} \\ -\widehat{D} \end{bmatrix}$$

so as to reduce its degree as much as possible. Repeat the process for all other columns and the result would be (X_{min}, Y_{min}).

With obvious changes, we may work with the case where (X, Y) is given and (D, N) is sought and get the dual results to Theorem 3.2.3.

Theorem 3.2.3 lays a foundation for stabilization and arbitrary pole assignment for any plant $G(s)$. One may get its left coprime fraction $D^{-1}(s)N(s)$

and specify a suitable $P(s)$ which reflects the designer's control specifications. Then, (3.32) is solved to get a solution (X, Y) and a stabilizer for $G(s)$ is obtained as $K(s) = YX^{-1}$ provided that X is nonsingular and YX^{-1} proper. This procedure actually does more than stabilization or pole assignment because it can assign an entire polynomial matrix $P(s)$ instead of just $\det P(s)$, the characteristic polynomial of the feedback system.

Example 3.2.3. Consider

$$G = \begin{bmatrix} \frac{1}{s} & 1 & 0 \\ \\ 0 & 0 & \frac{1}{s} \end{bmatrix}.$$

Suppose that a controller in the unity feedback system is to be found such that the resulting closed-loop denominator is

$$P(s) = \begin{bmatrix} (s+1)^2 & 0 \\ 0 & s+1 \end{bmatrix}.$$

We write

$$G = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & s & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is left coprime. Take $K = K_1 K_2$ with

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, it follows that

$$GK_1 = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}^{-1} I_2$$

is decoupled. Let

$$K_2 = \begin{bmatrix} \frac{b}{s+a} & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} (s+a) & 0 \\ 0 & 1 \end{bmatrix}^{-1}.$$

Then, (3.32) becomes

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} s+a & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} (s+1)^2 & 0 \\ 0 & s+1 \end{bmatrix},$$

whose solution is $a = 2$, $b = 1$; and $c = 1$ so that the controller is

$$K = K_1 K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+2} & 0 \\ \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which is also proper. ◊

In general, a solution (X, Y) from Diophantine equation (3.32) may not guarantee properness of YX^{-1}. One has to pay a special attention to this problem. Besides, for practical use, systematic numerical algorithms would be desirable, especially, to be included in control system CAD toolbox. We now present such an algorithm.

Given a $p \times m$ proper plant $G(s)$, we obtain its left right fraction $D^{-1}N = G$ with D row reduced. Let $\nu_i, i = 1, 2, \ldots, p$, be the row degrees of $D(s)$ and $\nu = \max\{\nu_1, \nu_2, \ldots, \nu_p\}$. Let μ be the largest controllability index of $G(s)$. We write

$$\begin{aligned} D(s) &= D_0 + D_1 s + \ldots + D_\nu s^\nu, \\ N(s) &= N_0 + N_1 s + \ldots + N_\nu s^\nu, \\ X(s) &= X_0 + X_1 s + \ldots + X_q x^q, \\ Y(s) &= Y_0 + Y_1 s + \ldots + Y_q x^q, \\ P(s) &= P_0 + P_1 s + \ldots + P_{\nu+q} s^{\nu+q}. \end{aligned}$$

Substituting these into (3.32) yields

$$\begin{bmatrix} D_0 & N_0 & 0 & 0 & \cdots & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_\nu & N_\nu & D_{\nu-1} & N_{\nu-1} & \cdots & 0 & 0 \\ 0 & 0 & D_\nu & N_\nu & \cdots & D_0 & N_0 \\ 0 & 0 & 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & D_{\nu-1} & N_{\nu-1} \\ 0 & 0 & 0 & 0 & \cdots & D_\nu & N_\nu \end{bmatrix} \begin{bmatrix} D_0 \\ N_0 \\ D_1 \\ N_1 \\ \vdots \\ D_\nu \\ N_\nu \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ \vdots \\ \vdots \\ P_{\nu+q} \end{bmatrix}. \tag{3.38}$$

Theorem 3.2.4. *Choose an integer q such that $q \geq \mu$ for proper $G(s)$ or $q \geq \mu - 1$ for strictly proper $G(s)$. Specify the closed-loop denominator matrix $P(s)$ to have the desired characteristic polynomial $p_c(s) = \det P(s)$ and characteristic structure, and to meet the condition that*

$$\lim_{s\to\infty} diag\{s^{-\nu_1}, s^{-\nu_2}, \ldots, s^{-\nu_p}\} P(s) diag\{s^{-q}, s^{-q}, \ldots, s^{-q}\}$$

exists and is nonsingular. Then, there exists a proper controller $K(s) = Y(s)X^{-1}(s)$ such that (3.32) holds true.

Proof. See Chen (1984). □

Quite often, one needs to investigate stability (we shall drop "internal" from now on) as well as its margins when the loop gain varies. In this regard, one may view $G(s) \in \mathbb{R}^{m\times m}(s)$ in Figure 3.10 as the series connection of the plant with the dynamic part of the controller, and we assume that there are no RHP pole-zero cancellations between these two components. Let $K(s) = kI_m$ for a real number k to examine system properties for various values of k. Recall from Theorem 3.2.1 that

$$p_c(s) = p_G(s)p_K(s)\det[I + G(s)K(s)],$$

where $p_c(s)$ is the pole or the characteristic polynomial of the closed-loop system, $p_G(s)$ and $p_K(s)$ are the pole polynomials of $G(s)$ and $K(s)$ respectively. Obviously, our present case leads to $p_K(s) = 1$ for $K(s) = kI$ and $p_c(s) = p_G(s)\det[I + kG(s)]$, or

$$\det[I + kG(s)] = \frac{p_c(s)}{p_G(s)}.$$

The principle of the argument is applied to get

$$\varDelta \arg\det(I + kG(s)) = 2\pi(P_c - P_o),$$

where P_c and P_o are the numbers of RHP roots of $p_c(s)$ and $p_G(s)$, respectively, where $p_G(s)$ is the same as the open-loop characteristic polynomial in the present case of static controller $K(s) = K$, so the notation P_o is justified. $\varDelta \arg$ denotes the change in the argument as s transverses the Nyquist contour once. As usual, the Nyquist contour consists of the straight-line segment from $(0, -jR)$ to $(0, jR)$ and the RHP semicircle joining these points, and by convention it is traversed clockwise. If $G(s)$ has any poles on the imaginary axis, the contour has to be indented into the left half-plane with a small semicircle surrounding each of them. For the closed-loop stability, we need $P_c = 0$, i.e.

$$\varDelta \arg\det(I + kG(s)) = -2\pi P_o.$$

Theorem 3.2.5. *The feedback system in Figure 3.10 is stable if and only if the Nyquist plot of* $\det[I+kG(s)]$ *encircles the origin* P_o *times anti-clockwise.*

To apply the above theorem, however, one has to draw the Nyquist plot of $\det[I + kG(s)]$ for each value of k which is under consideration. This is inconvenient, and different from the SISO case where one can know stability for all values of k once the Nyquist plot of $G(s)$ is drawn.

Let $\lambda_i(s)$ be an eigenvalue of $G(s)$. Then, $k\lambda_i(s)$ is an eigenvalue of $kG(s)$ and $(1 + k\lambda_i(s))$ is that of $(I + kG(s))$. One gets

$$\det[I + kG(s)] = \prod_i [1 + k\lambda_i(s)],$$

and

$$\varDelta \arg\det(I + kG(s)) = \sum_i \varDelta \arg[1 + k\lambda_i(s)]. \tag{3.39}$$

The Nyquist plot of $\lambda_i(s)$ are called the *characteristic loci* of $G(s)$. They have m branches for an $m \times m$ plant $G(s)$. It follows from (3.39) that we can infer closed-loop stability by counting the total number of encirclements of the point $(-1 + j0)$ made by the characteristic loci $k\lambda_i(jw)$ of $kG(s)$, which is as powerful as in the classical SISO Nyquist criterion as it is now enough to draw $\lambda_i(jw)$ once, usually for $k = 1$.

Theorem 3.2.6. (Multivariable Nyquist Theorem) *Suppose that $G(s)$ has P_o unstable poles. Then the feedback system in Figure 3.10 with $K(s) = kI$ is stable if and only if the characteristic loci of $kG(s)$, taken together, encircle the point, $-1 + j0$, P_o times anti-clockwise.*

Example 3.2.4. Let

$$G(s) = \frac{0.5}{(s+1)(s+2)} \begin{bmatrix} 2s+1 & 1 \\ 1 & 2s+3 \end{bmatrix}.$$

Its characteristic loci are found to be

$$\lambda_1(jw) = \tfrac{1}{jw+2},$$
$$\lambda_2(jw) = \tfrac{1}{jw+1},$$

which are shown in Figure 3.12. The open loop is stable, i.e. $P_o = 0$. For the closed-loop system to be stable, there should be zero net encirclements of the critical point $(-\frac{1}{k} + j0)$ made by the loci. Thus, one infers that

- for $-\infty < \frac{-1}{k} < 0$, there is no encirclement and the closed-loop is stable;
- for $0 < \frac{-1}{k} < 1$, there are one or two clockwise encirclements, implying closed-loop instability; and
- for $\frac{-1}{k} > 1$, there is no encirclement, and closed-loop stability is ensured.

◇

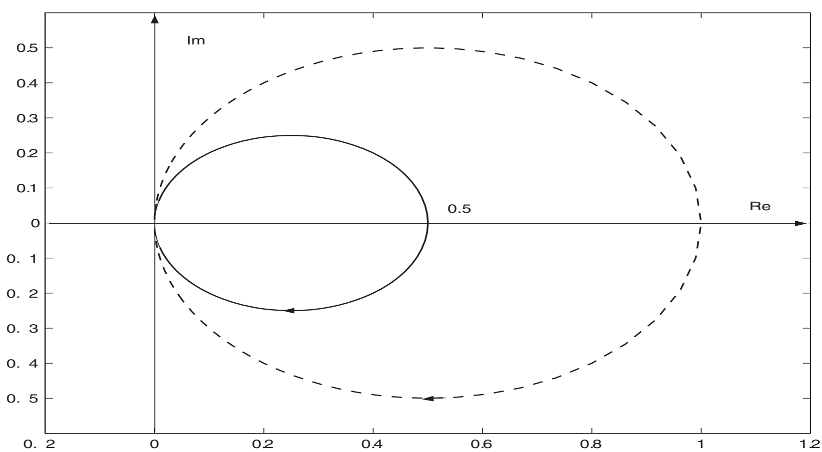


Fig. 3.12. Characteristic loci
(—— λ_1, - - - λ_2)

It is noted that the computation of characteristic loci is quite involved in general. For the special case of diagonally dominant systems, however, this is not necessary, and only the so-called Gershgorin bands need to be drawn.

Theorem 3.2.7. (Gershgorin's Theorem): *Let Z be a complex matrix of $m \times m$ size, then the eigenvalues of Z lie in the union of the m circles, each with centre z_{ii} and row radius:*

$$\sum_{j=1, j\neq i}^{m} |z_{ij}|, i = 1, 2, \dots, m.$$

They also lie in the union of circles, each with centre z_{ii} and column radius:

$$\sum_{j=1, j\neq i}^{m} |z_{ji}|, i = 1, 2, \dots, m.$$

Proof. Let λ be an eigenvalue of Z and $v = (v_1, v_2, \cdots, v_m)^T$ the corresponding eigenvector, then

$$Zv = \lambda v, \text{ (totally m rows)}.$$

Consider only the $i-th$ row for which $|v_i| \geq |v_j|$ for all $j \neq i$,

$$\sum_{j=1}^{m} z_{ij} v_j = \lambda v_i,$$

or

$$\lambda v_i - z_{ii} v_i = \sum_{j=1, j\neq i}^{m} z_{ij} v_j.$$

It follows that

$$|\lambda - z_{ii}| = |\sum_{j=1, j\neq i}^{m} z_{ij} \frac{v_j}{v_i}| \leq \sum_{j=1, j\neq i}^{m} |z_{ij}| \cdot |\frac{v_j}{v_i}| \leq \sum_{j=1, j\neq i}^{m} |z_{ij}|,$$

hence, the result. □

Consider now a proper rational matrix $G(s)$. Its elements are functions of a complex variable s. The *Nyquist array* of $G(s)$ is an array of graphs, the (i, j) graph of which is the Nyquist plot of $g_{ij}(s)$. For any i, at each $s = jw$, a circle of either row radius:

$$\sum_{j=1, j\neq i}^{m} |g_{ij}(jw)|,$$

or column radius

$$\sum_{j=1, j\neq i}^{m} |g_{ji}(jw)|,$$

is superimposed on $g_{ii}(jw)$. The "bands" obtained in this way are called the Gershgorin bands. For illustration, Figure 3.13 shows the Nyquist array and Gershgorin bands of

$$G(s) = \begin{bmatrix} \frac{s+2}{2s+1} & \frac{s+1}{2s+1} \\ \frac{s+1}{2s+1} & \frac{s+2}{2s+1} \end{bmatrix}. \tag{3.40}$$

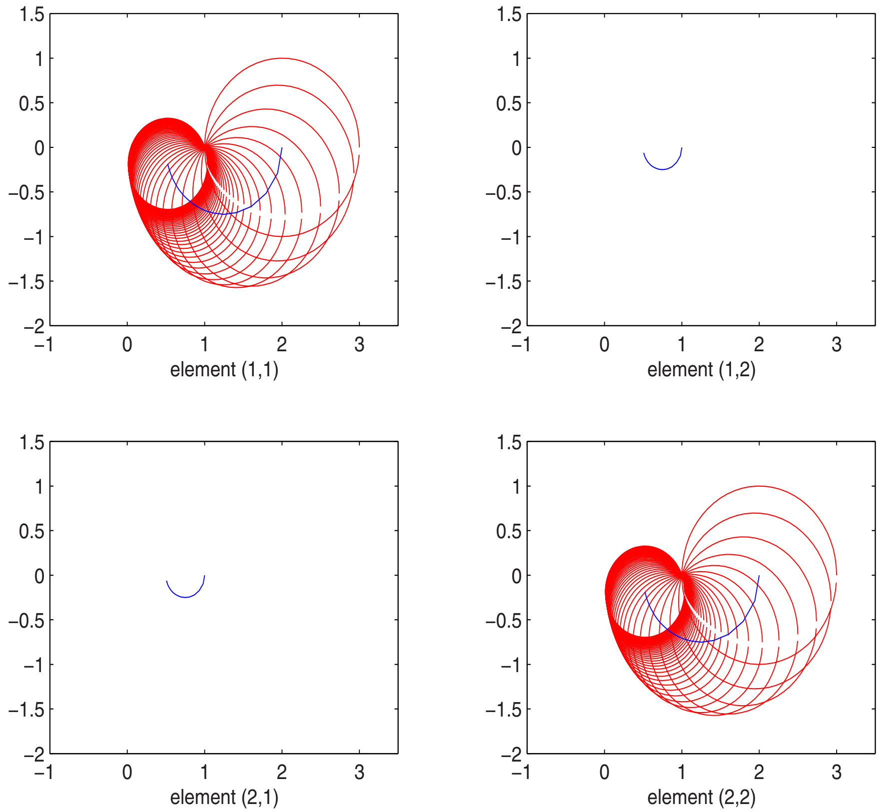


Fig. 3.13. Nyquist Array with Gershgorin Bands

By the Gershgorin's theorem, for a given $s = jw$, the eigenvalues (characteristic loci) $\lambda(jw)$ lie in the Gershgorin circles of $G(jw)$. When s traverses the Nyquist contour, the union of the Gershgorin bands "traps" the union of the characteristic loci. If all the Gershgorin bands exclude the point $(1 + j0)$, then it is sufficient to infer closed-loop stability by counting the encircles of $(1 + j0)$ by the Gershgorin bands, since in this case, these encirclements are the same as those by the characteristic loci. In this case, $[I + G]$ is diagonal dominant, a concept which we now introduces formally.

Definition 3.2.1. *(Diagonal Dominance) A $m \times m$ proper rational matrix $G(s)$ is called row diagonally dominant (on the Nyquist contour) if*

$$|g_{ii}(jw)| > \sum_{j=1, j\neq i}^{m} |g_{ij}(jw)|, i = 1, 2, \ldots, m;$$

and column diagonally dominant if

$$|g_{ii}(jw)| > \sum_{j=1, j\neq i}^{m} |g_{ji}(jw)|, i = 1, 2, \ldots, m,$$

assuming that $G(s)$ has no poles on the Nyquist contour.

Graphical Test for Dominance can be easily developed. Draw the Gershgorin bands for $G(s)$. If the banks with row radius exclude the origin, then $G(s)$ is row diagonally dominant. If it is the case with column radius, $G(s)$ is column diagonally dominant. A greater degree of dominance corresponds to narrower Gershgorin bands and more closely resembles m decoupled SISO systems. For $G(s)$ in (3.40), Figure 3.13 indicates that $G(s)$ is diagonally dominant, which can be confirmed analytically by noting $|jw+2| > |jw+1|$ for any real w.

For diagonally dominant $G(s)$, we have the following stability criterion.

Theorem 3.2.8. (Diagonally dominant Systems) *Consider the unity feedback system with the loop gain $L(s) = G(s)K$, where $G(s)$ is square with P_o unstable poles (multiplicities included), $K = diag\{k_i\}$, and*

$$|g_{ii}(s) + \frac{1}{k_i}| > \sum_{j\neq i} |g_{ji}(s)|,$$

for each i and for all s on the Nyquist contour, that is $I + GK$ is column diagonally dominant. Let the Nyquist plot of $g_{ii}(s)$ encircle the point, $(-1/k_i + j0)$, N_i times anticlockwise. Then, the system is stable if and only if

$$\sum_{i=1}^{m} N_i = P_o.$$

Proof. Due to the assumed column diagonal dominance, the following three numbers are equal:

- the total encirclements of $(-1/k_i + j0)$ made by $G_{ii}(jw)$ for all i, $N = \sum_{i=1}^{m} N_i$,
- the total encirclements of $(-1/k_i + j0)$ made by Gershgorin bands,
- the total encirclements of $(-1/k_i + j0)$ made by characteristic loci.

Hence, the result follows from the generalized Nyquist Stability Criterion. □

For $G(s)$ in (3.40), we have $P_o = 0$. It follows from the theorem that the closed-loop is stable if the net encirclement of $(-1/k_i + j0)$ made by the Gershgorin bands of G is zero. By inspection of Figure 3.13, this is the case when $k_1 > 0$ and $k_2 > 0$.

Several remarks on Theorem 3.2.8 are made as follows.

- Note that in the theorem, the condition on diagonal dominance has been imposed. In this case, individual Nyquist plots of $g_{ii}(s)$ can be used to assess closed-loop stability. The design can be carried out loop by loop, each for one $g_{ii}(s)$, like a sequence of SISO systems.
- Due to diagonal dominance, we have allowed K to be diagonal with different diagonal elements so that we can adjust gains for each loop. This is stronger than that in the multivariable Nyquist theorem where only $k = kI$ is allowed and the same gain must be used for all loops.
- If $(I+GK)$ is not diagonally dominant, some Gershgorin band may overlap the origin, the characteristic loci of $(I + GK)$ *may or may not* encircle the origin, an hence we cannot infer stability solely by inspecting Gershgorin bands.

3.3 Plant Uncertainty and Stability Robustness

Most control designs are based on some plant model. A model can never be a perfect match to a real plant and there must be modelling errors, which are also called model uncertainties or perturbations. Due to such uncertainties, a controller which works well for a design model may not necessarily work equally well for the real plant. It is thus of great importance to represent uncertainties as accurately as possible, analyze their effects on control systems, and take them into control design considerations.

To represent uncertainties requires the introduction of singular values. Singular values are very useful in matrix analysis and control engineering, and they are a good measure of the "size"of the matrix. Let A be a complex matrix. The positive square roots of the eigenvalues of $G^H G$ or GG^H, whichever has smaller size, are called the singular values of A, are denoted by $\sigma_i(A)$, where A^H is the conjugate and transpose of A. A nice thing with singular values is that we actually need not compute them directly from eigenvalue calculation but through unitary transformation (numerically stable and efficient), as stated below.

Lemma 3.3.1. (Singular Value Decomposition) *Let $A \in \mathbb{C}^{p\times m}$. There are unitary matrices $U \in \mathbb{C}^{p\times p}$ and $V \in \mathbb{C}^{m\times m}$ such that*

$$A = U\Sigma V^H,$$

where $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \Sigma_1 = diag\{\sigma_1, \sigma_2, \cdots, \sigma_r\}, \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0,$ *and* $r = \min\{p, m\}$.

The following notations are widely adopted:

$$\overline{\sigma}(A) = \sigma_{max}(A) = \sigma_1 = \text{ the largest singular value of A},$$

and

$$\underline{\sigma}(A) = \sigma_{min}(A) = \sigma_r = \text{ the smallest singular value of A}.$$

Let $\|x\|$ denote the Euclidean vector norm for a vector x. Then the induced matrix norm from the vector norm, $\|\cdot\|$, is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

and is given by the spectral norm:

$$\|A\| = \overline{\sigma}(A). \tag{3.41}$$

Lemma 3.3.2. *Suppose that A and B are square matrices. Then*
(i) Rank A = number of non-zero singular values of A;
(ii) A is nonsingular if and only if $\underline{\sigma}(A) > 0$;
(iii) $\overline{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$ if A is nonsingular;
(iv) $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$;
(v) $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\underline{\sigma}(B)$;
(vi) $\underline{\sigma}(A+B) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$; and
(vii) $\overline{\sigma}(AB) < 1$ or $\overline{\sigma}(BA) < 1$ if and only if $\delta\overline{\sigma}(A) < 1$, assuming that B are all matrices such that $\overline{\sigma}(B) \leq \delta$.

Consider now a transfer function matrix $G(s)$. For a particular s, say, $s = jw$ for a fixed w, $G(jw)$ is a complex matrix, and the matrix norm, $\|G(jw)\| = \overline{\sigma}(G(jw))$, measures the largest gain of the system at this particular frequency. The largest gain of the system over all frequencies w may be measured by a operation norm, $L_\infty - norm$:

$$\|G\|_\infty = \sup_w \overline{\sigma}(G(jw)). \tag{3.42}$$

If we plot $\overline{\sigma}(G(jw))$ versus w, then $\|G\|_\infty$ is simply the peak value of $\overline{\sigma}(G(jw))$.

3.3.1 Unstructured Uncertainty

Let $G_0(s)$ be a nominal transfer matrix, which is a best estimate, in some sense, of the true plant behavior. Let $G(s)$ denote the true transfer matrix of the plant. The following are three most commonly used uncertainty models:

- Additive Perturbation: $G(s) = G_0(s) + \Delta(s)$, (3.43)
- Input Multiplicative Perturbation: $G(s) = G_0(s)(I + \Delta(s))$, (3.44)
- Output Multiplicative Perturbation: $G(s) = (I + \Delta(s))G_0(s)$. (3.45)

Two kinds of multiplicative perturbations (input or output) are required as matrix multiplications are not commutative. These uncertainty descriptions are said to be unstructured if the only restriction on the perturbations Δ is on their "size", which is usually measured by

$$\overline{\sigma}(\Delta(jw)) \leq \delta(jw), \tag{3.46}$$

where $\delta(\cdot)$ is a positive scalar function. Note that δ represents the absolute magnitude of model errors in the additive perturbation case and the relative magnitude in the multiplicative perturbation cases.

The design problem in an uncertain environment consists in selecting a controller $K(s)$ in a unity feedback system so as to ensure stability as well as satisfactory performances not only for the nominal plant G_0 but also for any G which belongs to the given uncertain model set. As for the basic stability requirement, the control system is called *robustly stable* if it is stable for all permissible G.

Theorem 3.3.1. (Unstructured Uncertainty) *Suppose that $K(s)$ stabilizes the nominal plant G_0 and that G and G_0 have the same number of unstable poles. Then, the closed-loop system in Figure 3.10 is robustly stable for the uncertainty meeting* (3.46) *if and only if*

(i) $\delta(jw)\overline{\sigma}(K(jw)[I + G_0(jw)K(jw)]^{-1}) < 1$, $\forall w \in \mathbb{R}$, *for $G(s)$ in (3.43);*
(ii) $\delta(jw)\overline{\sigma}(K(jw)G_0(jw)[I + K(jw)G_0(jw)]^{-1}) < 1$, $\forall w \in \mathbb{R}$, *for $G(s)$ in (3.44);*
(iii) $\delta(jw)\overline{\sigma}(G_0(jw)K(jw)[I + G_0(jw)K(jw)]^{-1}) < 1$, $\forall w \in \mathbb{R}$, *for $G(s)$ in (3.45).*

Proof. We will prove (i) only as other two cases are similar. Since the closed-loop is stable for $\Delta(s) = 0$ (nominally stabilized), and since G and G_0 have been assumed to have the same number of unstable poles, it follows from Theorem 3.2.5 that the closed-loop remains stable as long as there is no change in the number of encirclements of the origin made by $\det(I + G(jw)K(jw))$ with respect to the nominal $det(I + G_0(jw)K(jw))$, which is equivalent to

$$\det[I + G(jw)K(jw)] \neq 0$$

for all $w \in \mathbb{R}$ and permissible G. Bring (3.43) in the above:

$$\begin{aligned} \det[I + (G_0 + \Delta)K] &= \det[I + G_0K + \Delta K] \\ &= \det[I + \Delta K(I + G_0K)^{-1}] \cdot \det[I + G_0K] \\ &\neq 0, \end{aligned} \tag{3.47}$$

which, due to $\det[I + G_0K] \neq 0$, is equivalent to

$$\det[I + \Delta K(I + G_0K)^{-1}] \neq 0. \tag{3.48}$$

By Lemma 3.3.2(ii) we can rewrite the above as

$$\underline{\sigma}[I + \Delta K(I + G_0 K)^{-1}] > 0. \tag{3.49}$$

By Lemma 3.3.2(vi) with $A = I$, $\underline{\sigma}(I) = 1$ and $B = \Delta K(I + G_0 K)^{-1}$, (3.49) is implied by

$$1 - \overline{\sigma}(\Delta K(I + GK)^{-1}) > 0,$$

or $\overline{\sigma}(\Delta K(I + G_0 K)^{-1}) < 1$, which, by Lemma 3.3.2(vii), is ensured by Condition (i) in the theorem.
Conversely, suppose that the condition is violated, i.e., for some w^*, there holds

$$\delta(jw^*)\overline{\sigma}(K(jw^*)[I + G_0(jw^*)K(jw^*]^{-1}) = 1. \tag{3.50}$$

Let $M = K(I + G_0 K)^{-1}$ and $M(jw)$ have the singular value decomposition:

$$M = U\Sigma V^H$$

with $\sigma_1 = \overline{\sigma}(M(jw^*))$ at the $(1,1)$ position of $\Sigma(jw^*)$. Choose Δ to be

$$\Delta = -\delta V U^H$$

which is permissible since $\overline{\sigma}(\Delta(jw)) \leq \delta(jw)$. Then, we have

$$\begin{aligned} \det[I + \Delta(jw^*)M(jw^*)] &= \det[I - \delta(jw^*)\Sigma(jw^*)] \\ &= \det(diag\{1 - \delta(jw^*)\overline{\sigma}(M(jw^*)), ?, \cdots, ?\}) \\ &= 0 \end{aligned}$$

where the last equality is due to (3.50). This in turn, by (3.47), implies

$$\det[I + G(jw^*)K(jw^*)] = 0,$$

for a permissible $G = G_0 + \Delta$. Hence jw^* is a pole of the corresponding closed-loop system and the system is unstable. □

3.3.2 Structured Uncertainties

In contrast to the unstructured uncertainties given in (3.46), in many cases, we may trace uncertainties into certain parts or elements of the system and lead to so-called structured uncertainties. Usually, there will be both structured and unstructured uncertainties. In order to capture them accurately, we adopt the following general and standard representation of uncertain systems. Let the plant have three sets of inputs and outputs.

- 1st set inputs: all manipulated variables.
- 2nd set inputs: all other external signals (disturbances/set points).
- 1st set outputs: all measured variables for feedback.
- 2nd set of outputs: all other outputs whose behaviors are of interest.

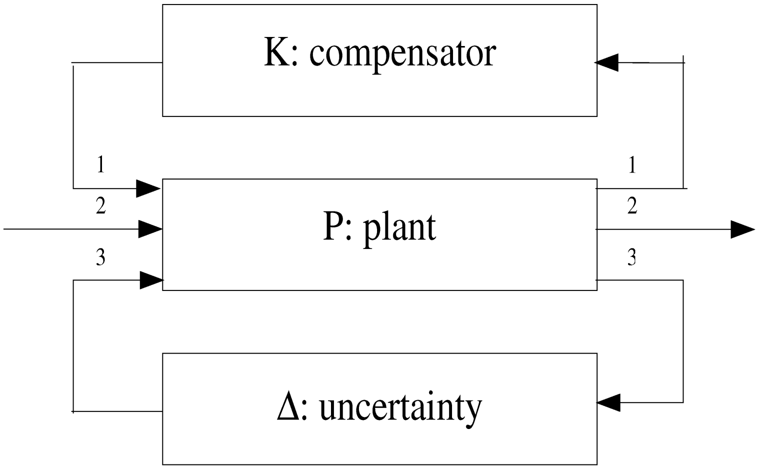


Fig. 3.14. Rearrangement of Uncertain System

The third set of inputs and outputs is novel and comes from uncertainty. For each uncertainty in the plant (either structured or unstructured), we take it outside the plant and assign it with one block. Collect them all together as a special system, which is around the plant and has a block-diagonal structure, on the diagonal are just those blocks which have been pulled out from inside the plant. Write

$$\Delta(s) = \mathrm{diag}\,\{\Delta_1(s), \cdots, \Delta_n(s)\}. \tag{3.51}$$

where Δ_i may be a scalar or a matrix. The resultant representation is shown in Figure 3.14, noting that the third set of inputs and outputs for plant is outputs and inputs for uncertainty. By putting suitable weights in loops, we can always normalize Δ such that

$$\|\Delta_i\|_\infty \le 1, \quad i = 1, 2, \cdots, n. \tag{3.52}$$

This normalization is used to formulate the standard problem.

If the compensator is already known, we can form a single system Q representing the closed-loop system consisting of G_0 and K with the first set of inputs and outputs disappeared, as shown in Figure 3.15, for which general robust stability conditions will be derived below. Define x, v and z according to Figure 3.15, then one has

$$\begin{bmatrix} y \\ x \end{bmatrix} = Q \begin{bmatrix} v \\ z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix}.$$

Suppose that a compensator $K(s)$ stabilizes a nominal model $G_0(s)$ (called nominal stabilization). Then, $Q(s)$ will be stable.

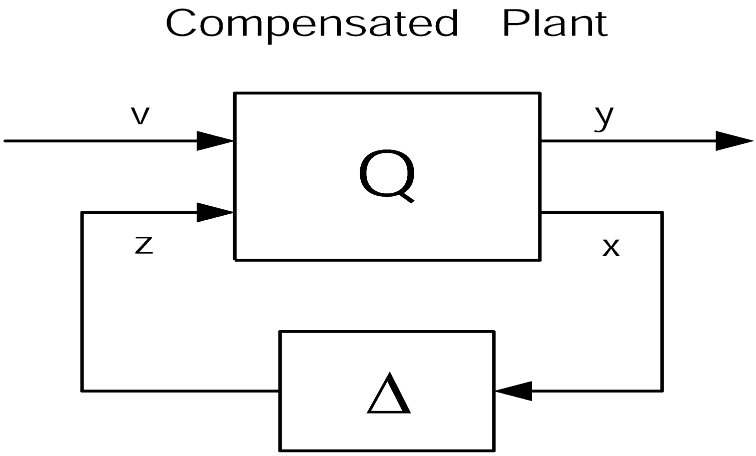


Fig. 3.15. Standard Representation of Uncertain System

Example 3.3.1. Consider a unity feedback system in Figure 3.10, where G has element-wise uncertainties:

$$|g_{ij}(s) - g_{ij}^0(s)| \leq |a_{ij}(s)|.|g_{ij}^0(s)|,$$

with the nominal plant $G_0 = \{g_{ij}^0\}$. We write

$$g_{ij}(s) = g_{ij}^0 + (g_{ij} - g_{ij}^0) := g_{ij}^0 + \delta_{ij},$$

to get elementwise the additive uncertainty $\Delta = \{\delta_{ij}\}$ and its bounds:

$$|\delta_{ij}| \leq |a_{ij}|.|g_{ij}^0| \triangleq \alpha_{ij}(s).$$

We can redraw the uncertain system into Figure 3.16, where

$$\begin{aligned} \widetilde{\Delta} &:= diag\{\widetilde{\Delta}_1, \widetilde{\Delta}_2, \widetilde{\Delta}_3, \widetilde{\Delta}_4\} \\ &:= diag\{\delta_{11}/\alpha_{11}, \delta_{21}/\alpha_{21}, \delta_{12}/\alpha_{12}, \delta_{22}/\alpha_{22}\} \end{aligned} \tag{3.53}$$

is the normalized uncertainty with $\|\widetilde{\Delta}_i\|_\infty \leq 1, i = 1, 2, 3, 4.$ ◊

Turn back to the standard representation of uncertain system in Figure 3.15. Suppose nominal stabilization. Our concern is whether or not the system remains stable for all permissible Δ. Since $Q(s)$ is stable and since Q_{11}, Q_{12} and Q_{21} are outside the feedback loop, to know whether the system will remain stable under permissible perturbations, we only need to check whether the (positive) feedback combination of Q_{22} and Δ will remain stable for all allowable Δ. In the case of the unstructured uncertainty, or $n = 1$, apply Theorem 3.3.1 with assignment of $G_0 = 0$, $K = -Q_{22}$, and $\delta = 1$, to get a necessary and sufficient condition for robust stability:

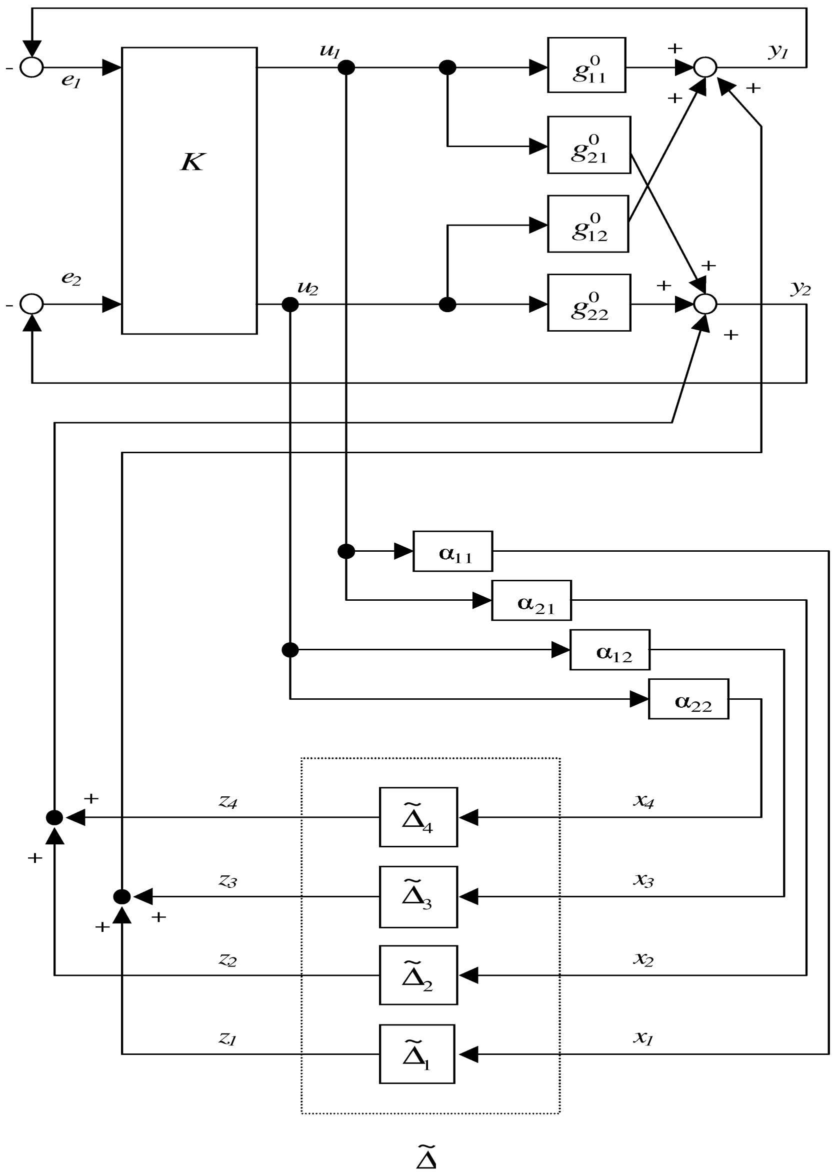


Fig. 3.16. Redrawing of Uncertain System into Standard Form

$$\overline{\sigma}(Q_{22}(jw)) < 1, \quad \forall w \in \mathbb{R},$$

or equivalently

$$\|Q_{22}\|_\infty < 1. \tag{3.54}$$

In general, however, a uncertainty $\varDelta$ may have more than one blocks in its standard representation in Figure 3.15, i.e. $n \geq 2$ for $\varDelta$ in (3.52). Then the condition (3.54) is only sufficient for robust stability, because most perturbations which satisfy $\|\varDelta\| \leq 1$ are no longer permissible. (3.54) is usually too conservative to use for such structured uncertainties.

Let BD_δ denote the set of stable, block-diagonal perturbations with a particular structure, and $\|\varDelta_j\| \leq \delta$. Suppose nominal stability of the system in Figure 3.15. It follows from the multivariable Nyquist theorem that the feedback system can be unstable if and only if

$$\det[I - Q_{22}(jw)\varDelta(jw)] = 0 \tag{3.55}$$

for some w and some $\varDelta \in BD_\delta$. We thus define the structured singular value $\mu(Q_{22}(jw))$ as follows:

$$\mu(Q_{22}(jw)) = \begin{cases} 0, \text{ if } \det[I - Q_{22}\varDelta] \neq 0, \text{ for any } \varDelta \in BD_\infty; \\ 1/(\min_{\varDelta \in BD_\infty}\{\overline{\sigma}(\varDelta(jw)) | \det[I - Q_{22}(jw)\varDelta(jw)] = 0\}), \text{ otherwise.} \end{cases} \tag{3.56}$$

In other words, one tries to find the smallest structured $\varDelta$ (in the sense of $\overline{\sigma}(\varDelta)$) which makes $\det(I - Q_{22}(jw)) = 0$, then, $\mu(Q_{22}) = \frac{1}{\overline{\sigma}(\varDelta)}$; If no $\varDelta$ meets (3.55), then $\mu(Q_{22}) = 0$.

Example 3.3.2. Let

$$Q_{22} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

The smallest (unstructured) full matrix perturbation $\varDelta$ which makes $\det(I - Q_{22}\varDelta) = 0$ is

$$\varDelta = \begin{bmatrix} 0.2 & 0.2 \\ -0.1 & -0.1 \end{bmatrix},$$

and

$$\overline{\sigma}(\varDelta) = 0.316; \quad \mu(Q_{22}) = 3.162.$$

On the other hand, the smallest (structured) diagonal matrix perturbation $\varDelta$ which makes $\det(I - Q_{22}\varDelta) = 0$ is

$$\varDelta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\overline{\sigma}(\varDelta) = 0.333; \quad \mu(Q_{22}) = 3.$$

◊

This example shows that the structured singular value $\mu(.)$ depends on the structure of Δ and that a larger perturbation in the structured form is usually allowed than unstructured one, before the matrix becomes singular.

One notes that if $\Delta = 0$, $I - Q_{22}\Delta = I$ is nonsingular. Thus, to obtain $\mu(Q_{22})$, one way is to gradually increase $\overline{\sigma}(\Delta)$ until we find $I - Q_{22}\Delta$ is singular. Besides, the sequence of Q_{22} and Δ in μ-definition does not matter since

$$\det(I - Q_{22}\Delta) = \det(I - \Delta Q_{22}).$$

The structured singular value $\mu(Q)$ depends on the structure of the set BD_δ as well as on Q. It has the following properties.

Lemma 3.3.3. *There hold*

(i) $\mu(\alpha Q) = |\alpha|\mu(Q)$, *$\alpha$ scalar;*
(ii) $\mu(I) = 1$;
(iii) $\mu(Q) = \overline{\sigma}(Q)$, *if BD_δ has only one block;*
(iv) Let diag $\{I_{k_i}\}$ *has the same block-diagonal structure as BD_δ, define $D =$ diag $\{d_i I_{k_i}\}$, $d_i > 0$, then $\mu(Q) = \mu(DQD^{-1}) \le \inf_D \overline{\sigma}(DQD^{-1})$; and*
(v) If $i \le 3$, i.e., there are no more than three blocks in Δ and Δ_i are independent of each others, then $\mu(Q) = \inf_D \overline{\sigma}(DQD^{-1})$.

With structured singular values, we can define

$$\|Q_{22}\|_\mu = \sup_w \mu(Q_{22}(jw)),$$

and state a robust stability theorem for structured uncertainty (Doyle,1982).

Theorem 3.3.2. (Structured Uncertainty) *The system shown in Figure 3.15 remains stable for all $\Delta \in BD_1$ if and only if*

$$\|Q_{22}\|_\mu < 1. \tag{3.57}$$

One sees that this condition is similar to (3.54) for the unstructured case. The only difference is that the singular value has been replaced here with the structured singular value. Under condition (3.57), one can show that $\max_\omega |\lambda_{\max}(Q_{22}\Delta)| < 1$ so that $(I - Q_{22}\Delta)$ can not be singular on ω axis and $\det(I - Q_{22}(jw)\Delta(jw)) \ne 0$, and the system is thus stable for all Δ meeting $\|\Delta_j\| \le 1$. If Δ has one block only, by Lemma 3.3.3(ii), (3.57) reduces to (3.54). It should be pointed out that the structured singular value is difficult to compute and we usually use the property (iv) in Lemma 3.6 to overestimate $\mu(Q)$.

Example 3.3.3. Continue Example 3.3.1, we have to find Q_{22} which meets $x = Q_{22}z$. One sees

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha_{11}u_1 \\ \alpha_{21}u_1 \\ \alpha_{12}u_2 \\ \alpha_{22}u_2 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \\ 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -K \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -K \left\{ G_0 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \right\},$$

giving

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -(I + KG_0)^{-1} K \begin{bmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{bmatrix} z,$$

so that

$$\begin{aligned} x &= - \begin{bmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \\ 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{bmatrix} (I + KG_0)^{-1} K \begin{bmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{bmatrix} z. \\ &:= Q_{22} z. \end{aligned}$$

The system remains stable if and only if

$$\|Q_{22}\|_\mu \leq 1,$$

where $\| \cdot \|_\mu$ is computed with respect to $\Delta = diag\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, and Δ_i are scalar. ◊

3.3.3 Quantitative Robust Stability Analysis

Theorems 3.3.1 and 3.3.2 are actually special cases of the small gain theorem. Essentially, they all say the same thing: if a feedback loop consists of stable systems and if the open-loop gain (in a suitable sense) is less than unity, then the closed-loop system is stable. These results are elegant and have been widely used in the area of robust control. Unfortunately, it could be very conservative in many cases. To reveal this conservativeness, one notes that descriptions in (3.46) for unstructured uncertainty and (3.51) and (3.52) for structured ones are actually the same for each individual block Δ_i which is restricted by its bound only, and allow much richer uncertainties to occur than what could exist in reality. In particular, phase changes in the uncertainties are not limited at all, and an infinite phase change is permissible, which is definitely unrealistic in practice. In real life, both gain and phase uncertainties would be finite. For example, a common practice in modelling is that a parametric plant model with some parameter uncertainty is in general valid over a working frequency range, and may become useless beyond the range due to unmoulded dynamics, noise and so on. This implies that phase uncertainty is much smaller in low frequencies than that in high frequencies range, and surely finite. For illustration, let us give a concrete example to show conservativeness of Theorem 3.3.1 and to motivate our sequel development.

Example 3.3.4. Let an uncertain plant be described by

$$G(s) = \frac{1+\delta_1}{s^2+(1+\delta_2)s+(1+\delta_3)}, \tag{3.58}$$

where $\delta_1 \in [-0.5, 0.5]$, $\delta_2 \in [-0.5, 0.5]$ and $\delta_3 \in [-0.5, 0.5]$. Suppose that the controller $K(s) = 1$. One may easily see that the closed-loop system is robustly stable, because its characteristic polynomial,

$$p(s) = s^2 + (1+\delta_2)s + 2 + \delta_1 + \delta_3, \tag{3.59}$$

always has positive coefficients and thus its roots lie in th open left half of the complex plane.

Now, Let us try the small gain theorem. The nominal plant is

$$G_0(s) = \frac{1}{s^2+s+1}. \tag{3.60}$$

we may express the uncertainty as an additive one:

$$\Delta(s) = \frac{1+\delta_1}{s^2+(1+\delta_2)s+(1+\delta_3)} - \frac{1}{s^2+s+1}. \tag{3.61}$$

Theorem 3.3.1 would say that the feedback system is robustly stable if and only if

$$|Q_{22}(j\omega)\Delta(j\omega)| < 1, \qquad \omega \in [0, \infty) \tag{3.62}$$

where

$$Q_{22}(s) := \frac{-K(s)}{1+K(s)G_0(s)} = -\frac{s^2+s+1}{s^2+s+2}. \tag{3.63}$$

We have

$$\underset{\omega}{sup}|Q_{22}(j\omega)\Delta(j\omega)| \geqslant \underset{\omega\to 0}{lim}|Q_{22}(j\omega)\Delta(j\omega)| = \frac{1}{2} \times |\frac{1+\delta_1}{1+\delta_3} - 1|. \tag{3.64}$$

If we choose $\delta_1 = 0.5$ and $\delta_3 = -0.5$, or the plant is perturbed to

$$G(s) = \frac{1.5}{s^2+s+0.5}, \tag{3.65}$$

there results in

$$\sup_{\omega}|Q_{22}(j\omega)\Delta(j\omega)| \geqslant 1, \tag{3.66}$$

violating (3.62) though the system is stable. This shows the conservativeness of (3.62) that assumes infinite phase change at any frequency, which is not true in this example. ◊

It should be pointed out that the structured uncertainty and the theory of structured singular values cater to the block diagonal structure of the uncertainty, and have nothing to do with further information on individual blocks such as breaking them into magnitude and phase. In the rest of this subsection, we will acknowledge the importance of limited phase uncertainty and incorporate such information into the robust stability conditions, in addition to the existing magnitude information, so as to remove conservativeness and make the stability criterion exact.

To be specific, we take the SISO unity feedback system for study. If the plant $G(s)$ and the controller $K(s)$ have no poles in the closed right half plane, then the Nyquist criterion tells that the closed-loop system is stable if and only if the Nyquist curve of $G(j\omega)K(j\omega)$ does not encircle the critical point, $(-1+j0)$. An uncertain plant is always represented by a set of models. Each model gives rise to a Nyquist curve and the set thus sweeps a band of Nyquist curves. Then from the Nyquist criterion, the closed-loop system is robustly stable if and only if the band of the Nyquist cluster for the uncertain plant and the controller does not encircle the point, $(-1+j0)$. Conversely, the system is not robustly stable if and only if there exits a Nyquist curve among the cluster which encircles -1 point, or there must exist a specific frequency ω_0 at which the frequency response region of the uncertain loop is such that both

$$\max_{G}\{|G(j\omega_0)K(j\omega_0)|\} \geqslant 1 \tag{3.67}$$

and

$$\min_{G}\{arg\{G(j\omega_0)K(j\omega_0)\}\} \leqslant -\pi \tag{3.68}$$

hold true. Note that

$$\max_{G}\{|G(j\omega_0)K(j\omega_0)|\} = |K(j\omega_0)|\max_{G}\{|G(j\omega_0)|\}, \tag{3.69}$$

and

$$\min_{G}\{arg\{G(j\omega_0)K(j\omega_0)\}\} = arg\{K(j\omega_0)\} + \min_{G}\{arg\{G(j\omega_0)\}\}. \tag{3.70}$$

Thus, we establish the following quantitative robust stability theorem.

Theorem 3.3.3. *Suppose stability of the uncertain plant and the controller. The uncertain closed-loop system is robustly stable if for any* $\omega \in \mathbb{R}$, *either*

$$|K(j\omega)| \leqslant \frac{1}{\underset{G}{max}\{|G(j\omega)|\}} \tag{3.71}$$

or

$$arg\{K(j\omega)\} > -\pi - \underset{G}{min}\{arg\{G(j\omega)\}\} \tag{3.72}$$

holds.

For a SISO uncertain system where the controller is unstable, we can always convert it to the standard representation in Figure 3.15. Suppose that the nominal plant is stabilized and that the uncertain plant is stable, too. Then, $Q_{22}(s)$ and $\Delta(s)$ are stable and Theorem 3.3.3 is applicable with obvious assignment of $K(s) = -Q_{22}(s)$ and $G(s) = \Delta(s)$.

Corollary 3.3.1. *For the system in Figure 3.15, if the nominal controlled system* $Q_{22}(s)$ *and the uncertainty* $\Delta(s)$ *are both stable, then the uncertain closed-loop system remains stable if for any* $\omega \in \mathbb{R}$, *either*

$$|Q_{22}(j\omega)| \leqslant \frac{1}{\underset{\Delta}{max}\{|\Delta(j\omega)|\}} \tag{3.73}$$

or

$$arg\{-Q_{22}(j\omega)\} > -\pi - \underset{\Delta}{min}\{arg\{\Delta(j\omega)\}\} \tag{3.74}$$

holds.

For analysis of a given uncertain control system, the robust stability criterion in (3.71) and (3.72) can be tested graphically as follows:

(i) Draw Bode plots for magnitude $1/max\{|G(j\omega)|\}$ and phase $min\{arg\{G(j\omega)\}\}$ respectively, according to the given structure of the uncertain plant;
(ii) Draw Bode plots for $K(j\omega)$ in the same diagram; and
(iii) Check if condition (3.71) or (3.72) is satisfied.

Similarly, one can graphically determine the boundaries of the Nyquest band of a given uncertain plant. Such boundaries will contain the worst case of the plant which can facilitate robust stability analysis and robust control design. Obviously, it is more powerful if analytical expressions for the uncertain plant boundaries can be derived. Though this is difficult in general, it is possible to do so for a class of typical processes, the uncertain second-order plus dead time model.

Let us consider

$$G(s) = \frac{b}{s^2 + a_1 s + a_2} e^{-Ls}, \tag{3.75}$$

where the uncertain parameters are $b \in [b^- \ b^+]$, $a_1 \in [a_1^-, a_1^+]$, $a_2 \in [a_2^-, a_2^+]$ and $L \in [L^-, L^+]$, $b^- > 0$, $a_1^- > 0$, $a_2^- > 0$ and $L^- > 0$. For a fixed ω, the gain is

$$|G(j\omega)| = b/\sqrt{(a_1\omega)^2 + (a_2 - \omega^2)^2}. \tag{3.76}$$

$max\{|G(j\omega)|\}$ and $min\{|G(j\omega)|\}$ can be obtained as

$$max\{|G(j\omega)|\} = \begin{cases} \frac{b^+}{\sqrt{(a_1^-\omega)^2 + (a_2^- - \omega^2)^2}}, & \omega \in \left[0, \sqrt{a_2^-}\right]; \\ \frac{b^+}{a_1^-\omega}, & \omega \in \left[\sqrt{a_2^-}, \sqrt{a_2^+}\right]; \\ \frac{b^+}{\sqrt{(a_1^-\omega)^2 + (a_2^+ - \omega^2)^2}}, & \omega \in \left[\sqrt{a_2^+}, \infty\right]; \end{cases} \tag{3.77}$$

and

$$min\{|G(j\omega)|\} = \begin{cases} \frac{b^-}{\sqrt{(a_1^+\omega)^2 + (a_2^+ - \omega^2)^2}}, & \omega \in \left[0, \sqrt{\frac{a_2^- + a_2^+}{2}}\right]; \\ \frac{b^-}{\sqrt{(a_1^+\omega)^2 + (\omega^2 - a_2^-)^2}}, & \omega \in \left[\sqrt{\frac{a_2^- + a_2^+}{2}}, \infty\right]. \end{cases} \tag{3.78}$$

In a similar way, the phase of the system is given as

$$arg(G(j\omega)) = -arg(a_2 - \omega^2 + ja_1\omega) - L\omega. \tag{3.79}$$

It follows that

$$min\{arg(G(j\omega))\} = \begin{cases} -arg(a_2^- - \omega^2 + ja_1^+\omega) - L^+\omega, & \omega \in \left[0, \sqrt{a_2^-}\right]; \\ -arg(ja_1^-\omega + a_2^- - \omega^2) - L^+\omega, & \omega \in \left[\sqrt{a_2^-}, \infty\right]; \end{cases}$$

and

$$max\{arg(G(j\omega))\} = \begin{cases} -arg(a_2^+ - \omega^2 + ja_1^-\omega) - L^-\omega, & \omega \in \left[0, \sqrt{a_2^+}\right]; \\ -arg(ja_1^+\omega + a_2^+ - \omega^2) - L^-\omega, & \omega \in \left[\sqrt{a_2^+}, \infty\right]. \end{cases} \tag{3.80}$$

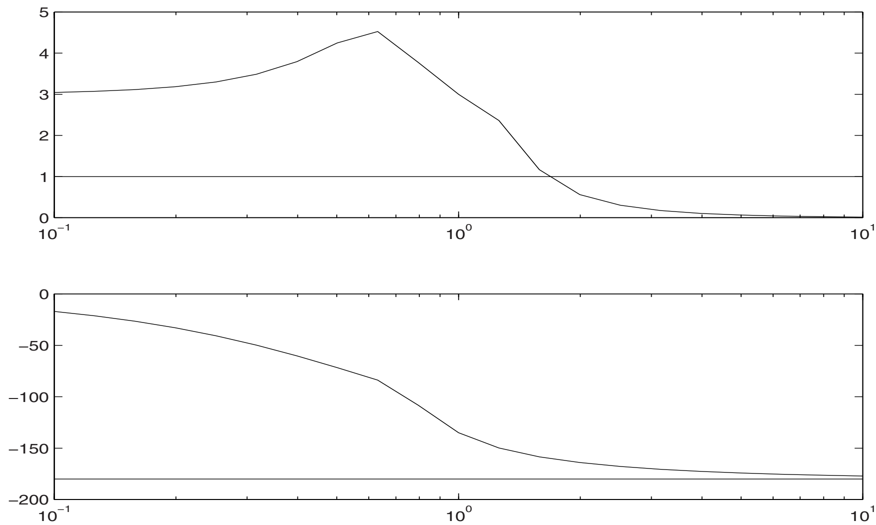


Fig. 3.17. The plot of $max\{|G(j\omega)|\}$ and $min\{arg\{G(j\omega)\}\}$ in Example 3.3.5

Example 3.3.5. In Example 3.3.4, the small gain theory fails to decide stability of the uncertain system given there. We now use Theorem 3.3.3, an exact robust stability criterion, with the help of the above precise uncertainty bounds to re-examine the system. Apply (3.77) and (3.80) to get

$$max\{|G(j\omega)|\} = \begin{cases} \frac{1.5}{\sqrt{(0.5\omega)^2+(0.5-\omega^2)^2}}, & \omega \in \left[0, \sqrt{0.5}\right]; \\ \frac{1.5}{0.5\omega}, & \omega \in \left[\sqrt{0.5}, \sqrt{1.5}\right]; \\ \frac{1.5}{\sqrt{(0.5\omega)^2+(1.5-\omega^2)^2}}, & \omega \in \left[\sqrt{1.5}, \infty\right]; \end{cases}$$

and

$$min\{arg(G(j\omega))\} = \begin{cases} -arg(0.5-\omega^2+j1.5\omega), & \omega \in \left[0, \sqrt{0.5}\right]; \\ -arg(0.5-\omega^2+j0.5\omega), & \omega \in \left[\sqrt{0.5}, \infty\right]. \end{cases}$$

with $K(j\omega) = 1$, though the gain condition (3.71) is violated when $\omega < 1.66$, see the gain plot of Figure 3.17. But the phase condition (3.72) will be satisfied for all frequencies, see the phase plot of Figure 3.17. Thus the uncertain system is robustly stable. $\Diamond$

3.4 Notes and References

The results in this chapter on internal stability using polynomial matrix fractions and frequency response can be found in many linear systems monographs (Rosenbrock, 1974; Callier and Desoer, 1982; Chen, 1984; Morari and

Zafiriou, 1989), and the exception is Section 1, which is based on Wang *et al.* (1999*b*). The materials on uncertainties and robustness in Section 3 are standard, and can be found in Maciejowski (1989) and Zhou *et al.* (1996), and the exception is the quantitative robust stability criterion, which is due to Wang *et al.* (2002).

4. State Space Approach

In this chapter we shall consider a system of the following state space form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{4.1}$$

If $x(0) = 0$ and the input and the output vectors have the same dimension m, these are related by the transfer function matrix:

$$y(s) = G(s)u(s) = C(sI - A)^{-1}Bu(s), \tag{4.2}$$

which may be expanded into

$$\begin{aligned} y_1(s) &= g_{11}(s)u_1(s) + g_{12}(s)u_2(s) + \cdots + g_{1m}(s)u_m(s), \\ y_2(s) &= g_{21}(s)u_1(s) + g_{22}(s)u_2(s) + \cdots + g_{2m}(s)u_m(s), \\ &\vdots \\ y_m(s) &= g_{m1}(s)u_1(s) + g_{m2}(s)u_2(s) + \cdots + g_{mm}(s)u_m(s). \end{aligned} \tag{4.3}$$

These equations are said to be *coupled*, since each individual input influences all of the outputs. If it is necessary to adjust one of the outputs without affecting any of the others, determining appropriate inputs $u_1, u_2, \ldots, u_m$ will be a difficult task in general. Consequently, there is considerable interest in designing control laws which remove this coupling, so that each input control *only* the corresponding output. A system of the form (4.1) is said to be (dynamically) *decoupled* (or non-interacting) if its transfer function matrix $G(s)$ is diagonal and non-singular, that is,

$$\begin{aligned} y_1(s) &= g_{11}(s)u_1(s), \\ y_2(s) &= g_{22}(s)u_2(s), \\ &\vdots \\ y_m(s) &= g_{mm}(s)u_m(s). \end{aligned} \tag{4.4}$$

and none of the $h_{ii}(s)$ are identically zero. Such a system may be viewed as consisting of m independent subsystems. Note that this definition depends upon the ordering of inputs and outputs, which is of course quite arbitrary. The definition will be extended to block-decoupling case in Section 4.5.

One example of a system requiring decoupling is a vertical take-off aircraft. The outputs of interest are pitch angle, horizontal position, and altitude, and the control variables consist of three different fan inputs. Since

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 115-128, 2003.

these are all coupled, the pilot must acquire considerable skill in order to simultaneously manipulate the three inputs and successfully control the aircraft. The system may be decoupled using state-variable feedback to provide the pilot with three independent and highly stable subsystems governing his pitch angle, horizontal position, and altitude.

It should be pointed out that dynamic decoupling is very demanding. A signal applied to input u_i must control output y_i and have *no* whatsoever effect on the other outputs. In many cases this requires a complex and highly sensitive control law, and in other cases it cannot be achieved at all (without additional compensation). It is useful, therefore, to consider a less stringent definition which involves only the steady-state behavior of the system.

A system of the form (4.2) is said to be *statically decoupled* if it is stable and its static gain matrix $G(0)$ is diagonal and non-singular. This means that for a step function input $u(t) = \alpha 1(t)$, where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_m]^T$ is constant, the outputs satisfy

$$\begin{aligned} \lim_{t\to\infty} y_1(t) &= g_{11}(0)\alpha_1, \\ \lim_{t\to\infty} y_2(t) &= g_{22}(0)\alpha_2, \\ &\vdots \\ \lim_{t\to\infty} y_m(t) &= g_{mm}(0)\alpha_m, \end{aligned} \tag{4.5}$$

where $g_{ii}(0) \neq 0, i = 1, 2, \ldots, m$. Again, note that the definition depends upon the ordering of inputs and outputs. A step change in one input u_i will in general cause transients to appear to all of the outputs, but (4.5) ensures that $y_j, j \neq i$, will be unchanged in a steady state. We will quickly solve the static decoupling problem in the next section and then concentrate on dynamic decoupling until section 4.5 from which we discuss block decoupling problem.

4.1 Static Decoupling

For the system:

$$\dot{x} = Ax + Bu, \tag{4.6}$$

$$y = Cx + Du, \tag{4.7}$$

we seek a control policy of the form

$$u = -Kx + Fr, \tag{4.8}$$

such that the $m \times m$ closed-loop transfer function given by

$$\begin{aligned} H(s) &= (C - DK)(sI - A + BK)^{-1}BF + DF \\ &= [(C - DK)(sI - A + BK)^{-1}B + D]F \end{aligned} \tag{4.9}$$

has a diagonal and non-singular $H(0)$, and the control system is stable.

Theorem 4.1.1. *The static decoupling problem by state feedback is solvable if and only if*

(i) (A, B) *is stabilizable; and*
(ii) *rank* $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + m.$

Proof. If (A, B) is stabilizable, one can find a K such that $(A - BK)$ is stable. Let us assume that such a K has been found. Consider

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \operatorname{rank}\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I_n & 0 \\ -K & I_m \end{bmatrix}\right] \\
&= \operatorname{rank}\begin{bmatrix} A - BK & B \\ C - DK & D \end{bmatrix} \\
&= \operatorname{rank}\left[\begin{bmatrix} I_n & 0 \\ (C - DK)(-A + BK)^{-1} & I_m \end{bmatrix}\begin{bmatrix} A - BK & B \\ C - DK & D \end{bmatrix}\right] \\
&= \operatorname{rank}\begin{bmatrix} A - BK & B \\ 0 & (C - DK)(-A + BK)^{-1}B + D \end{bmatrix}. \qquad (4.10)
\end{aligned}$$

Note that $(A - BK)$ is non-singular since it is stable. Thus,

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + m$$

implies that $\operatorname{rank}[(C - DK)(-A + BK)^{-1}B + D] = m$. With F chosen as $F = [(C - DK)(-A + BK)^{-1}B + D]^{-1}$, one gets $H(0) = I$. Hence static decoupling is possible. Conversely, if static decoupling is possible, (A, B) should be stabilizable (trivially). And

$$\operatorname{rank}[H(0)] = \operatorname{rank}[(C - DK)(-A + BK)^{-1}BF + DF] = m.$$

This requires that both F and $[(C - DK)(-A + BK)^{-1}B + D]$ are non-singular. Thus we have, again by (4.10), that

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A - BK & B \\ 0 & (C - DK)(-A + BK)^{-1}B + D \end{bmatrix} = n + m,$$

from which the necessity follows. □

Design for Static Decoupling. Assume that conditions (i) and (ii) in Theorem 4.1.1 are satisfied, thus the problem is solvable. We proceed as follows.

(i) Design a K such that the state feedback system is stable, i.e, $\det(sI - A + BK)$ has all its roots in the left half plane;
(ii) Obtain F as $F = [(C - DK)(-A + BK)^{-1}B + D]^{-1}$; and
(iii) Form the feedback control law as $u = -Kx + Fr$.

Example 4.1.1. Consider the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 1 & \beta \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

One can easily check that

$$\text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 5,$$

and the system is stabilizable. Hence static decoupling is possible. Since the system is stable, let $K = 0$. Choose

$$\begin{aligned} F &= [-CA^{-1}B]^{-1} \\ &= (1/36) \begin{bmatrix} -6 & -17 \\ 6 & 11 \end{bmatrix}, \text{when} \quad \beta = 1. \end{aligned}$$

The resulting closed-loop transfer function is

$$H(s) = \begin{bmatrix} 6(s+6) & -s(6s+25) \\ 6s(s+6) & (11s^2+66s+36) \end{bmatrix} \frac{1}{(s+1)(s+2)(s+3)6}.$$

Clearly, $H(0) = I$ as it should be. ◇

4.2 Dynamic Decoupling

In this section we consider dynamic decoupling of the system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{4.11}$$

where $x \in R^n, u, y \in R^m, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{m\times n}$, by the control law:

$$u = -Kx + Fr. \tag{4.12}$$

This problem is called decoupling by state feedback.

When the control law (4.12) is employed, the resultant system becomes

$$\dot{x} = (A - BK)x + BFr, \tag{4.13}$$

$$y = Cx, \tag{4.14}$$

and the transfer function matrix is given by

$$H(s) = C(sI - A + BK)^{-1}BF. \tag{4.15}$$

Therefore decoupling by state feedback requires one to find the control matrices K and F which make $H(s)$ diagonal and non-singular. Before treating the decoupling problem, we first consider the relation between the transfer function $H(s)$ and that of the plant (4.11).

$$G(s) = C(sI - A)^{-1}B. \tag{4.16}$$

Proposition 4.2.1. *The transfer function matrix $H(s)$ in (4.15) is related to $G(s)$ in (4.16) by*

$$\begin{aligned} H(s) &= G(s)\left[I - K(sI - A + BK)^{-1}B\right]F \\ &= G(s)\left[I + K(sI - A)^{-1}B\right]^{-1}F. \end{aligned} \tag{4.17}$$

Proof. One notes that

$$\begin{aligned} H(s) &= C(sI - A + BK)^{-1}BF \\ &= C(sI - A)^{-1}[(sI - A + BK) - BK](sI - A + BK)^{-1}BF \\ &= C(sI - A)^{-1}[I - BK(sI - A + BK)^{-1}]BF \\ &= C(sI - A)^{-1}[B - BK(sI - A + BK)^{-1}B]F \\ &= C(sI - A)^{-1}B[I - K(sI - A + BK)^{-1}B]F \\ &= G(s)[I - K(sI - A + BK)^{-1}B]F. \end{aligned}$$

Consider now

$$\begin{aligned} &[I - K(sI - A + BK)^{-1}B][I + K(sI - A)^{-1}B] \\ &= I - K(sI - A + BK)^{-1}(sI - A + BK - BK)(sI - A)^{-1}B \\ &\quad +K(sI - A)^{-1}B - K(sI - A + BK)^{-1}BK(sI - A)^{-1}B \\ &= I, \end{aligned}$$

which implies that

$$[I - K(sI - A + BK)^{-1}B] = [I + K(sI - A)^{-1}B]^{-1}, \tag{4.18}$$

and the proposition is thus proved. □

Proposition 4.2.1 indicates that controlling the system (4.11) with the control law (4.12) is equivalent to compensating for the system (4.11) serially by using the compensator

$$H_c(s) = [I - K(sI - A + BK)^{-1}B]F, \tag{4.19}$$

as shown in Figure 4.1. This compensator can be represented by the state space equations:

$$\dot{x}_c = (A - BK)x_c + BFr, \tag{4.20}$$

$$y_c = -Kx_c + Fr, \tag{4.21}$$

where r is the input to the compensator and y_c is its output.

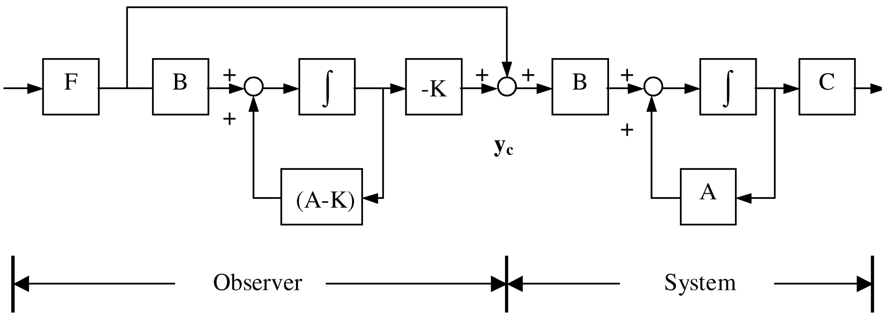


Fig. 4.1. Series Compensator

Let

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix},$$

and define integers $\sigma_i, i = 1, 2, \cdots, m$, by

$$\sigma_i = \begin{cases} \min \left(j | c_i^T A^{j-1} B \neq 0^T, \quad j = 1, 2, \cdots, n-1 \right); \\ \\ n-1; \text{ if } c_i^T A^{j-1} B = 0^T, \quad j = 1, 2, \cdots, n. \end{cases} \tag{4.22}$$

To actually evaluate σ_i, we start with $i = 1, c_i^T = c_1^T$. Let $j = 1$, then $c_i^T A^{j-1} B = c_1^T A^0 B = c_1^T B$. If $c_1^T B = 0$, then $\sigma_1 = 1$; otherwise, increase j to 2, if $c_1^T A^{j-1} B = c_1 AB = 0$, then $\sigma_1 = 2$; otherwise, increase j again until $j = n$. Repeat this procedure for $i = 2, 3, \cdots, m$.

With $\sigma_i, i = 1, 2, \cdots, m$, we can define

$$B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{bmatrix}, \tag{4.23}$$

and

$$C^* = \begin{bmatrix} c_1 A^{\sigma_1} \\ c_2 A^{\sigma_2} \\ \vdots \\ c_m A^{\sigma_m} \end{bmatrix}. \tag{4.24}$$

Theorem 4.2.1. *There exists a control law of the form (4.12) to decouple the system (4.11) if and only if the matrix B^* is non-singular. If this is the case, by choosing*

$$F = (B^*)^{-1},$$
$$K = (B^*)^{-1}C^*,$$

the resultant feedback system has the transfer function matrix:

$$H(s) = diag\{s^{-\sigma_1}, s^{-\sigma_2}, \cdots, s^{-\sigma_m}\}.$$

Proof. Necessity is proved first. Noting that

$$(sI - A)^{-1} = s^{-1}I + As^{-2} + A^2s^{-3} + \cdots, \quad \|A/s\| < 1,$$

the *i-th* row of the transfer function $G(s)$ of (4.16) can be expanded in polynomials of s^{-1} as

$$\begin{aligned} g_i(s)^T &= c_i^T(sI - A)^{-1}B = c_i^T Bs^{-1} + c_i^T ABs^{-2} + \cdots \\ &= c_i^T A^{\sigma_i - 1}Bs^{-\sigma_i} + c_i^T A^{\sigma_i}Bs^{-(\sigma_i+1)} + \cdots \\ &= s^{-\sigma_i}(c_i^T A^{\sigma_i-1}B + c_i^T A^{\sigma_i}Bs^{-1} + c_i^T A^{\sigma_i+1}Bs^{-2} + \cdots) \\ &= s^{-\sigma_i}\{c_i^T A^{\sigma_i-1}B + c_i^T A^{\sigma_i}(sI - A)^{-1}B\}. \end{aligned} \tag{4.25}$$

Using B^* of (4.23), $G(s)$ can be written as

$$G(s) = \begin{bmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{bmatrix} [B^* + C^*(sI - A)^{-1}B]. \tag{4.26}$$

Thus from Proposition 4.2.1 , the transfer function matrix of the feedback system is given by

$$\begin{aligned} H(s) &= G(s)[I - K(sI - A + BK)^{-1}B]F \\ &= \mathrm{diag}(s^{-\sigma_1}, s^{-\sigma_2}, \ldots, s^{-\sigma_m})[B^* + C^*(sI - A)^{-1}B] \\ &\quad \times [I - K(sI - A + BK)^{-1}B]F. \end{aligned} \tag{4.27}$$

For decoupling, $H(s)$ is non-singular, so are $[B^* + C^*(sI - A)^{-1}B][I - K(sI - A + BK)^{-1}B]F$, and its coefficient matrix for s^0, B^*F. Thus the non-singularity of B^* is a necessary condition for decoupling.

The sufficiency is proved by constructing the control law to decouple the system as follows. If B^* is non-singular, let K and F be chosen as

$$K = B^{*-1}C^*, \tag{4.28}$$
$$F = B^{*-1}. \tag{4.29}$$

It follows from Proposition 4.2.1 that

$$\begin{aligned} H(s) &= \mathrm{diag}\,(s^{-\sigma_1}, s^{-\sigma_2}, \ldots, s^{-\sigma_m})[B^* + C^*(sI - A)^{-1}B] \\ &\quad \times [F^{-1} + F^{-1}K(sI - A)^{-1}B]^{-1}. \end{aligned} \tag{4.30}$$

Substituting (4.29) and (4.28) into (4.30) gives

$$H(s) = \mathrm{diag}(s^{-\sigma_1}, s^{-\sigma_2}, \ldots, s^{-\sigma_m}), \tag{4.31}$$

and the system is decoupled. □

The form of the decoupled system (4.31) having $s^{-\sigma_i}$ as the diagonal elements is called an integrator decoupled system.

Example 4.2.1. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x.$$

One checks

$$c_1^T A^0 B = c_1^T B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq 0,$$

yielding $\sigma_1 = 1$; And

$$c_2^T A^0 B = c_2^T B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0,$$
$$c_2^T A^1 B = c_2^T AB = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

giving $\sigma_2 = 2$. But one has

$$B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \end{bmatrix} = \begin{bmatrix} c_1^T B \\ c_2^T AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which is singular, and the system cannot be decoupled.

It is noted that the B-matrix of the above system is of rank defect. It is trivial to show from the feedback system transfer function matrix $H(s)$ in (4.15) that a necessary condition for decoupling is that both C and B have full rank. $\diamondsuit$

Example 4.2.2. Consider the 2 by 2 system described by

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -3 & -3 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} x.$$

It is readily checked that

$$c_1^T A^0 B = c_1^T B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \textit{or} \quad \sigma_1 = 1;$$
$$c_2^T B = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad c_2^T AB = \begin{bmatrix} 4 & 3 \end{bmatrix}, \quad \textit{or} \quad \sigma_2 = 2,$$

so that

$$B^* = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

Since B^* is non-singular, the system can be decoupled by state feedback, $u = -Kx + Fr$. As

$$C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \end{bmatrix} = \begin{bmatrix} c_1^T A \\ c_2^T A^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix},$$

it follows from Theorem 4.1.1 that

$$F = B^{*-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -4 & 1 \end{bmatrix},$$

and

$$K = B^{*-1}C^* = \begin{bmatrix} -1 & 0 & 0 \\ \frac{5}{3} & \frac{4}{3} & 3 \end{bmatrix}.$$

The resulting closed-loop transfer function is

$$H(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}. \qquad \diamondsuit$$

Example 4.2.3. The linearized equations governing the equatorial motion of a satellite in a circular orbit is

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x,$$

where the outputs and inputs are radial and tangential

$$y = \begin{bmatrix} r \\ \theta \end{bmatrix}, \quad u = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}.$$

These are coupled by the orbital dynamics, as indicated by the transfer function matrix

$$H(s) = \begin{bmatrix} \dfrac{1}{s^2+\omega^2} & \dfrac{2\omega}{s(s^2+\omega^2)} \\ \dfrac{-2\omega}{s(s^2+\omega^2)} & \dfrac{s^2-3\omega^2}{s^2(s^2+\omega^2)} \end{bmatrix}.$$

To determine whether this system can be decoupled, we compute

$$c_1^T B = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad c_1^T AB = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \sigma_1 = 2;$$

$$c_2^T B = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad c_2^T AB = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \sigma_2 = 2.$$

Note that

$$B^* = \begin{bmatrix} c_1^T AB \\ c_2^T AB \end{bmatrix} = \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix} = I$$

is non-singular. Thus decoupling can be achieved with

$$u(t) = -B^{*-1}C^* x(t) + B^{*-1}v(t) = -\begin{bmatrix} 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x(t) + v(t),$$

which results in the closed-loop system:

$$\dot{x} = (A - BB^{*-1}C^*)x + Bv = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{bmatrix} x + \begin{bmatrix} 0\,0 \\ 1\,0 \\ 0\,0 \\ 0\,1 \end{bmatrix} v,$$

$$y = Cx = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix} x.$$

Note that this decoupled system is in the controllable canonical form, and we may apply additional feedback:

$$v = Kx + \bar{v} = \begin{bmatrix} k_1 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix} x + \begin{bmatrix} \bar{v}_r \\ \bar{v}_\theta \end{bmatrix},$$

to have

$$\dot{x} = (A - BB^{*-1}C^* + BK)x + \bar{v} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k_3 & k_4 \end{bmatrix} x + \begin{bmatrix} 0\,0 \\ 1\,0 \\ 0\,0 \\ 0\,1 \end{bmatrix} \bar{v},$$

$$y = Cx = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix} x.$$

If we choose the feedback gains $k_1 = k_3 = -1$ and $k_2 = k_4 = -2$, each of the two independent subsystems will be stable with a double pole at $s = -1$. The transfer function matrix may easily be shown to be

$$H(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix}.$$

This closed-loop system is not only stable (perturbations from the nominal trajectory will always decay to zero), but adjustments to r and θ can be made independently. ♢

In general, however, the decoupled system will not be in any canonical forms. How can pole placement be done together with decoupling?

4.3 Poles and Zeros of Decoupled Systems

In this section, the control gain matrices K and F are determined so that the decoupled system has pre-assigned poles. To this end, let

$$\bar{n} = \sum_{i=1}^{m} \sigma_i$$

be the number of the desired feedback system poles to be assigned. We split them into m subsets, $\lambda_{ij}, j = 1, 2, \cdots, \sigma_i$ for each i, $i = 1, 2, \cdots, m$, and form

$$\phi_i(s) = \sum_{j=1}^{\sigma_i} (s - \lambda_{ij}), \quad i = 1, 2, \cdots, m.$$

For each i, we express $\phi_i(s)$ as

$$\phi_i(s) = s^{\sigma_i} + \gamma_{i1} s^{\sigma_i - 1} + \gamma_{i2} s^{\sigma_i - 2} + \cdots + \gamma_{i\sigma_i},$$

and define

$$\phi_i(A) = A^{\sigma_i} + \gamma_{i1} A^{\sigma_i - 1} + \gamma_{i2} A^{\sigma_i - 2} + \cdots + \gamma_{i\sigma_i} I,$$

and

$$C^{**} = \begin{bmatrix} c_1^{**T} \\ c_2^{**T} \\ \vdots \\ c_m^{**T} \end{bmatrix} = \begin{bmatrix} c_1^T \phi_1(A) \\ c_2^T \phi_2(A) \\ \vdots \\ c_m^T \phi_m(A) \end{bmatrix}.$$

Note that $C^{**} = C^*$ if $\gamma_{ij} = 0$ for all i and j, or equivalently, $\lambda_{ij} = 0$, for all i and j.

Theorem 4.3.1. *Suppose that the system (4.11) can be decoupled by state feedback. If K and F are chosen as*

$$\begin{aligned} F &= (B^*)^{-1}, \\ K &= (B^*)^{-1} C^{**}, \end{aligned} \tag{4.32}$$

then the resultant feedback system has the transfer function given by

$$H(s) = diag \left\{ \frac{1}{\phi_1(s)}, \frac{1}{\phi_2(s)}, \cdots, \frac{1}{\phi_m(s)} \right\}. \tag{4.33}$$

Proof. Let the *i-th* row vector of the transfer function matrix of (4.16) be denoted by $g_i^T(s)$. Multiplying it by $(s^{\sigma_i} + \cdots + \gamma_{i\sigma_i})$ yields from (4.25)

$$\begin{aligned}
&(s^{\sigma_i} + \gamma_{i1}s^{\sigma_i - 1} + \cdots + \gamma_{i\sigma_i})g_i^T(s) \\
&= (1 + \gamma_{i1}s^{-1} + \gamma_{i2}s^{-2} + \cdots + \gamma_{i\sigma_i}s^{-\sigma_i}) \\
&\quad (c_i^T A^{\sigma_i - 1}B + c_i^T A^{\sigma_i}Bs^{-1} + c_i^T A^{\sigma_i + 1}Bs^{-2} + \cdots) \\
&= c_i^T A^{\sigma_i - 1}B + (c_i^T A^{\sigma_i}B + \gamma_{i1}c_i^T A^{\sigma_i - 1}B + \gamma_{i2}c_i^T A^{\sigma_i - 2}B + \cdots)s^{-1} \\
&\quad + (c_i^T A^{\sigma_i + 1}B + \gamma_{i1}c_i^T A^{\sigma_i}B + \gamma_{i2}c_i^T A^{\sigma_i - 1}B + \cdots)s^{-2} + \cdots \\
&= c_i^T A^{\sigma_i - 1}B + c_i^{**T}Bs^{-1} + c_i^{**T}ABs^{-2} + \cdots \\
&= c_i^T A^{\sigma_i - 1}B + c_i^{**T}(Is^{-1} + As^{-2} + A^2 s^{-3} + \cdots)B \\
&= c_i^T A^{\sigma_i - 1}B + c_i^{**T}(sI - A)^{-1}B. \qquad (4.34)
\end{aligned}$$

Dividing both sides of (4.34) by $(s^{\sigma_i} + \gamma_{i1}s^{\sigma_i-1} + \cdots + \gamma_{i\sigma_i})$, $G(s)$ is given by

$$\begin{aligned}
G(s) &= \text{diag}\,[(s^{\sigma_1} + \gamma_{11}s^{\sigma_i - 1} + \cdots + \gamma_{1\sigma_i})^{-1}, (s^{\sigma_2} + \gamma_{21}s^{\sigma_2 - 1} + \cdots + \gamma_{2\sigma_2})^{-1}, \\
&\quad \ldots, (s^{\sigma_m} + \gamma_{m1}s^{\sigma_m - 1} + \cdots + \gamma_{m\sigma_m})^{-1}][B^* + C^{**}(sI - A)^{-1}B]. \qquad (4.35)
\end{aligned}$$

Then it follows from (4.17) and (4.35) that

$$\begin{aligned}
H(s) &= \text{diag}[(s^{\sigma_1} + \gamma_{11}s^{\sigma_1 - 1} + \cdots + \gamma_{1\sigma_1})^{-1}, \ldots, (s^{\sigma_m} + \gamma_{m1}s^{\sigma_m - 1} + \cdots \\
&\quad + \gamma_{m\sigma_m})^{-1}][B^* + C^{**}(sI - A)^{-1}B][F^{-1} + F^{-1}K(sI - A)^{-1}B]^{-1}. \qquad (4.36)
\end{aligned}$$

Thus by choosing K and F as (4.32), the transfer function matrix of the closed loop system is given by (4.33). □

Example 4.3.1. Consider the system:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x.$$

We have $c_1^T B = \begin{bmatrix} 1 & 0 \end{bmatrix}$, which yields $\sigma_1 = 1$ and $c_1^T A = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, and $c_2^T B = \begin{bmatrix} 0 & 1 \end{bmatrix}$, which yields $\sigma_2 = 1$ and $c_2^T A = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$. Therefore, B^* and C^* are

$$B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C^* = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}.$$

Since B^* is non-singular, the system can be decoupled. Using Theorem 4.1.1, the control law has

$$\begin{aligned}
F &= B^{*-1} = I, \\
K &= B^{*-1}C^* = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix},
\end{aligned}$$

which will lead to the integrator decoupled closed-loop:

$$H(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}.$$

However, if we want to decouple the system to get

$$H(s) = \mathrm{diag}[(s+1)^{-1}, (s+2)^{-1}],$$

then, we have $\gamma_{11} = 1, \gamma_{21} = 2$. (4.32) becomes

$$\begin{aligned} K &= \begin{bmatrix} c_1^T A \\ c_2^T A \end{bmatrix} + \begin{bmatrix} \gamma_{11} c_1^T \\ \gamma_{21} c_2^T \end{bmatrix} \\ &= C^* + \begin{bmatrix} 1 c_1^T \\ 2 c_2^T \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \end{aligned}$$

and $F = I$ remains the same as before. ♢

In the above example, a $3rd$-order system has been decoupled to give a $2nd$-order system. Thus a pole-zero cancellation has taken place. In the next theorem the zero positions are considered in the decoupling.

Theorem 4.3.2. *The linear system (4.11) can be decoupled by the feedback control law (4.12) with the i-th diagonal element of the transfer function of the decoupled system having a numerator $n_i(s)$ if and only if the i-th row of $G(s)$, denoted by $g_i^T(s)$, is represented by*

$$g_i^T(s) = c_i^T (sI - A)^{-1} B = n_i(s) \bar{c}_i^T (sI - A)^{-1} B, \quad i = 1, \dots, m, \tag{4.37}$$

and

$$\bar{B}^* = \begin{bmatrix} \bar{c}_1^T A^{(\bar{\sigma}_1 - 1)} B \\ \bar{c}_2^T A^{(\bar{\sigma}_2 - 1)} B \\ \vdots \\ \bar{c}_m^T A^{(\bar{\sigma}_m - 1)} B \end{bmatrix} \tag{4.38}$$

is non-singular, where $\bar{\sigma}_i$ is

$$\bar{\sigma}_i = \begin{cases} min(j | \bar{c}_i^T A^{j-1} B \neq 0^T, \quad j = 1, 2, \cdots, n-1); \\ n - 1 : \bar{c}_i^T A^{j-1} B = 0^T, \quad j = 1, 2, \cdots, n. \end{cases}$$

Proof. Sufficiency is proved first. Let

$$\bar{C} = \begin{bmatrix} \bar{c}_1^T \\ \vdots \\ \bar{c}_m^T \end{bmatrix}.$$

Then it follows from (4.37), the transfer function matrix $G(s)$ is

$$G(s) = \mathrm{diag}[n_1(s), n_2(s), \dots, n_m(s)]\bar{C}(sI - A)^{-1}B. \tag{4.39}$$

By Theorem 4.2.1 and (4.38), $(A, B, \bar{C})$ can be decoupled to give

$$H(s) = \mathrm{diag}\,[n_1(s), n_2(s), \dots, n_m(s)]\mathrm{diag}\,[d_1^{-1}(s), \dots, d_m^{-1}(s)]. \tag{4.40}$$

Therefore, the i-th component of the decoupled transfer function is represented by $n_i(s)/d_i(s)$ and $d_i(s)$ is determined to have no common factors with $n_i(s)$. Thus sufficiency is proved.

Necessity is proved next. When the diagonal element of the decoupled system is given by (4.40), the i-th row vector $h_i^T(s)$ of the decoupled transfer function is

$$\begin{aligned} h_i^T(s) &= c_i^T(sI - A + BK)^{-1}BF \\ &= n_i(s)/d_i(s)e_i^T, \end{aligned} \tag{4.41}$$

which implies that

$$h_i^T(s) = n_i(s)\bar{c}_i^T(sI - A + BK)^{-1}BF. \tag{4.42}$$

On the other hand, it follows from (4.17), (4.41) and (4.42) that

$$\begin{aligned} &c_i^T(sT - A)^{-1}B[I + K(sI - A)^{-1}B]^{-1}F \\ &= n_i(s)\bar{c}_i^T(sI - A)^{-1}B[I + K(sI - A)^{-1}B]F. \end{aligned}$$

(4.39) is then derived by dividing both sides of the above by $[I + K(sI - A)^{-1}B]^{-1}F$ from the right side. □

The above theorem indicates that the numerator terms in the decoupled system are given by the properties of the original system and cannot be changed.

4.4 Notes and References

The problem of decoupling linear time-invariant multivariable systems received considerable attention in the system theoretic literature for several decades. Much of this attention was directed toward decoupling by state feedback (Morgan, 1964; Falb and Wolovich, 1967; Hautus and Heymann, 1983; Descusse, 1991; Descusse *et al.*, 1988; Wonham, 1986). More general block decoupling were investigated by Koussiouris (1979), Koussiouris (1980), Commault *et al.* (1991), Descusse (1991), Williams and Antsaklis (1986). Sections 2 and 3 are based on Falb and Wolovich (1967), Wolovich and Falb (1969). The algebraic approach has been exclusively taken in this chapter. For the geometric approach, see Wonham (1986), Camarta and Martínez-García (2001).

5. Polynomial Matrix Approach

In contrast to the preceding chapter where a state space model of the plant and state feedback controller are exclusively utilized for decoupling problem, we will employ unity output feedback compensation to decouple the plant in this chapter. Polynomial matrix fractions as an input-output representation of the plant appear a natural and effective approach to dealing with output feedback decoupling problem. The resulting output feedback controller can be directly implemented without any need to construct a state estimator. In particular, we will formulate a general decoupling problem and give some preliminary results in Section 1. We start our journey with the diagonal decoupling problem for square plants in Section 2 and then extend the results to the general block decoupling case for possible non-square plants in Section 3. A unified and independent solution is also presented in Section 4. In all the cases, stability is included in the discussion. A necessary and sufficient condition for solvability of the given problem is first given, and the set of all compensators solving the problem is then characterized. Performance limitations of decoupled systems are compared with a synthesis without decoupling. Numerous examples are presented to illustrate the relevant concepts and results.

5.1 Problem Formulation and Preliminaries

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}[s]$ be the ring of polynomials, $\mathbb{R}(s)$ the field of rational functions, $\mathbb{R}_p(s)$ the ring of proper rational functions, and $\mathbb{R}_{sp}(s)$ the ring of strictly proper rational functions, respectively, all with coefficients in $\mathbb{R}$. Consider an unity output feedback system shown in Figure 5.1, where $G \in \mathbb{R}_p^{m\times l}(s)$, the set of $m \times l$ proper rational matrices, is a given plant and $K \in \mathbb{R}_p^{l\times m}(s)$ is a compensator. Let $D^{-1}N = G$ and $N_k D_k^{-1} = K$ be, respectively, left and right coprime polynomial matrix fractions. A list of integers $(m_1, m_2, \cdots, m_v)$ is called a partition of m if $m_i \geq 1$, $i = 1, 2, \cdots, v$, and $\Sigma_{i=1}^{v} m_i = m$. The system in Figure 5.1 is said to be internally stable (Chapter 3) if $det[DD_k + NN_k]$ has all its roots in $\mathbb{C}^-$, the open left half of the complex plane, and to be block-decoupled with respect to the partition $(m_1, m_2, \cdots, m_v)$ if the closed-loop transfer function matrix between r and y is nonsingular and block-diagonal, i.e.,

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 129-185, 2003.

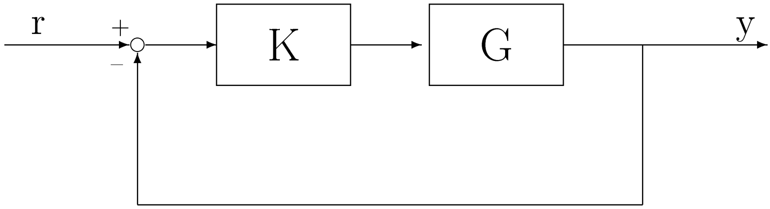


Fig. 5.1. Unity Output Feedback System

$$\begin{aligned} T &= GK[I+GK]^{-1} \\ &= \begin{bmatrix} T_{11} & & & 0 \\ & T_{22} & & \\ & & \ddots & \\ 0 & & & T_{vv} \end{bmatrix} \\ &= block\ diag\{T_{11}, T_{22}, \cdots, T_{vv}\}, \end{aligned} \tag{5.1}$$

with $T_{ii} \in \mathbb{R}_p^{m_i \times m_i}(s)$ being nonsingular for all i, $i = 1, 2, \cdots, v$. The case of $v = m$ and $m_1 = m_2 = \cdots = m_v = 1$ is called diagonal decoupling. If the list of positive integers $\{m_i\}$ is obviously known from the context and no confusion arises, we denote by $\mathbb{R}_d^{m \times m}[s]$, $\mathbb{R}_d^{m \times m}(s)$, $\mathbb{R}_{dp}^{m \times m}(s)$ and $\mathbb{R}_{dsp}^{m \times m}(s)$ the set of block diagonal and nonsingular elements in $\mathbb{R}^{m \times m}[s]$, $\mathbb{R}^{m \times m}(s)$, $\mathbb{R}_p^{m \times m}(s)$ and $\mathbb{R}_{sp}^{m \times m}(s)$, respectively. The noninteracting control problem to be solved in this chapter is stated as follows.

(P). *Let a plant $G \in \mathbb{R}_p^{m \times l}(s)$ and a partition (m_i) be given. Find a controller $K \in \mathbb{R}_p^{l \times m}(s)$ such that the resulting system in Figure 5.1 is internally stable and block decoupled.*

The plant G has to have rank m in order for T to be nonsingular. Therefore, throughout the paper, we assume that the plant G has full row rank. We define zeros and poles of a transfer matrix G in terms of its Smith-McMillan form (Chapter 2). A pole or zero is called a LHP one if it is in $\mathbb{C}^-$, and a RHP one if it is in $\mathbb{C}^+$, the closed right half of the complex plane. The following concept of a stability factorization will result in a simplification of **(P)**.

Definition 5.1.1. *Let $G \in \mathbb{R}^{m \times l}(s)$ be of full row rank. A factorization of G in the form of $G = [G^+ \ \ 0]G^-$ is called a stability factorization if $G^+ \in \mathbb{R}^{m \times m}(s)$ and $G^- \in \mathbb{R}^{l \times l}(s)$ are nonsingular, the zeros and poles of G^+ are precisely the RHP zeros and poles of G, and the zeros and poles of G^- are precisely the LHP zeros and poles of G, counting multiplicities.*

Note that $G^+$and G^- may be non-proper, even if G is proper. A stability factorization can easily be constructed from the Smith-McMillan form of G.

Lemma 5.1.1. *Any full row rank matrix $G \in \mathbb{R}^{m\times l}(s)$ has a stability factorization $G = [G^+ \ \ 0]G^-$. The matrix G^+ is unique up to a multiplication by an unimodular polynomial matrix from the right. In the case of square G, G^- is unique up to a multiplication by an unimodular polynomial matrix from the left.*

Proof. It follows from Chapter 2 that there are unimodular polynomial matrices U_1 and U_2 such that

$$G = U_1 S U_2,$$

where $S = [diag\{r_i\} \ \ 0]$ with $r_i \in \mathbb{R}(s)$ is the Smith-McMillan form of G. Factorize each r_i as r_i^+ and r_i^- such that all zeros and poles of r_i^+ are in $\mathbb{C}^+$, while all zeros and poles of r_i^- lie in $\mathbb{C}^-$. We then obtain a stability factorization of G as $G = [G^+ \ \ 0]G^-$, where

$$G^+ = U_1 diag\{r_i^+\},$$

and

$$G^- = block\ diag\{diag\{r_i^-\}, I\}U_2.$$

Consider now any two stability factorizations

$$G = [G_1^+ \ \ 0]G_1^- = [G_2^+ \ \ 0]G_2^-.$$

We have

$$[G_1^+ \ \ 0] = [G_2^+ \ \ 0]G_2^-(G_1^-)^{-1}.$$

Define $U = G_2^-(G_1^-)^{-1}$ and partition it as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

with $U_{11} \in \mathbb{R}^{m\times m}(s)$, so that

$$[G_1^+ \ \ 0] = [G_2^+ U_{11} \ \ G_2^+ U_{12}]$$

and

$$G_1^+ = G_2^+ U_{11}. \tag{5.2}$$

Note that all the distinct poles of U_{11} must be the poles of $U = G_2^-(G_1^-)^{-1}$ which is stable. So U_{11} can only have stable poles if any, and by (5.2) these poles must be also the poles of G_1^+ because they cannot be cancelled by G_2^+. But G_1^+ cannot have any stable poles by definition, and thus U_{11} must not have any pole. By a similar argument using $G_2^+ = G_1^+ U_{11}^{-1}$, U_{11}^{-1} must not have any pole either. Hence, U_{11} is unimodular. For square G, $[G^+ \ \ 0]$ reduces to G^+ only and $U_{11} = U$. $UG_1^- = G_2^-$ and the proof is completed. □

Note that for nonsquare full row rank G, G^- is highly non-unique: its last $l-m$ rows are arbitrary subject to the constraint that no poles and zeros are introduced. The following special stability factorization will be required later.

Corollary 5.1.1. *There exist unique square polynomial matrices N_g^+ and D_g^+ and a matrix G^- such that $G_g^+ := N_g^+(D_g^+)^{-1}$ and G^- define a stability factorization, N_g^+ is in the Hermite column form and D_g^+ is in the Hermite row form.*

Proof. Let $G = [G_1^+ \ \ 0]G_1^-$ be a stability factorization and $G_1^+ = N^+(D^+)^{-1}$ be a coprime polynomial matrix fraction. There exists a unique unimodular polynomial matrix U such that $N_g^+ := N^+U$ is in the required form (Theorem 2.2.2 of Chapter 2). Moreover, there exists a unique unimodular polynomial matrix V such that $D_g^+ := V(D^+U)$ is in the required form. □

Let us introduce further notations. Consider a rational matrix R and assume that $D_l^{-1}N_l$ and $N_rD_r^{-1}$ are coprime polynomial matrix fractions such that $D_l^{-1}N_l = N_rD_r^{-1} = R$. The McMillan degree $\delta(R)$ of R (Definition 2.3.2 of Chapter 2) is given by $\delta(R) := deg(det(D_l)) = deg(det(D_r))$. The McMillan RHP degree of R is defined to be the number of all the RHP poles of R (counting multiplicities), and is given by $\delta^+(R) := deg(det(D_l^+)) = deg(det(D_r^+))$, where $D_l = D_l^+D_l^-$ and $D_r = D_r^+D_r^-$ are stability factorizations. It is obvious that $\delta^+(G) = \delta^+(G^+) = \delta(G^+)$ for a stability factorization $G = [G^+ \ \ 0]G^-$. Let G and K be two rational matrices of compatible size. The series connection GK is said to have no pole-zero cancellations if the equality $\delta(G)+\delta(K) = \delta(GK)$ holds. Similarly, the equality $\delta^+(G)+\delta^+(K) = \delta^+(GK)$ means that GK has no RHP pole-zero cancellations.

Consider **(P)** now. Assume first that **(P)** is solvable, then the closed-loop transfer matrix given in (5.1) is nonsingular and block-diagonal. Its inverse is

$$T^{-1} = I + (GK)^{-1}.$$

Thus, we have $GK \in \mathbb{R}_d^{m\times m}(s)$. The internal stability of the closed-loop system ensures that GK has no RHP pole-zero cancellations. Conversely, if there is a $K \in \mathbb{R}^{l\times m}(s)$ such that $GK \in \mathbb{R}_d^{m\times m}(s)$ and it has no RHP pole-zero cancellations, then we can always choose a set of stable polynomial matrices $P_i \in \mathbb{R}^{m_i\times m_i}[s]$ such that $GK\, block\, diag\{P_i^{-1}\}$ is strictly proper and $K\, block\, diag\{P_i^{-1}\}$ is proper. It follows from Chapter 3 that there is a proper rational matrix $block\, diag\{R_i\}$ such that it internally stabilizes $GK\, block\, diag\{P_i^{-1}\}$. From the above construction, it is obvious that $\bar{K} = K\, block\, diag\{P_i^{-1}R_i\}$ is a solution to **(P)**. Therefore, we have established the following theorem.

Theorem 5.1.1. *For a plant $G \in \mathbb{R}_p^{m\times l}(s)$,* **(P)** *is solvable if and only if there is a $K \in \mathbb{R}^{l\times m}(s)$ such that GK is block-diagonal and nonsingular, and has no RHP pole-zero cancellations.*

Theorem 5.1.1 means that **(P)** is equivalent to the decoupling problem for a plant G by precompensator K without RHP pole-zero cancellations in GK, a great simplification from the closed-loop problem to the open-loop problem. It is noted that in the latter case, K is not required to be proper.

The skew prime polynomial matrices were introduced by Wolovich (1978) and have appeared in linear system theory for solving various output regulation and/or tracking problems. It turns out that our solvability conditions for **(P)** also include a skew primeness one.

Definition 5.1.2. *Let a full row rank $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $D \in \mathbb{R}^{m\times m}[s]$ be given. If there are $\bar{N} \in \mathbb{R}^{m\times l}[s]$ and $\bar{D} \in \mathbb{R}^{l\times l}[s]$ such that*

$$DN = \bar{N}\bar{D},$$

where D and $\bar{N}$ are left coprime, and N and $\bar{D}$ right coprime, then N and D are called as externally skew prime and $\bar{N}$ and $\bar{D}$ as internally skew prime.

Algorithms for constructing $\bar{N}$ and $\bar{D}$ from N and D have been given in Wolovich (1978) and Wang, Sun and Zhou (1989). In particular, if N and D have no common zeros, then they are externally skew prime. In the sequel development, we will make frequent use of the following lemma.

Lemma 5.1.2. *Let A, B, C, D be all polynomial matrices such that $AB^{-1}C = D$. If $B^{-1}C$ (resp. AB^{-1}) is coprime, then AB^{-1} (resp. $B^{-1}C$) is a polynomial matrix.*

Proof: If AB^{-1} is not a polynomial matrix, let $\tilde{A}\tilde{B}^{-1}$ be a coprime polynomial matrix fraction of AB^{-1} with $deg(det\tilde{B}) > 0$, then $D = AB^{-1}C = \tilde{A}\tilde{B}^{-1}C$ with $\tilde{A}\tilde{B}^{-1}$ and $\tilde{B}^{-}C$ both coprime. It follows (Theorem 2.13 of Chapter 2) that D is a rational matrix with its Mcmillan degree $\delta(D) = deg(det\tilde{B})$ nonzero, contradicting the assumption that D is a polynomial matrix. The proof is completed. □

5.2 Diagonally Decoupling for Square Plants

Let us consider first the simplest case of **(P)**: diagonal decoupling for square plants, which is abbreviated as **(PS)** for ease of frequent future reference. Then G should be square and nonsingular. Our task, by Theorem 5.1.1, is to find a nonsingular K such that GK is nonsingular and diagonal and has no RHP pole-zero cancellations.

5.2.1 Solution

The solution K is shown in the following theorem to be representable as the product of the plant inverse and a diagonal nonsingular matrix.

Theorem 5.2.1. **(PS)** *is solvable if and only if there is a* $T \in \mathbf{R}_d^{m\times m}(s)$ *such that* GK *with* $K = G^{-1}T$ *has no RHP pole-zero cancellations.*

PROOF: If **(PS)** is solvable, it follows from Theorem 5.1.1 that there is a K such that GK has no RHP pole-zero cancellations and $GK = T$ for some $T \in \mathbb{R}_d^{m\times m}$. Then $K = G^{-1}T$. On the other hand, if there is a $T \in \mathbb{R}_d^{m\times m}$ such that GK with $K = G^{-1}T$ has no RHP pole-zero cancellations, then $GK = T$ is diagonal and nonsingular, and by Theorem 5.1.1, **(PS)** is solvable. The proof is completed. □

The concept of strict adjoints of a polynomial matrix was introduced by Hammer and Khargonekar (1984). Let $P, P_{ra} \in \mathbb{R}^{m\times m}[s]$ be nonsingular, P_{ra} is said to be a right strict adjoint of P whenever the following conditions are satisfied: (i) $PP_{ra} \in \mathbf{R}_d^{m\times m}[s]$, and (ii) if any nonsingular $Q \in \mathbb{R}^{m\times m}[s]$ is such that $PQ \in \mathbb{R}_d^{m\times m}[s]$, then P_{ra} is a left divisor of Q, i.e., $Q = P_{ra}Q_1$ for some $Q_1 \in \mathbf{R}^{m\times m}[s]$.

Lemma 5.2.1. *Let* $P \in \mathbb{R}^{m\times m}[s]$ *be nonsingular, then* P *has a right strict adjoint.*

A right strict adjoint of P can be constructed as follows. The inverse P^{-1} of P is first computed. Let $(P^{-1})_i$ be the i-th column of P^{-1} and c_i be the least common multiple of denominators of all elements in $(P^{-1})_i$, then it can be shown (see the general case in Lemma 5.3.1 and its proof) that $P_{ra} = P^{-1}diag\{c_i\}$ is a right strict adjoint of P. The concept of left strict adjoints can be defined in a dual way and the dual result to Lemma 5.2.1 also holds.

Consider now the plant G in details. By Lemma 5.1.1, there is a stability factorization for the nonsingular G:

$$G = G^+G^-. \tag{5.3}$$

We further factorize G^+ as

$$G^+ = N^+(D^+)^{-1}, \tag{5.4}$$

where N^+ and D^+ are nosingular polynomial matrices and right coprime. By Lemma 5.2.1, N^+ has a right strict adjoint N_{ra}^+ such that $N^+N_{ra}^+ \in \mathbb{R}_d^{m\times m}[s]$. Further, if N_{ra}^+ and D^+ are externally skew prime, there exists a pair of internally skew prime polynomial matrices $\bar{N}_{ra}^+$ and $\bar{D}^+$ such that

$$D^+N_{ra}^+ = \bar{N}_{ra}^+\bar{D}^+. \tag{5.5}$$

Let $\bar{D}_{la}^+$ be a left strict adjoint of $\bar{D}^+$, we can now state the main solvability conditions for **(PS)**.

Theorem 5.2.2. (PS) *is solvable if and only if N^+_{ra} and D^+ are externally skew prime and $N^+N^+_{ra}$ and $\bar{D}^+_{la}\bar{D}^+$ are coprime.*

Proof. Sufficiency: If the conditions in the theorem are satisfied, then we take K as

$$K = G^{-1}T, \tag{5.6}$$

where $T \in \mathbb{R}^{m\times m}_d(s)$ is given by the coprime polynomial matrix fraction

$$T = N^+N^+_{ra}(\bar{D}^+_{la}\bar{D}^+)^{-1}. \tag{5.7}$$

This K can be reduced as

$$\begin{aligned} K &= (G^-)^{-1}D^+(N^+)^{-1}N^+N^+_{ra}(\bar{D}^+_{la}\bar{D}^+)^{-1} \\ &= (G^-)^{-1}\bar{N}_{ra}\bar{D}^+(\bar{D}^+_{la}\bar{D}^+)^{-1} \\ &= (G^-)^{-1}\bar{N}^+_{ra}(\bar{D}^+_{la})^{-1}. \end{aligned} \tag{5.8}$$

Then from (5.3) and (5.4) we obtain

$$\delta^+(G) = deg(det(D^+)). \tag{5.9}$$

(5.7) and (5.8) imply that

$$\delta^+(T) = deg(det(\bar{D}^+)) + deg(det(\bar{D}^+_{la})), \tag{5.10}$$

and

$$\delta^+(K) \le deg(det(\bar{D}^+_{la})). \tag{5.11}$$

It follows from the skew primeness equation (5.5) that

$$deg(det(\bar{D}^+)) = deg(det(D^+)). \tag{5.12}$$

Collecting (5.9)-5.12) yields

$$\delta^+(T) \ge deg^+(G) + deg^+(K), \tag{5.13}$$

which means that GK has no RHP pole-zero cancellations since $GK = T$. By Theorem 5.2.1, **(PS)** is solvable.

Necessity: If **(PS)** is solvable, then by Theorem 5.2.1, there is a $T \in \mathbb{R}^{m\times m}_d(s)$ such that GK with $K = G^{-1}T$ has no RHP pole-zero cancellations. Let $T = N_TD_T^{-1}$, with $N_T, D_T \in \mathbb{R}^{m\times m}_d[s]$, be coprime, K then becomes

$$K = G^{-1}T = (G^-)^{-1}D^+(N^+)^{-1}N_TD_T^{-1}. \tag{5.14}$$

We claim that no RHP pole-zero cancellations in GK imply that $(N^+)^{-1}N_T$ is a polynomial matrix. Suppose, conversely, that $(N^+)^{-1}N_T$ is a rational matrix. Let N^+_1 be a greatest common left divisor of N^+ and N_T, then

$(N_2^+)^{-1}N_{T2}$ with $N_2^+ = (N_1^+)^{-1}N^+$ and $N_{T2} = (N_1^+)^{-1}N_T$ is a coprime polynomial matrix fraction and $(N_2^+)^{-1}$ is not cancelled in $G^{-1}T$. Therefore, there must be such a representation

$$(\tilde{N}_2^+)^{-1}Q = (G^-)^{-1}D^+(N_2^+)^{-1}$$

with $det(\tilde{N}_2^+) = det(N_2^+)$ that $\tilde{N}_2^+$ is a divisor of the denominator of a left coprime polynomial matrix fraction of K, that is,

$$K = (D_K\tilde{N}_2^+)^{-1}N_K$$

with $D_K\tilde{N}_2^+$ and N_K left coprime. But then this unstable $\tilde{N}_2^+$ will disappear in GK because

$$GK = [(\tilde{N}_2^+)^{-1}Q(N_1^+)^{-1}]^{-1}[(D_K\tilde{N}_2^+)^{-1}N_K] = N_1^+Q^{-1}D_K^{-1}N_K.$$

This contradicts the assumption of no RHP pole-zero cancellations in GK. Thus, we have

$$(N^+)^{-1}N_T = P$$

or equivalently

$$N_T = N^+P$$

for some polynomial matrix P. Since $N_T \in \mathbb{R}_d^{m\times m}[s]$, we obtain

$$N_T = N^+N_{ra}^+P_1, \tag{5.15}$$

where N_{ra}^+ is a right strict adjoint of N^+ and $P_1 \in \mathbb{R}_d^{m\times m}[s]$. Substituting (5.15) into (5.14) gives

$$K = (G^-)^{-1}D^+N_{ra}^+D_T^{-1}P_1.$$

The coprimeness of N_T and D_T implies that N_{ra}^+ and D_T are right coprime. Construct a dual left coprime polynomial matrix fraction from $N_{ra}^+D_T^{-1}$ as

$$\tilde{D}_T^{-1}\tilde{N}_{ra}^+ = N_{ra}^+D_T^{-1} \tag{5.16}$$

with

$$det(\tilde{N}_{ra}^+) = a_1\, det(N_{ra}^+), \quad a_1 \in \mathbb{R}. \tag{5.17}$$

K becomes

$$K = (G^-)^{-1}D^+\tilde{D}_T^{-1}\tilde{N}_{ra}^+P_1.$$

Similarly, no RHP pole-zero cancellations in GK ensure that there is a $\tilde{P}_2 \in \mathbb{R}^{m\times m}[s]$ such that

$$\tilde{D}_T = \tilde{P}_2 D^+. \tag{5.18}$$

Substituting (5.18) into (5.16) yields

$$D_T = \bar{P}_2 (\bar{N}_{ra}^+)^{-1} D^+ N_{ra}^+, \tag{5.19}$$

where

$$\bar{P}_2 (\bar{N}_{ra}^+)^{-1} = (\tilde{N}_{ra}^+)^{-1} \tilde{P}_2 \tag{5.20}$$

is right coprime with

$$det(\bar{N}_{ra}^+) = a_2\, det(\tilde{N}_{ra}^+), \quad a_2 \in \mathbb{R}, \tag{5.21}$$

and

$$\bar{N}_{ra}^+ \text{ and } D^+ \text{ are left coprime.} \tag{5.22}$$

Equations (5.21) and (5.22) hold due to the left coprimeness of $\tilde{D}_T$ and $\tilde{N}_{ra}^+$. Because D_T is a polynomial matrix and $\bar{P}_2$ and $\bar{N}_{ra}^+$ are right coprime, it follows from (5.19) that

$$(\bar{N}_{ra}^+)^{-1} D^+ N_{ra}^+ = \bar{D}^+ \tag{5.23}$$

for some polynomial matrix $\bar{D}^+$. Equation (5.23), together with (5.17), (5.21), and (5.22), implies that N_{ra}^+ and D^+ are externally skew prime and $\bar{N}_{ra}^+$ and $\bar{D}^+$ just constitute their dual internally skew prime pair. With (5.23), (5.19) is simplified into

$$D_T = \bar{P}_2 \bar{D}^+.$$

Again, it follows from $D_T \in \mathbb{R}_d^{m\times m}[s]$ that

$$D_T = P_2 \bar{D}_{la}^+ \bar{D}^+, \tag{5.24}$$

where $\bar{D}_{la}^+$ is a left strict adjoint of $\bar{D}^+$ and $P_2 \in \mathbb{R}_d^{m\times m}[s]$. From (5.15) and (5.24), one sees that $N^+ N_{ra}^+$ and $\bar{D}_{la}^+ \bar{D}^+$ are, respectively, divisors of *coprime* N_T and D_T and thus they are certainly coprime. The proof is completed. □

Specializing Theorem 5.2.2 to the case of no coinciding RHP pole and zeros, we obtain the following corollary.

Corollary 5.2.1. *For a nonsingular plant G, if $det(N^+)$ and $det(D^+)$ are coprime, that is, the plant has no RHP poles coinciding with zeros, then* **(PS)** *is solvable.*

Proof. From the inverse matrix construction, we obtain $N^+ adj\{N^+\} = det(N^+) I$. It follows from the definition of strict adjoints that

$$N^+ N_{ra}^+ P = det(N^+) I \tag{5.25}$$

for some $P \in \mathbb{R}_d^{m\times m}[s]$. Hence, when $det(N^+)$ and $det(D^+)$ are coprime, so are $det(N_{ra}^+)$ and $det(D^+)$ and in this case N_{ra}^+ and D^+ are externally skew prime. From the skew primeness equation (5.5), one has

$$det(\bar{D}^+) = a_1\, det(D^+), \quad a_1 \in \mathbb{R}. \tag{5.26}$$

A similar equation to (5.25) also holds for $\bar{D}^+$ as

$$P' \bar{D}_{la}^+ \bar{D}^+ = det(\bar{D}^+) I \tag{5.27}$$

for some $P' \in \mathbb{R}_d^{m\times m}[s]$. Under (5.25), (5.26) and (5.27), the coprimeness of $det(N^+)$ and $det(D^+)$ implies that $N^+ N_{ra}^+$ and $\bar{D}_{la}^+ \bar{D}^+$ are also coprime and, by Theorem 5.2.2, **(PS)** is solvable. □

Apart from the solvability conditions, one needs to characterize decoupling compensators and achievable loop maps. The following corollary is presented for this purpose.

Corollary 5.2.2. *If a nonsingular plant G satisfies the conditions of Theorem 5.2.2, then*

(i) all the compensators which decouple G with internal stabilizability are given by

$$K = (G^-)^{-1} \bar{N}_{ra}^+ (\bar{D}_{la}^+)^{-1} K_2, \tag{5.28}$$

where $K_2 \in \mathbb{R}_d^{m\times m}(s)$ is such that $K_1 K_2$ with $K_1 = N^+ N_{ra}^+ (\bar{D}_{la}^+ \bar{D}^+)^{-1}$ has no RHP pole-zero cancellations.

(ii) all the achievable open-loop maps GK under decoupling with internal stabilizability can be expressed by

$$GK = K_1 K_2,$$

where K_1 and K_2 are the same as in (i).

Proof. It follows from the proof of Theorem 5.2.2 that every K which decouples G without RHP pole-zero cancellations has the form

$$K = G^{-1} N_T D_T^{-1},$$

where $N_T = N^+ N_{ra}^+ P_1$, $D_T = P_2 \bar{D}_{la}^+ \bar{D}^+$ for some $P_1, P_2 \in \mathbb{R}_d^{m\times m}[s]$, and N_T and D_T are coprime. Define $K_2 = P_1 P_2^{-1}$, then $N_T D_T^{-1} = K_1 K_2$. The coprimeness of N_T and D_T implies that $K_1 K_2$ has no RHP pole-zero cancellations. Simple calculations yield

$$K = G^{-1} K_1 K_2 = (G^-)^{-1} \bar{N}_{ra}^+ (\bar{D}_{la}^+)^{-1} K_2,$$

which is (5.28). In addition, we have $GK = K_1 K_2$ as well, and hence the corollary follows. □

From the point of view of closed-loop system design, all remains to do is to construct a diagonal compensator $\tilde{K}$ for the decoupled plant $\tilde{G} = GG^{-1}K_1 = K_1$. It becomes a sequence of SISO design problems for which effective methods are available.

The decoupling compensators constructed so far usually result in LHP pole-zero cancellations $(G^-(G^-)^{-1})$ so that arbitrary pole assignment is not achievable. However, by imposing the stronger constraint, no pole-zero cancellations in GK, instead of only RHP ones, we can directly obtain the following result and the proof is omitted because of its similarity to that of Theorem 5.2.2.

Theorem 5.2.3. *For a nonsingular plant G, the following hold true.*

(i) There is a proper compensator in a unity output feedback system such that it decouples G and achieves arbitrary closed-loop pole assignment if and only if N_{ra} and D are externally skew prime and NN_{ra} and $\bar{D}_{la}\bar{D}$ are coprime, where ND^{-1} is a right coprime polynomial matrix fraction of G, $\bar{N}_{ra}$ and $\bar{D}$ are a dual internally skew prime pair of the externally skew prime N_{ra} and D such that $DN_{ra} = \bar{N}_{ra}\bar{D}$, and N_{ra} and $\bar{D}_{la}$ are, respectively, right and left strict adjoints of N and $\bar{D}$.

(ii) Under the conditions of (i), all the compensators which decouple G and preserve arbitrary pole assignability are given by

$$K = \bar{N}_{ra}(\bar{D}_{la})^{-1}K_2,$$

where $K_2 \in \mathbb{R}_d^{m\times m}(s)$ is such that K_1K_2 with $K_1 = NN_{ra}(\bar{D}_{la}\bar{D})^{-1}$ has no pole-zero cancellations.

(iii) All the achievable open-loop maps under decoupling with arbitrary pole assignability can be expressed by

$$GK = K_1K_2,$$

where K_1 and K_2 are defined as in (ii).

We now present two examples to illustrate our results.

Example 5.2.1. Consider the plant:

$$G = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix},$$

which is the transfer matrix of the state-space model example in Linnemann and Maier (1990) for $\alpha = 1$. G is factorized as

$$G = G^+G^-,$$

where

$$G^+ = \begin{bmatrix} 1 & 1 \\ \frac{1}{s-1} & \frac{s}{s-1} \end{bmatrix},$$

and

$$G^- = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

From G^+, we construct one of its right coprime polynomial matrix fractions as

$$G^+ = N^+(D^+)^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}.$$

One sees that G has a common zero and pole at $s = 1$, which is unstable. Compute $(N^+)^{-1}$ as

$$(N^+)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{-1}{s-1} & 1 \end{bmatrix},$$

then,

$$N^+_{ra} = (N^+)^{-1} \, diag\{(s-1), 1\} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Obviously, N^+_{ra} and D^+ are externally skew prime since $det(N^+_{ra}) \in \mathbb{R}$ has no root at all. One easily checks that

$$\bar{D}^+ = \begin{bmatrix} s & -1 \\ -1 & 1 \end{bmatrix}$$

and $\bar{N}^+_{ra} = I_2$ satisfy the skew primeness equation

$$D^+ N^+_{ra} = \bar{N}^+_{ra} \bar{D}^+,$$

and they constitute a dual skew prime pair. A left strict adjoint of $\bar{D}^+$ is similarly obtained as

$$\bar{D}^+_{la} = \begin{bmatrix} 1 & 1 \\ 1 & s \end{bmatrix}.$$

Hence, we have

$$N^+ N^+_{ra} = \begin{bmatrix} s-1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\bar{D}^+_{la} \bar{D}^+ = \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}.$$

They are not coprime. **(PS)** for this plant is thus not solvable. ◊

Example 5.2.2. Consider now a 3×3 plant:

$$G = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} & 0 \\ \frac{1}{s(s+1)} & \frac{1}{s(s+2)} & 0 \\ \frac{1}{(s-1)(s+3)} & \frac{1}{(s-1)(s+4)} & \frac{1}{(s-1)(s+5)} \end{bmatrix}.$$

A stability factorization of G is given by

$$G = G^+G^- = \begin{bmatrix} s & 1 & 0 \\ \frac{1}{s} & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 \\ \frac{1}{s+3} & \frac{1}{s+4} & \frac{1}{s+5} \end{bmatrix}.$$

Again, G has a zero coinciding with a pole at $s = 1$. We could go step by step following Theorem 5.2.2. But, one notes that this G^+ has a special block-diagonal structure and it enables us to decompose the original **(PS)** into several smaller ones. In fact, the proof of Theorem 5.2.1 shows that G^- of a plant plays no role in solvability of **(PS)**, and we can always take K as $K = (G^-)^{-1}\tilde{K}$ to cancel it without affecting the solvability of **(PS)**. Furthermore, if G^+ has a block-diagonal structure $G^+ = block\,diag\{G_i^+\}$ and there is a solution $\tilde{K}_i$ to **(PS)** for each G_i^+, then $K = (G^-)^{-1} block\,diag\{\tilde{K}_i\}$ will be a solution to **(PS)** for the original plant G. Therefore, we obtain the following sufficient condition for **(PS)**. ♢

Theorem 5.2.4. *If a nonsingular plant G has a stability factorization $G = G^+G^-$ with $G^+ = block\,diag\{G_i^+\}$ and for each G_i^+* **(PS)** *is solvable, then* **(PS)** *for G is also solvable. In particular, if $G^+ = diag\{g_i^+\}$, then* **(PS)** *for G is solvable.*

Now turn back to the example, we have

$$G^+ = block\,diag\{G_1^+, G_2^+\},$$

where $G_2^+ = 1/(s-1)$ is scalar, for which **(PS)** is of course solvable, and

$$G_1^+ = \begin{bmatrix} s & 1 \\ \frac{1}{s} & \frac{1}{s} \end{bmatrix}.$$

The zero of G_1^+ at $s = 1$ is different from its pole at $s = 0$, and by Corollary 5.2.1, **(PS)** for G_1^+ is also solvable. It follows from Theorem 5.2.4 that **(PS)** for G is thus solvable. ♢

5.2.2 An Alternative Condition

This subsection presents an alternative solvability condition which involves transfer function matrices only and requires no coprime factorizations of polynomial matrices.

Given an $m \times m$ nonsingular proper rational transfer function matrix G, let the set of all the common RHP poles and zeros of G be $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$. Let $\delta_p(G)$ denote the McMillan degree of a transfer function G at the RHP pole p, i.e., multiplicity of the pole at p. For simplification, we denote $\overline{\delta}_p(G) = \delta_p(G) + \delta_p(G^{-1})$. G_d is called a decoupler for G if GG_d is decoupled.

Definition 5.2.1. *$G_{d,min}$ is called a minimal C^+- decoupler for G if (i) $GG_{d,min}$ is decoupled; (ii) For any decoupler G_d for G, there holds $\overline{\delta}_\gamma(G_{d,min}) \leq \overline{\delta}_\gamma(G_d)$ for any $\gamma \in \Gamma$.*

Lemma 5.2.2. *(**Existence and uniqueness of** $G_{d,min}$) For any nonsingular G , (i) there exists at least a $G_{d,min}$; (ii) the set of the $G_{d,min}$ which have different zero and pole structure at $\gamma \in \Gamma$ is finite, each number is unique up to right multiplication by a decoupled matrix which has neither pole nor zero at any $\gamma \in \Gamma$, and all the members have the same $\overline{\delta}_\gamma(G_{d,min})$.*

Proof. (i) Obviously, $G_d = G^{-1}$ is a decoupler for any $m \times m$ nonsingular G. One can see that $G_d D$ is still a decoupler for any decoupled $D \in \mathbb{R}_d^{m \times m}$. For each $\gamma \in \Gamma$, we perform the following search to find $G_{d,min}$ from an initial decoupler G_d. Let l_p (*resp.* l_z) be the greatest multiplicity of the pole of all the elements of G_d (*resp.* G_d^{-1}) at $\gamma \in \Gamma$. Suppose that $l_j, j = 1, 2, \cdots, m$, are integers, $D = diag\{(s-\gamma)^{l_1}, (s-\gamma)^{l_2}, \cdots, (s-\gamma)^{l_m}\}$ and $L = \{(l_1, l_2, \cdots, l_m) | -l_z \leq l_i \leq l_p,\ i = 1, 2, \cdots, m\}$. If $(l_1, l_2, \cdots, l_m) \notin L$, then there exists at least an l_j^* such that either (a) $l_j^* > l_p$ or (b) $l_j^* < -l_z$ holds. Suppose the case (a) (the case (b) can be treated with obvious changes). Then, $Gdiag\{(s-\gamma)^{l_1}, \cdots, (s-\gamma)^{l_{j-1}}, (s-\gamma)^{l_p}, (s-\gamma)^{l_{j+1}}, \cdots, (s-\gamma)^{l_m}\}$ cannot have any pole-zero cancellation with $diag\{1, \cdots, 1, (s-\gamma)^{l_j^* - l_p}, 1, \cdots, 1\}$ and thus $\overline{\delta}_r(GD)$ with $(l_1, l_2, \cdots, l_m) = (l_1, \cdots, l_{j-1}, l_j^*, l_{j+1}, \cdots, l_m)$ is strictly greater than $\overline{\delta}_r(GD)$ with $(l_1, l_2, \cdots, l_m) = (l_1, \cdots, l_{j-1}, l_p, l_{j+1}, \cdots, l_m)$. It thus follows that a minimum $\overline{\delta}_r(GD)$ will only come from some D with $(l_1, l_2, \cdots, l_m) \in L$ if it exists. Further, the combinations of such l_j in the range L are finite and the unique minimum $\overline{\delta}_\gamma(G_d D)$ must be achieved with one or a few combinations of l_j in the range and the corresponding D will yield $G_{d,min} = G_d D$.

(ii) From the above construction, one sees that the number of such $G_{d,min}$ for a given initial G_d is finite and all of them have the same $\overline{\delta}_r(G_{d,min})$. Let now $\tilde{G}_{d,min}$ be an arbitrary minimal C^+-decoupler for G, which may not come from the above search. Then, $G\tilde{G}_{d,min} = D_1$, and any $G_{d,min}^i$ from the above search also meets $GG_{d,min}^i = D_2$, where D_1 and D_2 are decoupled. This gives rise to $\tilde{G}_{d,min} = G_{d,min}^i D_3$ for a decoupled $D_3 = D_1 D_2^{-1}$. Factorize D_3 as $D_3 = D_{3\gamma} D_{3\bar{\gamma}}$, where decoupled $D_{3\gamma}$ contains all the poles and zeros of D_3 at all $\gamma \in \Gamma$ and $D_{3\bar{\gamma}}$ has no pole or zero at any $\gamma \in \Gamma$. Then, we have $\tilde{G}_{d,min} = (G_{d,min}^i D_{3\gamma}) D_{3\bar{\gamma}}$ and $\overline{\delta}_\gamma(\tilde{G}_{d,min}) = \overline{\delta}_\gamma(G_{d,min}^i D_{3,\gamma})$. It follows that $G_{d,min}^i D_{3\gamma}$ is a minimal C^+-decoupler. Further, $G_{d,min}^i D_{3\gamma}$ falls into the search in (i) and thus it is itself a member of $G_{d,min}$, say, $G^j_{d,min}$. Therefore, we conclude that $\tilde{G}_{d,min} = G^j_{d,min} D_{3\gamma}$. The proof is completed. □

Remark 5.2.1. Note that the locations of poles of a multivariable system must be the same as those of its elements (Chapter 2), and that the zeros of G are the poles of G^{-1}. Thus, we can find the common RHP zeros and poles of G by examining the poles of the elements of G and G^{-1}. This observation together with the proof of Lemma 5.2.2 naturally gives rise to the following constructive procedure for finding $G_{d,min}$ from G.

- **Step 1:** Calculate $G_d = G^{-1}$ and determine the set Γ from pole locations of elements of G and G^{-1}. If $\Gamma = \phi$, i.e. empty set, take $G_{d,min} = G_d$; otherwise, proceed to step 2.
- **Step 2:** For each $\gamma \in \Gamma$, find $l_{p\gamma}$ and $l_{z\gamma}$ from pole multiplicity of elements of G_d and G_d^{-1}. Let $D_\gamma = diag\{(s-\gamma)^{l_1}, (s-\gamma)^{l_2}, \cdots, (s-\gamma)^{l_m}\}$, tabulate $\overline{\delta}_\gamma(G_d D_\gamma)$ for all possible combinations of l_j such that $-l_{z\gamma} \le l_j \le l_{p\gamma}$ and determine the minimum $\overline{\delta}_\gamma(G D_\gamma)$ and the corresponding D^*_γ.
- **Step 3:** Set $G_{d,min} = G_d D^*_{\gamma_1} D^*_{\gamma_2} \cdots D^*_{\gamma_n}$.

Example 5.2.3. Let

$$G = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix},$$

which is the same as in Example 5.2.1. It follows that

$$G^{-1} = \begin{bmatrix} \frac{s(s+1)}{s-1} & -(s+1) \\ -\frac{s+2}{s-1} & (s+2) \end{bmatrix},$$

for which we can easily find that the only common RHP pole and zero of G is at 1 i.e., $\Gamma = \{1\}$. Take $G_d = G^{-1}$ and obviously we have $l_{p\gamma} = 1$ and $l_{z\gamma} = 1$. For $D = diag\{(s-1)^{l_1}, (s-1)^{l_2}\}$, where $-1 \le l_1, l_2 \le 1$, the search for $G_{d,min}$ is tabulated in Table 5.1, from which two $G_{d,min}$ are found as

$$G_{d,min1} = \begin{bmatrix} \frac{s(s+1)}{s-1} & -\frac{s+1}{s-1} \\ -\frac{s+2}{s-1} & \frac{s+2}{s-1} \end{bmatrix},$$

and

$$G_{d,min2} = \begin{bmatrix} s(s+1) & -(s+1) \\ -(s+2) & (s+2) \end{bmatrix}.$$

If we start our procedure with a very different $\tilde{G}_d$:

$$\tilde{G}_d = \begin{bmatrix} 1 & -\frac{s+1}{s+2} \\ -\frac{s+2}{s(s+1)} & 1 \end{bmatrix},$$

with $l_{p\gamma} = 0$. And we have

$$\tilde{G}_d^{-1} = \begin{bmatrix} \frac{s}{s-1} & \frac{s(s+1)}{(s+2)(s-1)} \\ \frac{s+2}{(s-1)(s+1)} & \frac{s}{s-1} \end{bmatrix},$$

so that $l_{z\gamma} = 1$. For $D = diag\{(s-1)^{l_1}, (s-1)^{l_2}\}$ with $-1 \le l_1, l_2 \le 0$, the search for $G_{d,min}$ is tabulated in Table 5.2, from which two $G_{d,min}$ are found as

$$\tilde{G}_{d,min1} = \begin{bmatrix} 1 & -\frac{s+1}{s+2} \\ -\frac{s+2}{s(s+1)} & 1 \end{bmatrix}, \tag{5.29}$$

and

$$\tilde{G}_{d,min2} = \begin{bmatrix} \frac{1}{s-1} & -\frac{s+1}{(s+2)(s-1)} \\ -\frac{s+2}{s(s+1)(s-1)} & \frac{1}{s-1} \end{bmatrix}. \tag{5.30}$$

Tables 5.1 and 5.2 verify that $\overline{\delta}_\gamma(G_{d,min})$ is unique for a given G. Although two sets of the $G_{d,min}$ obtained above look different, actually they are related to each other by

$$G_{d,min1} = \tilde{G}_{d,min2} \begin{bmatrix} s(s+1) & 0 \\ 0 & (s+2) \end{bmatrix}^{-1},$$

and

$$G_{d,min2} = \tilde{G}_{d,min1} \begin{bmatrix} \frac{1}{s(s+1)} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}^{-1},$$

which is in agreement with (ii) of Lemma 5.2.2. ◇

Lemma 5.2.3. *If G_d is a solution to* **(PS)** *for G, then there exists a $G_{d,min}$ such that (i) $G_d = G_{d,min}D$, where $D \in \mathbb{R}_d^{m\times m}$ is decoupled; and (ii) $\delta_\gamma(G_d) = \delta_\gamma(G_{d,min}) + \delta_\gamma(D)$, where $\gamma \in \Gamma$.*

Proof. Take $G_{d,min} = (G^-)^{-1}\overline{N}_{ra}^{+}(\overline{D}_{la}^{+})^{-1}$ as in (5.28) and the lemma follows from Corollary 5.2.2. □

Theorem 5.2.5. *For a nonsingular G,* **(PS)** *is solvable if and only if there exists a $G_{d,min}$ which has no pole-zero cancellation with G at any $\gamma \in \Gamma \subset \mathbb{C}^+$.*

Table 5.1. Search for $G_{d,min}$ in Example 5.2.3

(l_1, l_2)	$K = G_d \begin{bmatrix} (s-1)^{l_1} & \\ & (s-1)^{l_2} \end{bmatrix}$	$\bar{\delta}_1(K)$
$(-1,-1)$	$\begin{bmatrix} \frac{s(s+1)}{(s-1)^2} & -\frac{s+1}{s-1} \\ -\frac{s+2}{(s+1)^2} & \frac{s+2}{s-1} \end{bmatrix}$	3
$(-1,0)$	$\begin{bmatrix} \frac{s(s+1)}{(s-1)^2} & -(s+1) \\ -\frac{s+2}{(s-1)^2} & s+2 \end{bmatrix}$	3
$(-1,1)$	$\begin{bmatrix} \frac{s(s+1)}{(s-1)^2} & -(s+1)(s-1) \\ -\frac{s+2}{(s-1)^2} & (s-1)(s+2) \end{bmatrix}$	4
$(0,-1)$	$\begin{bmatrix} \frac{s(s+1)}{s-1} & -\frac{s+1}{s-1} \\ -\frac{s+2}{s-1} & \frac{s+2}{s-1} \end{bmatrix}$	1
$(0,0)$	$\begin{bmatrix} \frac{s(s+1)}{s-1} & -(s+1) \\ -\frac{s+2}{s-1} & s+2 \end{bmatrix}$	2
$(0,1)$	$\begin{bmatrix} \frac{s(s+1)}{s-1} & -(s+1)(s-1) \\ -\frac{s+2}{s-1} & (s-1)(s+2) \end{bmatrix}$	3
$(1,-1)$	$\begin{bmatrix} s(s+1) & -\frac{s+1}{s-1} \\ -(s+2) & \frac{s+2}{s-1} \end{bmatrix}$	2
$(1,0)$	$\begin{bmatrix} s(s+1) & -(s+1) \\ -(s+2) & s+2 \end{bmatrix}$	1
$(1,1)$	$\begin{bmatrix} s(s+1) & -(s+1)(s-1) \\ -(s+2) & (s+2)(s-1) \end{bmatrix}$	2

Proof. Sufficiency: By the definition of $G_{d,min}$, $GG_{d,min}$ is decoupled. The theorem will follow if $G_{d,min}$ has no pole-zero cancellation with G at any $\rho \in \mathbb{C}^+$ but $\rho \notin \Gamma$. Then, such ρ can be a pole or a zero of G but not both. If ρ be a pole (resp. zero) of G, then ρ must be a zero (resp. pole) of $G_{d,min}$ for the cancellation to occur. Form $G_d = \frac{1}{(s-\rho)^l} G_{d,min}$ (resp. $(s-\rho)^l G_{d,min}$) for a sufficiently large integer l such that G_d no longer has any zero (resp. pole) at ρ. This G_d has no pole-zero cancellation with G at ρ. Repeat this elimination of pole-zero cancellation for all possible ρ and the resultant G_d will have no pole-zero cancellation with G at $\rho \in \mathbb{C}^+$ but $\rho \notin \Gamma$.

Necessity: It follows from Theorem 5.1.1 that if **(PS)** is solvable, then there is a G_d such that GG_d is decoupled and meets

$$\delta_\gamma(G) + \delta_\gamma(G_d) = \delta_\gamma(GG_d), \quad \gamma \in \Gamma. \tag{5.31}$$

Table 5.2. Search for $G_{d,min}$ in Example 5.2.3

(l_1, l_2)	$K = G_d \begin{bmatrix} (s-1)^{l_1} & \\ & (s-1)^{l_2} \end{bmatrix}$	$\overline{\delta}_1(K)$
$(-1,-1)$	$\begin{bmatrix} \frac{1}{s-1} & -\frac{s+1}{(s+2)(s-1)} \\ -\frac{s+2}{s(s+1)(s-1)} & \frac{1}{s-1} \end{bmatrix}$	1
$(-1,0)$	$\begin{bmatrix} \frac{1}{s-1} & -\frac{s+1}{s+2} \\ -\frac{s+2}{s(s+1)(s-1)} & 1 \end{bmatrix}$	2
$(0,-1)$	$\begin{bmatrix} 1 & -\frac{s+1}{(s-1)(s+2)} \\ -\frac{s+2}{s(s+1)} & \frac{1}{s-1} \end{bmatrix}$	2
$(0,0)$	$\begin{bmatrix} 1 & -\frac{s+1}{s+2} \\ -\frac{s+2}{s(s+1)} & 1 \end{bmatrix}$	1

By Lemma 5.2.3, for G_d there exists a $G_{d,min}$ such that $G_d = G_{d,min}D$ and

$$\delta_\gamma(G_d) = \delta_\gamma(G_{d,min}) + \delta_\gamma(D). \tag{5.32}$$

Substituting (5.32) into (5.31) yields

$$\delta_\gamma(G) + \delta_\gamma(G_{d,min}) + \delta_\gamma(D) = \delta_\gamma(GG_{d,min}D), \tag{5.33}$$

implying that G and $G_{d,min}$ have no pole-zero cancellation at $\gamma \in \Gamma$. This completes the proof. □

Example 5.2.4. Consider the same G as in Example 5.2.3 with $\Gamma = \{1\}$. For $G_{d,min1}$ in (5.29), one sees

$$GG_{d,min1} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 & -\frac{s+1}{s+2} \\ -\frac{s+2}{s(s+1)} & 1 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s(s+1)} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix},$$

$\delta_1(G) = 1$, $\delta_1(G_{d,min1}) = 0$, and $\delta_1(GG_{d,min1}) = 0$, $\delta_1(G) + \delta_1(G_{d,min1}) \neq \delta_1(GG_{d,min1})$. There is a pole-zero cancelation between G and $G_{d,min1}$ at $\gamma = 1$.

For $G_{d,min2}$ in (5.30), it follows that

$$\begin{aligned} GG_{d,min2} &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & -\frac{s+1}{(s+2)(s-1)} \\ -\frac{s+2}{s(s+1)(s-1)} & \frac{1}{s-1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s(s+1)} & 0 \\ 0 & \frac{1}{(s+2)(s-1)} \end{bmatrix}, \end{aligned}$$

$\delta_1(G) = 1$, $\delta_1(G_{d,min2}) = 1$, and $\delta_1(GG_{d,min1}) = 1$, so that $\delta_1(G) + \delta_1(G_{d,min1}) \neq \delta_1(GG_{d,min1})$. There is a pole-zero cancellation between G

and $G_{d,min2}$ at $\gamma = 1$ too. Thus, none of $G_{d,min}$ can meet the condition in Theorem 5.2.5 and so **(PS)** for this plant is not solvable. $\diamondsuit$

Example 5.2.5. Let

$$G(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{1}{s(s+1)} & \frac{1}{s(s+2)} \end{bmatrix}.$$

Choose

$$G_d = G^{-1} = \begin{bmatrix} \frac{s+1}{s-1} & -\frac{s(s+1)}{s-1} \\ -\frac{s+2}{s-1} & \frac{s^2(s+2)}{s-1} \end{bmatrix}.$$

It is readily seen that the only common RHP pole and zero of $G(s)$ is 1, $l_{z\gamma} = 0$ and $l_{p\gamma} = 1$. From Table 5.3, one can see that $G_{d,min}$ is not unique. For

$$G_{d,min1} = \begin{bmatrix} \frac{s+1}{s-1} & -\frac{s(s+1)}{s-1} \\ -\frac{s+2}{s-1} & \frac{s^2(s+2)}{s-1} \end{bmatrix},$$

we have

$$GG_{d,min1} = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{1}{s(s+1)} & \frac{1}{s(s+2)} \end{bmatrix} \begin{bmatrix} \frac{s+1}{s-1} & -\frac{s(s+1)}{s-1} \\ -\frac{s+2}{s-1} & \frac{s^2(s+2)}{s-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\delta_1(G) = 0$, $\delta_1(G_{d,min1}) = 1$, and $\delta_1(GG_{d,min1}) = 0$, so $\delta_1(G) + \delta_1(G_{d,min1}) \neq \delta_1(GG_{d,min1})$. There is a pole-zero cancelation between G and $G_{d,min1}$ at $\gamma = 1$.

For

$$G_{d,min2} = \begin{bmatrix} s+1 & -s(s-1) \\ -(s+2) & s^2(s+2) \end{bmatrix},$$

we have

$$GG_{d,min2} = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{1}{s(s+1)} & \frac{1}{s(s+2)} \end{bmatrix} \begin{bmatrix} s+1 & -s(s+1) \\ -(s+2) & s^2(s+2) \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix},$$

$\delta_1(G) = 0$, $\delta_1(G_{d,min2}) = 0$, and $\delta_1(GG_{d,min2}) = 0$, so that $\delta_1(G) + \delta_1(G_{d,min2}) = \delta_1(GG_{d,min2})$. There is no pole-zero cancelation between G and $G_{d,min2}$ at $\gamma = 1$, and **(PS)** for this $G(s)$ is solvable. $\diamondsuit$

Example 5.2.6. Let

$$G(s) = \begin{bmatrix} \frac{s}{s-1} & -\frac{1}{s-1} & \frac{s}{s-1} \\ -1 & 1 & -s \\ 0 & 0 & s-1 \end{bmatrix}.$$

Table 5.3. Search for $G_{d,min}$ in Example 5.2.5

(l_1, l_2)	$K = G_d \begin{bmatrix} (s-1)^{l_1} & \\ & (s-1)^{l_2} \end{bmatrix}$	$\overline{\delta}_1(K)$
$(0,0)$	$\begin{bmatrix} \frac{s+1}{s-1} & \frac{s(s+1)}{s-1} \\ -\frac{s+2}{s-1} & \frac{s^2(s+2)}{s-1} \end{bmatrix}$	1
$(1,0)$	$\begin{bmatrix} s+1 & -\frac{s(s+1)}{s-1} \\ -(s+2) & \frac{s^2(s+2)}{s-1} \end{bmatrix}$	2
$(0,1)$	$\begin{bmatrix} \frac{s+1}{s-1} & -s(s-1) \\ -\frac{s+2}{s-1} & s^2(s+2) \end{bmatrix}$	2
$(1,1)$	$\begin{bmatrix} s+1 & -s(s+1) \\ -(s+2) & s^2(s+2) \end{bmatrix}$	1

From Table 5.4, there are two $G_{d,min}$. For

$$G_{d,min1} = \begin{bmatrix} 1 & \frac{1}{s} & 0 \\ 1 & 1 & s \\ 0 & 0 & 1 \end{bmatrix},$$

we calculate

$$GG_{d,min1} = \begin{bmatrix} \frac{s}{s-1} & -\frac{1}{s-1} & \frac{s}{s-1} \\ -1 & 1 & -s \\ 0 & 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{s} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{s-1}{s} & 0 \\ 0 & 0 & s-1 \end{bmatrix},$$

$\delta_1(G) = 1$, $\delta_1(G_{d,min1}) = 0$, and $\delta_1(GG_{d,min1}) = 0$, $\delta_1(G) + \delta_1(G_{d,min1}) \neq \delta_1(GG_{d,min1})$. There is a pole-zero cancelation between G and $G_{d,min1}$ at $\gamma = 1$.

For

$$G_{d,min2} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & 0 \\ \frac{1}{s-1} & \frac{1}{s-1} & s \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$GG_{d,min2} = \begin{bmatrix} \frac{s}{s-1} & -\frac{1}{s-1} & \frac{s}{s-1} \\ -1 & 1 & -s \\ 0 & 0 & s-1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & 0 \\ \frac{1}{s-1} & \frac{1}{s-1} & s \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & s-1 \end{bmatrix},$$

$\delta_1(G) = 1$, $\delta_1(G_{d,min2}) = 1$, and $\delta_1(GG_{d,min2}) = 1$, $\delta_1(G) + \delta_1(G_{d,min2}) \neq \delta_1(GG_{d,min2})$. There is a pole-zero cancelation between G and $G_{d,min2}$ at $\gamma = 1$, too. So **(PS)** for the plant is not solvable. $\diamondsuit$

Table 5.4. Search for $G_{d,min}$ in Example 5.2.6

(l_1, l_2, l_3)	$K = G_d \begin{bmatrix} (s-1)^{l_1} & & \\ & (s-1)^{l_2} & \\ & & (s-1)^{l_3} \end{bmatrix}$	$\bar{\delta}_1(K)$
$(-1,-1,-1)$	$\begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & 0 \\ \frac{1}{s-1} & \frac{1}{s-1} & \frac{s}{s-1} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$	2
$(-1,-1,0)$	$\begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & 0 \\ \frac{1}{s-1} & \frac{1}{s-1} & s \\ 0 & 0 & 1 \end{bmatrix}$	1
$(-1,0,-1)$	$\begin{bmatrix} -\frac{1}{s-1} & \frac{1}{s} & 0 \\ \frac{1}{s-1} & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$	3
$(-1,0,0)$	$\begin{bmatrix} \frac{1}{s-1} & \frac{1}{s} & 0 \\ \frac{1}{s-1} & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$	2
$(0,-1,-1)$	$\begin{bmatrix} 1 & \frac{1}{s(s-1)} & 0 \\ 1 & \frac{1}{s-1} & \frac{s}{s-1} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$	3
$(0,-1,0)$	$\begin{bmatrix} 1 & \frac{1}{s(s-1)} & 0 \\ 1 & \frac{1}{s-1} & s \\ 0 & 0 & 1 \end{bmatrix}$	2
$(0,0,-1)$	$\begin{bmatrix} 1 & \frac{1}{s} & 0 \\ 1 & 1 & \frac{s}{s-1} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$	2
$(0,0,0)$	$\begin{bmatrix} 1 & \frac{1}{s} & 0 \\ 1 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$	1

5.3 Block Decoupling for General Plants

In the preceding section, we have established necessary and sufficient conditions for the solvability of the diagonal decoupling problem with internal stability for unity output feedback systems with square plants. The conditions are only concerned with G^+ of a given plant, and they consist of the externally skew primeness of N_{ra}^+ and D^+ and the coprimeness of $N^+N_{ra}^+$ and $\bar{D}_{la}^+\bar{D}^+$ over the ring of polynomials. The problem, however, is solved only for square plants. A difficulty arising from a nonsquare plant is that its right inverses are not unique and Theorem 5.2.1 is no longer valid in this case. Another generalization needed is to consider block-diagonalization with internal stability. These problems form the original **(P)** and will be solved in the present section.

Proposition 5.3.1. *Let $G \in \mathbb{R}_{sp}^{m\times l}(s)$ be a given plant of full row rank and $G = [G^+ \ \ 0]G^-$ be a stability factorization. Then* **(P)** *is solvable if and only if there exists a $K_1 \in \mathbf{R}^{m\times m}(s)$ such that G^+K_1 is block diagonal, nonsingular and has no pole-zero cancellations, i.e.,*

$$G^+K_1 \in \mathbb{R}_d^{m\times m}(s), \tag{5.34}$$

and

$$\delta(G^+) + \delta(K_1) = \delta(G^+K_1). \tag{5.35}$$

Proof. Assume that K solves **(P)**. Then by Theorem 5.1.1 GK is block diagonal and has no RHP pole-zero cancellations. Let I_m be the $m \times m$ identity matrix and

$$K_1 := [I_m \ \ 0]G^-K.$$

Then $GK = G^+K_1$ (which implies (5.34)) and $\delta^+(K) \geq \delta^+(K_1)$. Moreover, G^+K_1 has no RHP pole-zero cancellations, because

$$\delta^+(G^+) + \delta^+(K_1) \leq \delta^+(G) + \delta^+(K) = \delta^+(GK) = \delta^+(G^+K_1).$$

Since G^+ has only RHP zeros and poles and G^+ and K_1 are both square, this means that G^+K_1 has no zero-pole cancellations at all, which is (5.35).

Conversely, if K_1 satisfies (5.34) and (5.35), then

$$K := (G^-)^{-1}\begin{bmatrix} K_1 \\ 0 \end{bmatrix} P^{-1}C \tag{5.36}$$

solves **(P)**, where $P \in \mathbb{R}_d^{m\times m}[s]$ is stable and such that $(G^-)^{-1}[K_1^T \ \ 0]^T P^{-1}$ is proper, and $C \in \mathbb{R}_{dp}^{m\times m}(s)$ internally stabilizes $G^+K_1P^{-1}$. □

Proposition 5.3.1 simplifies **(P)** into the following *open-loop* block decoupling problem with stabilizability for a *square* and *nonsingular* rational matrix having only RHP zeros and poles. Note again that K_1 is not required to be proper.

(PO). *Let a nonsingular $G^+ \in \mathbb{R}^{m\times m}(s)$ and a partition be given. Find a $K_1 \in \mathbb{R}^{m\times m}(s)$ such that G^+K_1 is block diagonal, nonsingular and has no pole-zero cancellations, i.e., (5.34) and (5.35) are satisfied.*

Example 5.3.1. Consider the plant given by

$$G = \begin{bmatrix} \frac{s}{s-1} & \frac{-1}{s-1} & \frac{s}{(s-1)(s+1)} & \frac{1}{s+1} \\ -1 & 1 & \frac{-s}{s+1} & 0 \\ 0 & 0 & \frac{s-1}{s+1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & s & s & -1 \\ 0 & 0 & s+1 & 0 \\ 0 & 0 & 0 & s+1 \end{bmatrix}^{-1}.$$

Its pole and zero polynomial is given by $(s-1)(s+1)^2$ and $(s-1)^2$, respectively. A stability factorization is defined by

$$G^+ = \begin{bmatrix} \frac{s}{s-1} & \frac{-1}{s-1} & \frac{s}{s-1} \\ -1 & 1 & -s \\ 0 & 0 & s-1 \end{bmatrix}, \quad G^- = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{s+1} \\ 0 & 1 & 0 & \frac{1}{s+1} \\ 0 & 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & 0 & \frac{1}{s+1} \end{bmatrix}.$$

By Proposition 5.3.1, **(P)** for G is equivalent to **(PO)** for G^+ for every partition. ◊

5.3.1 Solvability

The notion of strict adjoints plays a vital rule in solving the diagonal decoupling problem. Its generalization to the block case is obviously required to solve the general problem **(PO)**.

Definition 5.3.1. *Let $P \in \mathbb{R}^{m\times m}[s]$ be a nonsingular polynomial matrix. A polynomial matrix $P_{rs} \in \mathbb{R}^{m\times m}[s]$ is said to be a right strict block adjoint of P whenever the following conditions are satisfied: (i) $PP_{rs} \in \mathbb{R}_d^{m\times m}[s]$; and (ii) any $P_r \in \mathbb{R}^{m\times m}[s]$ satisfying $PP_r \in \mathbb{R}_d^{m\times m}[s]$ is a right multiple of P_{rs}, i.e., $P_r = P_{rs}P_d$ for some polynomial matrix $P_d \in \mathbb{R}_d^{m\times m}[s]$.*

A right strict block adjoint of P can be determined by partitioning P^{-1} into

$$P^{-1} = [\tilde{P}_1 \ \tilde{P}_2 \ \cdots \ \tilde{P}_v],$$

where $\tilde{P}_i \in \mathbb{R}^{m\times m_i}(s)$. Let $A_i \in \mathbb{R}^{m\times m_i}[s]$ and $B_i \in \mathbb{R}^{m_i\times m_i}[s]$ be right coprime such that $\tilde{P}_i = A_i B_i^{-1}, \quad i = 1, 2, \cdots, v$. The following lemma shows that

$$A := [A_1 \quad A_2 \quad \cdots \quad A_v] \tag{5.37}$$

is a right strict block adjoint of P.

Lemma 5.3.1. *Any nonsingular polynomial matrix P has a right strict block adjoint with respect to every partition. A particular right strict block adjoint of P is given by $P_{rs} = A$, as defined in (5.37). For a fixed partition, right strict block adjoints of P are uniquely determined up to multiplication by a block diagonal unimodular matrix from the right.*

Proof. By the above construction there holds $PA = B$ with

$$B := block\,diag\{B_1, B_2, \cdots, B_v\} \in \mathbb{R}_d^{m\times m}[s].$$

Assume now that a nonsingular $P_r \in \mathbb{R}^{m\times m}[s]$ satisfies $PP_r = Q = block\,diag\{Q_i\} \in \mathbb{R}_d^{m\times m}[s]$. Then the equality:

$$P_r = P^{-1}Q = [A_1B_1^{-1}Q_1 \ \ A_2B_2^{-1}Q_2 \ \ \cdots \ \ A_vB_v^{-1}Q_v],$$

implies that $A_iB_i^{-1}Q_i$ is a polynomial matrix for each i. Since $A_iB_i^{-1}$ is coprime, by Lemma 5.1.2, $P_i := B_i^{-1}Q_i$ is a polynomial matrix. This means that $P_r = AP_d$ with $P_d = block\,diag\{P_i\} \in \mathbb{R}_d^{m\times m}[s]$. Therefore, A is a right strict block adjoint of P.

Let P_{rs} and P'_{rs} be both right strict block adjoints of P. Then there exist $P_{d1}, P_{d2} \in \mathbb{R}_d^{m\times m}[s]$ such that $P_{rs} = P'_{rs}P_{d1}$ and $P'_{rs} = P_{rs}P_{d2}$. This implies $P_{rs} = P_{rs}P_{d2}P_{d1}$, so that P_{d1} and P_{d2} are both unimodular. □

We will also need left strict block adjoints which are defined, like Definition 5.3.1 with the obvious modifications. A dual result to Lemma 5.3.1 will then also hold.

Example 5.3.2. Consider the following nonsingular polynomial matrix P with its inverse,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & s & 0 \\ 0 & s & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} \frac{s}{s-1} & \frac{-1}{s-1} & 0 \\ \frac{-1}{s-1} & \frac{1}{s-1} & 0 \\ \frac{s}{s-1} & \frac{-s}{s-1} & 1 \end{bmatrix}.$$

Let the partition be defined by $m_1 = 1$ and $m_2 = 2$. Then

$$\tilde{P}_1 = \begin{bmatrix} \frac{s}{s-1} \\ \frac{-1}{s-1} \\ \frac{s}{s-1} \end{bmatrix} = \begin{bmatrix} s \\ -1 \\ s \end{bmatrix} (s-1)^{-1} = A_1B_1^{-1},$$

and

$$\tilde{P}_2 = \begin{bmatrix} \frac{-1}{s-1} & 0 \\ \frac{1}{s-1} & 0 \\ \frac{-s}{s-1} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = A_2B_2^{-1}.$$

Thus a right strict block adjoint $P_{rs}(1,2)$ with respect to the partition $(1,2)$ is given by

$$P_{rs}(1,2) = \begin{bmatrix} s & -1 & 0 \\ -1 & 1 & 0 \\ s & -s & 1 \end{bmatrix}.$$

Using the same approach, a right strict block adjoint $P_{rs}(2,1)$ with respect to the partition $(2,1)$ can be computed to be

$$P_{rs}(2,1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -s & 1 \end{bmatrix}.$$

$\diamondsuit$

We now turn back to **(PO)** as defined before and choose right coprime polynomial matrices N^+ and D^+ such that

$$G^+ = N^+(D^+)^{-1}. \tag{5.38}$$

The matrix N^+ is nonsingular, and, by Lemma 5.3.1, it has a right strict block adjoint N^+_{rs}. For the special case that D^+ is the identity matrix (which means that G is stable), $K_1 = N^+_{rs}$ solves the problem. Using the same approach for the general case, we have to guarantee that the order of multiplication of N^+_{rs} and $(D^+)^{-1}$ can be appropriately interchanged. Let us assume that N^+_{rs} and D^+ are externally skew prime. Then there exist $\bar{N}^+_{rs}$ and $\bar{D}^+$ such that

$$N^+_{rs}(\bar{D}^+)^{-1} = (D^+)^{-1}\bar{N}^+_{rs}, \tag{5.39}$$

with both fractions coprime. Let $\bar{D}^+_{ls}$ be a left strict block adjoint of $\bar{D}^+$ and $K_1 := \bar{N}^+_{rs}(\bar{D}^+_{ls})^{-1}$. Then we have

$$G^+K_1 = N^+N^+_{rs}(\bar{D}^+_{ls}\bar{D}^+)^{-1},$$

which is block diagonal. Let us assume that the above fraction is coprime. Then there holds

$$\begin{aligned} \delta(G^+K_1) &= deg(det(\bar{D}^+_{ls})) + deg(det(\bar{D}^+)) \\ &= deg(det(\bar{D}^+_{ls})) + deg(det(D^+)) \\ &\geq \delta(K_1) + \delta(G^+). \end{aligned}$$

Hence there are no pole-zero cancellations, and K_1 solves **(PO)**. The following proposition shows that the assumptions made for the above solution of the problem are also necessary for solvability of **(PO)**.

Proposition 5.3.2. *Let a nonsingular $G^+ \in \mathbb{R}^{m\times m}(s)$ and a partition be given.* **(PO)** *is solvable if and only if N^+_{rs} and D^+ are externally skew prime and, moreover, $N^+N^+_{rs}$ and $\bar{D}^+_{ls}\bar{D}^+$ are right coprime.*

Proof. Sufficiency has been shown above. The proof of necessity is more involved, but follows the same lines as the proof of Theorem 5.2.2. At those places where that proof uses the equality $N_dD_d^{-1} = D_d^{-1}N_d$ for diagonal polynomial matrices we have to introduce block diagonal and coprime polynomial matrices with $N_dD_d^{-1} = \tilde{D}_d^{-1}\tilde{N}_d$. The details are omitted. $\square$

Propositions 5.3.1 and 5.3.2 directly imply the following solvability condition for the block-decoupling problem with stability using unity output feedback.

Theorem 5.3.1. *Let $G \in \mathbb{R}_{sp}^{m\times l}(s)$ be a given plant of full row rank and $G = [G^+ \ \ 0]G^-$ be a stability factorization. Then* **(P)** *is solvable if and only if N_{rs}^+ and D^+ are externally skew prime and, moreover, $N^+N_{rs}^+$ and $\bar{D}_{ls}^+\bar{D}^+$ are right coprime.*

Let us recapitulate the main steps to check solvability of **(P)** and to compute a controller K solving **(P)**: obtain a stability factorization $G = [G^+ \ \ 0]G^-$ using the Smith-McMillan form of the plant; determine a polynomial fraction as in (5.38), using the Smith-McMillan form of G^+, for instance; construct a right strict block adjoint N_{rs}^+ of N^+ as described in Lemma 5.3.1; check if N_{rs}^+ and D^+ are externally skew prime and determine $\bar{D}^+$ and $\bar{N}_{rs}^+$ satisfying (5.39); determine a left strict block adjoint $\bar{D}_{ls}^+$ of $\bar{D}^+$ as described in Lemma 5.3.1; check if $N^+N_{rs}^+$ and $\bar{D}_{ls}^+\bar{D}^+$ are right coprime; determine $K_1 = \bar{N}_{rs}^+(\bar{D}_{ls}^+)^{-1}$; find a stable polynomial matrix $P \in \mathbb{R}_d^{m\times m}[s]$ such that $(G^-)^{-1}[K_1^T \ \ 0]^T P^{-1}$ is proper; determine $C \in \mathbb{R}_{dp}^{m\times m}(s)$ which internally stabilizes the block diagonal $G^+K_1P^{-1}$; calculate K from (5.36).

Example 5.3.3. Consider the plant of Example 5.3.1. **(P)** is solvable for G if and only if **(PO)** is solvable for

$$G^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s-1) & 0 \\ 0 & 0 & (s-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & s & s \\ 0 & 0 & 1 \end{bmatrix}^{-1} = N^+(D^+)^{-1}.$$

Note that G^+ has a zero of multiplicity 2 at $s = 1$ coinciding with a single pole at $s = 1$. Since N^+ is diagonal, we can take $N_{rs}^+ = I$ for any partition. Hence N_{rs}^+ and D^+ are externally skew prime and $\bar{N}_{rs}^+ = I$ and $\bar{D}^+ = D^+$ satisfy (5.39). In order to determine the solvability of **(PO)** for this example, we need only to check if $N^+N_{rs}^+ = N^+$ and $\bar{D}_{ls}^+\bar{D}^+ = D_{ls}^+D^+$ are right coprime, where D_{ls}^+ is a left strict block adjoint of D^+. We have $(D^+)^T = P$, where P is given in Example 5.3.2, and $D_{ls}^+D^+ = ((D^+)^T(D_{ls}^+)^T)^T$. Hence, for the partitions $(2,1)$ and $(1,2)$, there holds

$$D_{ls}^+(2,1)D^+ = (P_{rs}(2,1))^T D^+ = \begin{bmatrix} 1 & 1 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$D_{ls}^+(1,2)D^+ = (P_{rs}(1,2))^T D^+ = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One easily sees that N^+ and $D_{ls}^+(2,1)D^+$ are right coprime but N^+ and $D_{ls}^+(1,2)D^+$ are not. Hence, by Theorem 5.3.1, **(P)** is solvable for the partition $(2,1)$ but not solvable for the partition $(1,2)$. For the partition $(2,1)$, the controller $K_1 = (D_{ls}^+(2,1))^{-1}$ solves **(PO)**, and the controller K defined by (5.36) solves **(P)**. ◊

5.3.2 Controller Parameterization

The following proposition characterizes the set of all compensators solving **(PO)**. Its proof is analogous to that of Corollary 5.2.2 and is therefore omitted.

Proposition 5.3.3. *Let $G^+ \in \mathbb{R}^{m\times m}(s)$ be nonsingular and let* **(PO)** *be solvable. Then K_1 solves* **(PO)** *if and only if*

$$K_1 = \bar{N}_{rs}^+(\bar{D}_{ls}^+)^{-1}K_d,$$

where $K_d \in \mathbb{R}_d^{m\times m}(s)$ has no pole-zero cancellations with $Q_d = N^+N_{rs}^+ \times (\bar{D}_{ls}^+\bar{D}^+)^{-1}$. An open-loop transfer function matrix $Q = G^+K_1$ under block decoupling with internal stabilizability is achievable if and only if $Q = Q_dK_d$.

The main difficulty in generalizing this result to a parameterization of controllers solving **(P)** comes from the fact that a non-square controller K will allow considerable additional degrees of freedom as compared with controllers solving **(PO)**. To illustrate this effect, consider the following simple example.

Example 5.3.4. Consider $G = [\frac{1}{s-1} \;\; 0]$ with $G^+ = \frac{1}{s-1}$ and $G^- = I_2$. G^+ is already decoupled, so that a controller K_1 solves **(PO)** if and only if it has no RHP pole-zero cancellations with G^+. For example, $\hat{K}_1 = \frac{4s}{s-1}$ solves **(PO)**. Now, all controllers $K = [\hat{K}_1 \;\; K_2]^T$ with stable K_2 clearly solve **(P)**. However, there are also unstable K_2 such that K solves **(P)**, e.g., $K = [\frac{4s}{s-1} \;\; \frac{1}{s-1}]^T$. The following theorem shows that unstable poles of K_2 do necessarily come from K_1. Note that some important closed-loop transfer matrices, such as $K(I+GK)^{-1}$, depend on K_2. Therefore, a parameterization of K_2 is of relevance, although it does not influence GK. ♢

Theorem 5.3.2. *Let $G = [G^+ \;\; 0]G^-$ be a stability factorization of a full row rank plant $G \in \mathbb{R}_{sp}^{m\times l}(s)$ and let* **(P)** *be solvable. Then $K \in \mathbb{R}_p^{l\times m}(s)$ solves* **(P)** *if and only if*

$$K = (G^-)^{-1}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix},$$

where K_1 has the form $K_1 = \bar{N}_{rs}^+(\bar{D}_{ls}^+)^{-1}K_d$ for some $K_d \in \mathbb{R}_d^{m\times m}(s)$ stabilizing $Q_d = N^+N_{rs}^+(\bar{D}_{ls}^+\bar{D}^+)^{-1}$, and K_2 has the form $K_2 = \tilde{K}_2D_1^{-1}$ for some stable $\tilde{K}_2 \in \mathbb{R}^{(l-m)\times m}(s)$ and a polynomial matrix D_1 such that $N_1D_1^{-1}$ is a right coprime polynomial matrix fraction of K_1.

Proof. Suppose that a K satisfies the conditions of the theorem. Then $GK = G^+K_1 = Q_dK_d \in \mathbb{R}_d^{m\times m}(s)$ and the closed-loop system is input-output stable as K_d stabilizes Q_d. This will imply internal stability if GK has no unstable pole-zero cancellations. Since there are no unstable pole-zero cancellations in $Q_d K_d$ and $K_1 K_d^{-1}$ solves **(PO)** for G^+, we have

$$\begin{aligned}\delta^+(GK) &= \delta^+(Q_d K_d)\\ &= \delta^+(Q_d) + \delta^+(K_d)\\ &= \delta^+(G^+(K_1 K_d^{-1})) + \delta^+(K_d)\\ &= \delta^+(G^+) + \delta^+(K_1 K_d^{-1}) + \delta^+(K_d)\\ &\geq \delta^+(G) + \delta^+(K_1).\end{aligned}$$

Moreover, due to the stability of $(G^-)^{-1}$ and $\tilde{K}_2$,

$$\delta^+(K) = \delta^+((G^-)^{-1} \begin{bmatrix} N_1 \\ \tilde{K}_2 \end{bmatrix} D_1^{-1}) \leq deg^+(D_1) = \delta^+(K_1).$$

Hence $\delta^+(GK) \geq \delta^+(G) + \delta^+(K)$, which means that there are no unstable pole-zero cancellations in GK.

Conversely, suppose that K solves **(P)** for G, and define $K_1 := [I \;\; 0]G^- K$ and $K_2 := [0 \;\; I]G^- K$. Then $GK = G^+ K_1$ and

$$\delta^+(GK) = \delta^+(G) + \delta^+(K). \tag{5.40}$$

The proof of Proposition 5.3.1 shows that K_1 solves **(PO)** for G^+, so that, by Proposition 5.3.3, K_1 has the form $K_1 = \bar{N}_{rs}^+(\bar{D}_{ls}^+)^{-1} K_d$ where $K_d \in \mathbb{R}_d^{m\times m}(s)$ has no pole-zero cancellations with Q_d. This implies that K_d stabilizes Q_d. Hence it remains to verify the properties of K_2. The equation (5.40) and $\delta^+(G^+ K_1) = \delta^+(G^+) + \delta^+(K_1)$ imply that $\delta^+(K_1) = \delta^+(K)$, which reduces to

$$\delta^+(K_1) = \delta^+(\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}).$$

Let $\tilde{N}$ and $\tilde{D}$ be right coprime such that

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \tilde{N}\tilde{D}^{-1} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{bmatrix} \tilde{D}^{-1}.$$

Since the fraction on the right hand side of $\tilde{N}_1 \tilde{D}^{-1} = N_1 D_1^{-1}$ is coprime, it follows that $\tilde{D} = D_1 \tilde{D}_2$ for some polynomial matrix $\tilde{D}_2$. The matrix $\tilde{D}_2$ is stable because

$$deg^+(det(\tilde{D}_2)) = deg^+(det(\tilde{D})) - deg^+(det(D_1)) = \delta^+(\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}) - \delta^+(K_1) = 0,$$

where deg^+ denotes the degree of the unstable part. Hence, $K_2 = \tilde{N}_2 \tilde{D}^{-1} = \tilde{K}_2 D_1^{-1}$ with $\tilde{K}_2 = \tilde{N}_2 \tilde{D}_2^{-1}$ being stable. □

It is well known that the achievable performance of a system is essentially determined by the RHP open-loop poles and zeros. Therefore, in view of our study in the following section, it is of interest to determine the RHP poles and zeros which are necessarily introduced by decoupling, i.e., the RHP poles and zeros of the coprime fraction $\tilde{K}_1 = \bar{N}_{rs}^+(\bar{D}_{ls}^+)^{-1}$.

Corollary 5.3.1. *The polynomials* $det(\bar{N}^+_{rs})$ *and* $det(\bar{D}^+_{ls})$ *divide* $[det(N^+(s))]^{m-1}$ *and* $[det(D^+(s))]^{m-1}$*, respectively. Moreover, the controller* $\tilde{K}_1 = \bar{N}^+_{rs}(\bar{D}^+_{ls})^{-1}$ *is stable (respectively, minimum phase) if and only if* D^+_g *(respectively,* N^+_g*) as defined in Corollary 5.1.1 is block diagonal.*

Proof. We have $det(\bar{N}^+_{rs}) = det(N^+_{rs})$, where N^+_{rs} is a right strict block adjoint of N^+. From the equation $N^+(adj(N^+)) = (det(N^+)I$ it follows that $adj(N^+)$ is a multiple of N^+_{rs} and that $det(adj(N^+)) = (det(N^+)^{m-1}$. Thus $det(\bar{N}^+_{rs})$ divides $[det(N^+(s))]^{m-1}$. Moreover, N^+_g is block diagonal if and only if N^+_{rs} is unimodular. This proves the first part of the corollary. The property of $\bar{D}^+_{ls}$ follows similarly. □

The corollary shows that it may be necessary to introduce additional RHP open-loop poles and zeros for decoupling. However these will not occur at completely new positions; only the multiplicity of individual poles and zeros of the plant might increase up to a factor $m-1$. The corollary also shows that no additional unstable poles and zeros need to be introduced for decoupling if and only if N^+_g and D^+_g are both block diagonal. Note that this condition means that G^+U is block diagonal for some unimodular matrix U, where G^+ is an arbitrary matrix taken from a stability factorization.

Example 5.3.5. Consider the plant of Example 5.3.3. For the partition $(2,1)$, it has been shown that **(PO)** is solvable. In addition, one can check that G^+U is block diagonal for the unimodular matrix $U = (D^+_{ls}(2,1))^{-1}$. Moreover, the controller K_1 given in Example 5.3.3 is free of unstable poles and zeros, as expected. ♢

Using the arguments of Doyle and Stein (1981), we intuitively expect that no achievable performance is lost by decoupling if N^+_g and D^+_g are both block diagonal. This conjecture is verified in the following section for a large class of performance measures.

5.3.3 Achievable Performance after Decoupling

In this section we assume that the plant G and partition (m_i) are such that the solvability conditions of Theorem 5.3.1 are satisfied. We study to what extend the remaining free parameters, as specified in Theorem 5.3.2, can be used to satisfy additional specifications. We determine a class of plants for which the achievable performance after decoupling is not inferior to the one in not necessarily decoupled systems.

It is well known (Freudenberg and Looze, 1988) that many performance specifications, such as disturbance rejection, tracking and robustness can be formulated in terms of $S = (I+GK)^{-1}$ and $T = GK(I+GK)^{-1}$. We consider performance measures J of the form

$$J(\begin{bmatrix} S \\ T \end{bmatrix}) : \mathbb{R}^{2m\times m}(s) \to \mathbf{R}.$$

satisfying

$$J(\begin{bmatrix} S \\ T \end{bmatrix}) \geq J(\begin{bmatrix} d(S) \\ d(T) \end{bmatrix}), \tag{5.41}$$

where the operation $d(M)$ extracts the diagonal blocks of size $m_i \times m_i$ from a matrix M, i.e., $d(M) := block\,diag\{M_{ii}\}$. The inequality (5.41) is satisfied for most measures of practical interest. Consider for instance the measure J_p defined by the H_p-norm of $[(W_1SW_2)^T \quad (W_3TW_4)^T]^T$, where $W_1, W_2, W_3, W_4 \in \mathbb{R}_d^{m\times m}(s)$ are stable weighting matrices. Dickman and Sivan (1985) have shown that J_∞ satisfies (5.41). For other performance measures, such as J_2, (5.41) is also easily verified. Let J^* be the optimal performance, i.e., the infimum of J over all stabilizing controllers, and J_d^* be the optimal decoupled performance, i.e., the infimum of J over all stabilizing and decoupling controllers. Then $J_d^* \geq J^*$, and no achievable performance (as measured by J) is lost by decoupling if J_d^* is equal to J^*. The results of this section will imply that this is the case provided the matrix G^+ in a stability factorization is block diagonal.

Some further notation is required. Let $\mathbf{T}$ ($\mathbf{T}_d$) be the set of complimentary sensitivity functions achievable by stabilizing (stabilizing and diagonalizing) controllers, i.e.,

$$\mathbf{T} := \{T = GK(I+GK)^{-1} : K \textit{ internally stabilizes } G\}$$

and

$$\mathbf{T}_d := \{T \in \mathbf{T} : T \in \mathbb{R}_d^{m\times m}(s)\}.$$

Note that the elements of $\mathbf{T}_d$ have been characterized in Theorem 5.3.2. Moreover, let

$$d(\mathbf{T}) := \{d(T) : T \in \mathbf{T},\ d(T) \textit{ nonsingular}\}.$$

In the definition of the above sets we allow the controllers to be non-proper. This is done in order to simplify the development and the notation. If the controllers are constrained to be proper, then zeros and poles at infinity have to be considered in addition to finite ones. It is obvious that $\mathbf{T}_d \subset d(\mathbf{T})$. The following theorem gives a sufficient (and almost necessary) condition for $\mathbf{T}_d = d(\mathbf{T})$ to hold. The proof is deferred to the end of this section.

Theorem 5.3.3. *The equality $\mathbf{T}_d = d(\mathbf{T})$ holds if the polynomial matrices N_g^+ and D_g^+, as defined in Corollary 5.1.1, are both block diagonal, i.e.,*

$$N_g^+ \in \mathbb{R}_d^{m\times m}[s], \quad D_g^+ \in \mathbb{R}_d^{m\times m}[s]. \tag{5.42}$$

Suppose in addition that $det(N^+)$ and $det(D^+)$ are coprime. Then $\mathbf{T}_d$ is equal to $d(\mathbf{T})$ if and only if (5.42) holds.

Note that (5.42) holds if and only if there exists a stability factorization with a block diagonal G^+. Moreover, by Corollary 5.3.1, (5.42) holds if and only if **(PO)** can be (easily) solved by the stable and minimum phase controller $K_1 = (G^-)^{-1}$. However, this does not mean that **(P)** can be solved by a stable and minimum phase controller, because stabilization might require unstable poles and zeros in K_d (Vidyasagar, 1985). For the special case that the plant is already block diagonal, Theorem 5.3.3 means that the cross-coupling terms in the controller are of no use for performance optimization with respect to J.

The assumption on the determinants is quite weak; it is satisfied for an open and dense set of plants. Nevertheless, at least from a theoretical point of view, it is of interest to analyze if this assumption is really required to prove the necessity of (5.42). We conjecture that this is not the case.

For a measure J satisfying (5.41), $\mathbf{T}_d = d(\mathbf{T})$ implies $J_d^* = J^*$. This proves the following corollary.

Corollary 5.3.2. *Let the plant satisfy (5.42) and the performance measure satisfy (5.41). Then there are no performance limitations imposed by decoupling, in the sense that* $J_d^* = J^*$ *holds.*

Youla, Jabr and Bongiorno (1976) have shown that the optimal controller of the H_2- problem leads to a decoupled system provided that the plant is square, stable and minimum phase. Corollary 5.3.2 thus generalizes their result to possibly unstable, non-minimum-phase, or nonsquare plants, to the block decoupling problem, and to a wide class of measures including J_∞ and J_2.

Example 5.3.6. Consider the plant of Example 5.3.1. We have already shown that **(P)** is solvable if $m_1 = 2$ and $m_2 = 1$. The matrix N^+ in Example 5.3.3 is already in Hermite form, so that $N_g^+ = N^+$. The Hermite form of D^+ is given by

$$D_g^+ = \begin{bmatrix} 1 & 1 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is block diagonal with respect to the partition $m_1 = 2$ and $m_2 = 1$. Thus we will not lose any H_∞- and H_2- performance (in the sense of J_∞ and J_2) by designing separate optimal controllers $K_{d,1}$ and $K_{d,2}$ for

$$G_1^+ = \begin{bmatrix} 1 & \frac{-1}{s-1} \\ 0 & 1 \end{bmatrix}$$

and $G_2^+ = s - 1$, respectively, and then using

$$K = (G_g^-)^{-1} \begin{bmatrix} K_{d,1} & 0 \\ 0 & K_{d,2} \\ 0 & 0 \end{bmatrix}$$

as a controller for G, where $[N_g^+(D_g^+)^{-1} \;\; 0]G_g^- = G$ is a stability factorization. Recall that, in this subsection, we allow the controller to be non-proper. A proper controller for this example is obtained by choosing stable polynomial matrices P_1 and P_2 such that $(G_g^-)^{-1}(block\,diag\{P_1, P_2\})^{-1}$ is proper, and designing proper optimal controllers $K_{d,1}$ and $K_{d,2}$ for the strictly proper systems $G_1^+P_1^{-1}$ and $G_2^+P_2^{-1}$, respectively, compare (5.36). ♢

Note that the condition (5.42) is far from being necessary for the H_∞ case. This is in contrast to the condition in Theorem 5.3.3 and comes from the fact that H_∞-norm of a block diagonal matrix is equal to the maximum of the norms of the diagonal blocks. Dickman and Sivan (1985) give an example illustrating this effect.

Proof of Theorem 5.3.3. Sufficiency: Suppose that N^+ and D^+ are both block diagonal. It suffices to prove the claim for $v = 2$. The general case then follows by using induction on v. Consider any stabilizing controller K for G and $T = GK(I+GK)^{-1}$. We have to construct another stabilizing controller $\bar{K}$ for G such that $\bar{T} := G\bar{K}(I + G\bar{K})^{-1} = d(T)$. The matrix G^+ is block diagonal, i.e., $G^+ = block\,diag\{G_1^+, G_2^+\}$. Since K stabilizes G, it follows that $\tilde{K} := [I \;\; 0]G^-K$ stabilizes G^+. Consider now Figure 5.2. Internal stability means (Chapter 3) that the four transfer function matrices $r \to z$, $r \to y$, $d \to z$, and $d \to y$, are stable. In particular, the transfer function matrices, $r_1 \to z_1$, $r_1 \to y_1$, $d_1 \to z_1$ and $d_1 \to y_1$, are stable. This implies that the encircled controller $K_{d,1}$ internally stabilizes G_1^+. Its transfer function is

$$K_{d,1} = \tilde{K}_{11} + \tilde{K}_{12}(I + \tilde{K}_{22}G_2^+)^{-1},$$

where $\tilde{K}_{ij} \in \mathbb{R}^{m_i \times m_j}(s)$ is the (i,j) block of $\tilde{K}$, $\; i,j = 1,2$. Similar arguments for the second channel show that

$$K_{d,2} = \tilde{K}_{22} + \tilde{K}_{21}(I + \tilde{K}_{11}G_1^+)^{-1}$$

internally stabilizes G_2^+. Hence, $K_d := block\,diag\{K_{d,1}, K_{d,2}\}$ stabilizes G^+, implying that $\bar{K} := (G^-)^{-1}[I \;\; 0]^T K_d$ stabilizes $G = [G^+ \;\; 0]G^-$. Moreover, we have $\bar{T} := G\bar{K}(I + G\bar{K})^{-1} = G^+K_d(I + G^+K_d)^{-1} = d(T)$.

Necessity: Assume first that N_g^+ is not block diagonal. Let $N_i^+ \in \mathbb{R}^{m_i \times m}[s]$ be the i-th block-row of N_g^+, L_i be the greatest common left divisor of the columns of N_i^+, and $L := block\,diag\{L_i\}$. Then, $N_g^+ = L\tilde{N}^+$ for some polynomial matrix $\tilde{N}^+$, which can be partitioned as follows:

$$\tilde{N}^+ = \begin{bmatrix} \tilde{N}_1^+ \\ \tilde{N}_2^+ \\ \vdots \\ \tilde{N}_v^+ \end{bmatrix}, \tilde{N}_i^+ = [0 \;\; N_i \;\; M_i],$$

where $\tilde{N}_i^+ \in \mathbb{R}^{m_i \times m}[s]$, $N_i \in \mathbb{R}^{m_i \times m_i}[s]$, and $M_i \in \mathbf{R}^{m_i \times p_i}[s]$; $\;\; p_i := \sum_{j=i+1}^{v} m_j$. Let k $(1 \le k < v)$ be the index such that $M_k \neq 0$ and $M_i = 0$

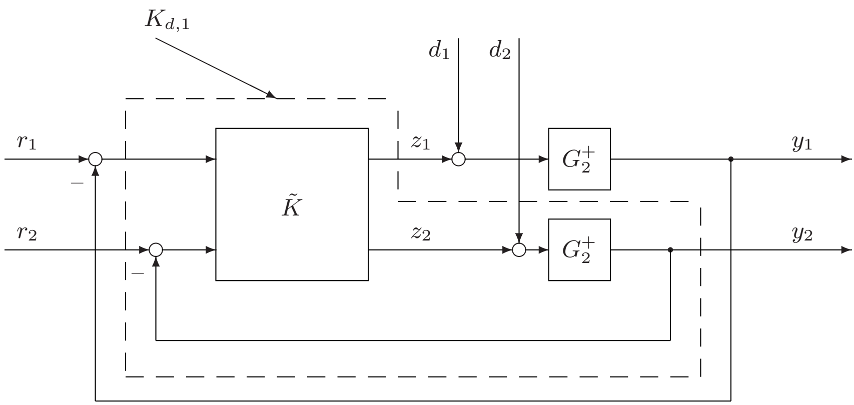


Fig. 5.2. Controller construction

for $i > k$. Since the degrees of the off-diagonal elements of a row in N_g^+ are strictly smaller than the degree of the corresponding diagonal element, the matrix N_k has at least one zero $s_0 \in \mathbb{C}^+$. For this zero there exist $y_0 \in \mathbf{C}^{m_i}$ and $z_0 \in \mathbf{C}^{p_i}$ such that

$$\begin{aligned} y_0^T N_k(s_0) &= 0, \\ y_0^T M_k(s_0) &\neq 0, \\ [0, y_0^T, z_0^T]\tilde{N}^+(s_0) &= 0. \end{aligned}$$

Consider now a right strict adjoint N_{ra}^+ of N^+. Since $\tilde{N}^+ N_{ra}^+$ is block diagonal, the latter equality implies

$$y_0^T \tilde{N}_k^+(s_0) N_{ra}^+(s_0) = 0.$$

Theorem 5.3.2 shows that, for any stabilizing and block decoupling controller $\bar{K}$ for G, we have $G\bar{K} = Q_d K_d$, where $Q_d = N^+ N_{ra}^+ (\bar{D}^+ \bar{D}^+)^{-1}$ and K_d stabilizes Q_d. The parameterization of all stabilizing controllers K_d for Q_d (Chapter 3) implies that each element $\bar{T}$ in $\mathbf{T}_d$ has the form

$$\bar{T} = N^+ N_{ra}^+ \bar{H}_d,$$

where $\bar{H}_d$ is some block diagonal stable rational matrix. Its k-th diagonal block $\bar{T}_k$ becomes $\bar{T}_k = L_k \bar{P}_k$, where $\bar{P}_k = \tilde{N}_k^+ N_{ra}^+ \bar{H}_d [0 \quad I \quad 0]^T$ satisfies $y_0^T \bar{P}_k(s_0) = 0$.

We now construct a particular stabilizing controller K for G such that the k-th diagonal block T_k of $T = GK(I + GK)^{-1}$ is of the form $T_k = L_k P_k$ with $y_0^T P_k(s_0) \neq 0$. This then implies that $d(T) \in d(\mathbf{T})$ but $d(T) \notin \mathbf{T}_d$, which will complete the proof.

Let $G = ND^{-1}$ be a right coprime polynomial matrix fraction of G with $N = N^+ N^-$ for some stable polynomial matrix N^-. It follows again from the parameterization of all stabilizing controllers for G that $\mathbf{T}$ is parameterized by

$$T = N(-\tilde{Y} + Q\tilde{D}),$$

where Q runs through all stable rational matrices, and the polynomial matrices $\tilde{Y}$ and $\tilde{D}$ satisfy the Bezout identity (Chapter 2). The k-th diagonal block of T has the form $T_k = L_k P_k$, where $P_k = \tilde{N}_k^+ N^-(-\tilde{Y}+Q\tilde{D})[0 \;\; I \;\; 0]^T$. Note that $N^-(s_0)$ is nonsingular, so that $y_0^T \tilde{N}_k^+(s_0)N^-(s_0) \neq 0$. We have $det(\tilde{D}) = det(D)$, so that $\tilde{D}(s_0)$ is nonsingular, by assumption. Hence $(-\tilde{Y}(s_0) + Q(s_0)\tilde{D}(s_0))[0 \;\; I \;\; 0]^T$ can be set to be an arbitrary matrix by a proper choice of $Q(s_0)$. Especially we can choose Q such that $y_0^T P_k(s_0) \neq 0$. This Q defines the required controller.

The necessity of the other condition (D^+ is block diagonal) can be proven using similar arguments and appropriate dual concepts. □

In this section, firstly, the block decoupling problem with internal stability by unity output feedback is constructively solved for possibly non-square plants of full row rank. Necessary and sufficient solvability conditions (Theorem 5.3.1) and a characterization of all controllers solving the problem (Theorem 5.3.2) are given. Secondly, a class of systems is specified for which there is no need to scarify achievable performance by putting an decoupling precompensator into the feedback loop (Theorem 5.3.3). These results together can be used to determine a suitable grouping of the outputs for decoupling, and to support the decision on whether to do the decoupling inside or outside the feedback loop.

5.4 A Unified Solution

The approach to the general decoupling problem **(P)**, which we have taken in the previous two sections, is first treating the square case and then extending to the non-square case. This is of course a natural way of research evolution and facilitates understanding. It is however possible and also advantageous to unify both cases and develop a solution in a general setting. Thus, in this section, we will give the unified approach to **(P)**, which is also independent of Sections 5.2 and 5.3. It may give neater results in some places (for example, you may compare the parameterization of all the solutions later). The two new concepts, least block multiples of a full rank polynomial matrix and skew prime polynomial matrices in $\mathbb{C}^+$, are introduced in resolution of the problem.

To unify square and non-square cases, we need to modify the stability factorization, $G = [G^+ \quad 0]G^-$. But $[G^+ \quad 0]$ is to be replaced by a single matrix and for notational clarity it is still be denoted by G^+.

Definition 5.1.1′. *Let $G \in \mathbb{R}^{m\times l}(s)$ have full row rank, and assume that there are a full row rank $G^+ \in \mathbb{R}^{m\times l}(s)$ and a nonsingular $G^- \in \mathbb{R}^{l\times l}(s)$*

such that

$$G = G^+ G^-. \tag{5.43}$$

(5.43) is called a stability factorization of G *if the zeros and poles of* G^+ *(respectively,* G^-*) are precisely the RHP (respectively, LHP) zeros and poles (counting multiplicities) of* G*. Such a* G^- *is then stable, minimum phase, and invertible.* G^+ *and* G^- *in a stability factorization are here called the greatest* C^+*-divisor and the greatest* C^-*-divisor of* G*, respectively.*

It follows from Lemma 5.1.1 that this new stability factorization exists for any full rank $G(s)$. As a convention, the stability factorization in this section is always referred to that in Definition 5.1.1$'$ above.

For a full row rank polynomial matrix $P \in \mathbb{R}^{m\times l}[s]$, denote by $\Delta(P)$ the m-th determinantal, the monic greatest common divisor of all $m \times m$ minors of $P(s)$. In terms of the Smith form, $P = U[\Lambda \;\; 0]V$ with U and V unimodular and Λ diagonal, there holds $\Delta(P) = det(\Lambda)$. For a square P, $\Delta(P)$ equals $det(P)$ up to a nonzero constant. Let $P = P^+P^-$ be a stability factorization as in Definition 5.1.1$'$, define $\Delta^+(P) := \Delta(P^+)$, $deg(P) := deg(\Delta(P))$, where the right hand side of the last equality means the degree of the polynomial $\Delta(P)$, and $deg^+(P) = deg(P^+) = deg(\Delta(P^+))$. P is called antistable if $\Delta(P) = \Delta(P^+)$, that is, P has all its zeros in $\mathbb{C}^+$.

Let $P \in \mathbb{R}^{m\times l}[s]$ be a full row rank polynomial matrix. The representation, $P = LU$, is said to be an LU-factorization of P if L is a nonsingular polynomial matrix and U is an irreducible polynomial matrix, i.e., $U(s)$ has full row rank for all $s \in \mathbb{C}$.

Lemma 5.4.1. *For an LU-factorization,* $P = LU$*, there holds* $\Delta(P) = \Delta(L)$*. In general, let* $P \in \mathbb{R}^{m\times l}[s]$ *and* $P_1 \in \mathbb{R}^{m\times k}[s]$ *be both of full row rank with* $k \geq m$ *such that* $P = P_1P_2$ *for some polynomial matrix* P_2*, then* $\Delta(P_1)$ *divides* $\Delta(P)$*.*

Proof. P has a Smith factorization, $P = U_1[\Lambda \;\; 0]U_2$, where U_1 and U_2 are unimodular with $\Lambda = diag\{p_i\}$. By definition, $\Delta(P) = \Delta(\Lambda)$. Express P as $P = LU$, where $L = U_1\Lambda$ and $U = [I \;\; 0]U_2$, then $P = LU$ is an LU-factorization and $\Delta(L) = \Delta(\Lambda) = \Delta(P)$. Now, let $P = \tilde{L}\tilde{U}$ be another LU-factorization, then $LU = \tilde{L}\tilde{U}$. If $L^{-1}\tilde{L} = U_3$ is unimodular, there will hold $\Delta(\tilde{L}) = \Delta(L) = \Delta(P)$, which then completes the proof. Let AB^{-1} be a coprime polynomial matrix fraction of U_3, then $U = AB^{-1}\tilde{U}$. Lemma 5.1.2 says that $\tilde{U}$ has B as its left divisor but $\tilde{U}$ is irreducible and B is thus unimodular. Consider dually $\tilde{U} = BA^{-1}U$, one concludes that A is unimodular, too. The second statement follows from applying the Binet-Cauchy formula (Gantmacher,1959) to the equality, $P = L_1\tilde{P}$, where $P_1 = L_1U_1$ is an LU-factorization and $\tilde{P} = U_1P_2$. □

A strange thing about nonsquare polynomial matrices is that an antistable P can have a stable divisor for itself. For instance, we have $[s \;\; 0] =$

$[s \;\; 0]diag\{1, s+1\}$. However, the following lemma shows that such a stable divisor has certain structure.

Lemma 5.4.2. *Let $P \in \mathbb{R}^{m\times l}[s]$ and $P_1 \in \mathbb{R}^{m\times l}[s]$ be both of full row rank such that $P = P_1 P_2$ for some stable and nonsingular polynomial matrix P_2. If P is antistable, then UP_2^{-1} is a polynomial matrix, where $P = LU$ is an LU-factorization.*

Proof. Assume conversely that it is not the case, let $\tilde{P}_2^{-1}\tilde{U}$ be a left coprime polynomial matrix fraction of UP_2^{-1} with $\tilde{P}_2$ stable and $deg(\tilde{P}_2)$ nonzero, it then follows that $L\tilde{P}_2^{-1}\tilde{U} = P_1$. By Lemma 5.1.2, $L\tilde{P}_2^{-1}$ is a polynomial matrix. This means that L has a stable divisor $\tilde{P}_2$ which is not unimodular, contradicting the fact that P is antistable. □

5.4.1 Skew Primeness in $\mathbb{C}^+$

The matrices $D \in \mathbb{R}^{m\times m}[s]$ and $N \in \mathbb{R}^{m\times l}[s]$ are called left coprime in $\mathbb{C}^+$, if

$$Rank[D(s) \;\; N(s)] = m, \quad for\ all\ s \in \mathbb{C}^+.$$

The notion of skew primeness needs to be generized, too.

Definition 5.4.1. *Let a full row rank $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $D \in \mathbb{R}^{m\times m}[s]$ be given. N and D are said to be externally skew prime in $\mathbb{C}^+$ if there are a full row rank $\bar{N} \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $\bar{D} \in \mathbb{R}^{l\times l}[s]$ such that*

$$DN = \bar{N}\bar{D} \tag{5.44}$$

with D and $\bar{N}$ left coprime in $\mathbb{C}^+$ and N and $\bar{D}$ right coprime in $\mathbb{C}^+$. Such $(\bar{N}, \bar{D})$ is refereed to as an internally skew prime pair of (N, D) in $\mathbb{C}^+$.

Note that the skew primeness in $\mathbb{C}^+$ and ordinary skew primeness (in $\mathbb{C}$) as defined by Wolovich (1978) differ only in relevant domains, and all tests and algorithms for ordinary skew prime polynomial matrices in Wolovich (1978) and Wang,Sun and Zhou (1989) thus apply to the case of skew primeness in $\mathbb{C}^+$ with the obvious modification of replacing $\mathbb{C}$ by $\mathbb{C}^+$. Our development does however require skew primeness in $\mathbb{C}^+$. The following lemma shows that, under certain conditions, skew primeness in $\mathbb{C}^+$ is equivalent to the ordinary skew primness.

Lemma 5.4.3. *Let a full row rank $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $D \in \mathbb{R}^{m\times m}[s]$ be both antistable, then N and D are externally skew prime in $\mathbb{C}^+$ if and only if they are externally skew prime (in $\mathbb{C}$).*

Proof. Assume that N and D are externally skew prime in $\mathbb{C}^+$, then there are internally skew prime $\bar{N}$ and $\bar{D}$ in $\mathbb{C}^+$. By performing a stability factorization for $\bar{D}^T$, we get $\bar{D} = \bar{D}^- \bar{D}^+$ with $\bar{D}^-$ stable and $\bar{D}^+$ antistable. (5.44) then becomes

$$DN = \tilde{N}\tilde{D},$$

where $\tilde{N} = \bar{N}\bar{D}^-$ and $\tilde{D} = \bar{D}^+$. The left coprimeness of D and $\bar{N}$ in $\mathbb{C}^+$ implies that

$$Rank[D(s) \;\; \tilde{N}(s)] = m, \quad for\ all\ s \in \mathbb{C}^+, \tag{5.45}$$

since $\bar{D}^-(s)$ is nonsingular for all $s \in \mathbb{C}^+$. Additionally, because D is antistable, we also have

$$Rank[D(s) \;\; \tilde{N}(s)] = m, \quad for\ all\ s \in \mathbb{C}^-. \tag{5.46}$$

(5.45) and (5.46) together mean that D and $\tilde{N}$ are left coprime. In a similar way, one can show that N and $\tilde{D}$ are right coprime, too. Thus, it follows that N and D are externally skew prime. The converse implication is trivial. The proof is completed. □

As one will see later, our solvability condition and compensator parameterization for **(P)** deal only with antistable N and D. Then, due to Lemma 5.4.3, a natural question which one is likely to ask is why the skew primeness in $\mathbb{C}^+$ is here introduced. The real reason is that in spite of the equivalence there do exist differences between the ordinary *internally* skew prime pair $(\tilde{N}, \tilde{D})$ and the internally skew prime pair $(\bar{N}, \bar{D})$ in $\mathbb{C}^+$, as seen in the above proof and will be further exhibited by the following example. Even more importantly, it will turn out that a general compensator which solves **(P)** is constructed from $(\bar{N}, \bar{D})$ but not from $(\tilde{N}, \tilde{D})$, and it therefore proves to be necessary to introduce skew prime polynomial matrices in $\mathbb{C}^+$.

Example 5.4.1. Let $N = [s \;\; 0]$ and $D = (s-1)$. If we set $\tilde{N} = [s \;\; 0]$ and $\tilde{D} = diag\{(s-1), 1\}$, then

$$DN = (s-1)[s \;\; 0] = [s \;\; 0]diag\{(s-1), 1\} = \tilde{N}\tilde{D}.$$

One also sees that D and $\tilde{N}$ are left coprime and N and $\tilde{D}$ are right coprime. $(\tilde{N}, \tilde{D})$ is thus an internally skew prime pair of (N, D). Further, for a pair $(\tilde{N}, \tilde{D})$ to be an internally skew prime one of (N, D), it must have certain properties. Take $\tilde{D}$ for comparison, it satisfies

$$\Delta(\tilde{D}) = \Delta(D) = (s-1). \tag{5.47}$$

On the other hand, it can be easily verified that $\bar{N} = [s \;\; 0]$ and $\bar{D} = diag\{(s-1), p\}$ with p being a stable polynomial constitute an internally skew prime pair of (N, D) in $\mathbb{C}^+$ (but not in $\mathbb{C}$) and $\bar{D}$ satisfies

$$\Delta(\bar{D}) = (s-1)p. \tag{5.48}$$

(5.47) and (5.48) clearly exhibit an essential difference between ordinary skew prime pairs and skew prime pairs in $\mathbb{C}^+$ for the same pair of externally skew prime polynomial matrices N and D even when they are both antistable. ♢

For an ordinary skew prime pair $(\tilde{N}, \tilde{D})$ of (N, D), we have that $\Delta(D) = \Delta(\tilde{D})$ and $\Delta(N) = \Delta(\tilde{N})$ since $N\tilde{D}^{-1} = D^{-1}\tilde{N}$ with both fractions coprime. In general, this does not hold for a skew prime pair in $\mathbb{C}^+$. The following lemma gives a test for skew primeness in $\mathbb{C}^+$ and characterizes zero polynomials of an internally skew prime pair in $\mathbb{C}^+$ in terms of those of a given externally skew pair.

Lemma 5.4.4. *Let a full row rank $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $D \in \mathbb{R}^{m\times m}[s]$ be both antistable, then*

(i) N and D are externally skew prime in $\mathbb{C}^+$ if and only if there are $\bar{N}$ and $\bar{D}$ such that (5.44) holds true with $\Delta^+(\bar{D}) = \Delta(D)$ and N and $\bar{D}$ right coprime in $\mathbb{C}^+$;

(ii) for an internally skew prime pair $(\bar{N}, \bar{D})$ in $\mathbb{C}^+$ of (N, D), there holds the equality $\Delta(\bar{N}) = \Delta(N)$; and

(iii) the equality $\Delta(\bar{D}) = \Delta(D)$ also holds if N is square. In this case, there is no difference between ordinary internally skew prime pairs and internally skew prime pairs in $\mathbb{C}^+$.

Proof. (i): For the sufficiency, we need only to show that D and $\bar{N}$ are left coprime in $\mathbb{C}^+$. This is true if there holds

$$deg(D) = deg^+(D_1), \tag{5.49}$$

where $D_1^{-1}\bar{N}_1$ is a coprime polynomial matrix fraction of $D^{-1}\bar{N}$. Let $N_1(\bar{D}_1)^{-1}$ be a coprime polynomial matrix fraction of $N(\bar{D})^{-1}$. As $N(\bar{D})^{-1}$ has been assumed to be coprime in $\mathbb{C}^+$, there holds

$$deg^+(\bar{D}_1) = deg^+(\bar{D}). \tag{5.50}$$

(5.44) can be rewritten as $N_1(\bar{D}_1)^{-1} = (D_1)^{-1}\bar{N}_1$ with both sides being coprime, implying that

$$deg^+(\bar{D}_1) = deg^+(D_1). \tag{5.51}$$

Collecting (5.50), (5.51), and the assumed condition, $deg^+(\bar{D}) = deg(D)$, gives (5.49).

For necessity, assume now that antistable matrices N and D are externally skew prime and a dual internally skew prime polynomial matrices in $\mathbb{C}^+$ are $\bar{N}$ and $\bar{D}$. Let R be a greatest common right divisor of N and $\bar{D}$ so that

$$\bar{D} = \bar{D}_1 R, \quad \text{and} \quad N = N_1 R. \tag{5.52}$$

Then, (5.44) becomes

$$DN_1 = \bar{N}\bar{D}_1, \tag{5.53}$$

where $N_1\bar{D}_1^{-1}$ is a right coprime. Moreover, D and $\bar{N}$ are coprime since they are coprime in $\mathbb{C}^+$ and D is antistable. Therefore, (5.53) means that $(\bar{N}, \bar{D}_1)$ is an internally skew prime pair of (N_1, D) and we thus have

$$\Delta(\bar{D}_1) = \Delta(D), \tag{5.54}$$

which implies that $\bar{D}_1$ is antistable, too. Since N and $\bar{D}$ are coprime in $\mathbb{C}^+$, their common divisor R must be stable and $\Delta^+(\bar{D}) = \Delta^+(\bar{D}_1 R) = \Delta^+(\bar{D}_1) = \Delta(\bar{D}_1)$. (5.54) then becomes $\Delta^+(\bar{D}) = \Delta(D)$.

(ii): It follows also from (5.53) that

$$\Delta(\bar{N}) = \Delta(N_1). \tag{5.55}$$

By Lemma 5.4.1, (5.52) implies that

$$\Delta(N_1) \;\; \textit{divides} \;\; \Delta(N). \tag{5.56}$$

By Lemma 5.4.2, (5.52) can be rewritten as $LP = N_1$, where $N = LU$ is a LU-factorization and $P = UR^{-1}$ is a polynomial matrix. It then follows again from Lemma 5.4.1 that

$$\Delta(L) = \Delta(N) \;\; \textit{divides} \;\; \Delta(N_1). \tag{5.57}$$

(5.55)-(5.57) together give the required result.

(iii): If N is square, then (5.52) implies that $\Delta(R)$ divides $\Delta(N)$ and R is antistable. But it has been shown in (i) that R is stable and it is thus unimodular. This, together with (5.54), implies that

$$\Delta(\bar{D}) = \Delta(\bar{D}_1 R) = \Delta(\bar{D}_1) = \Delta(D).$$

The proof is completed. □

5.4.2 Least Block Multiples

The notion of strict adjoints, as introduced by Hammer and Khargonekar (1984) played an important role in the solution of diagonal decoupling problem (Section 2). Recall that for a nonsingular matrix $P \in \mathbb{R}^{m\times m}[s]$, a polynomial matrix P_{ra} is said to be a right strict adjoint of P if $PP_{ra} \in \mathbb{R}_d^{m\times m}[s]$ and P_{ra} is a left divisor of any polynomial matrix P_a for which $PP_a \in \mathbb{R}_d^{m\times m}[s]$, where the partition is $m_i = 1, i = 1, 2, \cdots, v$. The following example suggests that this notion can not be extended to the nonsquare case, and motivates the alternative concept of least block multiples, which will be formally introduced in Definition 5.4.2 below.

Example 5.4.2. Consider a polynomial matrix be $P = [s-1 \;\; s]$. If we take $P_1 = [1 \;\; 0]^T$ and $P_2 = [0 \;\; 1]^T$, then both PP_1 and PP_2 are diagonal and nonsingular. However, there exists no polynomial matrix P_{ra} such that PP_{ra} is nonzero and P_{ra} a common divisor of both P_1 and P_2. ♢

One can however consider products PP_i instead of P_i themselves. It is easily verified that PP_1 and PP_2 are both multiples of PP_3 with $P_3 = [-1 \;\; 1]^T$. It turns out that such a PP_3 can always be constructed so that the following definition makes sense.

Definition 5.4.2. *Let $P \in \mathbb{R}^{m\times l}[s]$ have full row rank, a block diagonal and nonsingular polynomial matrix $P_m \in \mathbb{R}_d^{m\times m}[s]$ is said to be a right block multiple of P with respect to the partition $\{m_i\}$ if there is a polynomial matrix $P_a \in \mathbb{R}^{l\times m}[s]$ such that $P_m = PP_a$. A right block multiple P_{rm} is further said to be a least right block multiple of P if for any right block multiple P_m of P with respect to the same partition, P_{rm} is a left divisor of P_m, i.e., $P_m = P_{rm}P_d$ for some $P_d \in \mathbb{R}_d^{m\times m}[s]$.*

In Example 5.4.2, $PP_3 = 1$ is a least right block multiple. In general, we need a procedure for constructing a least right block multiple in a systematic way. Given a full row rank $P \in \mathbb{R}^{m\times l}[s]$ and a partition $\{m_i\}$, we express P as an LU-factorization

$$P = LU,$$

where $L \in \mathbb{R}^{m\times m}[s]$ is nonsingular and $U \in \mathbb{R}^{m\times l}[s]$ is irreducible so that there is a polynomial matrix X such that

$$LX = I.$$

L^{-1} is then partitioned as

$$L^{-1} = [\tilde{L}_1 \; \tilde{L}_2 \; \cdots \; \tilde{L}_v],$$

where $\tilde{L}_i \in \mathbb{R}^{m\times m_i}(s), i = 1, 2, \cdots, v$. We further factorize each $\tilde{L}_i$ as a right coprime polynomial matrix fraction

$$\tilde{L}_i = A_i B_i^{-1}, \quad i = 1, 2, \cdots, v,$$

where $A_i \in \mathbb{R}^{m\times m_i}[s]$ and $B_i \in \mathbb{R}^{m_i\times m_i}[s]$. Then, L^{-1} becomes

$$L^{-1} = AB^{-1},$$

where

$$A = [A_1 \;\; A_2 \;\; \cdots \;\; A_v],$$

and

$$B = block\, diag\{B_1 \;\; B_2 \;\; \cdots \;\; B_v\}. \tag{5.58}$$

The following lemma shows that B is just a least right block multiple.

Lemma 5.4.5. *Let P be a full row rank polynomial matrix, then P has a least right block multiple with respect to every partition. A particular least right block multiples of P is given by $P_{rm} = B$, as defined in (5.58). Further, for a fixed partition, each diagonal block in a least right block multiple of P is right unimodular equivalent to the corresponding diagonal block in all other least right block multiples of P.*

Proof. Take $P_{ra} = XA \in \mathbb{R}^{l \times m}[s]$, it then follows that $PP_{ra} = PXA = LA = B$ so that B is a right block multiple of P. Assume now that P_m be an arbitrary right block multiple of P. This means that $P_m = PP_a$ for some polynomial matrix P_a, equivalently,

$$UP_a = L^{-1}P_m = [A_1B_1^{-1}P_{m11} \quad A_2B_2^{-1}P_{m22} \quad \cdots \quad A_vB_v^{-1}P_{mvv}],$$

where $P_m = block\,diag\{P_{mii}\}$. One sees from the above equation that $A_iB_i^{-1}P_{mii}$ are polynomial matrices, but as $A_iB_i^{-1}$ are coprime, $B_i^{-1}P_{mii}$ are thus polynomial matrices, or $P_m = BP_d$ for some block diagonal polynomial matrix P_d. Therefore, B is a left divisor of any right block multiple of P and by definition it is a least right block multiple of P. Let P_{rm} and P'_{rm} be both least right block multiples of P with respect to the same partition, it follows from the definition that $P_{rm} = P'_{rm}P_1$ and $P'_{rm} = P_{rm}P_2$ for some block diagonal polynomial matrices P_1 and P_2, implying that $P_{rm} = P_{rm}P_2P_1$ and both P_1 and P_2 are unimodular. Since they are block diagonal, all the diagonal blocks of P_1 and P_2 are unimodular, too. The proof is completed. □

Example 5.4.3. Consider a 3×4 polynomial matrix

$$P = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & s & 0 & s+1 \\ 0 & s & 1 & s+1 \end{bmatrix}.$$

An LU-factorization is given by $P = LU$ with

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 1 & s & 0 \\ 0 & s & 1 \end{bmatrix}.$$

Its inverse is

$$L^{-1} = \begin{bmatrix} \frac{s}{s-1} & \frac{-1}{s-1} & 0 \\ \frac{-1}{s-1} & \frac{1}{s-1} & 0 \\ \frac{s}{s-1} & \frac{-s}{s-1} & 1 \end{bmatrix}.$$

Let first the partition be $m_1 = 2$ and $m_2 = 1$, then the corresponding submatrices $\tilde{L}_1$ and $\tilde{L}_2$ of L^{-1} are of sizes 3×2 and 3×1, respectively. Their coprime polynomial matrix fractions are constructed as follows

$$\tilde{L}_1 = \begin{bmatrix} \frac{s}{s-1} & \frac{-1}{s-1} \\ \frac{-1}{s-1} & \frac{1}{s-1} \\ \frac{s}{s-1} & \frac{-s}{s-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & s-1 \end{bmatrix}^{-1} = A_1 B_1^{-1},$$

and

$$\tilde{L}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [1]^{-1} = A_2 B_2^{-1}.$$

We then obtain a least right block multiple $P_{rm}(2,1)$ with respect to the partition $\{2,1\}$ as

$$P_{rm}(2,1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

On the other hand, if a different partition, $m_1 = 1, m_2 = 2$, is considered, then the relevant submatrices and their coprime fractions become

$$\tilde{L}_1 = \begin{bmatrix} \frac{s}{s-1} \\ \frac{-1}{s-1} \\ \frac{s}{s-1} \end{bmatrix} = \begin{bmatrix} s \\ -1 \\ s \end{bmatrix} (s-1)^{-1} = A_1 B_1^{-1},$$

and

$$\tilde{L}_2 = \begin{bmatrix} \frac{-1}{s-1} & 0 \\ \frac{1}{s-1} & 0 \\ \frac{-s}{s-1} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = A_2 B_2^{-1}.$$

A least right block multiple $P_{rm}(1,2)$ with respect to the partition $\{1,2\}$ is given by

$$P_{rm}(1,2) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \diamondsuit$$

We are now interested in the zero property of least right block multiples. In Example 5.4.3, there hold $\Delta(P) = (s-1)$, $\Delta(P_{rm}(2,1)) = (s-1)$, and $\Delta(P_{rm}(1,2)) = (s-1)^2$. The following lemma relates the zero polynomial of P_{rm} to that of P.

Lemma 5.4.6. *Let P_{rm} be a least right block multiple of a full row rank polynomial matrix P, then, $\Delta(P)$ divides $\Delta(P_{rm})$ and $\Delta(P_{rm})$ divides $(\Delta(P))^m$.*

Proof. By definition, for any least right block multiple P_{rm} there is a polynomial matrix P_{ra} such that $PP_{ra} = P_{rm}$, which implies from Lemma 5.4.1 that $\Delta(P)$ divides $\Delta(P_{rm})$. Let $P = LU$ be an LU-factorization and X a polynomial matrix such that $UX = I$. Take $P_a = X\,adj(L)$, where $adj(L)$ is the adjoint of L, that is, $L\,adj(L) = det(L)\,I$, It then follows that $det(L)\,I$ is a right block multiple of P with respect to any partition as $PP_a = LUX\,adj(L) = det(L)\,I$. Therefore, there is a block polynomial matrix P_1 such that $det(L)\,I = P_{rm}P_1$. By Lemma 5.4.1, $\Delta(P_{rm})$ divides $\Delta(det(L) \cdot I_m) = (\Delta(L))^m = (\Delta(P))^m$, noting that $det(L)$ differs from $\Delta(L)$ only by a non-zero constant. Hence, the result. □

Lemma 5.4.6 shows that P_{rm} does not have any zeros different from those of P. Especially, P_{rm} will be stable (respectively, antistable) if P has this property. Example 5.4.3 demonstrates that multiplicity of a zero of P_{rm} may be greater than the corresponding multiplicity in P.

If the diagonal partition is taken into account, that is, $m_i = 1, i = 1, 2, \cdots, v, v = m$, one can expect a natural connection between least right block multiples and strict right adjoints. The following corollary serves this purpose.

Corollary 5.4.1. *For a nonsingular polynomial matrix P, P_{ra} is a right strict adjoint of P if and only if PP_{ra} is a least right block multiple of P with respect to the partition, $m_1 = m_2 = \cdots = m_v = 1$.*

Proof. If P_{ra} is a strict right adjoint of P, then $PP_{ra} \in \mathbb{R}_d^{m\times m}[s]$ and for any P_a such that $PP_a = P_m$ is diagonal and nonsingular, there holds $P_a = P_{ra}P_1$ for some diagonal polynomial matrix P_1. It also means that $P_m = PP_{ra}P_1$ and $P_{rm} = PP_{ra}$ is thus a least right diagonal multiple of P. Conversely, assume that P_{rm} is a least right diagonal multiple of P, then $PP_{ra} = P_{rm}$ for some polynomial matrix P_{ra}. Furthermore, for any P_a such that $PP_a = P_m$ is diagonal and nonsingular, we have $P_m = P_{rm}P_1$ for some diagonal polynomial matrix P_1. This implies that $PP_a = PP_{ra}P_1$, i.e., $P_a = P_{ra}P_1$, and then P_{ra} is a strict right adjoint of P. The proof is completed. □

Let P_a be a polynomial matrix such that $PP_a = P_m$ be a right block multiple of a full row rank polynomial matrix P. Then, *in general*, P_a can *not* be expressed as $P_a = P_{ra}P_1$ for any P_{ra} such that $PP_{ra} = P_{rm}$ is a least right block multiple of P, as shown by the following example. Let $P = [1 \;\; s]$, set $P_a = [0 \;\; 1]^T$, then $PP_a = s$. It is obvious that $P_{rm} = 1$. Take $P_{ra}^* = [1 \;\; 0]^T$ and $Z = [s \;\; -1]^T$, then a general solution of the equation $PP_{ra} = P_{rm}$ for P_{ra} is $P_{ra} = P_{ra}^* + Zw = [(1 + sw) \;\; w]^T$. One can see that $P_a = P_{ra}p$ will never hold for any polynomials w and p. In fact, P_a can be only expressed as $P_a = P_{ra}^*p + Zw$ with $p = s$ and $w = -1$. This is why least block multiples are introduced and the numerator polynomial matrix of a general decoupling compensator has the form in (5.94), which is much less nice than $\bar{N}_k = \bar{N}_{ra}^+\bar{N}_{q1}$ if the latter were true.

It is obvious that the concept of least left block multiples for a full column rank polynomial matrix can be defined in a dual way and the dual results to least right block multiples all hold.

5.4.3 Solvability Conditions

Let us consider **(P)** now. Combining Theorem 5.1 with the stability factorization in (5.43), we can obtain the following simplification.

Theorem 5.4.1. *For any full row rank plant $G \in \mathbb{R}_p^{m\times l}(s)$, there is a K such that GK is block diagonal and nonsingular, and has no RHP pole-zero cancellations if and only if there is a K^+ which can do so with G replaced by G^+, where $G = G^+G^-$ is a stability factorization.*

Proof. For a stability factorization $G = G^+G^-$, we have

$$\delta^+(G) = \delta^+(G^+). \tag{5.59}$$

Let K and K^+ be related by

$$K^+ = G^-K, \quad or, \quad K = (G^-)^{-1}K^+,$$

then, as G^- is nonsingular, stable, and minimum phase, K and K^+ satisfy

$$\delta^+(K) = \delta^+(K^+). \tag{5.60}$$

GK is nonsingular and block diagonal if and only if so is G^+K^+ since $GK = G^+G^-K = G^+K^+$, which also means that

$$\delta^+(GK) = \delta^+(G^+K^+). \tag{5.61}$$

It follows from (5.59)-(5.61) that

$$\delta^+(GK) = \delta^+(G) + \delta^+(K)$$

if and only if

$$\delta^+(G^+K^+) = \delta^+(G^+) + \delta^+(K^+).$$

The proof is completed. □

In view of Theorem 5.4.1, the greatest $\mathbb{C}^-$-divisor G^- of a plant G has no effect on the solvability of **(P)**. Instead, we need only to consider the greatest $\mathbb{C}^+$-divisor of G^+. By Definition 5.1.1′, $G^+ \in \mathbb{R}^{m\times l}(s)$ has full row rank. We further factorize G^+ as

$$G^+ = (D^+)^{-1}N^+,$$

where $D^+ \in \mathbb{R}^{m\times m}[s]$ is nonsingular, $N^+ \in \mathbb{R}^{m\times l}[s]$ has full row rank, and D^+ and N^+ are both antistable. By a dual statement to Lemma 5.4.5, there is a least left block multiple $D^+_{lm} \in \mathbb{R}_d^{m\times m}[s]$ of D^+ such that

$$D_{la}^+ D^+ = D_{lm}^+$$

for some nonsingular $D_{la}^+ \in \mathbb{R}^{m\times m}[s]$. It follows from Lemmas 5.4.1 and 5.4.6 that D_{la}^+ are antistable. Now, if N^+ and D_{la}^+ are externally skew prime in $\mathbb{C}^+$, then there exist internally skew prime $\bar{N}^+ \in \mathbb{R}^{m\times l}[s]$ and $\bar{D}_{la}^+ \in \mathbb{R}^{l\times l}[s]$ in $\mathbb{C}^+$ such that

$$D_{la}^+ N^+ = \bar{N}^+ \bar{D}_{la}^+$$

with $\bar{N}^+$ of full row rank and $\bar{D}_{la}^+$ nonsingular. Let $\bar{N}_{rm}^+$ be a least right block multiple of $\bar{N}^+$. We are now in a position to state the main solvability condition for **(P)**.

Theorem 5.4.2. *For a full row rank plant G,* **(P)** *is solvable if and only if N^+ and D_{la}^+ are externally skew prime in $\mathbb{C}^+$ and D_{lm}^+ and $\bar{N}_{rm}^+$ are left coprime.*

Proof: In the proof of necessity, we need the following lemmas.

Lemma 5.4.7. *Let A, B, C, D, and N be all polynomial matrices such that $B^{-1}AC^{-1} = D^{-1}N$ with $B^{-1}A$ and $D^{-1}N$ both coprime, then B is a right divisor of D, that is, $D = D_1 B$ for some polynomial matrix D_1.*

Proof. Let $\tilde{C}^{-1}\tilde{A}$ be a left coprime polynomial matrix fraction of AC^{-1}, it then follows (Chapter 2) that $(\tilde{C}B)^{-1}\tilde{A}$ is left coprime and thus $D = U\tilde{C}B$ for some unimodular polynomial matrix U. Hence the result. □

Lemma 5.4.8. *Let $(D^+)^{-1}N^+$ and $N_k D_k^{-1}$ be, respectively, coprime polynomial matrix fractions of G^+ and K, G^+K have no unstable pole-zero cancellations, and D^+ and N^+ be both antistable, then $(D^+)^{-1}(N^+N_k)$ is left coprime and $(N^+N_k)(D_k)^{-1}$ is right coprime in $\mathbb{C}^+$.*

Proof. Assume conversely that $(D^+)^{-1}(N^+N_k)$ is not coprime, then a coprime one $(\tilde{D}^+)^{-1}\tilde{N}$ of it will satisfy

$$deg\,(\tilde{D}^+) < deg\,(D^+).$$

We therefore have

$$\begin{aligned}\delta^+(G^+K) &= \delta^+((\tilde{D}^+)^{-1}\tilde{N}D_k^{-1})\\ &\le deg\,(\tilde{D}^+) + deg^+\,(D_k)\\ &< deg\,(D^+) + deg^+\,(D_k)\\ &= \delta^+(G^+) + \delta^+(K),\end{aligned}$$

which contradicts the assumption that G^+K has no unstable pole-zero cancellations. $(D^+)^{-1}(N^+N_k)$ is thus left coprime. The coprimeness of $(N^+N_k)(D_k)^{-1}$ in $\mathbb{C}^+$ follows similar lines. The proof is completed. □

According to Theorem 5.4.1, we can assume, without loss of generality, that the plant is such that $G = G^+$.

NECESSITY: Let $N_k D_k^{-1}$ and $D_q^{-1} N_q$ be coprime polynomial matrix fractions of K and $Q = G^+ K$, respectively. Then, we have

$$Q = D_q^{-1} N_q = (D^+)^{-1} N^+ N_k D_k^{-1}. \tag{5.62}$$

If K solves **(P)** for G^+, it follows from Theorem 5.4.1 that K is such that $G^+ K$ is block diagonal and has no unstable pole-zero cancellations. The latter fact further implies by Lemma 5.4.8 that $(D^+)^{-1}(N^+ N_k)$ is coprime and Lemma 5.4.7 then says that

$$D_q = D_1 D^+ \tag{5.63}$$

for some polynomial matrix D_1. Since Q is block diagonal, we can have a coprime polynomial matrix fraction of it with its numerator and denominator being both block diagonal, and assume that $D_q^{-1} N_q$ is just such a fraction. By a dual result to Lemma 5.4.5, (5.63) for a block diagonal D_q implies that

$$D_q = D_{q1} D_{lm}^+ = D_{q1} D_{la}^+ D^+, \tag{5.64}$$

where D_{q1} is block diagonal polynomial matrix, D_{lm}^+ is a least left block multiple of D^+, and D_{la}^+ is an antistable polynomial matrix such that $D_{la}^+ D^+ = D_{lm}^+$. Substituting (5.64) into (5.62) gives

$$D_{la}^+ N^+ N_k D_k^{-1} = \bar{N}_q \bar{D}_{q1}^{-1}, \tag{5.65}$$

where

$$\bar{N}_q \bar{D}_{q1}^{-1} = D_{q1}^{-1} N_q \tag{5.66}$$

is block diagonal and polynomial matrices $\bar{N}_q$ and $\bar{D}_{q1}$ are chosen such that they are both block diagonal and right coprime. Since $\bar{N}_q \bar{D}_{q1}^{-1}$ is coprime and (5.65) holds, $\bar{D}_{q1}$ is a left divisor of D_k, i.e.,

$$D_k = \bar{D}_{q1} D_{k1}, \tag{5.67}$$

for some polynomial matrix D_{k1}. Substituting (5.67) into (5.65) yields

$$D_{la}^+ N^+ \bar{D}_{k1}^{-1} \bar{N}_k = \bar{N}_q, \tag{5.68}$$

where

$$\bar{D}_{k1}^{-1} \bar{N}_k = N_k D_{k1}^{-1} \tag{5.69}$$

with the left hand side of (5.69) being a left coprime polynomial matrix fraction. Note from (5.68) that this coprimeness, together with the fact that $\bar{N}_q$ is a polynomial matrix, implies, by Lemma 5.1.2, that

$$D_{la}^{+}N^{+}\bar{D}_{k1}^{-1}=\bar{N}^{+} \tag{5.70}$$

is a polynomial matrix and of full row rank. One knows from Lemma 5.4.8 that no unstable pole-zero cancellations in $G^{+}K$ implies that $N^{+}N_k$ and D_k are right coprime in $\mathbb{C}^{+}$ and this also ensures that

$$N^{+} \text{ and } \bar{D}_{k1} \text{ are right coprime in } \mathbb{C}^{+} \tag{5.71}$$

because $N^{+}N_kD_k^{-1} = N^{+}N_kD_{k1}^{-1}\bar{D}_{q1}^{-1} = N^{+}\bar{D}_{k1}^{-1}\bar{N}_k\bar{D}_{q1}^{-1}$ and the involved fractions $N_kD_k^{-1}$, $N_kD_{k1}^{-1}$, and $\bar{D}_{k1}^{-1}\bar{N}_k$ are all coprime there. The detailed proof of (5.71) is similar to that of Lemma 5.4.8 and is therefore omitted.

We now proceed to show that

$$\Delta(D_{la}^{+})=\Delta^{+}(\bar{D}_{k1}). \tag{5.72}$$

It follows from (5.64) and (5.66) that

$$\Delta^{+}(D_q)=\Delta^{+}(D_{q1})\Delta(D_{la}^{+})\Delta(D^{+}), \tag{5.73}$$

and

$$\Delta^{+}(D_{q1})=\Delta^{+}(\bar{D}_{q1}). \tag{5.74}$$

No unstable pole-zero cancellations in $G^{+}K$ mean that

$$\Delta^{+}(D_q)=\Delta^{+}(D_k)\Delta(D^{+}). \tag{5.75}$$

Combining (5.73)-(5.75) gives

$$\Delta^{+}(D_k)=\Delta^{+}(\bar{D}_{q1})\Delta(D_{la}^{+}). \tag{5.76}$$

Further, (5.67) gives

$$\Delta^{+}(D_k)=\Delta^{+}(\bar{D}_{q1})\Delta^{+}(D_{k1}). \tag{5.77}$$

Since both fractions in (5.69) are coprime, we also obtain

$$\Delta^{+}(D_{k1})=\Delta^{+}(\bar{D}_{k1}). \tag{5.78}$$

Collecting (5.76)-(5.78) yields (5.72). By Lemma 5.4.4, (5.70)-(5.72) together mean that N^{+} and D_{la}^{+} are externally skew prime in $\mathbb{C}^{+}$.

With (5.70), (5.68) becomes

$$\bar{N}^{+}\bar{N}_k=\bar{N}_q. \tag{5.79}$$

Since $\bar{N}_q$ is block diagonal, it follows from Lemma 5.4.5 that there is some $\bar{N}_{q1}\in\mathbb{R}_d^{m\times m}[s]$ such that

$$\bar{N}_{rm}^{+}\bar{N}_{q1}=\bar{N}_q, \tag{5.80}$$

where $\bar{N}^+_{rm}$ is a least right block multiple of $\bar{N}^+$. From (5.64), (5.66), and (5.80), $Q = G^+K$ can be expressed as

$$\begin{aligned} Q &= D_q^{-1}N_q = (D_{q1}D^+_{lm})^{-1}N_q = (D^+_{lm})^{-1}D_{q1}^{-1}N_q \\ &= (D^+_{lm})^{-1}\bar{N}_q\bar{D}_{q1}^{-1} = (D^+_{lm})^{-1}\bar{N}^+_{rm}\bar{N}_{q1}\bar{D}_{q1}^{-1}. \end{aligned} \tag{5.81}$$

Because $D_q^{-1}N_q$ is coprime, so is $(D^+_{lm})^{-1}\bar{N}^+_{rm}$. The proof is completed. □

SUFFICIENCY: If the conditions in the theorem are satisfied, there is an internally skew prime pair $(\bar{N}^+, \bar{D}^+_{la})$ in $\mathbb{C}^+$ such that

$$D^+_{la}N^+ = \bar{N}^+\bar{D}^+_{la} \tag{5.82}$$

with N^+ and $\bar{D}^+_{la}$ right coprime in $\mathbb{C}^+$, and

$$\Delta(D^+_{la}) = \Delta^+(\bar{D}^+_{la}). \tag{5.83}$$

Let $\bar{N}^+_{rm}$ be a least right block multiple of $\bar{N}^+$ such that

$$\bar{N}^+\bar{N}^+_{ra} = \bar{N}^+_{rm} \tag{5.84}$$

for some polynomial matrix $\bar{N}^+_{ra}$. We then take

$$K_1 = (\bar{D}^+_{la})^{-1}\bar{N}^+_{ra}. \tag{5.85}$$

as a candidate for the required compensator K. One sees that

$$\delta^+(G) = deg(D^+), \tag{5.86}$$

and

$$\delta^+(K_1) \le deg^+(\bar{D}^+_{la}). \tag{5.87}$$

By (5.82) and (5.84), the loop transfer matrix can be rewritten as

$$\begin{aligned} Q_1 = GK_1 &= (D^+)^{-1}N^+(\bar{D}^+_{la})^{-1}\bar{N}^+_{ra} \\ &= (D^+)^{-1}(D^+_{la})^{-1}\bar{N}^+\bar{N}^+_{ra} = (D^+_{la}D^+)^{-1}\bar{N}^+\bar{N}^+_{ra} \\ &= (D^+_{lm})^{-1}\bar{N}^+_{rm}, \end{aligned} \tag{5.88}$$

which is block decoupled. The assumed left coprimeness of D^+_{lm} and $\bar{N}_{rm}$ implies that

$$\begin{aligned} \delta^+(Q_1) = deg(D^+_{lm}) &= deg(D^+) + deg(D^+_{la}) \\ &= deg(D^+) + deg^+(\bar{D}^+_{la}), \end{aligned} \tag{5.89}$$

where the last equality follows from (5.83). Therefore, collecting (5.86), (5.87), and (5.89) gives

$$\delta^+(Q_1) \ge \delta^+(G) + \delta^+(K_1), \tag{5.90}$$

which ensures that no unstable pole-zero cancellations will occur in GK_1. Thus, by Theorem 5.4.1, **(P)** is solvable for G^+. The sufficiency is proven. □

Let us take an example for illustration.

Example 5.4.4. Let a 3×4 plant be given by

$$G = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} & 0 & \frac{2}{s+4} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+2)} & 0 & \frac{s+1}{(s-1)(s+4)} \\ 0 & \frac{s}{(s-1)(s+2)} & \frac{1}{(s-1)(s+3)} & \frac{s+1}{(s-1)(s+4)} \end{bmatrix}.$$

A stability factorization of it is $G = G^+G^-$, where $G^- = diag^{-1}\{(s+1), (s+2), (s+3), (s+4)\}$ and

$$G^+ = \begin{bmatrix} 1 & 1 & 0 & 2 \\ \frac{1}{s-1} & \frac{s}{s-1} & 0 & \frac{s+1}{s-1} \\ 0 & \frac{s}{s-1} & \frac{1}{s-1} & \frac{s+1}{s-1} \end{bmatrix}.$$

We get a left coprime polynomial matrix fraction of G^+ as

$$G^+ = (D^+)^{-1}N^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (s-1) & 0 \\ 0 & 0 & (s-1) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & s & 0 & s+1 \\ 0 & s & 1 & s+1 \end{bmatrix}.$$

G^+ has a pole of multiplicity 2 at $s = 1$ coinciding with a single zero at $s = 1$. For this example, since D^+ is itself diagonal, we can invariably take $D^+_{lm} = D^+$ and $D^+_{la} = I$ for any partition so that N^+ and D^+_{la} are always externally skew prime in $\mathbb{C}^+$. Obviously, $\bar{N}^+ = N^+$ and $\bar{D}^+_{la} = I$ constitute a dual internally skew prime pair of N^+ and D^+_{la} in $\mathbb{C}^+$. In order to determine the solvability of **(P)** for this example, we need only to check if $D^+_{lm} = D^+$ and $\bar{N}^+_{rm} = N^+_{rm}$ are left coprime, where N^+_{rm} is a least right block multiple of N^+ with respect to a given partition. For the partition, $m_1 = 2$, and $m_2 = 1$, Example 5.4.3 gives a least right block multiple $N_{rm}(2,1)$ as

$$N^+_{rm}(2,1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One easily sees that D^+ and $N^+_{rm}(2,1)$ are left coprime and **(P)** is thus solvable for the partition $\{2,1\}$. On the other hand, if the partition is changed to $m_1 = 1$, and $m_2 = 2$, then again from Example 5.4.3, a least right block multiple $N_{rm}(1,2)$ corresponding to the present partition is

$$N^+_{rm}(1,2) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, D^+ and $N^+_{rm}(1,2)$ are not left coprime and **(P)** is thus not solvable for the partition $\{1,2\}$. $\diamondsuit$

It has been shown in Section 5.2 that a sufficient condition for solvability of the diagonal decoupling problem with stability for a square plant is that the plant has no unstable poles coinciding with zeros and plants generically satisfy this condition. The same result is established for the general block decoupling case with arbitrary plants as follows.

Corollary 5.4.2. *If a full row rank plant G has no unstable poles coinciding with zeros, then* **(P)** *is solvable for every partition.*

Proof. It follows from $D_{la}^{+}D^{+} = D_{lm}^{+}$ that $\Delta(D_{la}^{+})$ divides $\Delta(D_{lm}^{+})$ and it further implies, by Lemma 5.4.6, that $\Delta(D_{lm}^{+})$ divides $(\Delta(D^{+}))^{m}$. Therefore, under the assumed condition, N^{+} and D_{la}^{+} have no zeros in common and they are externally skew prime in $\mathbb{C}$ and so are in $\mathbb{C}^{+}$. It follows from Lemmas 5.4.6 and 5.4.4 that $\Delta(D_{lm}^{+})$ divides $(\Delta(D^{+}))^{m}$ and $\Delta(\bar{N}_{rm}^{+})$ divides $(\Delta(\bar{N}^{+}))^{m} = (\Delta(N^{+}))^{m}$, so that *square* polynomial matrices D_{lm}^{+} and $\bar{N}_{rm}^{+}$ have no zeros in common and they are thus left coprime. Hence, by Theorem 5.4.2, **(P)** is solvable. The proof is completed. □

In fact, for the case considered in the above corollary, we can directly construct a decoupling compensator as follows. Let $N^{+} = L^{+}U$ be an LU-factorization and X a polynomial matrix such that $UX = I$, then take K as

$$K = (G^{-})^{-1}X\,adj(L^{+})(adj(D^{+}))^{-1}.$$

Simple calculations yield

$$GK = a(\Delta(D^{+}))^{-1}\Delta(N^{+})I, \quad a \in \mathbb{R},$$

so that GK is block decoupled for any partition. Since $\Delta(D^{+})$ and $\Delta(N^{+})$ have been assumed to be coprime in $\mathbb{C}^{+}$, there are no unstable pole-zero cancellations in GK. The disadvantage with this decoupling compensator is that its order may be much higher than necessary. In general, lower order ones can be obtained from the parameterization of all decoupling compensators, which will be presented in the next section.

In the extreme case of one block partition, i.e., $v = 1, m_1 = m$, our **(P)** defects to nothing than stabilization and the conditions in Theorem 5.4.2 should hold automatically for any plant G. Indeed, in this case, $D_{lm}^{+} = D^{+}$, $D_{la}^{+} = I$, $\bar{N}_{rm}^{+} = \bar{N}^{+} = N^{+}$. The solvability conditions are then reduced to ones that N^{+} and I are externally skew prime in $\mathbb{C}^{+}$ and D^{+} and N^{+} are left coprime, which, of course are satisfied for any G^{+}.

5.4.4 Characterization of Solutions

Besides the solvability conditions, it is usually required to characterize all decoupling compensators and achievable loop transfer function matrices so that free parameters can be used to achieve other performance specifications.

Proposition 5.4.1. *If $G^+ = (D^+)^{-1}N^+$ satisfies the conditions in Theorem 5.4.2, then all compensators which block-decouple G^+ and at the same time maintain the internal stabilizability are given by*

$$K = (\bar{D}^+_{la})^{-1}(\bar{N}^+_{ra}\bar{N}_{q1} + ZW)(\bar{D}_{q1})^{-1}, \tag{5.91}$$

where $\bar{D}^+_{la}$ and $\bar{N}^+_{ra}$ are defined as above, polynomial matrices $\bar{N}_{q1}$ and $\bar{D}_{q1}$ are both block diagonal and right coprime, $K_2 = \bar{N}_{q1}(\bar{D}_{q1})^{-1}$ is coprime and has no pole-zero cancellations with $Q_1 = (D^+_{lm})^{-1}\bar{N}^+_{rm}$, $\bar{N}^+_{ra}$ and $\bar{N}^+_{rm}$ satisfy $\bar{N}^+\bar{N}^+_{ra} = \bar{N}^+_{rm}$, Z is a minimal polynomial base of the right null space of $\bar{N}^+$ (Forney,1975) and of size $l \times (l-m)$, and W is an arbitrary polynomial matrix of size $(l-m) \times m$.

Proof. In the proof of necessity of Theorem 5.4.2, one sees from (5.67), (5.69), (5.70) and (5.82) that every K which block-decouples G^+ and maintains internal stabilizability cab be expressed as

$$\begin{aligned} K &= N_k D_k^{-1} = N_k(\bar{D}_{q1}D_{k1})^{-1} \\ &= N_k D_{k1}^{-1}\bar{D}_{q1}^{-1} = \bar{D}_{k1}^{-1}\bar{N}_k\bar{D}_{q1}^{-1} \\ &= (\bar{D}^+_{la})^{-1}\bar{N}_k\bar{D}_{q1}^{-1}. \end{aligned} \tag{5.92}$$

It follows from (5.79) and (5.80) that $\bar{N}_k$ satisfies

$$\bar{N}^+\bar{N}_k = \bar{N}^+_{rm}\bar{N}_{q1}. \tag{5.93}$$

Since $\bar{N}^+_{ra}\bar{N}_{q1}$ is obviously a particular solution of (5.93) for $\bar{N}_k$ and ZW is a general solution of $\bar{N}^+\bar{N}_k = 0$, a general $\bar{N}_k$ which satisfies (5.93) is then given by

$$\bar{N}_k = \bar{N}^+_{ra}\bar{N}_{q1} + ZW, \tag{5.94}$$

which is then substituted to (5.92) to yield (5.91). Furthermore, (5.81) can be expressed as

$$Q = D_q^{-1}N_q = (D^+_{lm})^{-1}\bar{N}^+_{rm}\bar{N}_{q1}\bar{D}_{q1}^{-1} = Q_1K_2. \tag{5.95}$$

As D_q and N_q are coprime and $\Delta(D_q) = \Delta(D^+_{lm})\Delta(\bar{D}_{q1})$ by (5.64) and (5.66), $\bar{N}_{q1}$ and $\bar{D}_{q1}$ are thus coprime and Q_1K_2 has no pole-zero cancellations. On the other hand, following a similar reasoning to the proof of sufficiency of Theorem 5.4.2 with K_1 in (5.85) replaced by K in (5.91), one can easily show that each K in (5.91) block-decouples G^+ and preserves the internal stabilizability. $\square$

Theorem 5.4.1 says that K is a solution to the block decoupling problem without unstable pole-zero cancellations for a plant G if and only if so is $K^+ = G^-K$ for G^+, where $G = G^+G^-$ is a stability factorization. But such K^+ and the resulting loop maps have been parameterized above. We thus obtain the following characterization of solutions for G.

Theorem 5.4.3. *Assume that a full row rank plant G and a partition be given, the notations, $G^+, G^-, D^+, N^+, D_{lm}^+, D_{la}^+, \bar{N}^+, \bar{D}_{la}^+$, and $\bar{N}_{rm}^+$ are defined as before, and the conditions in Theorem 5.4.2 are satisfied. In addition, let $\bar{N}_{ra}^+$ be a polynomial matrix such that $\bar{N}^+\bar{N}_{ra}^+ = \bar{N}_{rm}^+$ and Z be an $l\times(l-m)$ minimal polynomial base of the right null space of $\bar{N}^+$, then*

(i) all compensators which block-decouple G with internal stabilizability are given by

$$K = (G^-)^{-1}(\bar{D}_{la}^+)^{-1}(\bar{N}_{ra}^+N_{k2} + ZW)D_{k2}^{-1},$$

where polynomial matrices N_{k2} and D_{k2} are both block diagonal with respect to the partition and right coprime, $K_2 = N_{k2}D_{k2}^{-1}$ is coprime and has no pole-zero cancellations with $Q_1 = (D_{lm}^+)^{-1}\bar{N}_{rm}^+$ and, W is an arbitrary polynomial matrix of size $(l-m)\times m$;

(ii) all achievable open-loop transfer function matrices $Q = GK$ under block decoupling with internal stabilizability can be expressed as

$$Q = Q_1K_2,$$

where Q_1 and K_2 are the same as in (i).

From the point of view of closed-loop system design, all remains to do is construct a block diagonal stabilizer K_2 for the decoupled plant Q_1. It becomes a sequence of reduced size design problems. In addition, Theorem 5.4.3 exhibits useful information about the zero and pole properties of a decoupled system. Because there are no unstable pole-zero cancellations between Q_1 and K_2, then Q_1 will thus entirely remain in each decoupled loop for a given plant G no matter what K_2 is chosen. It follows from Lemmas 5.4.6 and 5.4.4 that $\Delta(D^+)$ divides $\Delta(D_{lm}^+)$ and $\Delta(N^+)$ divides $\Delta(\bar{N}_{rm}^+)$, but Q_1 introduces neither zeros nor poles different from those of the plant. What may happen is that multiplicities of zeros or poles of Q_1 are greater than those of G. It is interesting to find out a class of plants for which the multiplicities are not increased, that is,

$$\Delta(D_{lm}^+) = \Delta(D^+) \tag{5.96}$$

and

$$\Delta(\bar{N}_{rm}^+) = \Delta(N^+) \tag{5.97}$$

are both satisfied.

Theorem 5.4.4. *Let a full row rank plant G and a partition be given, then the following statements are equivalent:*

(i) **(P)** *is solvable, and both (5.96) and (5.97) hold true.*

(ii) There exists an irreducible polynomial matrix U such that $(D_{lm}^+)^{-1}(\bar{N}_{rm}^+U)$ is a left coprime polynomial matrix fraction of G^+.

(iii) There is a stable and minimum phase compensator K such that GK is block decoupled and $\Delta(N^+) = \Delta^+(N^+N_k)$, where $N_k D_k^{-1}$ is a coprime polynomial matrix fraction of G^-K.

Proof. $(i) \Rightarrow (ii)$: If **(P)** is solvable and both (5.96) and (5.97) hold true, then $\Delta(D_{la}^+) = 1$ and D_{la}^+ is unimodular. The $\mathbb{C}^+$-skew primeness equation, $D_{la}^+ N^+ = \bar{N}^+ \bar{D}_{la}^+$, can be thus rewritten as $N^+ = P\bar{D}_{la}^+$, where $P = (D_{la}^+)^{-1}\bar{N}^+$ is a polynomial matrix. Note that N^+ is antistable, and $\bar{D}_{la}^+$ is stable since, by Lemma 5.4.4, $\Delta^+(\bar{D}_{la}^+) = \Delta(D_{la}^+) = 1$. It then follows from Lemma 5.4.2 that $U_1(\bar{D}_{la}^+)^{-1} = P_1$ is a polynomial matrix, where $N^+ = L^+U_1$ is an LU-factorization. Consider now the equality, $(D^+)^{-1}N^+(\bar{D}_{la}^+)^{-1}\bar{N}_{ra}^+ = (D_{lm}^+)^{-1}\bar{N}_{rm}^+$. It can be rewritten as

$$(D^+)^{-1}L^+P_1\bar{N}_{ra}^+ = (D_{lm}^+)^{-1}\bar{N}_{rm}^+. \tag{5.98}$$

Using (5.96) and (5.97) with $\Delta(L^+) = \Delta(N^+)$ and the coprimeness of D_{lm}^+ and $\bar{N}_{rm}^+$, one concludes that $U_2 = P_1\bar{N}_{ra}^+$ is a unimodular polynomial matrix. Take U as $U = U_2^{-1}U_1$, then U is an irreducible polynomial matrix. Postmultiplying (5.98) by U yields

$$(D_{lm}^+)^{-1}\bar{N}_{rm}^+U = (D^+)^{-1}L^+U_1 = (D^+)^{-1}N^+ = G^+. \tag{5.99}$$

In view of (5.96) and (5.99) with D^+ and N^+ left coprime, $(D_{lm}^+)^{-1}(\bar{N}_{rm}^+U)$ is thus left coprime, too.

$(ii) \Rightarrow (iii)$: Let X be a polynomial matrix such that $UX = I$, then X has neither zeros nor poles and $K = (G^-)^{-1}X$ is stable and minimum phase. The resulting loop transfer matrix becomes

$$Q = GK = (D_{lm}^+)^{-1}\bar{N}_{rm}^+UG^-(G^-)^{-1}X = (D_{lm}^+)^{-1}\bar{N}_{rm}^+,$$

which is block decoupled. In this case, $N^+ = \bar{N}_{rm}^+U$ and $N_k = X$. Hence, $\Delta(N^+) = \Delta(\bar{N}_{rm}^+) = \Delta^+(N^+N_k)$.

$(iii) \Rightarrow (i)$: Let $N^+ = L^+U_1$ be an LU-factorization, then (iii) implies that the *square* matrix $U_1N_kD_k^{-1}$ is stable and minimum phase and

$$[(D^+)^{-1}L^+][U_1N_kD_k^{-1}] = GK \tag{5.100}$$

has no unstable pole-zero cancellations. Such a K block-decouples G while maintaining stabilizability and the resulting loop transfer matrix is thus characterized in Theorem 5.4.3 so that

$$\Delta(\bar{N}_{rm}^+)\Delta^+(N_{k2}) = \Delta^+(N_q), \tag{5.101}$$

where $D_q^{-1}N_q$ is a coprime polynomial matrix fraction of $Q = GK$. It follows from (5.100) that $\Delta^+(N_q)$ divides $\Delta^+(N^+N_k) = \Delta(N^+)$, where the equality is the given condition. This, together with (5.101), further implies that

$\Delta(\bar{N}^+_{rm})$ divides $\Delta(N^+)$. But by Lemmas 5.4.4 and 5.4.6, $\Delta(N^+)$ divides $\Delta(\bar{N}^+_{rm})$, too. Hence, (5.97) holds true. (5.96) can be proven in a similar way by noting from (5.100) that $\Delta^+(D_q)$ divides $\Delta(D^+)\Delta^+(D_k) = \Delta(D^+)$ as K is stable. The proof is completed. □

Example 5.4.5. Consider again the plant in Example 5.4.4. For the partition $\{2, 1\}$, Example 5.4.4 has shown that **(P)** is solvable. In addition, one easily sees that $\Delta(\bar{N}^+_{rm}) = (s-1) = \Delta(N^+)$ and $\Delta(D^+_{lm}) = \Delta(D^+) = (s-1)^2$. Therefore, the conditions in (i) of Theorem 5.4.4 are satisfied. G^+ can be expressed as

$$G^+ = (D^+_{lm})^{-1}\bar{N}^+_{rm}U =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (s-1) & 0 \\ 0 & 0 & (s-1) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & s-1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & s & 1 & s+1 \end{bmatrix},$$

which is in the form given by (ii) of Theorem 5.4.4. Take K as

$$K = (G^-)^{-1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -s & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then K is stable and minimum phase and

$$GK = (D^+_{lm})^{-1}\bar{N}^+_{rm} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{s-1} & 1 & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix},$$

which is block decoupled with respect to the partition $\{2, 1\}$. One also sees that $N^+ = N^+_{rm}U$ and $N^+N_k = N^+_{rm}$ so that $\Delta(N^+) = \Delta^+(N^+N_k) = \Delta(N^+_{rm})$. Thus, (iii) of Theorem 5.4.4 also holds, indeed. ♢

It should be pointed out that (i) or (ii) of Theorem 5.4.4 implies that there is a stable and minimum phase compensator K such that GK is block decoupled and has no unstable pole-zero cancellations, but the converse is not true in general. For instance, consider the plant:

$$\begin{aligned} G &= \begin{bmatrix} \frac{s}{s-1} & \frac{1}{s-1} & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} s-1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} s & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= (D^+)^{-1}N^+, \end{aligned}$$

for the partition $\{1, 1\}$. Take K as

$$K = \begin{bmatrix} 1 & -1 \\ 0 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} = N_k D_k^{-1},$$

which is stable and minimum phase. One can verify that GK is decoupled and has no unstable pole-zero cancellations. Simple calculations show that $\Delta(N^+) = s$ but $\Delta^+(N^+N_k) = s^2$ so that $\Delta(N^+) \neq \Delta^+(N^+N_k)$ for this K. Are there any other decoupling K which makes the equality

$$\Delta(N^+) = \Delta^+(N^+N_k) \tag{5.102}$$

hold? The answer is No! To see this, one notes from Corollary 5.4.2 that **(P)** is solvable for the given plant and the partition $\{1, 1\}$, and (5.101) thus applies with $\bar{N}_{rm}^+ = N_{rm}^+ = diag\{s, s\}$. It follows that $deg^+(N^+N_k) \geq deg^+(N_q) \geq deg^+(N_{rm}^+) = 2 > 1 = deg(N^+)$. Therefore, among all the compensators K such that GK is block decoupled without unstable pole-zero cancellations, there is no one satisfying (5.102). One concludes from this example that the condition that there is a stable and minimum phase compensator which block-decouples the plant with internal stabilizability is only necessary but not sufficient for either (i) or (ii) of Theorem 5.4.4 to hold. In general, the additional condition (5.102) is needed to make it equivalent to (i) or (ii) of that theorem. However, if the plant is square, (5.102) will be always satisfied provided K is of minimum phase.

Corollary 5.4.3. *If the plant G is square, (iii) of Theorem 5.4.4 can be replaced by the following: there is a stable and minimum phase compensator K such that GK is block decoupled.*

The decoupling compensators constructed so far usually result in stable pole-zero cancellations, $G^-(G^-)^{-1}$, so that arbitrary pole assignment is not achievable. However, by imposing the stronger constraint, no pole-zero cancellations in GK, instead of only RHP ones, we can directly obtain the following result and the proof is omitted because of its similarity to those of Theorems 5.4.2 and 5.4.3.

Theorem 5.4.5. *Given a full row rank plant $G \in \mathbb{R}_p^{m\times l}(s)$ and a partition $\{m_i\}$, let $D^{-1}N = G$ be a left coprime polynomial matrix fraction and D_{lm} a least left block multiple of D with respect to the partition such that $D_{lm} = D_{la}D$ for some polynomial matrix D_{la}. If N and D_{la} are externally skew prime, construct internally skew prime polynomial matrices $\tilde{N}$ and $\tilde{D}_{la}$ such that $D_{la}N = \tilde{N}\tilde{D}_{la}$. And let $\tilde{N}_{rm}$ be a least right block multiple of $\tilde{N}$ with respect to the partition such that $\tilde{N}\tilde{N}_{ra} = \tilde{N}_{rm}$ for some polynomial matrix $\tilde{N}_{ra}$ and Z a $l \times (l-m)$ minimal polynomial base of the right null space of $\tilde{N}$, then the following hold true:*

(i) there is a proper compensator K in a unity output feedback system such that GK is nonsingular and block diagonal with respect to the partition

and arbitrary closed-loop pole assignment is achievable if and only if N *and* D_{la} *are externally skew prime and* D_{lm} *and* $\tilde{N}_{rm}$ *are left coprime;*

(*ii*) *under the conditions of* (*i*), *all compensators which block-decouple* G *and preserve arbitrary pole assignability are given by*

$$K = \tilde{D}_{la}^{-1}(\tilde{N}_{ra}N_{k2} + ZW)D_{k2}^{-1},$$

where polynomial matrices N_{k2} *and* D_{k2} *are both block diagonal with respect to the partition and right coprime,* Q_1K_2 *has no pole-zero cancellations with* $Q_1 = D_{lm}^{-1}\tilde{N}_{rm}$ *and* $K_2 = N_{k2}D_{k2}^{-1}$, *and* W *is an arbitrary polynomial matrix of size* $(l-m) \times m$; *And*

(*iii*) *all achievable open-loop maps under block decoupling with arbitrary pole assignability can be expressed as*

$$Q = Q_1K_2,$$

where Q_1 *and* K_2 *are the same as in* (*ii*).

In this section, the block decoupling problem with internal stability for unity output feedback systems has been considered in a general setting and completely solved. Necessary and sufficient conditions for its solvability are given and are only concerned with the greatest C^+-divisor of a given plant. The characterization of decoupled systems is presented and it shows that a general compensator which solves the problem consists mainly of two parts, one being the decoupling part which makes the plant block decoupled without unstable pole-zero cancellations, the other the stabilizing part which is block diagonal and makes the block decoupled plant internally stable and the whole compensator proper. The block decoupled plant (the plant plus the decoupling part) need not introduce different zeros or poles from those of the plant itself and what may happen is that zero or pole multiplicities of the former are greater than those of the latter. Two new concepts, least block multiples of a full rank polynomial matrix and skew prime polynomial matrices in $\mathbb{C}^+$, are introduced and they have played important roles, respectively, in the problem solvability and compensator parameterization.

5.5 Notes and References

The existence condition for decoupling by unity output feedback compensation is that the plant has full row rank and it is very simple and well known. But the problem of decoupling in such a configuration while maintaining internal stability of the decoupled system appears to be much more difficult. Early papers on the subject (Safonov and Chen, 1982; Vardulakis, 1987; Peng, 1990; Desoer, 1990; Lin and Hsieh, 1991) consider the special case that the set of RHP zeros of the system is disjoint from the set of poles. They show that the diagonal decoupling problem is solvable under this assumption and provide a parameterization of controllers solving the problem This assumption is relaxed and the diagonal decoupling problem with stability is solved by Linnemann and Maier (1990) for 2×2 plants, and by Wang (1992) for square plants, along with a controller parameterization. For the same problem, alternative necessary and sufficient conditions are also derived by Lin (1997) based on transfer matrices and residues, and in Wang and Yang (2002) using the minimal C^{+}-decoupler of the plant. The results of Wang (1992) are generalized to the **block** decoupling problem for possibly **non-square** plants by Linnemann and Wang (1993) with additional discussion on the performance limitations in the decoupled system compared with the performance which is achievable by not necessarily decoupling controllers. A unified and independent solution for the general case is also presented in Wang (1993). The main tools used in this chapter are skew primeness (Wolovich, 1978), strict adjoints (Hammer and Khargonekar, 1984), stability factorizations (Wang, 1992), and their extensions to non-square plants and block diagonalization.

6. Transfer Matrix Approach

In the preceding two chapters, the plants which we have considered assume no time delay and for such a kind of plants the complete solution has been developed in both state-space and polynomial matrix settings. However, time delay often exists in industrial processes and other systems, some of which will be highlighted in Section 1. We need an entirely new approach to decoupling problem for plants with time delays. Time delay imposes a serious difficulty in theoretical development and thus the theory of time delay systems is much less matured than delay-free systems. In the present and next chapters, we will address decoupling problem for plants with time delay and our emphasis is to develop decoupling methodologies with necessary theoretical supports as well as the controller design details for possible practical applications. This chapter will consider conventional unity output feedback configuration whereas the next chapter will exclusively deal with time delay compensation schemes.

In this chapter, several typical plants with time delay will be briefly described and their transfer matrices with time delay given in Section 1 as the motivation of subsequent discussions on delay systems. The new decoupling equations are derived in a transfer function matrix setting for delay plants (Section 2), and achievable performance after decoupling is analyzed (Section 3) where characterization of the unavoidable time delays and non-minimum phase zeros that are inherent in a feedback loop is given. In Section 4, a design method is presented. The objective is to design a controller of lowest complexity which can achieve fastest loops with acceptable overshoot and minimum loop interactions. Stability and robustness are analyzed in Section 5. Section 6 shows why multivariable PID control often fails when the number of inputs/outputs is getting bigger. More simulation is given in Section 7 and the conclusion is drawn in Section 8.

6.1 Plants with Time Delay

In the literature of process control, there are many 2×2 process models. But usually they are subsystems of actual processes. A typical process unit in industry has number of inputs/output well beyond 2, some can reach more than 10. And it usually has some time delay. A number of industrial

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processes are introduced in this section to show the presence of time delay and will be used in simulation later to substantiate the effectiveness of the proposed decoupling method for high performance control.

Example 6.1.1. Tyreus (1982) studied a sidestream column separating a ternary mixture, where a feed containing 10 percent benzene, 45 percent toluene and 45 percent o-xylene is separated in a single column into three product streams: a benzene distillate with 5 percent toluene impurity, a liquid toluene sidestream with 5 percent benzene and 6 percent xylene impurities, and a xylene bottom with 5 percent toluene impurity. The transfer function matrix for the column is

$$G(s) = \begin{bmatrix} \frac{1.986e^{-0.71s}}{66.7s+1} & \frac{-5.24e^{-60s}}{400s+1} & \frac{-5.984e^{-2.24s}}{14.29s+1} \\ \frac{-0.0204e^{-0.59s}}{(7.14s+1)^2} & \frac{0.33e^{-0.68s}}{(2.38s+1)^2} & \frac{-2.38e^{-0.42s}}{(1.43s+1)^2} \\ \frac{-0.374e^{-7.75s}}{22.22s+1} & \frac{11.3e^{-3.79s}}{(21.74s+1)^2} & \frac{9.811e^{-1.59s}}{11.36s+1} \end{bmatrix}, \tag{6.1}$$

where the controlled and manipulated variables are y_1 (toluene impurity in the distillate); y_2 (benzene impurity in the sidestream); y_3 (toluene impurity in the bottom); u_1 (reflux ratio); u_2 (sidestream flow rate); u_3 (reboil duty). ♢

Example 6.1.2. Doukas and Luyben (1978) studied the dynamics of a distillation column producing a liquid sidestream product, with the objective of maintaining four composition specifications on the three product streams. The Tyreus process in (6.1) is a 3×3 subsystem of it. The transfer function matrix for the 4×4 model is

$$G(s) = \begin{bmatrix} \frac{-11.3e^{-3.79s}}{(21.74s+1)^2} & \frac{0.374e^{-7.75s}}{22.2s+1} & \frac{-9.811e^{-1.59s}}{11.36s+1} & \frac{-2.37e^{-27.33s}}{33.3s+1} \\ \frac{5.24e^{-60s}}{400s+1} & \frac{-1.986e^{-0.71s}}{66.67s+1} & \frac{5.984e^{-2.24s}}{14.29s+1} & \frac{0.422e^{-8.72s}}{(250s+1)^2}; \\ \frac{-0.33e^{-0.68s}}{(2.38s+1)^2} & \frac{0.0204e^{-0.59s}}{(7.14s+1)^2} & \frac{2.38e^{-0.42s}}{(1.43s+1)^2} & 0.513e^{-s} \\ \frac{4.48e^{-0.52s}}{11.11s+1} & \frac{-0.176e^{-0.48s}}{(6.90s+1)^2} & \frac{-11.67e^{-1.91s}}{12.19s+1} & 15.54e^{-s} \end{bmatrix}, \tag{6.2}$$

where the controlled and manipulated variables are y_1 (toluene impurity in the bottom); y_2 (toluene impurity in the distillate); y_3 (benzene impurity in the sidestream); y_4 (xylene impurity in the sidestream); u_1 (sidestream flow rate); u_2 (reflux ratio); u_3 (reboil duty); and u_4 (side draw location). ♢

Example 6.1.3. Alatiqi and Luyben (1986) presented the results of a quantitative study of the dynamics of two alternative distillation systems for separating ternary mixtures that contain small amounts (less than 20%) of the intermediate component in the feed. The process transfer function matrix of the complex sidestream column/stripper distillation process is given by

$$G(s)=\begin{bmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} & \frac{-0.49e^{-5s}}{22s+1} \\ \frac{-4.17e^{-4s}}{45s+1} & \frac{6.93e^{-1.01s}}{44.6s+1} & \frac{-0.05e^{-5s}}{34.5s+1} & \frac{1.53e^{-2.8s}}{48s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} & \frac{-5.48e^{-0.5s}}{15s+1} \\ \frac{-11.18e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{14.04e^{-0.02s}}{(45s+1)(10s+1)} & \frac{-0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{bmatrix}, \tag{6.3}$$

where the four controlled variables are y_1 (mole fraction of the first component in the distillate); y_2 (mole fraction of the third component in the bottom); y_3 (mole fraction of the second component in the sidestream); y_4 (temperature difference); and the four manipulated variables are u_1 (reflux flow rate); u_2 (main column reboiler heat-transfer rate); u_3 (stripper reboiler heat-transfer rate); and u_4 (liquid draw rate from main column to stripper). Although the complex sidestream column/stripper configuration is more energy efficient than the conventional two-column sequential "light-out-first" configuration, it presents a challenging 4×4 multivariable control problem. ♢

Example 6.1.4. Ammonia is commercially produced by synthesis of the raw material gas (Hydrogen, H_2, Nitrogen, N_2) via a reversible reaction catalytically with great energy consumption for making raw gases. The recycled un-reacted raw gas makes the whole ammonia process more difficult to operate and control. In general, ammonia process consists of two big sessions. One is reformer session to produce reactant gas (H_2, N_2). Another is conversion session to synthesize ammonia. Many disturbances and frequent change of feed gas rate affect a modern ammonia plant. Moreover, these disturbances always propagate throughout the entire plant. It has been common requirements to improve operating stability, increase material and energy efficiency and prevent disturbances from echoing throughout the plant. A transfer function matrix of the reformer session was identified as

$$G(s)=\begin{bmatrix} \frac{-0.562e^{-4s}}{8s+1} & \frac{-0.01e^{-s}}{3s+1} & \frac{0.378e^{-4s}}{8s+1} \\ \frac{0.0135e^{-19s}}{5.6s+1} & 0 & \frac{-0.0159e^{-21s}}{10s+1} \\ \frac{0.002e^{-10s}}{7s+1} & \frac{-0.00125e^{-4s}}{3s+1} & \frac{0.12e^{-7.5s}}{10s+1} \end{bmatrix}, \tag{6.4}$$

where the controlled variables and manipulated variables are y_1 (the primary reformer coil outlet temperature); y_2 (the secondary reformer methane leakage) and y_3 (the H_2/N_2 ratio); u_1 (the process natural gas flow rate); u_2 (the process air flow rate); and u_3 (the fuel natural gas flow rate). ♢

Example 6.1.5. The purpose of a depropanizer column is to separate propane from the feed which comes from deethanizer column. The operation target of this unit is to keep the top butane concentration low than 5% and bottom propane concentration low than 5%. The main controlled variables includes y_1 (top butane concentration), y_2 (bottom propane concentration) and y_3

(column DP flooding). The main manipulated variables are u_1 (column top reflux flow), u_2 (Column bottom steam flow), u_3 (column overhead pressure). The 3×3 transfer function matrix model was found as

$$G(s) = \begin{bmatrix} \frac{-0.26978e^{-27.5s}}{97.5s+1} & \frac{1.978e^{-53.5s}}{118.5s+1} & \frac{0.07724e^{-56s}}{96s+1} \\ \frac{0.4881e^{-117s}}{56s+1} & \frac{-5.26e^{-26.5s}}{58.5s+1} & \frac{0.19996e^{-35s}}{51s+1} \\ \frac{0.6e^{-16.5s}}{40.5s+1} & \frac{5.5e^{-15.5s}}{19.5s+1} & \frac{-0.5e^{-17s}}{18s+1} \end{bmatrix}. \quad \diamondsuit \tag{6.5}$$

We have so far highlighted several typical complex industrial processes that appear in the literature or that we encountered during our industrial control practice in developing and implementing a multivariable controller. One should note that the complexity of the above processes do not come from the dynamics of the individual channels of the processes as clearly they are mostly nothing more than first or second order plus time delay form, but is from the multivariable interactions existing between the various variables. The more interactive loops are involved, the more difficult it is to compensate for these interactions. The difficulty of interaction handling and the demand for better decoupling is evidenced by the recent survey (Kong, 1995) from the leading international control companies such as Fish-Rosemount, Yokogawa and Foxboro, all of whom ranked poor decoupling as the principal common control problem in industry. Poor decoupling is universal and in most cases, the closure of a multivariable feedback loop brings more interactions to the controlled variable than the open-loop operation, thus leaving the benefit of the close-loop control only as mere preventing the controlled variable from flowing away. Therefore, we are well motivated to consider decoupling problem for plants with time delay.

6.2 Decoupling

Consider the conventional unity feedback control system in Figure 6.1, where G represents the transfer matrix of the plant and K the multivariable controller. Here, G is assumed to be square and nonsingular:

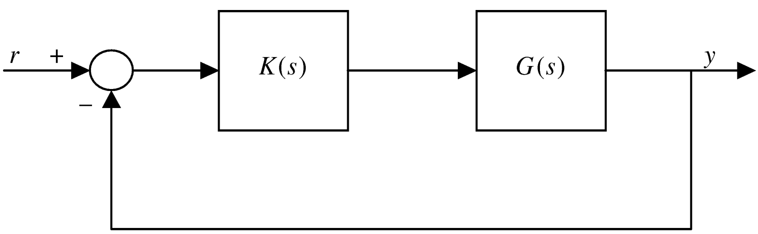


Fig. 6.1. Unity Output Feedback System

$$G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & \ddots & \vdots \\ g_{m1}(s) & \cdots & g_{mm}(s) \end{bmatrix}, \tag{6.6}$$

where

$$g_{ij}(s) = g_{ij0}(s)e^{-L_{ij}s},$$

and $g_{ij0}(s)$ are strictly proper, stable scalar rational functions and L_{ij} are non-negative constants, representing time delays. The controller $K(s)$ is an $m \times m$ full-cross coupled multivariable transfer matrix:

$$K(s) = \begin{bmatrix} k_{11}(s) & \cdots & k_{1m}(s) \\ \vdots & \ddots & \vdots \\ k_{m1}(s) & \cdots & k_{mm}(s) \end{bmatrix}, \tag{6.7}$$

where

$$k_{ij}(s) = k_{ij0}(s)e^{-\theta_{ij}s},$$

θ_{ij} are non-negative constants, and $k_{ij0}(s)$ are scalar proper rational functions. $k_{ij0}(s)$ should be stable except a pole at $s = 0$, and for simplicity, it is called stable in this chapter (referred to the controller only). This pole is desired to have integral control. Assume that the plant has no zero at the origin. Then, the integral control will lead to zero steady state errors in response to step inputs provided that the closed-loop is stable. It is also assumed that a proper input-output pairing has been achieved in $G(s)$ such that none of the m principal minors of $G(s)$ (the ith principle minor is the determinant of a $(m-1) \times (m-1)$ matrix obtained by deleting the ith row and ith column of $G(s)$) are zero.

Our task here is to find a $K(s)$ such that the closed loop transfer matrix from the reference vector r to the output vector y:

$$H(s) = G(s)K(s)\left[I + G(s)K(s)\right]^{-1}, \tag{6.8}$$

is decoupled, that is, $H(s)$ is diagonal and nonsingular. One sees that

$$\begin{aligned} H^{-1} &= \left[I + GK\right](GK)^{-1} \\ &= (GK)^{-1} + I, \end{aligned} \tag{6.9}$$

and the closed-loop $H(s)$ is decoupled if and only if the open loop $G(s)K(s)$ is decoupled, i.e.,

$$GK = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_m \end{bmatrix} = \mathrm{diag}\{q_{ii}\}, i = 1, 2, \ldots, m. \tag{6.10}$$

For each column of GK, we have

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix} k_i = \begin{bmatrix} 0 \cdots 0 \; q_{ii} \; 0 \cdots 0 \end{bmatrix}^T, \quad i = 1, 2, \ldots, m, \tag{6.11}$$

which is equivalent to

$$\begin{bmatrix} g_1 \\ \vdots \\ g_{i-1} \\ g_{i+1} \\ \vdots \\ g_m \end{bmatrix} k_i = 0, \quad i = 1, 2, \ldots, m, \tag{6.12}$$

and

$$g_i k_i = q_{ii} \neq 0, \quad i = 1, 2, \cdots, m, \tag{6.13}$$

For any i, we can solve (6.12) to obtain $k_{1,i}, \cdots, k_{i-1,i}, k_{i+1,i}, \cdots, k_{m,i}$, in terms of k_{ii} as

$$\begin{bmatrix} k_{1i} \\ \vdots \\ k_{i-1,i} \\ k_{i+1,i} \\ \vdots \\ k_{mi} \end{bmatrix} = \begin{bmatrix} \psi_{1i} \\ \vdots \\ \psi_{i-1,i} \\ \psi_{i+1,i} \\ \vdots \\ \psi_{mi} \end{bmatrix} k_{ii}, \qquad \forall\, i \in \mathbf{m}, \tag{6.14}$$

where $\mathbf{m} = \{1, 2, \cdots, m\}$ and

$$\begin{bmatrix} \psi_{1i} \\ \vdots \\ \psi_{i-1,i} \\ \psi_{i+1,i} \\ \vdots \\ \psi_{mi} \end{bmatrix} \triangleq - \begin{bmatrix} g_{11} & \cdots & g_{1,i-1} & g_{1,i+1} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{i-1,1} & \cdots & g_{i-1,i-1} & g_{i-1,i+1} & \cdots & g_{i-1,m} \\ g_{i+1,1} & \cdots & g_{i+1,i-1} & g_{i+1,i+1} & \cdots & g_{i+1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{m,i-1} & g_{m,i+1} & \cdots & g_{mm} \end{bmatrix}^{-1} \begin{bmatrix} g_{1i} \\ \vdots \\ g_{i-1,i} \\ g_{i+1,i} \\ \vdots \\ g_{mi} \end{bmatrix}. \tag{6.15}$$

The ith diagonal element of GK is $\tilde{g}_{ii} k_{ii}$, where

$$\tilde{g}_{ii} = g_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{m} g_{ik} \psi_{ki}. \tag{6.16}$$

One notes that for a given i, the resulting diagonal element of GK is independent of the controller off-diagonal elements but contains only the controller diagonal element k_{ii}.

To further simplify the above result, let G^{ij} be the cofactor corresponding to g_{ij} in G. It follows from linear algebra (Noble, 1969) that the inverse of G can be given as $G^{-1} = \frac{\operatorname{adj} G}{|G|}$, where $\operatorname{adj} G = \left[G^{ji}\right]$. This also means

$$\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{mm} \end{bmatrix} \begin{bmatrix} G^{11} & \cdots & G^{m1} \\ \vdots & \ddots & \vdots \\ G^{1m} & \cdots & G^{mm} \end{bmatrix} = |G| I_m. \tag{6.17}$$

Equation (6.17) can be equivalently put into the following two relations:

$$\begin{bmatrix} g_{11} & \cdots & g_{1,i-1} & g_{1,i+1} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{i-1,1} & \cdots & g_{i-1,i-1} & g_{i-1,i+1} & \cdots & g_{i-1,m} \\ g_{i+1,1} & \cdots & g_{i+1,i-1} & g_{i+1,i+1} & \cdots & g_{i+1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{m,i-1} & g_{m,i+1} & \cdots & g_{mm} \end{bmatrix} \begin{bmatrix} G^{i1} \\ \vdots \\ G^{i,i-1} \\ G^{i,i+1} \\ \vdots \\ G^{im} \end{bmatrix} = - \begin{bmatrix} g_{1i} \\ \vdots \\ g_{i-1,i} \\ g_{i+1,i} \\ \vdots \\ g_{mi} \end{bmatrix} G^{ii}, \quad \forall\, i \in \mathbf{m}, \tag{6.18}$$

and

$$\sum_{k=1}^{m} g_{ik} G^{ik} = |G|, \qquad \forall i \in \mathbf{m}.$$

Substituting (6.18) to (6.15) yields

$$\begin{bmatrix} \psi_{1i} \\ \vdots \\ \psi_{i-1,i} \\ \psi_{i+1,i} \\ \vdots \\ \psi_{mi} \end{bmatrix} = - \begin{bmatrix} g_{11} & \cdots & g_{1,i-1} & g_{1,i+1} & \cdots & g_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{i-1,1} & \cdots & g_{i-1,i-1} & g_{i-1,i+1} & \cdots & g_{i-1,m} \\ g_{i+1,1} & \cdots & g_{i+1,i-1} & g_{i+1,i+1} & \cdots & g_{i+1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{m,i-1} & g_{m,i+1} & \cdots & g_{mm} \end{bmatrix}^{-1} \begin{bmatrix} g_{1i} \\ \vdots \\ g_{i-1,i} \\ g_{i+1,i} \\ \vdots \\ g_{mi} \end{bmatrix} = \frac{1}{G^{ii}} \begin{bmatrix} G^{i1} \\ \vdots \\ G^{i,i-1} \\ G^{i,i+1} \\ \vdots \\ G^{im} \end{bmatrix},$$

or

$$\psi_{ji} = \frac{G^{ij}}{G^{ii}}, \qquad \forall\, i, j \in \mathbf{m}, j \neq i. \tag{6.19}$$

By (6.19), (6.14) and (6.16) become respectively

$$k_{ji} = \frac{G^{ij}}{G^{ii}} k_{ii}, \qquad \forall\, i, j \in \mathbf{m}, j \neq i, \tag{6.20}$$

and

$$\tilde{g}_{ii} = g_{ii} + \sum_{\substack{k=1\\k\neq i}}^{m} g_{ik}\frac{G^{ik}}{G^{ii}} = \frac{1}{G^{ii}}\sum_{k=1}^{m} g_{ik}G^{ik} = \frac{|G|}{G^{ii}}, \qquad \forall i \in \mathbf{m}. \tag{6.21}$$

Thus, the decoupled open-loop transfer function matrix is given by

$$GK = \operatorname{diag}\{\tilde{g}_{ii}k_{ii}\} = \operatorname{diag}\left\{\frac{|G|}{G^{ii}}k_{ii}, \quad i = 1, 2, \cdots, m\right\}. \tag{6.22}$$

Before proceeding further, it is interesting to note that $\frac{G^{ii}}{|G|}$ is the $(i,\ i)$-th element of G^{-1}. And it follows from Bristol (1966) that the (i,i)-th element of the relative gain array for G is given by $\lambda_{ii} = g_{ii}\frac{G^{ii}}{|G|}$. Hence, $\tilde{g}_{ii}$ is linked to g_{ii} by $\tilde{g}_{ii} = \lambda_{ii}^{-1} g_{ii}$.

To demonstrate how to find $\tilde{g}_{ii}$ and ψ_{ji}, take the following as an example:

$$G(s) = \begin{bmatrix} \frac{1}{s+2}e^{-2s} & \frac{-1}{s+2}e^{-6s} \\ \frac{s-0.5}{(s+2)^2}e^{-3s} & \frac{(s-0.5)^2}{2(s+2)^3}e^{-8s} \end{bmatrix}. \tag{6.23}$$

Simple calculations give

$$|G| = \frac{2(s-0.5)(s+2)e^{-9s} + (s-0.5)^2e^{-10s}}{2(s+2)^4},$$

$$G^{11} = \frac{(s-0.5)^2e^{-8s}}{2(s+2)^3}, \qquad G^{21} = \frac{e^{-6s}}{s+2},$$

$$G^{22} = \frac{e^{-2s}}{s+2}, \qquad G^{12} = -\frac{(s-0.5)e^{-3s}}{(s+2)^2}.$$

It follows from (6.21) that the decoupled loops have their equivalent processes as

$$\tilde{g}_{11} = \frac{2(s+2)e^{-s} + (s-0.5)e^{-2s}}{(s-0.5)(s+2)},$$

$$\tilde{g}_{22} = \frac{2(s-0.5)(s+2)e^{-7s} + (s-0.5)^2e^{-8s}}{2(s+2)^3},$$

respectively. By (6.19), ψ_{21} and ψ_{12} are found as

$$\psi_{21} = -\frac{2(s+2)}{s-0.5}e^{5s}, \qquad \psi_{12} = e^{-4s}.$$

6.3 Achievable Performance

A natural question to ask after decoupling the plant is what the achievable performance of the decoupled system is. It is well known that the performance limitations are imposed by the non-minimum phase part of the system. This requires us to derive the characterization of the unavoidable time delays and non-minimum phase zeros that are inherent in any feedback loop of the decoupled plants. Furthermore, to be applicable in practice, we also have to find a realizable controller, preferentially of lowest complexity, which can yield the best achievable performance. Note from the preceding section that the exact decoupler is likely to be very complex and thus not practical for industrial implementation. Rather, one would prefer to have a controller of normal type in the format of rational function plus possible time delay to get near-decoupling performance. This near-decoupling would usually benefit individual loop performance as there is a trade-off between loop-decoupling and loop performance in most cases. These considerations lead us to the following three-stage decoupling design approach.

(i) Derive the best achievable objective function for each decoupled closed-loop transfer function $h_{ri}(s)$ based on the process dynamic characteristics. Let the performance specifications of the decoupled closed-loop system are formulated as m objective closed-loop transfer functions h_{ri} from the set-point r_i to output y_i, $\forall\, i \in \mathbf{m}$. Then, the ith open-loop transfer function q_{rii} corresponding to the objective closed-loop transfer function h_{ri} is given by

$$q_{rii} = \frac{h_{ri}}{1 - h_{ri}}. \tag{6.24}$$

(ii) Match the actual loop $\tilde{g}_{ii}k_{ii}$ in (6.22) to the objective loop:

$$\tilde{g}_{ii}(s)k_{ii}(s) = q_{rii}(s),$$

to find the ideal diagonal elements of the controller as

$$k_{ii}^{IDEAL}(s) = \tilde{g}_{ii}^{-1}(s)q_{rii}(s) = \frac{G^{ii}(s)}{|G(s)|}q_{rii}(s), \tag{6.25}$$

and the corresponding ideal decoupling off-diagonal elements $k_{ji}(s)$ of the controller as

$$k_{ji}^{IDEAL}(s) = \frac{G^{ij}(s)}{G^{ii}(s)}k_{ii}^{IDEAL} = \frac{G^{ij}(s)}{|G(s)|}q_{rii}(s). \tag{6.26}$$

The ideal controller is generally highly complicated and/or difficult to realize.

(iii) Approximate the ideal controller $k_{ij}(s)$ element-wisely with a simplest rational function plus possible delay with a pre-specified accuracy.

In the present section, we will only consider analytical characteristics of the achievable objective loop transfer function whereas the rest of the above approach will be addressed in the next section, after which there will be stability and robustness analysis and simulation results.

The achievable control performance for a given process depends on its characteristics. Thus, the objective transfer functions cannot be chosen arbitrarily. Assume that the ith loop control specifications are expressed in terms of a desired closed-loop transfer function:

$$h_{ri}(s) = \frac{\omega_{ni}^2 e^{-L_i s}}{(s^2 + 2\omega_{ni}\xi_i s + \omega_{ni}^2)(\frac{1}{N_i\omega_{ni}}s + 1)^{\nu_i}} \prod_{z\in\mathbf{Z}_i} \left(\frac{z-s}{z+s}\right)^{n_i(z)}, \quad i \in \mathbf{m}. \tag{6.27}$$

Notice that time delays and non-minimum phase zeros are inherent characteristic of a process and can not be altered by any feedback control. They are reflected in (6.27) respectively by $e^{-L_i s}$ and $\prod_{z\in\mathbf{Z}_i}\left(\frac{z-s}{z+s}\right)^{n_i(z)}$. The factor of the standard 2nd-order transfer function represents usual performance requirement such as overshoot and settling time for the minimum phase and assignable part of the loop. The term $(\frac{1}{N_i\omega_{ni}}s + 1)^{\nu_i}$ is to provide necessary high frequency roll-off rate required by decoupling and properness of the controller.

Firstly, consider the time delay L_i of the ith loop. As time delay imposes performance limitation in a control loop, to achieve best performance and to make the closed-loop response fastest, the time delay L_i in the ith loop in (6.27) should be chosen as the smallest possible value. If only (6.22) is concerned, it seems that L_i can be chosen as the time delay in $\tilde{g}_{ii}$ as k_{ii} cannot contain pure prediction for realizability. However, decoupling may require this delay to be increased. To see this, assume that the system interaction from ith input to jth output, g_{ji}, has smaller time delay than that of the transfer function from the jth input to the jth output. Then, complete elimination of the interaction, g_{ji}, by a realizable k_{ji} is impossible, unless the ith loop delay is increased by a certain amount.

It is thus interesting to know when decoupling requires further delay in a certain loop and by what amount. To address this problem, consider the process $G(s)$ given by (6.6). A general expression for non-zero G^{ij} and $|G|$ will be

$$\phi(s) = \sum_{l=1}^{M} \phi_l(s) e^{-\alpha_l s}, \quad \alpha_l > 0, \tag{6.28}$$

where $\phi_l(s)$, $l = 1, \cdots, M$, are non-zero scalar rational transfer functions. Define the time delay for $\phi(s)$ in (6.28) as

$$\boldsymbol{\tau}(\phi(s)) = \min_{l=1,\cdots,M}(\alpha_l).$$

It is easy to verify $\boldsymbol{\tau}(\phi_1\phi_2) = \boldsymbol{\tau}(\phi_1) + \boldsymbol{\tau}(\phi_2)$ for any non-zero $\phi_1(s)$ and $\phi_2(s)$. And it is obvious that for any realizable and non-zero $\phi(s)$, $\boldsymbol{\tau}(\phi(s))$ can not be negative. It follows from (6.26) that

$$|G|k_{ji}^{IDEAL} = G^{ij} q_{rii}, \qquad \forall\, i, j \in \mathbf{m}. \tag{6.29}$$

Thus, a realizable controller requires

$$\boldsymbol{\tau}(G^{ji}) + \boldsymbol{\tau}(q_{rii}) \geq \boldsymbol{\tau}(|G|), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,$$

where $\mathbf{J}_i \triangleq \{j \in \mathbf{m} \mid G^{ij} \neq 0\}$, or

$$\boldsymbol{\tau}(q_{rii}) \geq \boldsymbol{\tau}(|G|) - \tau_i, \qquad \forall\, i \in \mathbf{m}, \tag{6.30}$$

where

$$\tau_i \triangleq \min_{j \in \mathbf{J}_i} \boldsymbol{\tau}(G^{ij}). \tag{6.31}$$

Therefore, the time delay L_i in the ith objective closed-loop transfer function is chosen as the minimum among all meeting (6.30):

$$L_i = \boldsymbol{\tau}(|G|) - \tau_i. \tag{6.32}$$

We have just seen how the various time delays in G^{ij} and $|G|$ limit the control system performance and affect the choice of the objective loop transfer function. In addition to time delays, zeros of G^{ij} and $|G|$ which lie in the closed right half plane (RHP) may also limit the control system performance. To address this problem, let $\mathbf{Z}^+_{|G|}$ be the set of RHP zeros of $|G|$.[1] For a stable and proper k_{ji}, $|k_{ji}(z)|$ is finite for all $\mathrm{Re}(z) > 0$. Thus, the left side of (6.29) will evaluate to 0 for each $z \in \mathbf{Z}^+_{|G|}$, and hence, the right side of (6.29), $G^{ij} q_{rii}$ must also interpolate to 0 for each $z \in \mathbf{Z}^+_{|G|}$. This implies that q_{rii} needs to have a RHP zero at z if z is itself not a RHP zero of G^{ij}. The situation may become more complicated when RHP zeros of $|G|$ coincide with those of G^{ij} and when multiplicity is considered.

In order to investigate what RHP zero must be included in q_{rii}, for a non-zero transfer function $\phi(s)$, let $\boldsymbol{\eta}_z(\phi)$ be a non-negative integer ν such that $\lim_{s \to z} \phi(s)/(s-z)^\nu$ exists and is non-zero. Thus, $\phi(s)$ has $\boldsymbol{\eta}_z(\phi)$ zeros at $s = z$ if $\boldsymbol{\eta}_z(\phi) > 0$, or no zeros if $\boldsymbol{\eta}_z(\phi) = 0$. It is easy to verify that $\boldsymbol{\eta}_z(\phi_1\phi_2) = \boldsymbol{\eta}_z(\phi_1) + \boldsymbol{\eta}_z(\phi_2)$ for any non-zero transfer functions $\phi_1(s)$ and $\phi_2(s)$. It then follows from (6.29) that a stable and proper $k_{ji}^{IDEAL}(s)$ requires

$$\boldsymbol{\eta}_z(G^{ij} q_{rii}) \geq \boldsymbol{\eta}_z(|G|), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,\ z \in \mathbf{Z}^+_{|G|},$$

[1] It is assumed in this chapter that $|G|$ has no zero at the origin so as to enable the inclusion of integral control in $K(s)$ for robust tracking and regulation with respect to step inputs.

or

$$\boldsymbol{\eta}_z(q_{rii}) \geq \boldsymbol{\eta}_z(|G|) - \eta_i(z), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbf{Z}^+_{|G|}, \tag{6.33}$$

where

$$\eta_i(z) \triangleq \min_{j \in \mathbf{J}_i} \boldsymbol{\eta}_z(G^{ij}). \tag{6.34}$$

Equation (6.33) implies that the ith decoupled open loop transfer function q_{rii} must contain the RHP zeros at each $\mathbf{Z}^+_{|G|}$ with multiplicity of $\boldsymbol{\eta}_z(|G|) - \eta_i(z)$ if $\boldsymbol{\eta}_z(|G|) - \eta_i(z) > 0$. Note that a feedback connection of a loop transfer function alter neither the existence nor the location of a zero. Thus the ith decoupled closed-loop transfer function h_{ri} should also contain these RHP zeros with exactly the same multiplicity. Note also from (6.29) and the assumed stability of G that q_{rii} need not include any other RHP zero except those of $|G|$; and for best performance of the closed-loop system it is undesirable to include any other RHP zeros in $q_{rii}(s)$ more than necessary. Therefore, $\mathbf{Z}_i$ and $n_i(z)$ in (6.27) are chosen respectively as

$$\mathbf{Z}_i = \mathbf{Z}^+_{|G|}, \tag{6.35}$$

and

$$n_i(z) = \boldsymbol{\eta}_z(|G|) - \eta_i(z). \tag{6.36}$$

We now consider the term $(\frac{1}{N_i \omega_{ni}} s + 1)^{\nu_i}$, which provides necessary high frequency roll-off rate required by decoupling and properness of the controller. N_i is usually chosen as $10 \sim 20$. To determine the integer ν_i, consider a non-zero transfer function $\phi(s)$, let $\boldsymbol{\nu}(\phi)$ be the smallest integer ν such that $\lim_{\omega \to \infty} |\phi(j\omega)/(j\omega)^{(1+\nu)}| = 0$. A transfer function ϕ is said to be proper if $\boldsymbol{\nu}(\phi) \geq 0$. It is easy to verify that $\boldsymbol{\nu}(\phi_1\phi_2) = \boldsymbol{\nu}(\phi_1) + \boldsymbol{\nu}(\phi_2)$ for any non-zero transfer functions $\phi_1(s)$ and $\phi_2(s)$. For realizability, all controller elements should be proper, i.e., $\boldsymbol{\nu}(k_{ij}^{IDEAL}) \geq 0$. It then follows from (6.29) that

$$\boldsymbol{\nu}(G^{ij} q_{rii}) \geq \boldsymbol{\nu}(|G|), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,$$

or

$$\boldsymbol{\nu}(q_{rii}) \geq \boldsymbol{\nu}(|G|) - \min_{j \in \mathbf{J}_i} \boldsymbol{\nu}(G^{ij}), \qquad \forall\, i \in \mathbf{m}. \tag{6.37}$$

Since the feedback connection of a loop transfer function does not alter its relative degree, i.e., $\boldsymbol{\nu}(q_{rii}) = \boldsymbol{\nu}(h_{ri}) = 2 + \nu_i$, (6.37) is equivalent to

$$\boldsymbol{\nu}(h_{ri}) = 2 + \nu_i \geq \boldsymbol{\nu}(|G|) - \min_{j \in \mathbf{J}_i} \boldsymbol{\nu}(G^{ij}),$$

and taking

$$\nu_i = \max\{0,\ \boldsymbol{\nu}(|G|) - \min_{j \in \mathbf{J}_i} \boldsymbol{\nu}(G^{ij}) - 2\} \tag{6.38}$$

ensures that the resultant k_{ij} are all proper and thus physically realizable.

Now all the necessary RHP zeros z, time delays L_i as well as ν_i, required by (6.27), have been determined. They are all feedback invariant. What remains to fix are the parameters ω_{ni} and ξ_i in (6.27). ξ_i determines the resonant peak magnitude M_{ri} of the frequency response of the closed-loop transfer function h_{ri}. The magnitude of the resonant peak gives an useful indication of the relative stability of the system (Ogata, 1997). A large resonant peak magnitude indicates the presence of a pair of dominant closed-loop poles with small damping ratio, which will yield an undesirable transient response. A small resonant peak magnitude, on the other hand, indicates the absence of a pair of dominant closed-loop poles with small damping ratio, meaning that the system is well damped. By ignoring the term $\frac{1}{(\frac{1}{N_i \omega_{ni}} s+1)^{\nu_i}}$ in the operating bandwidth, the magnitude of the ith closed-loop transfer function h_{ri} in (6.27) can be calculated as

$$|h_{ri}(j\omega)| = \frac{1}{\sqrt{(1 - \frac{\omega^2}{\omega_{ni}^2})^2 + (2\xi_i \frac{\omega}{\omega_{ni}})^2}}, \tag{6.39}$$

and for $0 \le \xi_i \le 0.707$, its maximum value M_{ri} is given by $\frac{1}{2\xi_i\sqrt{1-\xi_i^2}}$. For $\xi_i > 0.707$, there is no resonance, or the value of M_{ri} is unity for $\xi_i > 0.707$. We choose ξ_i to meet the specification that the closed-loop resonant peak magnitude is less than M_{ri} (where $M_{ri} > 1$), i.e.,

$$\frac{1}{2\xi_i\sqrt{1-\xi_i^2}} < M_{ri}. \tag{6.40}$$

A ξ value of 0.707 is widely used as it results in acceptable overshoot for a step change in the reference input and quite good damping. The overshoot becomes excessive for values of $\xi_i < 0.4$ (Ogata, 1997).

The parameter ω_{ni} in (6.27) is directly related to ω_{bi}, the bandwidth of the feedback loop, which is defined as the lowest frequency such that

$$|h_{ri}(j\omega_{bi})| = |h_{ri}(0)|/\sqrt{2},$$

and is given by

$$\omega_{bi} \approx \omega_{ni}\sqrt{\sqrt{(2\xi_i^2 - 1)^2 + 1} - (2\xi_i^2 - 1)}$$

for $h_{ri}(s)$ in (6.27). The loop bandwidth ω_{bi} is usually close to the 0 dB gain cross-over frequency ω_{gi}, at which $|q_{rii}(j\omega_{gi})| = 1$. Maciejowski (1989) shows that for typical, acceptable designs we can estimate the loop bandwidth in terms of the cross-over frequency by

$$\omega_{gi} \leq \omega_{bi} \leq 2\omega_{gi}, \tag{6.41}$$

and thus, we have

$$\omega_{ni} \approx \frac{\beta_i \omega_{gi}}{\sqrt{\sqrt{(2\xi_i^2 - 1)^2 + 1} - (2\xi_i^2 - 1)}}, \tag{6.42}$$

where $\beta_i \in [1, 2]$ with a default of 1.5. Essentially, to get the benefits of feedback control we want the loop gain, $|q_{rii}(j\omega)|$, to be large within certain frequency range. However, due to time delays, RHP zeros, unmodeled high frequency dynamics and limits on the manipulated variables, the loop gain has to drop below one at and above some frequency ω_{gi}. Astrom (1999) derived the following inequality for the gain cross-over frequency ω_g to meet

$$\arg P_{nmp}(j\omega_g) \geq -\pi + \phi_m - n_g\pi/2,$$

where P_{nmp} is the non-minimum phase part of the process, ϕ_m is the required phase margin and n_g is the slope of the open loop at the cross-over frequency. It is generally accepted that a desired loop shape for $|q_{rii}(j\omega)|$ should typically has a slope of about -1 around the desired cross-over frequency, with preferably a steeper slope before and after the cross-over. Notice that the non-minimum phase part of the ith loop is given by $e^{-L_i s} \prod_{z \in \mathbf{Z}_i} \left(\frac{z-s}{z+s}\right)^{n_i(z)}$, thus we require

$$\sum_{z \in \mathbf{Z}_i} n_i(z) \arctan\left(\frac{z - j\omega_{gi}}{z + j\omega_{gi}}\right) - L_i\omega_{gi} \geq -\pi/2 + \phi_{mi}, \tag{6.43}$$

and a straightforward method to assess the cross-over frequencies that can be obtained for ith loop is simply to plot the left hand side of (6.43) and determine when the inequality holds. And subsequently, ω_{ni} can be determined from (6.42). One notes that for $\phi_{mi} = 35^0$, ω_{gi} becomes $1/L$ if there is only a time delay L, or $z/2$ if there is a single real RHP zero at z. And for $\xi_i = 0.707$, the parameter ω_{ni} becomes β_i/L_i or, $\beta_i z/2$.

To consider the performance limitation imposed by input constraints, we use a frequency-by-frequency analysis. Assume that the inputs have to meet $|u_i(j\omega)| \leq u_{max,i}$ at each frequency and that the model has been scaled such that the reference signal in each channel satisfies $|r_i(j\omega)| \leq 1$. For a reference signal in the ith channel, $r = [0, \cdots, r_i, \cdots, 0]^T$, the ideally controlled variable y would be $y = [0, \cdots, h_{ri}r_i, \cdots, 0]^T$, and thus the corresponding manipulated variable becomes

$$u = G^{-1}y = G^{-1}[0, \cdots, h_{ri}r_i, \cdots, 0]^T,$$

i.e., elementwise as

$$u_j = \frac{G^{ij}}{\det(G)} h_{ri}r_i.$$

For the worst case references ($|r_i(j\omega)| = 1$), we requires

$$\left|\frac{G^{ij}}{\det(G)} h_{ri}\right| \le u_{max,j}, \qquad j \in \mathbf{J}_i,$$

or

$$|h_{ri}| \le \min_{j \in \mathbf{J}_i} \{ \left|\frac{\det(G)}{G^{ij}}\right| u_{max,j} \}.$$

For $\omega = 0$, the above yields $\min_{j\in\mathbf{J}_i}\{|\frac{det(G(0))}{G^{ij}(0)}|\} > u_{max,j}^{-1}$, which is necessary for tracking a unity step reference in the ith channel with zero steady state error. To derive an inequality on ω_{ni} imposed by input constraints, let $\omega = \omega_{ni}$ and notice from (6.39) that $h_{ri}(j\omega_{ni}) \approx \frac{1}{2\xi_i}$, we require

$$\frac{1}{2\xi_i} \le \min_{j \in \mathbf{J}_i} \{ |\frac{|G(j\omega_{ni})|}{G^{ij}(j\omega_{ni})}| u_{max,j} \}. \tag{6.44}$$

A straightforward method to determine the achievable ω_{ni} for ith loop under the given input constraints is simply to plot the right hand side of (6.44).

6.4 Design Procedure

In principle, once the objective loop performance, i.e., $q_{rii}(s)$, is determined, the decoupling controller $K(s)$ can be calculated from (6.25) and (6.26). This ideal controller decouples the process with the closed-loop transfer function for each decoupled subsystem specified in (6.27). For a multivariable multi-delay process in (6.6), the controller so found is usually highly complicated and difficult to implement. We have to apply model reduction to find a much simpler yet good approximation to the above theoretical controller. It should be also pointed out that for a general multivariable process in (6.6), the determination of the objective loop transfer function based on the exact $|G|$ and $G^{ij}(s)$ is actually very much involved. Therefore, the goal of this section is, with the aid of model reduction, to ease the computation effort involved in the determination of loop transfer function and more importantly, seek simplest possible controllers of general rational transfer function plus delay form, which can approximate the ideal controller such that user-specified requirements on loop transfer function and decoupling are met.

Model Reduction. Consider a scalar system of possibly complicated dynamics with available transfer function $\phi(s)$ or frequency response $\phi(j\omega)$. The problem at hand is to find a nth-order rational function plus dead time model:

$$\hat{\phi}(s) = \hat{\phi}_0(s) e^{-Ls} = \frac{b_n s^n + \cdots + b_1 s + 1}{a_n s^n + \cdots + a_1 s + a_0} e^{-Ls}, \quad L \ge 0, \tag{6.45}$$

such that the approximation error e defined by

$$e = \frac{1}{N} \sum_{k=1}^{N} \left| W(j\omega_k)(\hat{\phi}(j\omega_k) - \phi(j\omega_k)) \right|^2, \tag{6.46}$$

is minimized, where $W(j\omega_k)$ serves as the weightings. Two very effective algorithms with stability preservation have been presented in Chapter 3, one using recursive least squares and another using the combination of the FFT and step response. In what follows in this section, we refer "the model reduction algorithm" to one of them, or anyone else which can produce a reliable reduced-order model in the form of (6.45).

To find the objective loop transfer function numerically, we apply the model reduction algorithm to $|G|$ and G^{ij} to obtain the corresponding reduced-order models (here, the weighting can be chosen as unity, i.e., $W(j\omega) = 1$). To get a model with lowest order and yet achieve a specified approximation accuracy, we start the model reduction algorithm with a model of order 2 and then increase the order gradually to find the smallest integer n such that the corresponding maximum relative error is less than a user-specified ϵ (usually $5\% \sim 15\%$), i.e.,

$$\max_{k=1,\cdots,N} \left| \frac{\hat{\phi}(j\omega_k) - \phi(j\omega_k)}{\phi(j\omega_k)} \right| \le \epsilon. \tag{6.47}$$

Algorithm 6.4.1 *Determine the objective loop performances $q_{rii}(s)$ given $G(s)$, M_{ri}, ϕ_{mi} and $u_{max,i}$.*

(i) *Apply the model reduction algorithm to $|G|$ and G^{ij}, $i, j \in \mathbf{m}$, to obtain the corresponding reduced-order models. Determine the set $\mathbf{Z}^{+}_{|G|}$. Let $i = 1$.*
(ii) *Find, from the corresponding reduced-order models, $\boldsymbol{\tau}(G^{ji})$ and $\boldsymbol{\nu}(G^{ji})$ for all $j \in \mathbf{m}$, $\boldsymbol{\tau}(|G|)$ and $\boldsymbol{\nu}(|G|)$ as well as $\boldsymbol{\eta}_z(G^{ij})$ for all $j \in \mathbf{m}$ and $\boldsymbol{\eta}_z(|G|)$ for each $z \in \mathbf{Z}^{+}_{|G|}$.*
(iii) *Obtain L_i from (6.32), $n_i(z)$ from (6.36) for each $z \in \mathbf{Z}^{+}_{|G|}$, and ν_i from (6.38). Take $N_i = 10 \sim 20$.*
(iv) *Find the largest ω_{ni} and the smallest ξ_i to meet (6.40), (6.42), (6.43) and (6.44).*
(v) *Form h_{ri} using (6.27) and (6.35). Calculate q_{rii} from (6.24). Let $i = i + 1$. If $i \le m$, go to step (ii); otherwise end.*

After q_{rii}, $i = 1, \cdots, m$, are determined, the ideal controller elements k_{ji}^{IDEAL} can be calculated from (6.25) and (6.26). Again, they are of complicated form and it is desirable to approximate them with a simpler controller $K(s) = [k_{ij}]$. Let $GK = Q = [q_{ij}]$ be the actual open loop transfer function matrix. Let the frequency range of interest be

$$\mathcal{D}_{\omega i} = \{ s \in \mathcal{C} \ | s = j\omega, \ -\omega_{mi} < \omega < \omega_{mi} \}, \tag{6.48}$$

where $\omega_{mi} > 0$ is chosen to be large enough such that the magnitude of q_{rii} drops well below unity, or roughly, ω_{mi} is well beyond the system bandwidth. To ensure the actual loop to be close enough to the ideal one, we make

$$\frac{|q_{ii}(s) - q_{rii}(s)|}{|q_{rii}(s)|} \leq \varepsilon_{di} < 1, \quad s \in \mathcal{D}_{\omega i}, \tag{6.49}$$

$$\frac{\sum_{j \neq i} |q_{ji}(s)|}{|q_{ii}(s)|} \leq \varepsilon_{oi} < 1, \quad s \in \mathcal{D}_{\omega i}, \tag{6.50}$$

where ε_{oi} and ε_{di} are the required performance specifications on the loop interactions and loop performance, respectively. The values of ε_{di} and ε_{oi} are recommended to be in the range of 10% $\sim$ 30%. Smaller values of ε_{di} and ε_{oi} give more stringent demand on the performance and generally result in higher order controller. One notes that although (6.50) only limits the system couplings to a certain extent, it will still lead to almost decoupling if the number of inputs/outputs is large. Roughly, if the number is 10, and total couplings from all other loops are limited to 30%, then each off-diagonal element will have relative gain less than 3%, so the system is almost decoupled. This will be confirmed in the later simulation where almost decoupled response is usually obtained with a $\varepsilon_{oi} = 20\%$.

The left hand sides of (6.49) and (6.50) can be bounded respectively as

$$\frac{|q_{ii} - q_{rii}|}{|q_{rii}|} = \frac{|\sum_{j=1}^{m} g_{ij}(k_{ji} - k_{ji}^{IDEAL})|}{|q_{rii}|} \leq \sum_{j=1}^{m} |\frac{g_{ij}}{q_{rii}}(k_{ji} - k_{ji}^{IDEAL})|.$$

and

$$\begin{aligned} \frac{\sum_{j \neq i} |q_{ji}|}{|q_{ii}|} &= \frac{\sum_{j \neq i} |\sum_{l=1}^{m} g_{jl}(k_{li} - k_{li}^{IDEAL})|}{|q_{ii}|} \\ &\leq \frac{\sum_{j=1}^{m} \sum_{l \neq i} |g_{lj}(k_{ji} - k_{ji}^{IDEAL})|}{|q_{ii}|} \\ &= \sum_{j=1}^{m} \left| \left\{ \sum_{l \neq i} |\frac{g_{lj}}{q_{ii}}| \right\} (k_{ji} - k_{ji}^{IDEAL}) \right|. \end{aligned}$$

One notes that (6.49) implies $|q_{ii}| \geq (1 - \varepsilon_{di})|q_{rii}|$, and thus, (6.49) and (6.50) are both satisfied if for each $s \in \mathcal{D}_{\omega i}$, there hold

$$\sum_{j=1}^{m} |\frac{g_{ij}(s)}{q_{rii}(s)}(k_{ji}(s) - k_{ji}^{IDEAL}(s))| \leq \epsilon_{di},$$

$$\sum_{j=1}^{m} \left| \left\{ \sum_{l \neq i} |\frac{g_{lj}(s)}{q_{rii}(s)}| \right\} (k_{ji}(s) - k_{ji}^{IDEAL}(s)) \right| \leq (1 - \epsilon_{di})\epsilon_{oi},$$

which can be combined into

$$\sum_{j=1}^{m} |W_{ji}(s)(k_{ji}(s) - k_{ji}^{IDEAL}(s))| \leq 1, \tag{6.51}$$

where

$$W_{ji}(s) \triangleq \max \left\{ |\frac{g_{ij}(s)}{\epsilon_{di}}|, \sum_{l \neq i} |\frac{g_{lj}(s)}{(1-\epsilon_{di})\epsilon_{oi}}| \right\} / |q_{rii}(s)|, \tag{6.52}$$

is a positive real valued function.

In order to satisfy (6.51), we propose to minimize the following weighted sum of square errors

$$\sum_{s \in \tilde{\mathcal{D}}_{\omega i}} \left| W_{ji}(s) \left(k_{ji}(s) - k_{ji}^{IDEAL}(s) \right) \right|^2, \tag{6.53}$$

where $\tilde{\mathcal{D}}_{\omega i}$ is a suitably chosen subset of $\mathcal{D}_{\omega i}$, and the controller elements k_{ji} are chosen to be a rational transfer function plus a time delay in the form of (6.45). The most important frequency range for closed-loop stability and robustness is a decade above and below ω_{gi}, the unity gain cross-over frequency of the ith open loop objective transfer function q_{rii}. Therefore, $\tilde{\mathcal{D}}_{\omega i}$ is chosen as the interval between $\underline{\omega}_i = \frac{\omega_{gi}}{10}$ and $\overline{\omega}_i = 10\omega_{gi}$. In general, the resultant controller may not be in the PID form, but this is necessary for high performance. Note also that a possible time delay is included, and this may look unusual. The presence of dead time in the controller could be necessary for decoupling and it may also result from model reduction, otherwise controller order would be even higher. Though our design needs more calculations and a controller of high-order, only simple computations are involved and can be made automatic. The controller designed is directly realizable.

For actual computations, k_{ji} is found again by the model reduction algorithm, with the weighting taken as $W(j\omega) = W_{ji}(j\omega)$ as in (6.52). When applying the model reduction algorithm, we always set $a_0 = 0$ in (6.45) to impose an integrator in $k_{ji}(s)$. Let n_{ji} be the order of the controller element k_{ji}. To find a simplest controller, the orders of the controller elements in the ith column, n_{ji}, $j = 1, \cdots, m$, are determined in the following way. Firstly, an initial estimate of $n_{ji} = 2$, $j = 1, \cdots, m$, are made and then the parameters in k_{ji}s are calculated with the model reduction algorithm. Equations (6.49) and (6.50) are checked. If they are satisfied, $k_{ji}(s)$, $j = 1, \cdots, m$ are taken as the ith column controller elements; otherwise, the following indices are evaluated

$$\rho_{ji} = \max_{s \in \tilde{\mathcal{D}}_{\omega i}} |W_{ji}(s)(k_{ji}(s) - k_{ji}^{IDEAL}(s))|. \tag{6.54}$$

and let

$$\mathbf{J}^* = \left\{ j^* \in \mathbf{m} \;\middle|\; \rho_{j^*i} = \max_{j \in \mathbf{m}} \rho_{ji} \right\}. \tag{6.55}$$

We increase n_{ji} for each $j \in \mathbf{J}^*$ by 1 and re-calculate the corresponding k_{ji} of increased order n_{ji} and check (6.49) and (6.50) again. This procedure repeats until (6.49) and (6.50) are satisfied.

Algorithm 6.4.2 *Find the ith column controller approximations $k_{ji}(s)$, $j = 1, \cdots, m$ to meet (6.49) and (6.50), given $G(s)$, $q_{rii}(s)$, ϵ_{di} and ϵ_{oi}.*

(i) Set the frequency range $\tilde{\mathcal{D}}_{\omega i}$ as $[\frac{\omega_{gi}}{10}, 10\omega_{gi}]$, where ω_{gi} is the gain cross-over frequency of q_{rii}. Evaluate $k_{ji}^{IDEAL}(s)$, $j = 1, \cdots, m$, from (6.25) and (6.26).
(ii) Calculate $W_{ji}(s)$, $j = 1, \cdots, m$, from (6.52). Set $n_{ji} = 2$, $j = 1, \cdots, m$ and $\mathbf{J}^* = \mathbf{m}$.
(iii) For each $j \in \mathbf{J}^*$, find n_{ji}-th order approximation, $k_{ji}(s)$, to $k_{ji}^{IDEAL}(s)$ with the model reduction algorithm.
(iv) Check if (6.49) and (6.50) are satisfied. If yes, then stop; otherwise, calculate ρ_{ji} for $j \in \mathbf{m}$ from (6.54) and find $\mathbf{J}^*$ from (6.55). Set $n_{ji} = n_{ji} + 1$ for each $j \in \mathbf{J}^*$. Go to Step (iii).

Remark 6.4.1. An estimate of the time delay in $k_{ji}(s)$ can be obtained as follows. One notes from (6.29) that

$$\boldsymbol{\tau}(|G|) + \boldsymbol{\tau}(k_{ji}^{IDEAL}) = \boldsymbol{\tau}(G^{ji}) + L_i.$$

By (6.32), we have

$$\boldsymbol{\tau}(k_{ji}^{IDEAL}) = \boldsymbol{\tau}(G^{ji}) - \tau_i, \tag{6.56}$$

where estimates of $\boldsymbol{\tau}(G^{ji})$ and τ_i have already been available from Algorithm 6.4.1. Simulation shows that (6.56) usually gives a very good estimate to the delay in k_{ji} and the search on the delay can thus be avoided.

Remark 6.4.2. It is noted that model reduction on the ideal controller is performed over $\tilde{\mathcal{D}}_{\omega i}$, which is spanned over a decade below and above the gain cross-over frequency of q_{rii}. The magnitude and phase of the roll-off term $(\frac{1}{N_i \omega_{ni}} s + 1)^{\nu_i}$ in (6.27) usually remains almost unchanged over that frequency range. Hence, although it is theoretically necessary to be included in (6.27), it has little influence in our practical design.

The entire controller design procedure is carried out by first applying Algorithm 6.4.1 to find the objective loop transfer function and subsequently applying Algorithm 6.4.2 to get all the controller columns. It is recaptured as follows for ease of reference.

Overall Procedure. *–Seek a single-loop controller $K(s)$ to meet the user-specified M_{ri}, ϕ_{mi}, $u_{max,i}$, ϵ_{di} and ϵ_{oi}, given a stable $m \times m$ process $G(s)$.*

(i) *Apply the model reduction algorithm to* $|G|$ *and* G^{ij}, $i, j \in \mathbf{m}$, *to obtain the corresponding reduced-order models. Determine the set* $\mathbf{Z}^{+}_{|G|}$. *Let* $i = 1$.

(ii) *Find, from the corresponding reduced-order models,* $\boldsymbol{\tau}(G^{ji})$ *and* $\boldsymbol{\nu}(G^{ji})$ *for all* $j \in \mathbf{m}$, $\boldsymbol{\tau}(|G|)$ *and* $\boldsymbol{\nu}(|G|)$, *as well as* $\boldsymbol{\eta}_z(G^{ij})$ *for all* $j \in \mathbf{m}$ *and* $\boldsymbol{\eta}_z(|G|)$ *for each* $z \in \mathbf{Z}^{+}_{|G|}$. *Obtain* L_i *from* (6.32), $n_i(z)$ *from* (6.36) *for each* $z \in \mathbf{Z}^{+}_{|G|}$, *and* ν_i *from* (6.38). *Take* $N_i = 10 \sim 20$.

(iii) *Find the largest* ω_{ni} *and the smallest* ξ_i *to meet* (6.40), (6.42), (6.43) *and* (6.44). *Form* h_{ri} *using* (6.27) *and* (6.35). *Calculate* q_{rii} *from* (6.24).

(iv) *Set the frequency range* $\tilde{\mathcal{D}}_{\omega i}$ *as* $[\frac{\omega_{gi}}{10}, 10\omega_{gi}]$, *where* ω_{gi} *is the gain cross-over frequency of* q_{rii}. *Evaluate* $k_{ji}^{IDEAL}(s)$, $j = 1, \cdots, m$, *from* (6.25) *and* (6.26). *Calculate* $W_{ji}(s)$, $j = 1, \cdots, m$, *from* (6.52). *Set* $n_{ji} = 2$, $j = 1, \cdots, m$ *and* $\mathbf{J}^* = \mathbf{m}$.

(v) *For each* $j \in \mathbf{J}^*$, *find* n_{ji}*-th order approximation,* $k_{ji}(s)$, *to* $k_{ji}^{IDEAL}(s)$ *with the model reduction algorithm.*

(vi) *Check if* (6.49) *and* (6.50) *are satisfied. If yes, go to step (vii); otherwise, calculate* ρ_{ji} *for* $j \in \mathbf{m}$ *from* (6.54) *and find* $\mathbf{J}^*$ *from* (6.55). *Set* $n_{ji} = n_{ji} + 1$ *for each* $j \in \mathbf{J}^*$. *Go to Step (v).*

(vii) *Set* $i = i + 1$, *if* $i \le m$, *go to Step (ii), otherwise end the design, where* $K(s) = [k_{ij}(s)]$ *is a realizable controller designed for implementation.*

6.5 Stability and Robustness Analysis

Let us consider first the nominal stability of the multivariable system in Figure 6.1, where $G(s)$ is not perturbed. As the process $G(s)$ is assumed to be stable and the controller $K(s)$ is designed to contain no poles in the right half of the complex plane except the origin, according to the generalized Nyquist theorem (Chapter 3) with the Nequist Contour at the origin indented into the right, the closed-loop system is stable if and only if the characteristic loci of $G(s)K(s) = Q(s)$, taken together, does not encircle the point -1.

Thus, in principle, with the generalized Nyquist theorem, the nominal stability of the system can be determined by calculating the characteristic loci of Q and counting their encirclements of the point -1. However, it is not easy to calculate the characteristic loci, especially for processes of high dimension. Notice that we are adopting a decoupling design. As a result, the compensated open-loop transfer function $Q(s)$ is likely to have a high degree of column dominance, and its Gershgorin bands will be generally narrow. By Gershgorin's theorem (Maciejowski, 1989), we know that the union of the Gershgorin bands "trap" the union of the characteristic loci, and we can assess closed-loop stability by counting the encirclements of -1 by the Gershgorin bands, if they excludes -1, since this tells us the number of encirclements made by the characteristic loci. Thus, the closed-loop system is stable if all the Gershgorin bands exclude and make no encirclements of the point -1.

Notice that the elements of $G(s)$ are assumed to be strictly proper and the elements of $K(s)$ are designed to be proper. Thus, q_{ji}, the elements of $Q(s) = G(s)K(s)$ are also strictly proper, i.e., $\lim_{s\to\infty} q_{ji}(s) = 0$. Let $\omega_{mi} > 0$ in (6.48) be chosen such that for all $\omega > \omega_{mi}$, $\sum_{j=1}^{m} |q_{ji}(j\omega)| < \varepsilon$ (where $0 < \varepsilon < 1$), or $\sum_{j=1}^{m} |q_{ji}(s)| < \varepsilon$ for $s \in \overline{\mathcal{D}}_{\omega i}$, where $\overline{\mathcal{D}}_{\omega i}$ denotes $\mathcal{D}_{\omega i}$'s compliment of the Nyquist contour D, i.e., $D = D_{\omega i} \cup \overline{\mathcal{D}}_{\omega i}$. We have the following theorem.

Theorem 6.5.1. *(Nominal Stability) Suppose* (6.49) *and* (6.50) *for* $s \in \mathcal{D}_{\omega i}$ *and all* $i \in \mathbf{m}$, *and suppose* $\sum_{j=1}^{m} |q_{ji}(s)| < 1$ *and* $|q_{rii}(s)| < 1$ *for* $s \in \overline{\mathcal{D}}_{\omega i}$ *and all* $i \in \mathbf{m}$,*. Then the closed-loop system in Figure 6.1 is nominally stable if*

$$h_{ri}(s)|(\epsilon_{di} + \epsilon_{oi} + \epsilon_{di}\epsilon_{oi}) < 1, \qquad s \in \mathcal{D}_{\omega i}.$$

Proof. The ith Gershgorin band of $Q(s)$, by definition, is

$$\boldsymbol{G}_{er}^{i} := \cup_{s\in D} G_{er}^{i}(s),$$

where

$$G_{er}^{i}(s) = \left\{ z \in \mathcal{C} \;\middle|\; |z - q_{ii}(s)| \le \sum_{j\neq i} |q_{ji}(s)| \right\}.$$

As (6.49) and (6.50) are satisfied for $s \in \mathcal{D}_{\omega i}$, we have for $s \in \mathcal{D}_{\omega i}$

$$|q_{ii}| = |q_{ii} - q_{rii} + q_{rii}| \le |q_{ii} - q_{rii}| + |q_{rii}| \le (1 + \epsilon_{di})|q_{rii}|,$$

and

$$|q_{ii} - q_{rii}| + \sum_{j\neq i} |q_{ji}| \le \epsilon_{di}|q_{rii}| + \epsilon_{oi}|q_{ii}| \le (\epsilon_{di} + \epsilon_{oi} + \epsilon_{oi}\epsilon_{di})|q_{rii}|. \tag{6.57}$$

Thus for $z \in G_{er}^{i}(s)$, where $s \in \mathcal{D}_{\omega i}$, we have

$$|z - q_{rii}| \le |q_{ii} - q_{rii}| + |z - q_{ii}| \le |q_{ii} - q_{rii}| + \sum_{j\neq i} |q_{ji}| \le (\epsilon_{di} + \epsilon_{oi} + \epsilon_{oi}\epsilon_{di})|q_{rii}|.$$

If the condition in the theorem holds, then

$$|1 + q_{rii}| - |z - q_{rii}| \ge |1 + q_{rii}| - (\epsilon_{oi} + \epsilon_{di} + \epsilon_{oi}\epsilon_{di})|q_{rii}| > 0,$$

which means that the distance from q_{rii} to the point -1 is greater that from q_{rii} to any point in $\boldsymbol{G}_{er}^{i}$ so that the Nequist curve of q_{rii} and the ith Gershgorin band $\boldsymbol{G}_{er}^{i}$ have the same encirclememts with respect to the point -1.

As for $s \in \overline{\mathcal{D}}_{\omega i}$ and $i \in \mathbf{m}$, we have

$$|z| - |q_{ii}| \leq |z - q_{ii}| \leq \sum_{j \neq i} |q_{ji}|, z \in \boldsymbol{G}^i_{er}(s),$$

and with $\sum_{j=1}^m |q_{ji}(s)| < 1$, it gives

$$|z| \leq |q_{ii}| + \sum_{j \neq i} |q_{ji}| < 1,$$

which implies that no encirclement can be made by the Gershgorin band. The same can be said for q_{rii}, due to the assumed $|q_{rii}(s)| < 1$ for $s \in \overline{\mathcal{D}}_{\omega i}$. Thus, the Gershgorin bands make the same number of encirclement of the point -1 as $Q_r = \text{diag}\{q_{rii}\}$ does, and the system is stable as $H_r = \text{diag}\{h_{ri}\}$ is so. $\square$

In the real world where the model cannot describe the process exactly, nominal stability is not sufficient. Robust stability of the closed-loop has to be addressed. The multivariable control system in Figure 6.1 is referred to as being robustly stable if the closed-loop is stable for all members of a process family. Let the actual process be denoted as $\tilde{G}(s)$ and the nominal process still be denoted as $G(s)$. The following assumptions are made.

Assumption 6.5.1 *The actual process $\hat{G}(s)$ and the nominal process $G(s)$ do not have any unstable poles, and they are both strictly proper.*

Consider the family of stable processes $\prod$ with elementwise norm-bounded uncertainty described by

$$\prod = \left\{ \hat{G} = [\hat{g}_{ij}] \,\middle|\, \left|\frac{\hat{g}_{ij}(j\omega) - g_{ij}(j\omega)}{g_{ij}(j\omega)}\right| = |\Delta_{ij}(j\omega)| \leq \gamma_{ij}(\omega) \right\}, \tag{6.58}$$

where $\gamma_{ij}(\omega)$ is the bound on the multiplicative uncertainty Δ_{ij}.

Let the perturbed open-loop transfer matrix $\hat{G}(s)K(s)$ be denoted as $\hat{Q}(s)$ and its elements by $\hat{q}_{ji}(s)$. As the elements of $K(s)$ are designed to be proper, $\hat{q}_{ji}$, are thus strictly proper, i.e., $\lim_{s \to \infty} \hat{q}_{ji}(s) = 0$. Let $\omega_{mi} > 0$ be a sufficiently large number such that for all $\omega > \omega_{mi}$, $\sum_{j=1}^m |\hat{q}_{ji}(j\omega)| < \epsilon$, (where $0 < \epsilon < 1$)) or $\sum_{j=1}^m |\hat{q}_{ji}(s)| < \epsilon$ for $s \in \overline{\mathcal{D}}_{\omega i}$. We have the following theorem.

Theorem 6.5.2. (Robust Stability) Suppose nominal stability of the closed-loop system in Figure 6.1 and $\sum_{j=1}^m |\hat{q}_{ji}(s)| < 1$ for $s \in \overline{\mathcal{D}}_{\omega i}$ and all $i \in \mathbf{m}$. Then the system is robustly stable for all processes in $\prod$ if

$$|h_{ri}(s)|(\epsilon_{di} + \epsilon_{oi} + \epsilon_{di}\epsilon_{oi}) + \frac{\sum_{l,j \in \mathbf{m}} |\gamma_{lj} g_{lj} k_{ji}|}{|1 + q_{rii}|} < 1, \qquad s \in \mathcal{D}_{\omega i}.$$

Proof. The ith Gershgorin bands $\hat{\boldsymbol{G}}^i_{er}$ of the actual open loop transfer matrix $\hat{Q}(s)$ is,

$$\hat{\boldsymbol{G}}^i_{er} = \cup_{s\in D}\hat{G}^i_{er}(s),$$

where

$$\hat{G}^i_{er}(s) = \left\{ z \in \mathcal{C} \;\middle|\; |z - \hat{q}_{ii}(s)| \le \sum_{j\neq i} |\hat{q}_{ji}(s)|, \right\}$$

For $z \in \hat{G}^i_{er}(s)$, where $s \in \mathcal{D}_{\omega i}$, we have

$$\begin{aligned}
|z - \hat{q}_{ii}| &\le |\sum_{j\neq i} \hat{q}_{ji}| = |\sum_{j\neq i}\sum_{l=1}^{m}(1+\Delta_{jl})g_{jl}k_{li}| \\
&\le |\sum_{j\neq i}\sum_{l=1}^{m} g_{jl}k_{li}| + |\sum_{j\neq i}\sum_{l=1}^{m} \Delta_{jl}g_{jl}k_{li}| \\
&= \sum_{j\neq i}|q_{ji}| + |\sum_{j\neq i}\sum_{l=1}^{m} \Delta_{jl}g_{jl}k_{li}|,
\end{aligned}$$

and

$$\begin{aligned}
|z - \hat{q}_{ii}| &\ge |z - q_{rii}| - |\hat{q}_{ii} - q_{rii}| \\
&= |z - q_{rii}| - |\sum_{l=1}^{m}(1+\Delta_{il})g_{il}k_{li} - q_{rii}| \\
&= |z - q_{rii}| - |q_{ii} + \sum_{l=1}^{m}\Delta_{il}g_{il}k_{li} - q_{rii}| \\
&\ge |z - q_{rii}| - |q_{ii} - q_{rii}| - |\sum_{l=1}^{m}\Delta_{il}g_{il}k_{li}|.
\end{aligned}$$

Thus, we have

$$|z - q_{rii}| \le |q_{ii} - q_{rii}| + \sum_{j\neq i}|q_{ji}| + \sum_{i,j\in\mathbf{m}} |\Delta_{il}g_{il}k_{li}|.$$

It follows from (6.57) that

$$|z - q_{rii}| \le (\epsilon_{di} + \epsilon_{oi} + \epsilon_{oi}\epsilon_{di})|q_{rii}| + \sum_{i,j\in\mathbf{m}} |\Delta_{il}g_{il}k_{li}|,$$

and

$$\begin{aligned}
|z+1| &= |(1+q_{rii}) + (z - q_{rii})| \ge |1+q_{rii}| - |z - q_{rii}| \\
&\ge |1+q_{rii}| - \left\{ (\epsilon_{oi} + \epsilon_{di} + \epsilon_{oi}\epsilon_{di})|q_{rii}| + \sum_{i,j\in\mathbf{m}} |\Delta_{il}g_{il}k_{li}| \right\},
\end{aligned}$$

which will be positive, or the portion of the Gershgorin bands $\boldsymbol{G}_{er}^{i}$ for all $s \in \mathcal{D}_{\omega i}$ will exclude the point -1 if

$$|1 + q_{rii}| - \left\{ (\epsilon_{oi} + \epsilon_{di} + \epsilon_{oi}\epsilon_{di})|q_{rii}| + \sum_{i,j \in \mathbf{m}} |\Delta_{il} g_{il} k_{li}| \right\} > 0,$$

or

$$|h_{ri}(s)|(\epsilon_{di} + \epsilon_{oi} + \epsilon_{di}\epsilon_{oi}) + \frac{\sum_{i,j \in \mathbf{m}} |\Delta_{il} g_{il} k_{li}|}{|1 + q_{rii}|} < 1,$$

and the above will be satisfied for all perturbation $|\Delta_{ij}(j\omega)| \leq \gamma_{ij}(\omega)$, $i, j \in \mathbf{m}$, if and only if

$$|h_{ri}(j\omega)|(\epsilon_{di} + \epsilon_{oi} + \epsilon_{di}\epsilon_{oi}) + \frac{\sum_{i,j \in \mathbf{m}} |\gamma_{il}(\omega) g_{il}(j\omega) k_{li}(j\omega)|}{|1 + q_{rii}(j\omega)|} < 1,$$

for all $-\omega_{mi} < \omega < \omega_{mi}$.

As $\sum_{j=1}^{m} |\hat{q}_{ji}(s)| < 1$, for $s \in \overline{\mathcal{D}}_{\omega i}$ and $i \in \mathbf{m}$, thus for $z \in \hat{G}_{er}^{i}(s)$, where $s \in \overline{\mathcal{D}}_{\omega i}$, we have

$$|z| - |\hat{q}_{ii}| \leq |z - \hat{q}_{ii}| \leq \sum_{j \neq i} |\hat{q}_{ji}|,$$

giving

$$|z| \leq |\hat{q}_{ii}| + \sum_{j \neq i} |\hat{q}_{ji}| < 1,$$

or

$$|z + 1| > 0.$$

Therefore, the Gershgorin bands will exclude -1 for all s on the Nyquist contour and will not change the number of encirclement of the point -1 compared with the nominal system. The system is robustly stable. $\square$

6.6 Why PID Is Not Adequate

PID control is widely used in industry, but could be inadequate when applying to very complex SISO processes. A frequently encountered case of PID failure is where dynamics of individual channels are still simple but as there are more output variables to be controlled, the interactions existing between these variables can give rise to very complicated equivalent loops for which PID control is difficult to handle and more importantly, the interaction for such a process is too hard to be compensated for by PID, leading to very bad decoupling performance and then causing poor loop performance, too, thus making the design fail. To control such multivariable processes, PID is not adequate any more and higher order controller with integrator is necessary. In this section, we will demonstrate the above observations in a great detail with a practical example.

PID Design. Among the numerous techniques currently available for the tuning of PID controllers for a SISO process, the Astrom-Hagglund-based method (Astrom and Hagglund, 1984) is known to work well in many situations. The key idea of the method is to move the critical point to a desired position on the complex plane, $\frac{1}{A_m}e^{-j(\pi-\phi_m)}$, through the parameters of the PID controller so that a combined gain and phase margin type of specifications are satisfied. If the PID controller is described by

$$k(s) = k_c(1 + \frac{1}{\tau_i s} + \tau_d s),$$

then the parameters are given by

$$\begin{aligned} k_c &= \frac{k_u}{A_m}\cos\phi_m, \\ \tau_d &= \frac{\tan\phi_m + \sqrt{\frac{4}{\beta} + \arctan^2\phi_m}}{2\omega_u}, \\ \tau_i &= \beta\tau_d, \end{aligned}$$

where k_u and ω_u are the ultimate gain and ultimate frequency of the process, respectively and β is usually chosen as 4. This method is here referred to as the SV-PID tuning.

For Multivariable PID controller design, it follows from (6.22) that $\tilde{g}_{ii}$ can be regarded as the equivalent process that k_{ii} faces. k_{ii} are thus designed with respect to the equivalent processes $\tilde{g}_{ii}$ with some suitable PID tuning method, say the above SV-PID tuning. The PID parameters of the controller off-diagonal elements are determined individually by solving (6.20) at chosen frequencies. This method is here referred to as the MV-PID tuning.

For illustration, we will study, in details, the Alatiqi process in Example 6.1.3 with $G(s)$ given in (6.3), and show that PID, even in its multivariable version, is usually not adequate for complex multivariable processes for good control performance.

1×1 **Plant.** Look first at one of 1×1 subsystem of (6.3):

$$g_{11}(s) = \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)}.$$

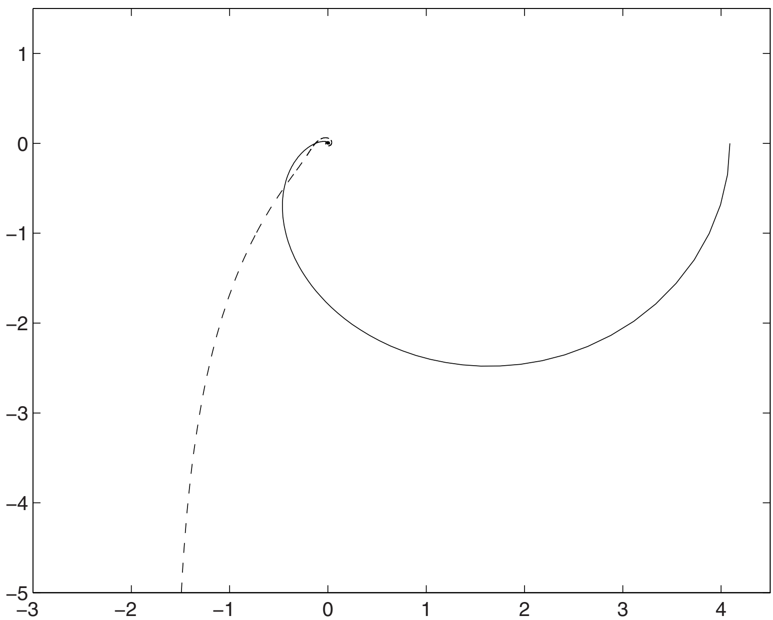


Fig. 6.2. Nyquist Curves (—— g_{11}; - - - $g_{11}k_{11}^{PID}$)

Its Nyquist curve is shown in Figure 6.2 in solid line. Like the other elements of the transfer matrix, $g_{11}(s)$ is basically in the form of first or second order plus time delay and simple PID controller is adequate to control it. Using the SV-PID tuning with $A_m = 2$ and $\Phi_m = \pi/4$ (the default settings throughout this chapter unless otherwise stated), a PID controller is obtained as

$$k_{11}^{PID}(s) = 2.8512 + \frac{0.0741}{s} + 9.1421s.$$

The resultant Nyquist curve of $g_{11}(s)k_{11}^{PID}(s)$ is depicted in Figure 6.2 in dashed line and the step response of the closed-loop system is given in Figure 6.3, showing that a PID controller is sufficient for this process and there will not be much benefits gained by using a more complex controller.

2×2 **Plant.** We now look at one of the 2×2 subsystem $G_{2\times 2}(s)$, obtained by taking the first and second rows and columns of $G(s)$, i.e.,

$$G_{2\times 2}(s) = \begin{bmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} \\ \frac{-4.17e^{-4s}}{45s+1} & \frac{6.93e^{-1.01s}}{44.6s+1} \end{bmatrix}.$$

The equivalent processes are

$$\tilde{g}_{11} = g_{11} - \frac{g_{12}g_{21}}{g_{22}} = \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} - \frac{3.83(44.6s+1)e^{-3.19s}}{(20s+1)(31.6s+1)(45s+1)}.$$

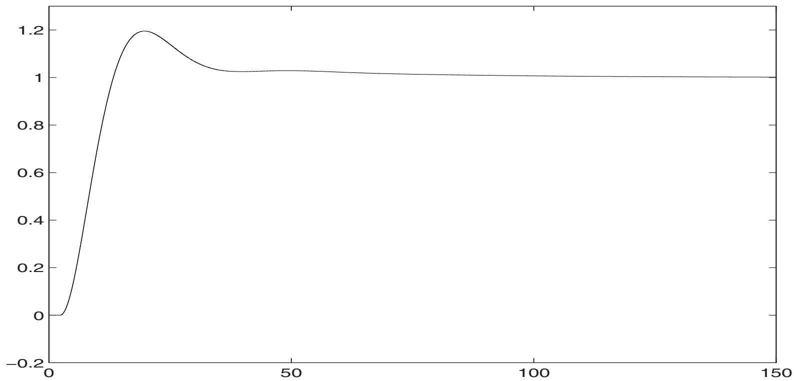


Fig. 6.3. Step Response for 1×1 System

and

$$\tilde{g}_{22} = g_{22} - \frac{g_{21}g_{12}}{g_{11}} = \frac{6.93e^{-1.01s}}{44.6s+1} - \frac{6.84(33s+1)(8.3s+1)e^{-2.9s}}{(31.6s+1)(20s+1)(45s+1)}.$$

Their Nyquist curves are shown in Figures 6.4(a) and 6.4(b), respectively. The SV-PID tuning gives the corresponding PID settings as

$$k_{11}^{PID}(s) = 3.1243 + \frac{0.2890}{s} + 6.7552s,$$

and

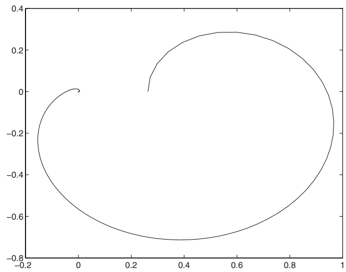
$$k_{22}^{PID}(s) = 1.0363 + \frac{0.2825}{s} + 0.7602s,$$

respectively. The resultant Nyquist curve of the loop transfer function $\tilde{g}_{11}(s)$ $k_{11}^{PID}(s)$ and $\tilde{g}_{22}(s)k_{22}^{PID}(s)$ are shown in Figures 6.4(c) and 6.4(d), respectively.

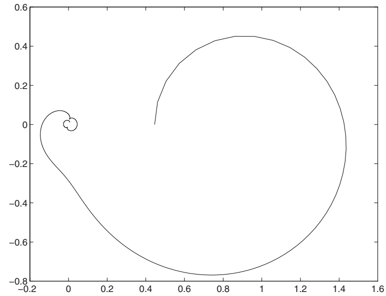
The MV-PID tuning yields

$$K^{PID}(s) = \begin{bmatrix} 3.1243 + \frac{0.2890}{s} + 6.7552s & -0.7896 + \frac{0.4383}{s} + 0.5459s \\ 1.7719 + \frac{0.1723}{s} - 2.9869 & 1.0363 + \frac{0.2825}{s} + 0.7602s \end{bmatrix},$$

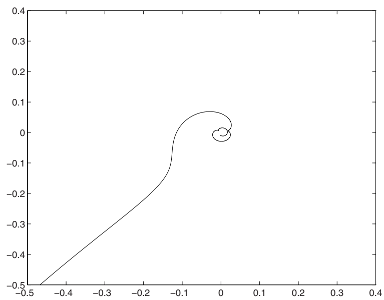
and the step response of the closed-loop system with this multivariable PID controller is shown in Figure 6.5 with dashed lines. One notes that for this 2 by 2 process, the equivalent processes $\tilde{g}_{11}(s)$ and $\tilde{g}_{22}(s)$, though no longer in simple first/second order plus delay form, is still in the capability of PID control. However, one also notes that the decoupling from loop 1 to loop 2 is far from satisfactory, and this is because the Nyquist curve of a PID controller is a vertical line in the complex plane and usually not sufficient to compensate for the interactions.



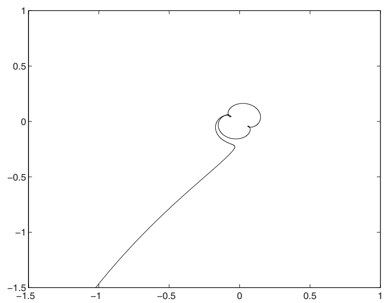
(a) Nyquist curve of $\tilde{g}_{11}$



(b) Nyquist curve of $\tilde{g}_{22}$



(c) Nyquist curve of $\tilde{g}_{11}k_{11}^{PID}$



(d) Nyquist curve of $\tilde{g}_{22}k_{22}^{PID}$

Fig. 6.4. Nyquist Curves

With the method presented in this chapter, a general controller is designed for this 2 by 2 process as $K^{HIGH}(s) = [k_{ij}^{HIGH}(s)]$, where

$$k_{11}^{HIGH}(s) = \frac{10210s^3 + 1625s^2 + 93.42s + 1}{841.2s^3 + 362.7s^2 + 0.4593s},$$

$$k_{12}^{HIGH}(s) = \frac{-231.6s^3 + 323.7s^2 + 50.83s + 1}{458.3s^3 + 349.8s^2 + 1.798s},$$

$$k_{21}^{HIGH}(s) = \frac{10070s^3 + 1584s^2 + 90.23s + 1}{1420s^3 + 590.3s^2 + 0.6502s}e^{-2.96s},$$

$$k_{22}^{HIGH}(s) = \frac{-543.7s^3 + 777.7s^2 + 60.68s + 1}{710.4s^3 + 541.1s^2 + 2.826s}e^{-1.089s},$$

and the corresponding step response is shown in Figure 6.5 with solid line. Compared with the PID performance, the high-order controller gives much

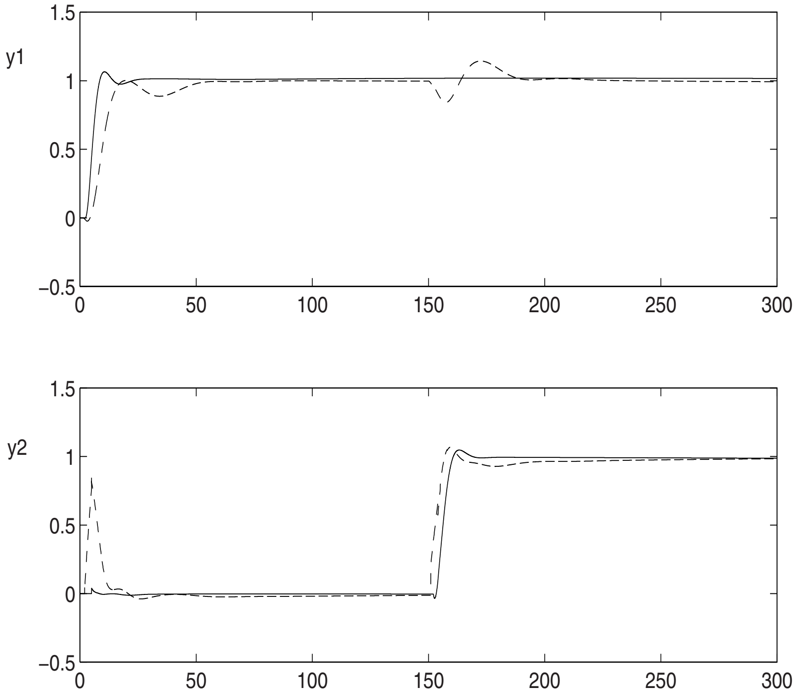


Fig. 6.5. Step Response for 2 × 2 System (—— Proposed; - - - MV-PID tuning;)

better loop and decoupling performance. According to our experience, multivariable PID controller can yield acceptable performance for many 2 by 2 industrial process, like this example. But yet, a controller of general type, if properly designed, can give tighter loop and decoupling performance.

3×3 **Plant.** We now proceed to a 3×3 subsystem obtained by deleting the 4th row and 4th column in $G(s)$ as

$$G_{3\times 3}(s) = \begin{bmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} \\ \frac{-4.17e^{-4s}}{45s+1} & \frac{6.93e^{-1.01s}}{44.6s+1} & \frac{-0.05e^{-5s}}{34.5s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} \end{bmatrix}.$$

To save space, we only pick up the second loop for illustration. The equivalent process that $k_{22}(s)$ faces is now

$$\tilde{g}_{22}(s) = \frac{|G|}{|G^{22}|} = \frac{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} \\ \frac{-4.17e^{-4s}}{45s+1} & \frac{6.93e^{-1.01s}}{44.6s+1} & \frac{-0.05e^{-5s}}{34.5s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} \end{vmatrix}}{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} \end{vmatrix}},$$

or,

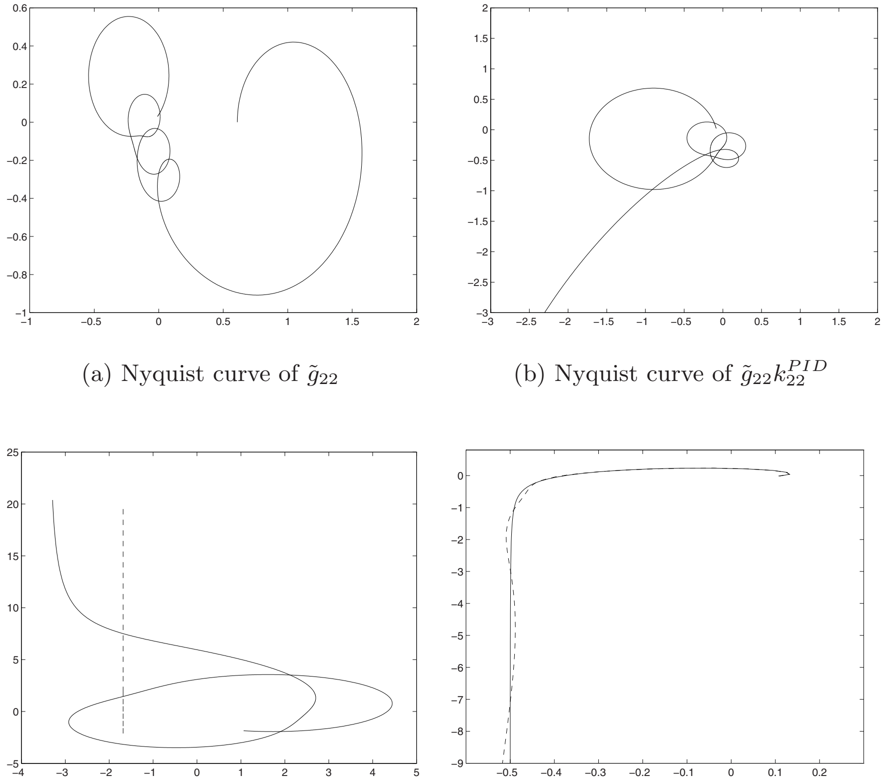
$$\begin{aligned} \tilde{g}_{22}(s) = \frac{6.93e^{-1.01s}}{44.6s+1} + \Bigg\{ & \frac{5.327e^{-15.4s}}{(21s+1)(45s+1)(13.3s+1)} \\ & - \frac{0.55e^{-19.88s}}{(31.6s+1)(20s+1)(34.5s+1)(13s+1)} \\ & + \frac{1.045e^{-14.98s}}{(33s+1)(8.3s+1)(34.5s+1)(13.3s+1)} \\ & - \frac{122.3e^{-2.9s}}{(31.6s+1)(20s+1)(45s+1)(18.5s+1)} \Bigg\} \\ & \cdot \left\{ \frac{18.85}{(33s+1)(8.3s+1)(18.5s+1)} - \frac{0.432e^{-15.08s}}{(21s+1)(13s+1)} \right\}^{-1} \end{aligned}$$

and its Nyquist curve is shown in Figure 6.6(a). The SV-PID tuning gives

$$k_{22}^{PID}(s) = 1.4929 + \frac{0.4006}{s} + 1.3907s.$$

The resultant loop Nyquist curves of $\tilde{g}_{22}(s)k_{22}^{PID}(s)$ is shown in Figure 6.6(b), from which we find that this controller $k_{22}^{PID}(s)$ fails to stabilize the equivalent process $\tilde{g}_{22}(s)$. Notice that in this case, the Nyquist curves of the equivalent process and loop transfer function are quite different from those of usual SISO processes and they can be improper, which means that the magnitude of the frequency response will increase and the Nyquist curve will not approach to the origin as the frequency goes to infinity. In the proposed method, an additional roll-off rate will be provided based on the process characteristic such that the loop transfer function becomes proper and approximates an objective transfer function. Obviously, these features cannot be obtained from PID controller. More importantly, it is the complexity of the Nyquist curve of the equivalent process over the medium frequency range that renders the PID controller to be inadequate, as it is generally difficult to shape the Nyquist curve of the equivalent process satisfactorily around this most important frequency range using a PID controller.

Even if one could be satisfied with such a loop performance, PID would be found to be yet inadequate to compensate for the interactions whose dynamics



(a) Nyquist curve of $\tilde{g}_{22}$

(b) Nyquist curve of $\tilde{g}_{22}k_{22}^{PID}$

(c) Nyquist curves (—— k_{32}^{IDEAL}; - - - k_{32}^{PID})

(d) Nyquist curves (—— q_{r22}; - - - $\tilde{g}_{22}k_{22}^{HIGH}$)

Fig. 6.6. Nyquist Curves

may also be very complicated and it would fail to achieve the decoupling. For an illustration, according to (6.20), the controller off-diagonal element, say $k_{32}(s)$, should be

$$k_{32}^{IDEAL}(s) = -\frac{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} \end{vmatrix}}{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} \end{vmatrix}} k_{22}(s), \tag{6.59}$$

with $k_{22}(s) = k_{22}^{PID}(s)$. The MV-PID design method gives the PID form of the off-diagonal controller elements $k_{32}^{PID}(s)$ as

$$k_{32}^{PID}(s) = -1.6825 - \frac{0.2152}{s} - 2.381s.$$

The Nyquist curve of the ideal decoupler $k_{32}^{IDEAL}(s)$ and its PID approximation k_{32}^{PID} are shown in Figure 6.6(c) in solid line and dash line, respectively, from which one sees that the Nyquist curve of the ideal decoupler $k_{32}(s)$ is quite complicated and thus its approximation by $k_{32}^{PID}(s)$ of PID form is inadequate. Actually, with the MV-PID tuned controller:

$$K^{PID}(s) = \begin{bmatrix} 5.976 + \frac{0.76}{s} + 11.7s & 2.16 + \frac{0.61}{s} + 2.72s & -0.825 + \frac{0.218}{s} + 0.994s \\ 2.59 + \frac{0.456}{s} - 5.87s & 1.49 + \frac{0.41}{s} + 1.39s & 0.56 + \frac{0.133}{s} + 0.328s \\ -9.25 - \frac{0.22}{s} + 5.8s & -1.68 - \frac{0.22}{s} - 2.4s & 1.169 + \frac{0.194}{s} + 1.76s \end{bmatrix},$$

the step response for the closed-loop system shown in Figure 6.7 with dash lines has gone unstable.

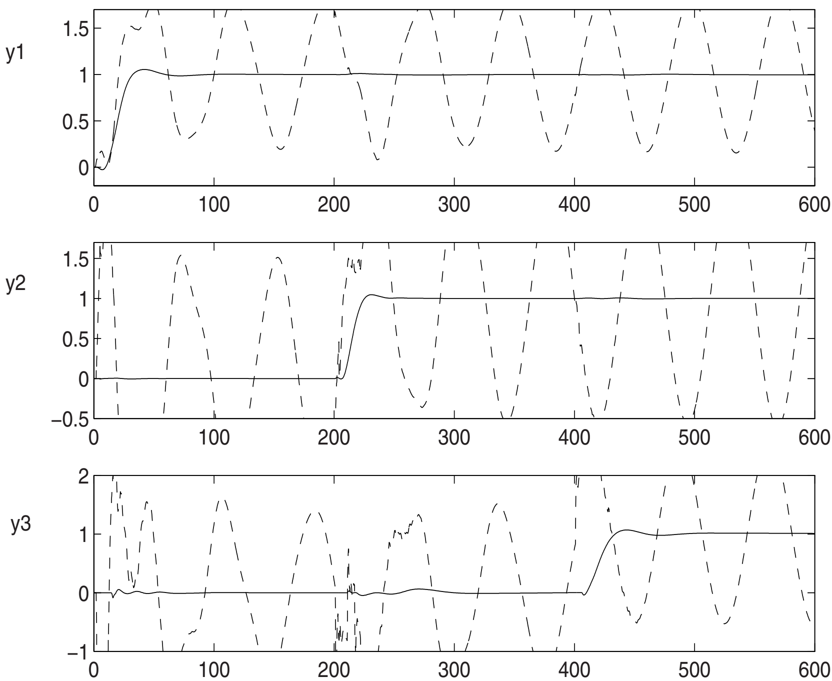


Fig. 6.7. Step Response for 3 × 3 System (—— Proposed; - - - MV-PID)

Generally, an equivalent process in a multivariable process can become highly complicated as the number of inputs and outputs increases. The Nyquist curve can exhibit strange shape especially around the cross-over frequency and thus PID controllers, which have only three adjustable parameters, are difficult to shape it satisfactorily or even fail to stabilize the equivalent processes. Of course, One can always de-tune the PID controllers, or reduce the gains to a sufficient degree to generate a stable closed-loop if the original process is stable. However, we have to have the sluggish response and poor performance, which might not be desirable or acceptable.

With our method, an objective loop transfer function for the second loop for this example is specified as

$$q_{r22}(s) = \frac{0.1790^2 e^{-6.62s}}{s^2 + 2 \times 0.1790 \times 0.707s + 0.1790^2(1 - e^{-6.62s})},$$

and a 6th-order controller, $k_{22}^{HIGH}(s)$, is designed as

$$k_{22}^{HIGH}(s) = \frac{7266s^6 - 3613s^5 + 1754s^4 - 1061s^3 + 626.7s^2 + 60.79s + 1}{59390s^6 + 35180s^5 + 22460s^4 + 5731s^3 + 1458s^2 + 8.781s},$$

such that $\tilde{g}_{22}(s)k_{22}^{HIGH}(s)$ approximates $q_{r22}(s)$. The Nyquist curve of the actual loop transfer function $\tilde{g}_{22}(s)k_{22}^{IDEAL}(s)$ and the ideal one $q_{r22}(s)$ are shown in Figure 6.6(d) with dashed line and solid line, respectively. We can see that $\tilde{g}_{22}(s)k_{22}^{HIGH}(s)$ with the 6th-order $k_{22}^{HIGH}(s)$ yields very close fitting to the ideal loop transfer function in frequency domain while generally PID controller is not capable of that.

To achieve decoupling, according to (6.20), the controller off-diagonal element, say $k_{32}(s)$, should be given by (6.59) with $k_{22}(s) = k_{22}^{HIGH}(s)$. The Nyquist curve of the ideal decoupler $k_{32}^{IDEAL}(s)$ is shown in Figure 6.8(a) with solid line, which is again a function that PID is difficult to approximate. A 6th-order approximation $k_{32}^{HIGH}(s)$ is obtained as

$$k_{32}^{HIGH}(s) = \frac{2110s^5 - 3671s^4 + 631s^3 + 1756s^2 + 68.43s + 1}{-39970s^5 - 19950s^4 - 8978s^3 - 2266s^2 - 17.41s} e^{-12.35s},$$

and its Nyquist curve is shown in Figure 6.8(a) with dashed line, from which we see that it approximates the ideal one very well over the interested frequency range. The other controller elements are obtained as

$$k_{11}^{HIGH}(s) = \frac{-4433s^4 - 1021s^3 + 679.6s^2 + 50.6s + 1}{10580s^4 + 2104s^3 + 1006s^2 + 6.468s},$$

$$k_{12}^{HIGH}(s) = \frac{373s^6 - 262s^5 + 153s^4 + 875s^3 + 260s^2 + 50.8s + 1}{55890s^6 + 19200s^5 + 16960s^4 + 2820s^3 + 943s^2 + 5.8s} e^{-2.03s},$$

$$k_{13}^{HIGH}(s) = \frac{-5094s^4 - 351.7s^3 + 501.5s^2 + 45.67s + 1}{231600s^4 + 47270s^3 + 17520s^2 + 110.4s},$$

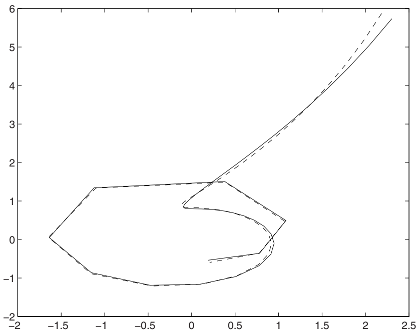
$$k_{21}^{HIGH}(s) = \frac{-4062s^4 - 1142s^3 + 666.5s^2 + 49.89s + 1}{17660s^4 + 3408s^3 + 1675s^2 + 10.8s} e^{-2.86s},$$

$$k_{23}^{HIGH}(s) = \frac{-1820s^3 + 752.3s^2 + 43.82s + 1}{118900s^3 + 27330s^2 + 184.3s} e^{-3.492s},$$

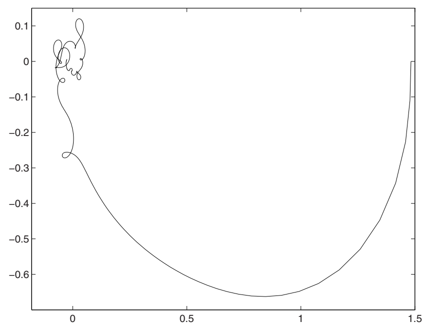
$$k_{31}^{HIGH}(s) = \frac{-19920s^4 + 3943s^3 + 1099s^2 + 60.08s + 1}{-31440s^4 - 6909s^3 - 3508s^2 - 22.31s} e^{-13.94s},$$

$$k_{33}^{HIGH}(s) = \frac{-24880s^3 + 7571s^2 + 434.8s + 1}{82100s^3 + 43020s^2 + 137.4s} e^{-5.085s}.$$

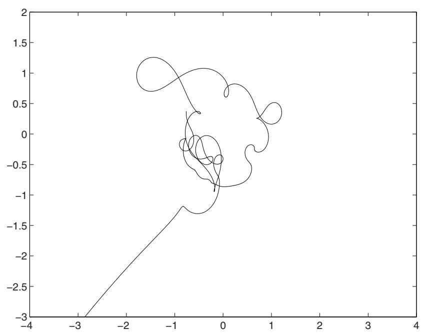
The time domain step response of the overall multivariable feedback system is shown in Figure 6.7 with solid lines, from which we see that the general controller gives very good loop and decoupling performance.



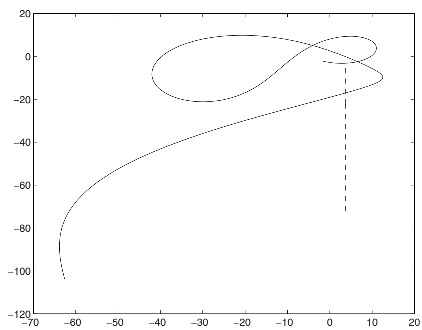
(a) Nyquist curves (—— k_{32}^{IDEAL}; - - - k_{32}^{HIGH})



(b) Nyquist curve of $\tilde{g}_{22}$



(c) Nyquist curve of $\tilde{g}_{22}k_{22}^{PID}$



(d) Nyquist curves (—— k_{32}^{IDEAL}; - - - k_{32}^{PID})

Fig. 6.8. Nyquist Curves

4×4 **Plant.** Now, we look at the original 4 by 4 process $G(s)$. Again, we pick up the second loop for illustration. Under decoupling, the equivalent process that $k_{22}(s)$ faces is

$$\tilde{g}_{22}(s) = \frac{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} & \frac{-0.49e^{-5s}}{22s+1} \\ \frac{-4.17e^{-4s}}{45s+1} & \frac{6.93e^{-1.01s}}{44.6s+1} & \frac{-0.05e^{-5s}}{34.5s+1} & \frac{1.53e^{-2.8s}}{48s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} & \frac{-5.48e^{-0.5s}}{15s+1} \\ \frac{-11.18e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{14.04e^{-0.02s}}{(45s+1)(10s+1)} & \frac{-0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{vmatrix}}{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} & \frac{-0.49e^{-5s}}{22s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} & \frac{-5.48e^{-0.5s}}{15s+1} \\ \frac{-11.18e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{-0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{vmatrix}}.$$

Its Nyquist curve is shown in Figure 6.8(b), and the complexity of the equivalent process is obvious. The SV-PID tuning gives the PID controller as

$$k_{22}^{PID}(s) = 3.5264 + \frac{0.7503}{s} + 4.1438s.$$

The resultant loop Nyquist curves of $\tilde{g}_{22}(s)k_{22}^{PID1}(s)$ is shown in Figure 6.8(c), from which we can expect very poor loop performance or instability.

Apart from the poor loop performance, PID is also found to be inadequate to achieve decoupling for this process. To achieve decoupling, (6.20) applies, say for $k_{32}(s)$, to get

$$k_{32}^{IDEAL}(s) = -\frac{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-6.36e^{-0.2s}}{(31.6s+1)(20s+1)} & \frac{-0.49e^{-5s}}{22s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{5.11e^{-11s}}{13.3s+1} & \frac{-5.48e^{-0.5s}}{15s+1} \\ \frac{-11.18e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{14.04e^{-0.02s}}{(45s+1)(10s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{vmatrix}}{\begin{vmatrix} \frac{4.09e^{-1.3s}}{(33s+1)(8.3s+1)} & \frac{-0.25e^{-0.4s}}{21s+1} & \frac{-0.49e^{-5s}}{22s+1} \\ \frac{-1.73e^{-17s}}{13s+1} & \frac{4.61e^{-1.02s}}{18.5s+1} & \frac{-5.48e^{-0.5s}}{15s+1} \\ \frac{-11.18e^{-2.6s}}{(43s+1)(6.5s+1)} & \frac{-0.1e^{-0.05s}}{(31.6s+1)(5s+1)} & \frac{4.49e^{-0.6s}}{(48s+1)(6.3s+1)} \end{vmatrix}} k_{22}(s), \tag{6.60}$$

with $k_{22}(s) = k_{22}^{PID}(s)$. The MV-PID design method gives the PID form of the off-diagonal controller elements $k_{32}^{PID}(s)$ as

$$k_{32}^{PID}(s) = -3.7326 + \frac{1.9152}{s} - 3.5149s.$$

The Nyquist curve of the ideal decoupler $k_{32}^{IDEAL}(s)$ and its PID approximation k_{32}^{PID} are shown in Figure 6.8(d) in solid line and dash line, respectively, indicating invalidation of the approximation. Actually, the closed-loop system with $K(s)$ being the multivariable PID controller designed with MV-PID is unstable.

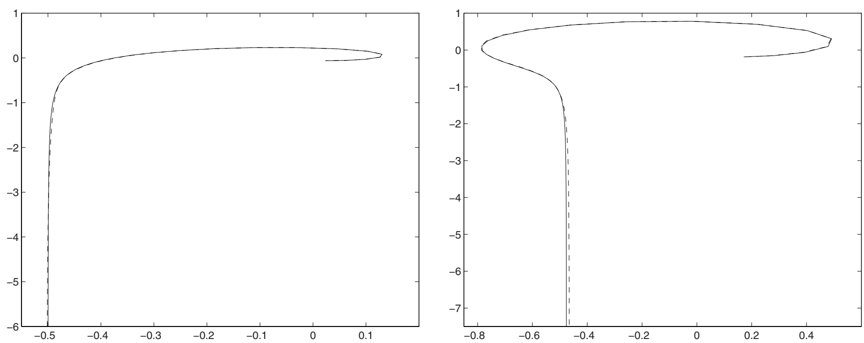
With our design procedure, an objective loop transfer function for the second loop for this plant is specified as

$$q_{r22}(s) = \frac{0.03242^2 e^{-36.54s}}{s^2 + 2 \times 0.03242 \times 0.707s + 0.03242^2(1 - e^{-36.54s})}.$$

and a 4th-order $k_{22}^{HIGH}(s)$:

$$k_{22}^{HIGH}(s) = \frac{-146.2s^4 - 121.3s^3 + 16.19s^2 + s + 0.009641}{15340s^4 + 1519s^3 + 114.8s^2 + 1.144s} e^{-25.35s},$$

is designed such that $\tilde{g}_{22}(s)k_{22}^{HIGH}(s)$ approximates $q_{r22}(s)$. The Nyquist curve of the actual loop transfer function $\tilde{g}_{22}(s)k_{22}^{HIGH}(s)$ and the ideal loop one $q_{r22}(s)$ are shown in Figure 6.9(a) with dash line and solid line, respectively, and they are different. Note that $k_{22}^{HIGH}(s)$ contains a time delay term. This can happen and is necessary in the multivariable case for decoupling, as shown before. If this delay is removed or reduced, or in other word, the time delay in the second decoupled loop transfer function q_{r22} is reduced, then one or some controller off-diagonal element will have time predictions $(e^{Ls}, L > 0)$, which is not realizable.



(a) Nyquist curves (—— q_{r22}; - - - $\tilde{g}_{22}k_{22}^{HIGH}$)

(b) Nyquist curves (—— k_{32}^{IDEAL}; - - - k_{32}^{HIGH})

Fig. 6.9. Nyquist Curves

The decoupling condition (6.20) determines the controller off-diagonal elements. For instance, $k_{32}(s)$ should be given by (6.60) with $k_{22}(s) = k_{22}^{HIGH}(s)$. The Nyquist curve of the ideal decoupler $k_{32}^{IDEAL}(s)$ is shown in Figure 6.9(b) with solid line, which is again a function that PID is difficult to approximate. A 5th-order approximation $k_{32}^{HIGH}(s)$ is obtained with the proposed method as

$$k_{32}(s) = \frac{66110s^5 - 841.8s^4 - 107.1s^3 + 216.1s^2 + s - 0.1403}{-619800s^5 - 102700s^4 - 10680s^3 - 545.2s^2 - 14.23s}e^{-1.683s},$$

and its Nyquist curve is shown in Figure 6.9(b) with dash line, from which we see that it approximates the ideal one very well over the interested frequency range. The remaining controller elements generated by the proposed method are listed in (6.61) to (6.74). The resultant time domain step response is shown in Figure 6.10, from which we see that the general controller gives excellent loop and decoupling performance.

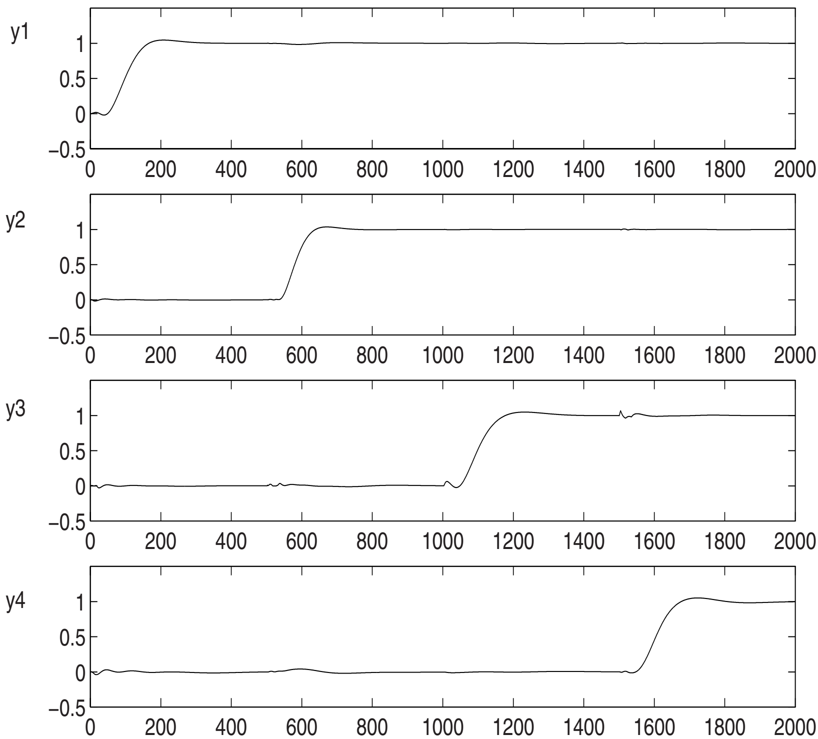


Fig. 6.10. Step response for Example 6.1.3

In view of the above case study, we observe that

(i) There are many complex multivariable processes. The complexity of the processes generally does not come from the complexity of the individual channels, but from the multivariable interactions, which may become very difficult to compensate for when the I/O number of the processes increases. The resultant equivalent processes can be extremely complicated even when all the individual channel dynamics are of simple first or second order plus time delay form.

(ii) For such complex multivariable processes, PID controller is usually inadequate, not only because the equivalent processes is difficult to be controlled by PID, but also the interactions are too difficult to be compensated for by PID.

(iii) Existing non-PID control are usually state-space based. Their specifications are generally quite different from those used in process control. Time delay is frequently encountered in process control while state-space based approach usually cannot handle the time delay easily. The decoupling requirement which is very important for process control is usually not considered explicitly in state-space based approaches. These explain for discrepancy between the many well developed state-space based multivariable designs (H_∞, LQG, ...) and the rare use of them in process control.

(iv) A general, effective and efficient design of non-PID control for complex multivariable process capable of achieving high performance, is thus needed and should have great potential for industrial applications. This chapter has presented such a method.

6.7 More Simulation

In this section, the design method presented in Sections 2-4 is applied to the five industrial processes described in Section 1 to show its effectiveness.

Example 6.1.1 (continued). Consider the Tyreus distillation column in (6.1). Let $\xi_i = 0.707$, i.e., $M_{ri} = 1$, and $\phi_{mi} = \pi/4$, $\epsilon_{di} = 0.2$, $\epsilon_{oi} = 0.2$, $i = 1, 2, 3$, the design procedure is activated.

(i) The application of the model reduction algorithm to $|G|$ and G^{ij} yields the following reduced-order models:

$$|G(s)| = \frac{-3724s^2 + 11260s + 54.77}{462800s^3 + 21080s^2 + 272.8s + 1} e^{-15.3s},$$

$$G^{11}(s) = \frac{-61.22s + 30.13}{490.4s^2 + 44.12s + 1} e^{-0.946s},$$

$$G^{12}(s) = \frac{-8.417s + 1.09}{-169.7s^2 + 13.06s + 1} e^{-11.47s},$$

$$G^{13}(s) = \frac{3.521s - 0.1071}{532.8s^2 + 45.3s + 1} e^{-14.77s},$$

$$G^{21}(s) = \frac{-2024s^2 - 27510s - 16.21}{274000s^3 + 20140s^2 + 449.3s + 1} e^{-14.22s},$$

$$G^{22}(s) = \frac{230.5s + 17.25}{1718s^2 + 92.27s + 1}e^{-5.847s},$$

$$G^{23}(s) = \frac{817.3s^2 - 4474s - 20.48}{504200s^3 + 22520s^2 + 288.9s + 1}e^{-14.81s},$$

$$G^{31}(s) = \frac{12630s^2 + 376.3s + 14.45}{68680s^3 + 8196s^2 + 420.1s + 1}e^{-8.905s},$$

$$G^{32}(s) = \frac{157.6s + 4.849}{2069s^2 + 97.88s + 1}e^{-4.334s},$$

$$G^{33}(s) = \frac{-348s^2 + 266.3s + 0.5485}{121800s^3 + 28450s^2 + 478.9s + 1}e^{-0.8209s}.$$

We got $\mathbf{Z}^+_{|G|} = \{3.027\}$ from $|G(s)|$. Set $i = 1$.

(ii) From the above reduced order models, we obtain $\boldsymbol{\tau}(|G|) = 15.3$, $\boldsymbol{\tau}(G^{11}) = 0.946$, $\boldsymbol{\tau}(G^{12}) = 11.47$, $\boldsymbol{\tau}(G^{13}) = 14.77$; for $z = 3.027$ in $\mathbf{Z}^+_{|G|}$, $\eta_z(|G|) = 1$ and $\eta_z(G^{1j}) = 0$, $j = 1, 2, 3$; $\tau_1 = 0.9463$ and $L_1 = 14.35$. And $n_1(z) = 1$.

(iii) Take $\xi_1 = 0.707$. With $\phi_{m1} = \pi/4$, ω_{g1} is obtained from (6.43) as $\omega_{g1} = 0.0529$, and ω_{n1} is determined from (6.42) as $\omega_{n1} = 0.0794$. h_{r1} is then formed from (6.27) as

$$h_{r1}(s) = \frac{0.0794^2(3.027 - s)e^{-14.35s}}{(s^2 + 2 \times 0.0794 \times 0.707s + 0.0794^2)(3.027 + s)},$$

and q_{r11} is calculated from (6.24).

(iv) Specify $\tilde{D}_{\omega 1}$ with $\underline{\omega}_1 = \frac{\omega_{g1}}{10}$ and $\overline{\omega}_1 = 10\omega_{g1}$. $k_{j1}^{IDEAL}(s)$, $j = 1, 2, 3$, are evaluated from (6.25) and (6.26). $W_{j1}(s)$, $j = 1, \cdots, m$, are found from (6.52). Set $n_{j1} = 2$, $j = 1, 2, 3$ and $\mathbf{J}^* = \{1, 2, 3\}$.

(v) For each $j \in \mathbf{J}^*$, the n_{j1}th-order approximation to $k_{j1}(s)$ is found with the model reduction algorithm as

$$k_{11}(s) = \frac{-689.7s^2 + 57.31s + 1}{710.8s^2 + 64.14s},$$

$$k_{21}(s) = \frac{20330s^2 + 235.2s + 1}{319900s^2 + 1400s}e^{-10.52s},$$

$$k_{31}(s) = \frac{-50380s^2 + 462.1s + 1}{-4709000s^2 - 74420s}e^{-13.82s}.$$

(vi) It is found that (6.49) and (6.50) are not satisfied. ρ_{11}, ρ_{21} and ρ_{31} are then calculated as 6.965, 6.858 and 4.400 from (6.54) and $\mathbf{J}^*$ is thus found as $\{1\}$ from (6.55). Set $n_{11} = 3$, i.e., the order of $\hat{k}_{11}(s)$ increases by 1. Go back to step (v). This procedure repeats until (6.49) and (6.50) are both satisfied and the following k_{11}, k_{21} and k_{31} are obtained:

$$k_{11}(s) = \frac{24.09s^3 - 8.787s^2 + s + 0.01918}{49.83s^3 + 6.991s^2 + 1.214s},$$

$$k_{21}(s) = \frac{-69.82s^3 + 10.68s^2 + s + 0.01277}{1041s^3 + 126.5s^2 + 22.49s} e^{-10.52s},$$

$$k_{31}(s) = \frac{537.2s^3 - 120.6s^2 + s + 0.07195}{-55630s^3 - 7997s^2 - 1186s} e^{-13.82s}.$$

(vii) We repeat the above process for $i = 2$ and find the 2nd-column of the controller as

$$k_{12}(s) = \frac{24260s^6 - 10870s^5 + 2651s^4 - 110s^3 + 55.22s^2 + s + 0.0006456}{-57140s^6 - 17230s^5 - 3013s^4 - 376.2s^3 - 23.97s^2 - 0.05722s} e^{-8.376s},$$

$$k_{22}(s) = \frac{-1386s^6 - 1555s^5 + 190.9s^4 + 105.1s^3 + 21.64s^2 + s + 0.0024}{120100s^6 + 20740s^5 + 8681s^4 + 717.5s^3 + 94.53s^2 + 0.2149s},$$

$$k_{32}(s) = \frac{37.42s^4 - 20.83s^3 + 6.675s^2 + s + 0.05205}{-2512s^4 - 878.2s^3 - 97.77s^2 - 3.945s} e^{-8.962s}.$$

For $i = 3$, we have

$$k_{13}(s) = \frac{-1430s^4 + 938.5s^3 + 29.26s^2 + s + 0.008427}{34560s^4 + 2527s^3 + 485.3s^2 + 1.039s} e^{-8.084s},$$

$$k_{23}(s) = \frac{-49.8s^3 + 6.178s^2 + s + 0.02832}{554.6s^3 + 62.99s^2 + 11.34s} e^{-3.513s},$$

$$k_{33}(s) = \frac{-32.71s^3 - 3.641s^2 + s + 0.03188}{4964s^3 + 411s^2 + 102.3s}.$$

If we restrict the controller to the PID type to approximate the ideal controller, the design procedure produces

$$K^{PID} = \begin{bmatrix} -0.0442 + \frac{0.0152}{s} + 1.2413s & -0.0095 - \frac{0.0475}{s} - 0.7917s & -0.0445 + \frac{0.0022}{s} + 0.2294s \\ 0.0062 + \frac{0.0006}{s} + 0.0715s & 0.0998 + \frac{0.0107}{s} - 0.4601s & 0.0198 + \frac{0.0021}{s} - 0.2532s \\ -0.0000538 + \frac{0.0356}{1000s} + 0.0188s & 0.0342 - \frac{0.0113}{s} - 0.2112s & -0.0073 + \frac{0.0005}{s} + 0.0298s. \end{bmatrix}.$$

The step response of the closed-loop system is shown in Figure 6.11, where the solid line is the response with $K(s)$ and the dash line with $K^{PID}(s)$, from which we can see that the performance of the general controller is very satisfactory while the PID performance is not. The response for the manipulated variables are shown in Figure 6.12, from which we see that the manipulated variables are quite small and smooth for the high-order controller while PID control generates big peaks from the derivative terms.

In order to see the robustness of the proposed method, we increase the static gains in all the elements in the process transfer matrix by 40%, i.e., each $g_{ij}(s)$ is perturbed to $1.4g_{ij}(s)$. The step response under such gain perturbations is shown in Figure 6.13 with dashdot lines, while the nominal

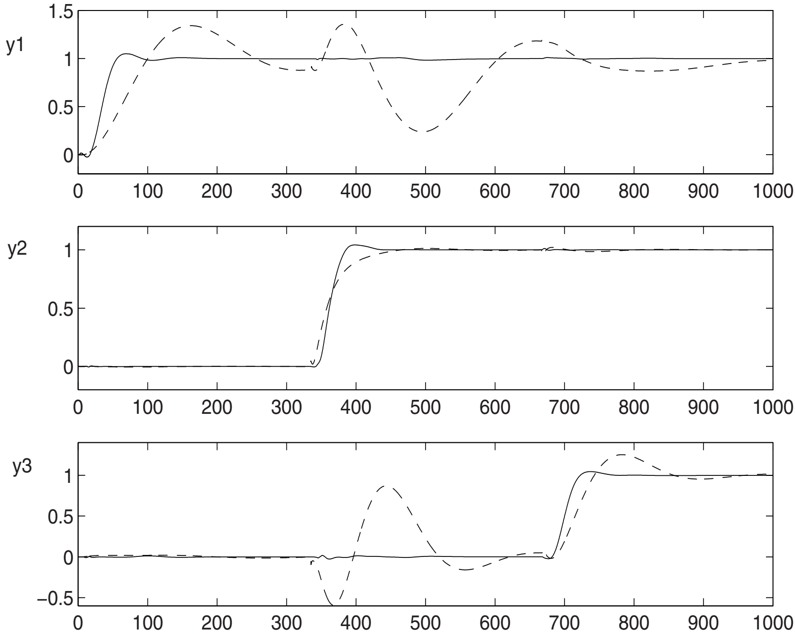


Fig. 6.11. Output Step Responses for Example 6.1.1 (—— High order; - - - PID)

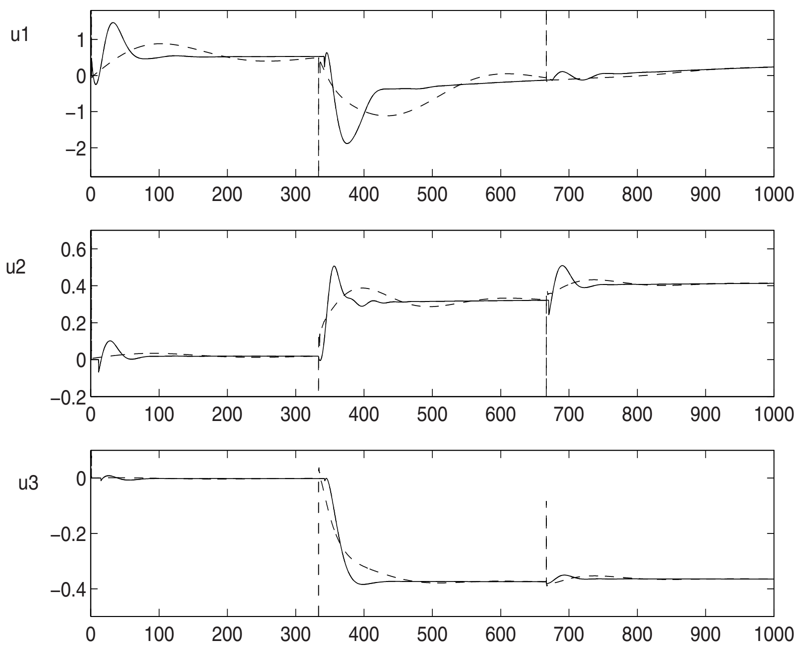


Fig. 6.12. Input Step Responses for Example 6.1.1C (—— High order; - - - PID)

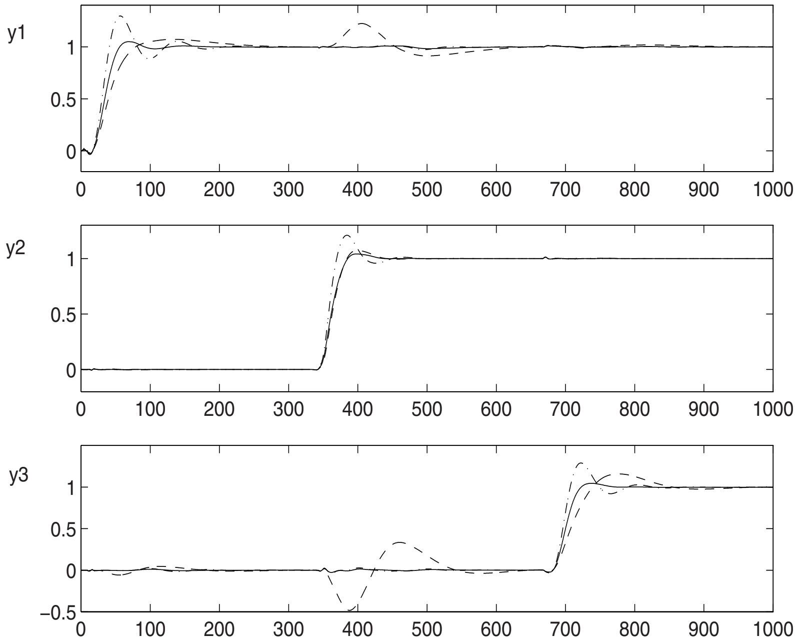


Fig. 6.13. Robustness for Example 6.1.1C (— nominal performance; $- \cdot - \cdot -\cdot$ under gains variation; - - - under time constants variation;)

performance is shown in solid lines. One notes that this gain variation does not affect the decoupling, which is expected. To introduce variation in the system dynamics, we perturb the nominal process by increasing the time constants in all $g_{ij}(s)$ by 40%. The corresponding step response is shown in Figure 6.13 with dashed lines, exhibiting the deterioration of both the loop and decoupling performance. However, such deterioration is reasonable as compared with the "size" of the perturbation.

Example 6.1.2 (continued). Consider the 4 by 4 Doukas process in (6.2). Following our method, the objective closed-loop transfer functions are determined as

$$h_{r1}(s) = \frac{0.04442^2 e^{-17.78s}}{s^2 + 2 \times 0.04442 \times 0.707s + 0.04442^2},$$

$$h_{r2}(s) = \frac{0.04527^2 e^{-17.45s}}{s^2 + 2 \times 0.04527 \times 0.707s + 0.04527^2},$$

$$h_{r3}(s) = \frac{0.04519^2 e^{-17.48s}}{s^2 + 2 \times 0.04519 \times 0.707s + 0.04519^2},$$

$$h_{r4}(s) = \frac{0.09668^2 e^{-8.17s}}{s^2 + 2 \times 0.09668 \times 0.707s + 0.09668^2}.$$

The resulting controller is given by

$$k_{11}(s) = \frac{-41.02s^3 + 5.685s^2 + 1s + 0.02819}{-565.3s^3 - 63.11s^2 - 11.35s} e^{-4.202s},$$

$$k_{12}(s) = \frac{-77.03s^3 + 7.328s^2 + 1s + 0.01236}{-1503s^3 - 147.2s^2 - 25.3s} e^{-9.929s},$$

$$k_{13}(s) = \frac{-658s^6 - 2209s^5 - 70.18s^4 + 173.4s^3 + 17.88s^2 + s + 0.005104}{-138600s^6 - 26980s^5 - 9990s^4 - 906.6s^3 - 97.3s^2 - 0.4822s} e^{-2.959s},$$

$$k_{14}(s) = \frac{1.299s^2 + s - 0.007045}{103.3s^2 + 18.46s},$$

$$k_{21}(s) = \frac{11620s^3 - 1709s^2 + s - 1.847}{89460s^3 + 5427s^2 + 597.3s},$$

$$k_{22}(s) = \frac{34.76s^3 - 9.855s^2 + s + 0.01976}{-63.31s^3 - 8.238s^2 - 1.377s} e^{-0.3349s},$$

$$k_{23}(s) = \frac{12730s^6 - 3075s^5 + 1485s^4 - 361.3s^3 + 42.57s^2 + s + 0.0008608}{84620s^6 + 15440s^5 + 4828s^4 + 484.7s^3 + 29.44s^2 + 0.09661s},$$

$$k_{24}(s) = \frac{4.94s^2 + 1s + 0.01047}{-66.5s^2 - 3.176s} e^{-11.05s},$$

$$k_{31}(s) = \frac{-20.65s^3 + 4.144s^2 + s + 0.028}{-3131s^3 - 305.5s^2 - 68.91s} e^{-6.645s},$$

$$k_{32}(s) = \frac{1597s^3 - 344s^2 + 1s + 0.1779}{191800s^3 + 24400s^2 + 3503s} e^{-15.17s},$$

$$k_{33}(s) = \frac{-247.2s^5 + 49.82s^4 + 2.14s^3 + 10.2s^2 + s + 0.1042}{20850s^5 + 4104s^4 + 1361s^3 + 109.6s^2 + 10.4s} e^{-12.71s},$$

$$k_{34}(s) = \frac{-0.5092s^2 + s + 0.02297}{-772.8s^2 - 16.49s} e^{-4.705s},$$

$$k_{41}(s) = \frac{-14.91s^3 - 2.164s^2 + s + 0.03425}{4279s^3 + 453.7s^2 + 90.39s} e^{-3.75s},$$

$$k_{42}(s) = \frac{12.65s^3 - 11.83s^2 + s + 0.001142}{24270s^3 + 1367s^2 + 388.5s},$$

$$k_{43}(s) = \frac{21.11s^4 + 17.06s^3 + 5.502s^2 + s + 0.1161}{2753s^4 + 446.6s^3 + 150s^2 + 10.8s} e^{-14.14s},$$

$$k_{44}(s) = \frac{-1.499s^2 + s + 0.08238}{165.2s^2 + 13.17s} e^{-2.63s}.$$

The step response is shown in Figure 6.14 with superior performance.

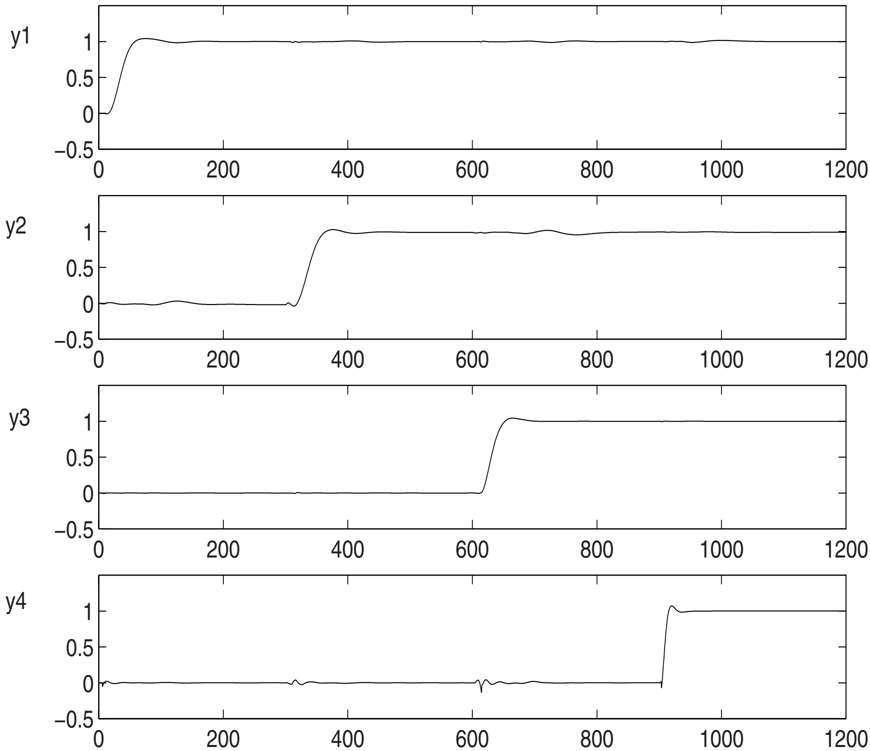


Fig. 6.14. Step Responses for Example 6.1.2C

Example 6.1.3C (continued). Consider the 4 by 4 Alatiqi process in (6.3). By the design procedure, the objective closed-loop transfer functions are

$$h_{r1}(s) = \frac{0.02708^2 e^{-43.75s}}{s^2 + 2 \times 0.02708 \times 0.707s + 0.02708^2},$$

$$h_{r2}(s) = \frac{0.03242^2 e^{-36.54s}}{s^2 + 2 \times 0.03242 \times 0.707s + 0.03242^2},$$

$$h_{r3}(s) = \frac{0.02667^2 e^{-44.42s}}{s^2 + 2 \times 0.02667 \times 0.707s + 0.02667^2},$$

$$h_{r4}(s) = \frac{0.02519^2 e^{-47.04s}}{s^2 + 2 \times 0.02519 \times 0.707s + 0.02519^2}.$$

And the controller is

$$k_{11}(s) = \frac{1526s^4 - 171s^3 + 14.62s^2 + s + 0.01487}{19500s^4 + 1616s^3 + 122.2s^2 + 1.879s}, \tag{6.61}$$

$$k_{12}(s) = \frac{-18920s^6 - 17290s^5 + 3341s^4 - 453.8s^3 - 1.136s^2 + s + 0.01}{791900s^6 + 247400s^5 + 33380s^4 + 2413s^3 + 91.32s^2 + 0.82s}, \tag{6.62}$$

$$k_{13}(s) = \frac{-12.21s^2 + s + 0.02426}{1982s^2 + 50.13s}, \tag{6.63}$$

$$k_{14}(s) = \frac{-123.4s^3 + 3.941s^2 + s - 0.01903}{2471s^3 + 114.7s^2 + 7.317s}e^{-1.958s}, \tag{6.64}$$

$$k_{21}(s) = \frac{1104s^4 - 107.1s^3 + 10.31s^2 + s + 0.02}{145000s^4 + 11950s^3 + 958.6s^2 + 13.55s}e^{-0.8466s}, \tag{6.65}$$

$$k_{23}(s) = \frac{-13.97s^2 + s + 0.04257}{20520s^2 + 392s}e^{-3.335s}, \tag{6.66}$$

$$k_{24}(s) = \frac{-553.9s^3 - 31.68s^2 + s + 0.1363}{-24360s^3 - 1085s^2 - 70.19s}e^{-24.2s}, \tag{6.67}$$

$$k_{31}(s) = \frac{21.66s^3 - 7.492s^2 + s + 0.03177}{395.4s^3 + 20.1s^2 + 1.61s}e^{-1.013s}, \tag{6.68}$$

$$k_{33}(s) = \frac{296.4s^3 - 23.09s^2 + s + 0.1881}{13050s^3 + 864.5s^2 + 54.53s}e^{-3.887s}, \tag{6.69}$$

$$k_{34}(s) = \frac{-122.6s^3 + 0.9794s^2 + s + 0.008617}{962.4s^3 + 40.13s^2 + 2.606s}e^{-3.358s}, \tag{6.70}$$

$$k_{41}(s) = \frac{19.56s^3 - 6.984s^2 + s + 0.03433}{542s^3 + 28.34s^2 + 2.217s}e^{-2.077s}, \tag{6.71}$$

$$k_{42}(s) = \frac{19180s^5 + 12170s^4 - 2071s^3 + 308.3s^2 + s - 0.2772}{-478900s^5 - 115500s^4 - 13660s^3 - 835.9s^2 - 24.39s}e^{-1.541s}, \tag{6.72}$$

$$k_{43}(s) = \frac{-20.23s^2 + s + 0.02799}{643.4s^2 + 29.45s}e^{-2.502s}, \tag{6.73}$$

$$k_{44}(s) = \frac{-76.07s^3 - 4.884s^2 + s + 0.00616}{1235s^3 + 51.16s^2 + 3.434s}. \tag{6.74}$$

The step response is shown in Figure 6.10.

Example 6.1.4 (continued). Consider the Ammonia process model (Yokogawa Ltd.) in (6.4). Following the procedure, we obtain

$$h_{r1}(s) = \frac{0.2057^2 e^{-5.760s}}{s^2 + 2 \times 0.2057 \times 0.707s + 0.2057^2},$$

$$h_{r2}(s) = \frac{0.05716^2 e^{-20.73s}}{s^2 + 2 \times 0.05716 \times 0.707s + 0.05716^2},$$

$$h_{r3}(s) = \frac{0.1353^2 e^{-8.760s}}{s^2 + 2 \times 0.1353 \times 0.707s + 0.1353^2};$$

and

$$k_{11}(s) = \frac{-0.6015s^2 + s + 0.09952}{-2.783s^2 - 1.409s} e^{-2s},$$

$$k_{12}(s) = \frac{3.685s^3 + 4.08s^2 + s + 0.08144}{5.537s^3 + 0.793s^2 + 0.1084s} e^{-1.861s},$$

$$k_{13}(s) = \frac{-0.1618s^3 + 1.721s^2 + s + 0.1044}{2.406s^3 + 0.8755s^2 + 0.2685s} e^{-2s},$$

$$k_{21}(s) = \frac{-0.0771s^3 + 0.5865s^2 + s + 0.1633}{-0.09085s^3 - 0.06274s^2 - 0.02637s} e^{-3.513s},$$

$$k_{22}(s) = \frac{-0.5531s^3 + 2.607s^2 + s + 0.1366}{-0.09555s^3 - 0.01373s^2 - 0.00194s} e^{-3.494s},$$

$$k_{23}(s) = \frac{4.332s^3 + 2.386s^2 + s - 0.1834}{0.1233s^3 + 0.04899s^2 + 0.01975s} e^{-3.936s},$$

$$k_{31}(s) = \frac{-0.5639s^2 + s + 0.05756}{-1.875s^2 - 1.051s},$$

$$k_{32}(s) = \frac{-4.014s^4 + 7.842s^3 + 4.624s^2 + s + 0.0617}{-7.849s^4 - 5.138s^3 - 0.7457s^2 - 0.08274s},$$

$$k_{33}(s) = \frac{-0.306s^3 + 2.29s^2 + s + 0.07093}{2.016s^3 + 0.6972s^2 + 0.2153s}.$$

The step response is shown in Figure 6.15.

Example 6.1.5 (continued). Consider the depropanizer process in (6.5). It follows from the design procedure that

$$h_{r1}(s) = \frac{0.006998^2(0.01495 - s)e^{-40.72s}}{(s^2 + 2 \times 0.006998 \times 0.707s + 0.006998^2)(0.01495 + s)},$$

$$h_{r2}(s) = \frac{0.006759^2(0.01495 - s)e^{-44.38s}}{(s^2 + 2 \times 0.006759 \times 0.707s + 0.006759^2)(0.01495 + s)},$$

$$h_{r3}(s) = \frac{0.007357^2(0.01495 - s)e^{-38.73s}}{(s^2 + 2 \times 0.007357 \times 0.707s + 0.007357^2)(0.01495 + s)};$$

and

$$k_{11}(s) = \frac{-53.59s^3 + 24.86s^2 + s + 0.005836}{4217s^3 + 85.86s^2 + 1.091s},$$

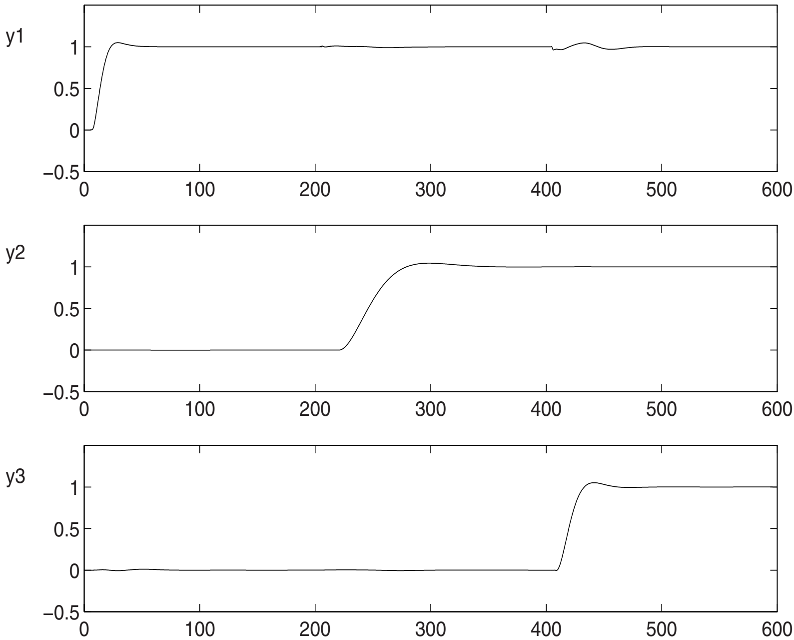


Fig. 6.15. Step Responses for Example 6.1.4C

$$k_{12}(s) = \frac{-63.61s^3 + 8.084s^2 + s + 0.008384}{7184s^3 + 151.5s^2 + 1.743s} e^{-27.47s},$$

$$k_{13}(s) = \frac{122.6s^3 - 6.025s^2 + s + 0.01103}{1.689 \times 10^{+004}s^3 + 333.5s^2 + 3.802s} e^{-33.82s},$$

$$k_{21}(s) = \frac{-18.72s^3 + 6.678s^2 + s + 0.00809}{2.37 \times 10^{+004}s^3 + 514.8s^2 + 6.357s} e^{-17.67s},$$

$$k_{22}(s) = \frac{26.31s^3 + 28.53s^2 + s + 0.006068}{7.887 \times 10^{+004}s^3 + 1640s^2 + 20.17s} e^{-11.81s},$$

$$k_{23}(s) = \frac{227.8s^3 + 4.625s^2 + s + 0.01424}{1.849 \times 10^{+005}s^3 + 3658s^2 + 42.94s} e^{-26.42s},$$

$$k_{31}(s) = \frac{27.48s^3 - 2.061s^2 + s + 0.008284}{1532s^3 + 32.7s^2 + 0.4058s} e^{-0.6552s},$$

$$k_{32}(s) = \frac{59.87s^3 - 1.332s^2 + s + 0.009638}{4699s^3 + 94.31s^2 + 1.06s} e^{-0s},$$

$$k_{33}(s) = \frac{101.5s^3 - 9.961s^2 + s + 0.002642}{7283s^3 + 154.2s^2 + 1.597s},$$

The step response is shown in Figure 6.16.

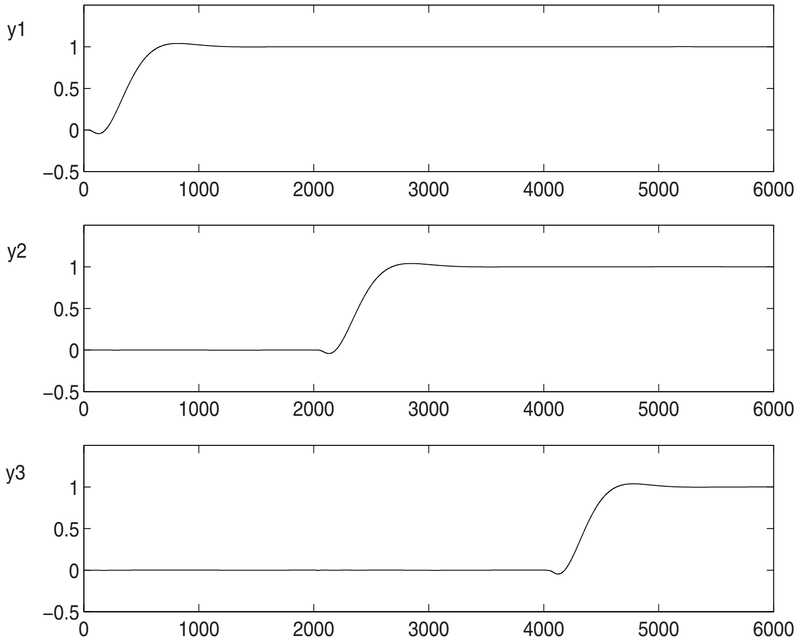


Fig. 6.16. Step Responses for Example 6.1.5C

6.8 Conclusions

In this chapter, a systematic design method is developed to achieve fastest loop speed with acceptable overshoot and minimum loop interactions with a controller of least complexity for complex multivariable processes. The fundamental relations for decoupling a multivariable process with time delay has been established. All the unavoidable time delays and non-minimum phase zeros that are inherent in a decoupled feedback loop are characterized. The objective loop transfer functions are then suitably specified to achieve fastest possible response taking into account the performance limitation imposed by the unavoidable non-minimum phase zeros and time delays. The ideal controller is obtained and it is generally a complicated irrational transfer matrix, for which model reduction is applied to fit it as a much simpler transfer matrix with its elements in rational transfer function plus delay form. To obtain a controller of least complexity and ensure the actual control performance, the orders of the rational parts of the controller elements are determined as the lowest such that a set of performance specifications on loop interaction and loop performance are satisfied. Stability and robustness are analyzed and substantial simulation shows that almost perfect control can be achieved.

6.9 Notes and References

For plants with time delay, frequency domain methodologies are popular, such as the inverse Nyquist array (Rosenbrock, 1974) and characteristic locus method (Macfarlane, 1980). Wang *et al.* (1997*a*) derived the decoupling conditions for delay systems. The approach presented in this chapter is based on Zhang (1999). In many cases, one replaces time delay by a rational function, say via Pade approximation, and then adopts delay-free designs (Morari and Zafiriou, 1989). Such approximations will deteriorate when time delay increases, and time delay compensation will be necessary to get high performance and a topic of the next chapter.

7. Delay Systems

Time delay is a very common phenomenon encountered in process and chemical industries. This is further complicated by multivariable nature of most plants in operations. Input-output loops in a multivariable plant usually have different time delays, and for a particular loop its output could be affected by all the inputs through likely different time delays. As a result, such a plant can be represented by a multivariable transfer function matrix having multiple (or different) time delays around a operating point. The presence of time delay in a feedback control loop could be a serious obstacle to good process operation. It prevents high gain of a feedback controller from being used, leading to sluggish system response. Such performance limitations can be released only by time delay compensation.

For single variable processes with delay, The Smith control scheme can remove the delay from the closed-loop characteristic equation and thus eases feedback control design and improves set-point response greatly. An alternative scheme is the internal model control (IMC) structure. They can be converted to each other. But the theory and design of the IMC is better developed than the Smith one. This chapter is to extend both the Smith and IMC schemes to the multivariable case. An essential difference between the SISO and MIMO with delay compensation is that for the SISO case the output behavior is completely determined by the delay-free design shifted by the plant time delay, whereas one can hardly tell the MIMO output performance from the delay-free design because the delays in the process could mix up the outputs of delay-free part to generate messy actual output response. Decoupling between the loops, in addition to serving its own purpose, can rectify such a problem. Therefore, we will consider multivariable delay compensation problem with decoupling in this chapter, and Sections 1 and 2 will address the IMC and Smith schemes, respectively.

7.1 The IMC Scheme

The class of systems to be considered here is square, stable, nonsingular MIMO linear ones. In process industry, most of the processes are open-loop stable. A process usually has unequal numbers of inputs and outputs, but normally a square subsystem will be selected from it for control purpose. A

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 237-266, 2003.

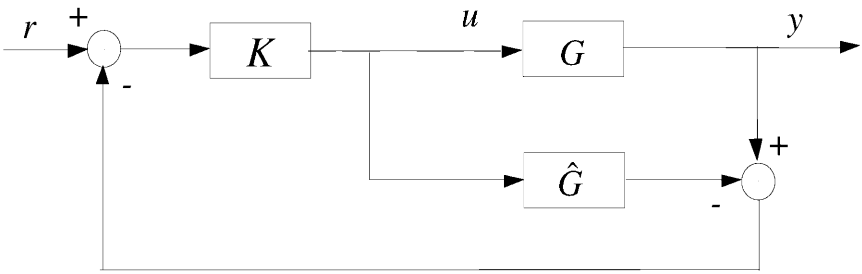


Fig. 7.1. Internal model control

singular process essentially cannot make independent control of each output and should not be used.

The IMC control system is depicted in Figure 7.1, where G and $\hat{G}$ represent the transfer function matrices of the plant and its model, respectively, and K the controller. $\hat{G}$ is assumed to be identical with G except our discussion on robust stability:

$$\hat{G}(s) = G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & \ddots & \vdots \\ g_{m1}(s) & \cdots & g_{mm}(s) \end{bmatrix}, \tag{7.1}$$

where

$$g_{ij}(s) = g_{ij0}(s)e^{-L_{ij}s},$$

and $g_{ij0}(s)$ are strictly proper, stable, and scalar rational functions and L_{ij} are non-negative constants. It is also assumed that a proper input-output pairing has been made for $G(s)$ such that none of the m principal minors of $G(s)$ (the ith principle minor is the determinant of a $(m-1)\times(m-1)$ matrix obtained by deleting the ith row and ith column of $G(s)$) is zero. Our task in this section is to characterize all the realizable controllers K and the resultant closed-loop transfer functions such that the IMC system is internally stable and decoupled.

The closed-loop transfer matrix H between y and r can be derived from Figure 7.1 as

$$H = GK[I + (G - \hat{G})K]^{-1},$$

which becomes

$$H = GK, \quad \text{if} \quad \hat{G} = G.$$

Thus, the closed-loop is decoupled if and only if GK is decoupled (diagonal and nonsingular) and the IMC system is internally stable if and only if

K is stable. Therefore, the problem becomes to characterize all stable and realizable controllers K and the resulting H such that $GK = H$ is decoupled.

The first question to ask is whether or not a solution exists. For a stable and nonsingular G, one may write it as

$$G(s) = \frac{N(s)}{d(s)},$$

where $d(s)$ is a least common denominator polynomial of $G(s)$. Now choose $K(s)$ as

$$K(s) = \frac{\mathrm{adj}\, N(s)}{\sigma(s)},$$

where $\mathrm{adj}\, N(s)$ is the adjoint matrix of $N(s)$ and $\sigma(s)$ is a stable polynomial and its degree is high enough to make the rational part of each k_{ij} proper. Then, this $K(s)$ is stable and realizable and it decouples $G(s)$ since

$$G(s)K(s) = \frac{|N(s)|}{d(s)\sigma(s)} I_m,$$

where $|\cdot|$ stands for the determinant of a matrix and I_m is the $m \times m$ identity matrix. Thus $K(s)$ is a solution. Nonsingularity of $G(s)$ is obviously also necessary in order for GK to be decoupled.

Theorem 7.1.1. *For a stable, square and multi-delay process $G(s)$, the decoupling problem with stability via the IMC is solvable if and only if $G(s)$ is non-singular.*

Let $\mathbf{m} = \{1, 2, \cdots, m\}$ and G^{ij} be the cofactor corresponding to g_{ij} in G. It has been shown in Chapter 6 that in order for GK to be decoupled, we have

$$GK = \mathrm{diag}\,\{\tilde{g}_{ii}k_{ii},\ i = 1, 2, \cdots, m\}, \tag{7.2}$$

where

$$\tilde{g}_{ii} = \frac{|G|}{G^{ii}}, \qquad \forall\, i \in \mathbf{m}; \tag{7.3}$$

and

$$k_{ji} = \psi_{ji} k_{ii}, \qquad \forall\, i, j \in \mathbf{m}, j \neq i, \tag{7.4}$$

where

$$\psi_{ji} = \frac{G^{ij}}{G^{ii}}, \qquad \forall\, i, j \in \mathbf{m}, j \neq i. \tag{7.5}$$

One notes that for a given i, the resulting diagonal element of GK is independent of the controller off-diagonal elements but contains only the controller diagonal element k_{ii}.

To demonstrate how to find $\tilde{g}_{ii}$ and ψ_{ji}, take the following as an example:

$$G(s) = \begin{bmatrix} \frac{1}{s+2}e^{-2s} & \frac{-1}{s+2}e^{-6s} \\ \frac{s-0.5}{(s+2)^2}e^{-3s} & \frac{(s-0.5)^2}{2(s+2)^3}e^{-8s} \end{bmatrix}. \tag{7.6}$$

Simple calculations give

$$|G| = \frac{2(s-0.5)(s+2)e^{-9s} + (s-0.5)^2 e^{-10s}}{2(s+2)^4},$$

$$G^{11} = \frac{(s-0.5)^2 e^{-8s}}{2(s+2)^3}, \qquad G^{21} = \frac{e^{-6s}}{s+2},$$

$$G^{22} = \frac{e^{-2s}}{s+2}, \qquad G^{12} = -\frac{(s-0.5)e^{-3s}}{(s+2)^2}.$$

It follows from (7.3) that the decoupled loops have their equivalent processes as

$$\tilde{g}_{11} = \frac{2(s+2)e^{-s} + (s-0.5)e^{-2s}}{(s-0.5)(s+2)},$$

$$\tilde{g}_{22} = \frac{2(s-0.5)(s+2)e^{-7s} + (s-0.5)^2 e^{-8s}}{2(s+2)^3},$$

respectively. By (7.5), ψ_{21} and ψ_{12} are found as

$$\psi_{21} = -\frac{2(s+2)}{s-0.5}e^{5s}, \qquad \psi_{12} = e^{-4s}.$$

At the first glance, each diagonal controller k_{ii} may be designed for $\tilde{g}_{ii}$ according to (7.2) and the off-diagonal controllers can then be determined from (7.4). Recall that for SISO IMC design, a scalar process $g(s)$ is first factored into $g(s) = g^+(s)g^-(s)$, where $g^+(s)$ contains all the time delay and non-minimum phase zeros. The controller $k(s)$ is then determined from $g(s)k(s) = g^+(s)f(s)$, where $f(s)$ is the IMC filter. This technique could seemingly be applied to the decoupled equivalent processes $\tilde{g}_{ii}$ to design k_{ii} directly, but it is actually not. The reasons are that in general $\tilde{g}_{ii}$ are not in the usual format of a rational function plus time delay, and more essentially a realizable k_{ii} derived from (7.2) based on $\tilde{g}_{ii}$ only may result in unrealizable or unstable off-diagonal controllers k_{ji}, $j \neq i$, In other words, realizable k_{ji}, $j \neq i$, may necessarily impose additional time delay and non-minimum phase zeros to k_{ii}.

7.1.1 Analysis

In what follows, we will develop the characterizations of time delays and non-minimum phase zeros on the decoupling controller and the resulting decoupled loops. For a nonsingular delay process $G(s)$ in (7.1), a general expression for $\tilde{g}_{ii}$ in (7.3) will be

$$\phi(s) = \frac{\sum_{k=0}^{M} n_k(s)e^{-\alpha_k s}}{d_0(s) + \sum_{l=1}^{N} d_l(s)e^{-\beta_l s}}, \tag{7.7}$$

where $n_k(s)$ and $d_l(s)$ are all non-zero scalar polynomials of s, $\alpha_0 < \alpha_1 < \cdots < \alpha_M$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_N$. Define the time delay for non-zero $\phi(s)$ in (7.7) as

$$\boldsymbol{\tau}(\phi(s)) = \alpha_0.$$

It is easy to verify $\boldsymbol{\tau}(\phi_1\phi_2) = \boldsymbol{\tau}(\phi_1) + \boldsymbol{\tau}(\phi_2)$ and $\boldsymbol{\tau}(\phi^{-1}(s)) = -\boldsymbol{\tau}(\phi(s))$ for any non-zero $\phi_1(s)$, $\phi_2(s)$ and $\phi(s)$. If $\boldsymbol{\tau}(\phi(s)) \geq 0$, it measures the time required for $\phi(s)$ to have a non-zero output in response to a step input. If $\boldsymbol{\tau}(\phi(s)) < 0$ then the system output will depend on the future values of input. It is obvious that for any realizable and non-zero $\phi(s)$, $\boldsymbol{\tau}(\phi(s))$ can not be negative. Therefore, a realizable K requires

$$\boldsymbol{\tau}(k_{ji}) \geq 0, \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i, \tag{7.8}$$

where $\mathbf{J}_i \triangleq \{j \in \mathbf{m} \mid G^{ij} \neq 0\}$. It follows from (7.4) that

$$\boldsymbol{\tau}(k_{ji}) = \boldsymbol{\tau}(\frac{G^{ij}}{G^{ii}}k_{ii}) = \boldsymbol{\tau}\left(G^{ij}\right) - \boldsymbol{\tau}(G^{ii}) + \boldsymbol{\tau}(k_{ii}), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i.$$

Thus, (7.8) is equivalent to

$$\boldsymbol{\tau}(k_{ii}) \geq \boldsymbol{\tau}(G^{ii}) - \boldsymbol{\tau}(G^{ij}), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i;$$

or

$$\boldsymbol{\tau}(k_{ii}) \geq \boldsymbol{\tau}(G^{ii}) - \tau_i, \qquad \forall\, i \in \mathbf{m}, \tag{7.9}$$

where

$$\tau_i \triangleq \min_{j\in\mathbf{J}_i} \boldsymbol{\tau}(G^{ij}). \tag{7.10}$$

Equation (7.9) is a characterization of the controller diagonal elements in terms of their time delays, which indicates the minimum amount of time delay that the *i-th* diagonal controller elements must contain. Consequently, the resulting *i-th* diagonal elements of $H = GK$, $h_{ii} = \tilde{g}_{ii}k_{ii}$, meet

$$\boldsymbol{\tau}(h_{ii}) = \boldsymbol{\tau}(\tilde{g}_{ii}) + \boldsymbol{\tau}(k_{ii}) \geq \boldsymbol{\tau}(\tilde{g}_{ii}) + \boldsymbol{\tau}(G^{ii}) - \tau_i, \qquad \forall\, i \in \mathbf{m},$$

which, by

$$\boldsymbol{\tau}(\tilde{g}_{ii}) = \boldsymbol{\tau}\left(\frac{|G|}{G^{ii}}\right) = \boldsymbol{\tau}(|G|) - \boldsymbol{\tau}(G^{ii}),$$

becomes

$$\boldsymbol{\tau}(h_{ii}) \geq \boldsymbol{\tau}(|G|) - \tau_i, \qquad \forall i \in \mathbf{m}. \tag{7.11}$$

Equation (7.11) is a characterization of the decoupled $i-th$ loop transfer function in terms of their time delays, which indicates the minimum amount of time delay that the $i-th$ decoupled loop transfer function must contain. One concludes from the above development that for a stable and realizable IMC controller K which decouples G, its diagonal elements are characterized by (7.9) on their time delays and they will uniquely determine the off-diagonal elements by (7.4). Equation (7.11) characterizes the time delays of the resultant loops.

For illustration, consider again the example in (7.6). It follows from the definition that the time delays for respective functions are $\boldsymbol{\tau}(|G|) = 9$, $\boldsymbol{\tau}(G^{11}) = 8$, $\boldsymbol{\tau}(G^{12}) = 3$, $\boldsymbol{\tau}(G^{21}) = 6$ and $\boldsymbol{\tau}(G^{22}) = 2$. The values for τ_1 and τ_2 can be calculated from (7.10) as

$$\tau_1 = \min\left\{\boldsymbol{\tau}(G^{11}), \boldsymbol{\tau}(G^{12})\right\} = \min\{8, 3\} = 3,$$

$$\tau_2 = \min\left\{\boldsymbol{\tau}(G^{21}), \boldsymbol{\tau}(G^{22})\right\} = \min\{6, 2\} = 2.$$

It follows from (7.9) that k_{11} and k_{22} are characterized on their time delays by

$$\boldsymbol{\tau}(k_{11}) \geq \boldsymbol{\tau}(G^{11}) - \tau_1 = 8 - 3 = 5,$$

$$\boldsymbol{\tau}(k_{22}) \geq \boldsymbol{\tau}(G^{22}) - \tau_2 = 2 - 2 = 0.$$

Also, h_{11} and h_{22} must meet

$$\boldsymbol{\tau}(h_{11}) \geq \boldsymbol{\tau}(|G|) - \tau_1 = 9 - 3 = 6,$$

$$\boldsymbol{\tau}(h_{22}) \geq \boldsymbol{\tau}(|G|) - \tau_2 = 9 - 2 = 7.$$

Consider now the performance limitation due to the non-minimum phase zeros. Denote by $\mathbb{C}^+$ the closed right half of the complex plane (RHP). For a non-zero transfer function $\phi(s)$, let $\mathbf{Z}_\phi^+$ be the set of all the RHP zeros of $\phi(s)$, i.e., $\mathbf{Z}_\phi^+ = \{z \in \mathbb{C}^+ \mid \phi(z) = 0\}$. Let $\eta_z(\phi)$ be an integer ν such that $\lim_{s\to z} \phi(s)/(s-z)^\nu$ exists and is non-zero. Thus, $\phi(s)$ has $\eta_z(\phi)$ zeros at $s = z$ if $\eta_z(\phi) > 0$, or $-\eta_z(\phi)$ poles if $\eta_z(\phi) < 0$, or neither poles nor zeros if $\eta_z(\phi) = 0$. It is easy to verify that $\eta_z(\phi_1\phi_2) = \eta_z(\phi_1) + \eta_z(\phi_2)$ and

$\eta_z(\phi^{-1}) = -\eta_z(\phi)$ for any non-zero transfer function $\phi_1(s)$, $\phi_2(s)$ and $\phi(s)$. Obviously, a non-zero transfer function $\phi(s)$ is stable if and only if $\eta_z(\phi) \geq 0$, $\forall\, z \in \mathbb{C}^+$.

A stable K thus requires

$$\eta_z(k_{ji}) \geq 0, \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,\ z \in \mathbb{C}^+. \tag{7.12}$$

It follows from (7.4) that

$$\eta_z(k_{ji}) = \eta_z\left(\frac{G^{ij}}{G^{ii}}k_{ii}\right) = \eta_z(G^{ij}) - \eta_z(G^{ii}) + \eta_z(k_{ii}),\ \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,\ z \in \mathbb{C}^+.$$

Thus (7.12) is equivalent to

$$\eta_z(k_{ii}) \geq \eta_z(G^{ii}) - \eta_z(G^{ij}), \qquad \forall\, i \in \mathbf{m},\ j \in \mathbf{J}_i,\ z \in \mathbb{C}^+,$$

or

$$\eta_z(k_{ii}) \geq \eta_z(G^{ii}) - \eta_i(z), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+, \tag{7.13}$$

where

$$\eta_i(z) \triangleq \min_{j \in \mathbf{J}_i} \eta_z(G^{ij}). \tag{7.14}$$

Since $G(s)$ is stable, so are G^{ij} . One sees that for $\forall\, i \in \mathbf{m}$ and $\forall\, z \in \mathbb{C}^+$, $\eta_i(z) \geq 0$. We have, for $\forall\, i \in \mathbf{m}$ and $\forall z\ \in \mathbb{C}^+$,

$$\eta_z(G^{ii}) - \eta_i(z) \leq \eta_z(G^{ii}),$$

and

$$\eta_z(G^{ii}) - \eta_i(z) = \eta_z(G^{ii}) - \min_{j \in \mathbf{J}_i} \eta_z(G^{ij}) \geq \eta_z(G^{ii}) - \eta_z(G^{ij})|_{j=i} = 0.$$

Thus, $\eta_z(G^{ii}) - \eta_i(z)$ is bounded by

$$0 \leq \eta_z(G^{ii}) - \eta_i(z) \leq \eta_z(G^{ii}), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+. \tag{7.15}$$

Equation (7.15) also implies that k_{ii} need not have any non-minimum phase zeros except at $z \in \mathbf{Z}^+_{G^{ii}}$. Therefore, by (7.13), k_{ii} is characterized on its non-minimum phase zeros by

$$\eta_z(k_{ii}) \geq \eta_z(G^{ii}) - \eta_i(z), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbf{Z}^+_{G^{ii}}, \tag{7.16}$$

with the right side bounded by (7.15).

Consequently, the resulting *i-th* diagonal elements of $H = GK$, $h_{ii} = \tilde{g}_{ii}k_{ii}$, meet

$$\eta_z(h_{ii}) = \eta_z(\tilde{g}_{ii}k_{ii}) = \eta_z(\tilde{g}_{ii}) + \eta_z(k_{ii}) \geq \eta_z(\tilde{g}_{ii}) + \eta_z(G^{ii}) - \eta_i(z)\ \forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+,$$

which, by

$$\eta_z(\tilde{g}_{ii}) = \eta_z\left(\frac{|G|}{G^{ii}}\right) = \eta_z(|G|) - \eta_z(G^{ii}),$$

becomes

$$\eta_z(h_{ii}) \geq \eta_z(|G|) - \eta_i(z), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+. \tag{7.17}$$

It is readily seen that for $\forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+$,

$$\eta_z(|G|) - \eta_i(z) \leq \eta_z(|G|),$$

and

$$\begin{aligned}\eta_z(|G|) - \eta_i(z) &= \eta_z\left(\frac{|G|}{(s-z)^{\eta_i(z)}}\right) = \eta_z\left(\frac{1}{(s-z)^{\eta_i(z)}}\sum_{j=1}^{m} g_{ij}G^{ij}\right)\\ &= \eta_z\left(\sum_{j=1}^{m} g_{ij}\frac{G^{ij}}{(s-z)^{\eta_i(z)}}\right) \geq 0.\end{aligned}$$

Thus, $\eta_z(|G|) - \eta_i(z)$ is bounded by

$$0 \leq \eta_z(|G|) - \eta_i(z) \leq \eta_z(|G|), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbb{C}^+. \tag{7.18}$$

Equation (7.18) also indicates that the ith closed-loop transfer function h_{ii} need not have any non-minimum phase zeros except at $z \in \mathbf{Z}^+_{|G|}$. Therefore, their characterization on non-minimum phase zeros are given by

$$\eta_z(h_{ii}) \geq \eta_z(|G|) - \eta_i(z), \qquad \forall\, i \in \mathbf{m},\ z \in \mathbf{Z}^+_{|G|}, \tag{7.19}$$

with the right side bounded by (7.18).

Consider the previous example again to demonstrate the method. It can be found that $\mathbf{Z}^+_{G^{11}} = \{0.5\}$, $\mathbf{Z}^+_{G^{22}} = \emptyset$, $\mathbf{Z}^+_{|G|} = \{0.5\}$, $\eta_z(|G|)|_{z=0.5} = 1$ and

$$\eta_z(G^{11})|_{z=0.5} = 2, \qquad \eta_z(G^{12})|_{z=0.5} = 1,$$

$$\eta_z(G^{21})|_{z=0.5} = 0, \qquad \eta_z(G^{22})|_{z=0.5} = 0.$$

By (7.14), one sees

$$\eta_1(z)|_{z=0.5} = \min\{\eta_z(G^{11})|_{z=0.5},\ \eta_z(G^{12})|_{z=0.5}\} = \min\{2, 1\} = 1,$$

$$\eta_2(z)|_{z=0.5} = \min\{\eta_z(G^{21})|_{z=0.5},\ \eta_z(G^{22})|_{z=0.5}\} = \min\{0, 0\} = 0.$$

Thus, k_{11} should then satisfy

$$\eta_z(k_{11})|_{z=0.5} \geq \eta_z(G^{11})|_{z=0.5} - \eta_1(z)|_{z=0.5} = 2 - 1 = 1.$$

As $\mathbf{Z}^+_{G^{22}} = \emptyset$, there will be no constraint for k_{22} on its non-minimum phase zeros. By (7.19), the resultant loops meet

$$\eta_z(h_{11})|_{z=0.5} \geq \eta_z(|G|)|_{z=0.5} - \eta_1(z)|_{z=0.5} = 1 - 1 = 0,$$

$$\eta_z(h_{22})|_{z=0.5} \geq \eta_z(|G|)|_{z=0.5} - \eta_2(z)|_{z=0.5} = 1 - 0 = 1,$$

i.e., the second loop must contain a non-minimum phase zero at $z = 0.5$ of multiplicity one while the first loop need not contain any non-minimum phase zero.

Theorem 7.1.2. *If the IMC system in Figure* 7.1 *is decoupled, stable and realizable, then* (i) *the controller K's diagonal elements are characterized by* (7.9) *on its time delays and by* (7.16) *on its RHP zeros, and they uniquely determine the controller's off-diagonal elements by* (6.20); (ii) *the diagonal elements of GK, $h_{ii} = \tilde{g}_{ii}k_{ii}$, are characterized by* (7.11) *on its time delays and* (7.19) *on its RHP zeros.*

For the example in (7.6), choose the controller diagonal elements as

$$k_{11} = \frac{s - 0.5}{s + \rho}e^{-5s}, \qquad k_{22} = \frac{1}{s + \rho},$$

where $\rho > 0$. It follows from the previous calculations that they satisfy the conditions of Theorem 7.1.2 with lower bounds in (7.9) and (7.16) exactly met. The off-diagonal controllers are then obtained from (7.4) and the complete controller is

$$K = \begin{bmatrix} \frac{s - 0.5}{s + \rho}e^{-5s} & \frac{1}{s + \rho}e^{-4s} \\ \frac{-2(s + 2)}{s + \rho} & \frac{1}{s + \rho} \end{bmatrix},$$

which is both realizable and stable and the resulting GK is calculated as

$$GK = \begin{bmatrix} \frac{2s + 4 + (s - 0.5)e^{-s}}{(s + \rho)(s + 2)}e^{-6s} & 0 \\ 0 & (s - 0.5)\frac{2s + 4 + (s - 0.5)e^{-s}}{2(s + \rho)(s + 2)^3}e^{-7s} \end{bmatrix},$$

whose diagonal elements fall in our previous characterization on time delays and non-minimum phase zeros, namely, the time delay for the decoupled loops is no less than 6 and 7, respectively, and loop two must contain a non-minimum phase zero at $s = 0.5$ of multiplicity 1.

However, if we choose the controller diagonal elements as

$$k_{11} = k_{22} = \frac{1}{s+\rho},$$

which violates the conditions of Theorem 7.1.2. It gives rise to

$$k_{21} = \frac{-2(s+2)}{(s+\rho)(s-0.5)} e^{5s},$$

which is neither realizable nor stable. The resultant h_{11} is given by

$$h_{11} = \tilde{g}_{11} k_{11} = \frac{2s+4+(s-0.5)e^{-s}}{(s+\rho)(s+2)(s-0.5)} e^{-s},$$

which is unstable, too. Such k_{ii} should be discarded.

A quite special phenomenon in such a decoupling control is that there might be unstable pole-zero cancellations in forming $\tilde{g}_{ii} k_{ii}$. In contrast to the normal intuition, this will not cause instability. Notice from the previous example that $\tilde{g}_{11}$ has a pole at $s = 0.5$ and k_{11} has at least one zero at the same location as $\eta_z(k_{11})|_{z=0.5} \geq 1$. Hence, an unstable pole-zero cancellation at $z = 0.5$ occurs in forming $\tilde{g}_{11} k_{11}$. However, the resulting closed-loop system is still stable because all the controller diagonal and off-diagonal elements are stable. In fact, as far as G and K are concerned, there is no unstable pole-zero cancellation since both G and K are stable.

7.1.2 Design

In this subsection, practical design of IMC is considered. We first briefly review the IMC design for SISO case. Let $g(s)$ be the transfer function model of a stable SISO process and $k(s)$ the controller. The resulting closed-loop transfer function in case of no process-model mismatch is $h(s)$, where $h(s) = g(s)k(s)$. It is quite clear that if $h(s)$ has been specified then $k(s)$ is determined. However $h(s)$ can not be specified arbitrarily. In IMC design, the process model $g(s)$ is first factored into $g(s) = g^+(s)g^-(s)$, where $g^+(s)$ contains all the time delay and RHP zeros. Then $h(s)$ is chosen as $h(s) = g^+(s)f(s)$, where $f(s)$ is the IMC filter, and consequently $k(s)$ is given by $k(s) = h(s)g^{-1}(s)$.

The factorizations which yield $g^+(s)$ are not unique. Holt and Morari (1985a,b) suggested that the following form be advantageous:

$$g^+(s) = e^{-Ls} \prod_{i=1}^{n} \left(\frac{z_i - s}{z_i + s} \right), \tag{7.20}$$

where L is the time delay and $z_1, z_2, \cdots, z_n$ are all the RHP zeros present in $g(s)$. The IMC filter $f(s)$ is usually chosen in the form:

$$f(s) = \frac{1}{(\tau s + 1)^l},$$

where τ and l are the filter time constant and order, respectively. The filter time constant τ is a tuning parameter for performance and robustness.

For multivariable decoupling IMC control, the closed-loop transfer function matrix $H(s)$ in case of no process-model mismatch is $H = \mathrm{diag}\{h_{ii}\} = \mathrm{diag}\{\tilde{g}_{ii}k_{ii}\}$ and thus the key point in design lies in specifying the m decoupled loop transfer function $h_{ii}(s)$. Unlike the SISO case, $h_{ii}(s)$ can not be chosen based solely on $\tilde{g}_{ii}(s)$ since this may cause an unrealizable and/or unstable controller. Instead, $h_{ii}(s)$ should be chosen according to Theorem 7.1.2. It is also obvious that for best performance of the IMC system and simplicity of the controller it is undesirable to include any time delays and non-minimum phase zeros in $h_{ii}(s)$ more than necessary, that is, only the minimum time delay in (7.11) and minimum multiplicity of unavoidable non-minimum phase zeros in (7.19) should be taken. This leads to

$$h_{ii} = e^{-(\boldsymbol{\tau}(|G|)-\tau_i)s} \prod_{z \in \mathbf{Z}^+_{|G|}} \left(\frac{z-s}{z+s}\right)^{\eta_z(|G|)-\eta_i(z)} f_i(s), \qquad \forall\, i \in \mathbf{m}, \quad (7.21)$$

where $f_i(s)$ is the *i-th* loop IMC filter. This choice of $h_{ii}(s)$ will then determine $k_{ii}(s)$ as

$$k_{ii}(s) = h_{ii}(s)\tilde{g}_{ii}^{-1}(s), \quad (7.22)$$

which in turn yields off-diagonal elements $k_{ji}(s)$ uniquely by (7.4).

Remark 7.1.1. Good control performance of the loops is what we seek apart from the loop-decoupling requirement. However, we do not explicitly define a performance measure and design the controller via the minimization (or maximization) of some measure. These are done in model predictive control, where the "size" of the error between the predicted output and a desired trajectory is minimized; and in LQG control, where a linear quadratic index is minimized to yield an optimal controller. These approaches have a different framework than ours. What we do here is to follow the IMC design approach with the requirement of decoupling for the control of multivariable multi-delay processes. Loop performance is simply observed with closed-loop step responses for each output, in terms of traditional performance specifications such as overshoot, rise time and settling time in comparison with the existing schemes of similar purposes. A link of the IMC design theory with the quantitative error measure is that the item with the non-minimum phase factor $g^+(s)$ in (7.20) used in IMC closed-loop transfer function in (7.21) can actually minimize the H-2 norm of the error signal in the presence of a step reference change (Morari and Zafiriou, 1989).

Remark 7.1.2. One may note that there is the case where good decoupling and performance may not be both achievable and one is reached at cost of another. For example, decoupling should never be used in flight control, instead, couplings are deliberately employed to boost performance. This case is however not the topic of the present paper. We here address applications where decoupling is required. Indeed, decoupling is usually required (Kong, 1995) in process control industry, for ease of process operations. Poor decoupling in closed-loop system could, in many cases, lead to poor diagonal loop performance. Thus, the design objective here is to decouple the loops at first and then to achieve as good performance as possible for each loop. Loop performance is considered to be good if best achievable performance is nearly reached among all decoupled and stable loops, subject to limitations imposed by delay, non-minimum phase zeros and bandwidth. Our design realizes this by selecting the objective loop transfer functions in (7.21) in which each factor reflects respective roles of delay, non-minimum phase zeros and bandwidth. It might be worth noting that decoupling and loop performance may not always be in conflict. It is shown (Linnemann and Wang, 1993) that for delay-free stable systems some optimal controllers will automatically result in decoupled loops though decoupling is not required in the problem formulation.

For a multivariable multi-delay process, the IMC controller designed by the above procedure is usually in the complicated form of (7.7), which is difficult to implement. Model reduction is then employed to find a much simpler rational function plus dead time approximation to the above theoretical controller. The topic of model reduction has been addressed in Chapter 2 in a great detail and two effective algorithms given there. We can utilize one of them for our present design.

Theoretically, if h_{ii} is specified as in (7.21) so that it contains no more time delays than necessary, then each column of K has at least one element whose time delay is zero. However, if the controller is obtained as a result of the model reduction, that element might have non-zero time delay though it is usually small. In such a case, it is reasonable to subtract it from all elements in the column to speed up the loop response if it is positive or to make the controller realizable if it is negative. This operation will not affect decoupling and stability. The above development is summarized into the following design procedure.

IMC design procedure. *Seek a controller $K(s)$ given $G(s)$*

- (i) *With $i = 1$ for loop one, apply model reduction to $|G|$ and all non-zero G^{ij}. Take time delays and non-minimum phase zeros (including multiplicities) from their reduced-order models. Find τ_i and $\eta_i(z)$ for all $z \in \mathbf{Z}^+_{|G|}$ from (7.10) and (7.14).*
- (ii) *Specify h_{ii} according to (7.21) and determine the diagonal controller k_{ii} from (7.22). Apply model reduction to k_{ii} to obtain $\hat{k}_{ii}$.*
- (iii) *Calculate the controller off-diagonal elements k_{ji} from (7.4). Apply model reduction to all k_{ji} to obtain $\hat{k}_{ji}$.*
- (iv) *Find the smallest time delay in $\hat{k}_{ji}$ and subtract it from all the time delays of $\hat{k}_{ji}$ if it is non-zero.*
- (v) *Repeat the above steps for all other loops.*

The IMC system in Figure 7.1 is referred to as the nominal case if $G = \hat{G}$. It is obvious that the IMC system is nominally stable if both the process G

and the controller K are stable, as discussed before. In the real world where the model does not represent the process exactly, nominal stability is not sufficient and robust stability of the IMC system has to be ensured. As a standard assumption in robust analysis, we assume that the nominal IMC system is stable (i.e., both $\hat{G}$ and K are stable). Suppose that the actual process G is also stable and is described by

$$G \in \prod = \{G : \overline{\sigma}\left([G(j\omega) - \hat{G}(j\omega)]\hat{G}^{-1}(j\omega)\right) = \overline{\sigma}\left(\Delta_m(j\omega)\right) \leq \tilde{l}_m(\omega)\},$$

where $\hat{l}_m(\omega)$ is the bound on the multiplicative uncertainty. It follows (Chapter 3) that the IMC system is robust stable if and only if

$$\overline{\sigma}(\hat{G}(j\omega)K(j\omega)) < l_m^{-1}(j\omega), \quad \forall\omega. \tag{7.23}$$

Let K^* be the ideal controller obtained from (7.22) and (7.4), we have $\hat{G}K^* = \mathrm{diag}\,\{h_{ii}\}$. Thus (7.23) becomes

$$\overline{\sigma}\left(\mathrm{diag}\,\{h_{ii}(j\omega)\}K^{*-1}(j\omega)K(j\omega)\right) < l_m^{-1}(j\omega), \quad \forall\omega.$$

One notes from (7.21) that $\mathrm{diag}\,\{h_{ii}(j\omega)\} = U\mathrm{diag}\,\{f_i(j\omega)\}$, where U is a unitary matrix. Noticing $\overline{\sigma}(UA) = \overline{\sigma}(A)$ for a unitary U, thus a sufficient and necessary robust stability condition is obtained as

$$\overline{\sigma}\left(\mathrm{diag}\,\{f_i(j\omega)\}K^{*-1}(j\omega)K(j\omega)\right) < l_m^{-1}(j\omega), \quad \forall\omega. \tag{7.24}$$

In practice, we can evaluate and plot the left hand side of (7.24) and compare with $l_m^{-1}(j\omega)$ to see whether robust stability is satisfied given the multiplicative uncertainty bound $l_m(j\omega)$. One also notes that if the error between the ideal controller K^* and its reduced-order model K are small, i.e., $K^* \approx K$, then the robust stability condition becomes

$$\overline{\sigma}\left(\mathrm{diag}\,\{f_i(j\omega)\}\right) < l_m^{-1}(j\omega), \quad \forall\omega.$$

which indicates that the IMC filter should be chosen such that its largest singular value is smaller than the inverse of the uncertainty magnitude for all ω.

7.1.3 Simulation

In this subsection, several simulation examples are given to show the effectiveness of the decoupling IMC design. The effects of time delays and non-minimum phase zeros are illustrated and control performance as well as its robustness is compared with the existing multivariable Smith predictor schemes. We intend to use commonly cited literature examples with real live background and try to explain technical points of the proposed method as much as possible. The first and third examples are well-known distillation column processes (Wood and Berry, 1973) and (Luyben, 1986*a*). The process in the second example is a modification of the Wood and Berry process where only the delays of its elements have been changed. This modification generates non-minimum phase zeros for the multivariable process, which is not available in the other two examples, and is used to illustrate relevant design issues.

Example 7.1.1. Consider the well-known Wood/Berry binary distillation column plant (Wood and Berry, 1973):

$$G(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}.$$

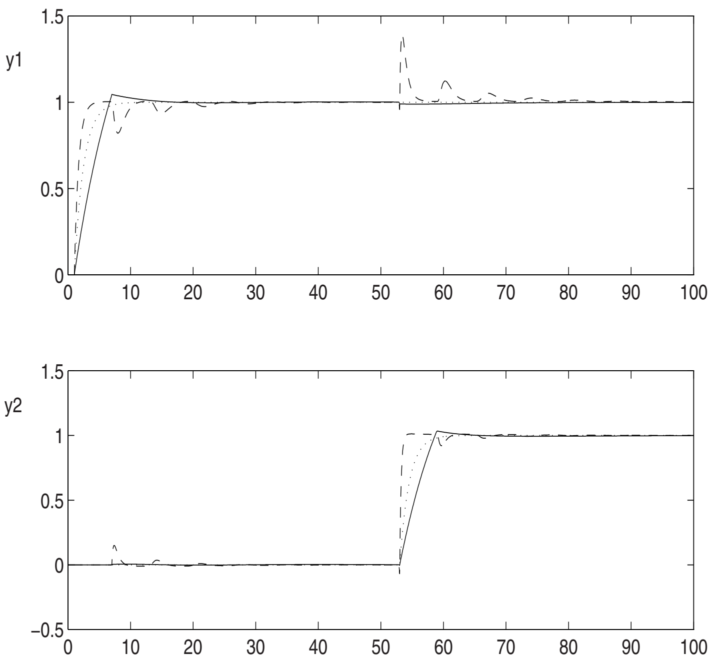


Fig. 7.2. Control Performance for Example 7.1.1
(—— IMC; - - - Jerome; $\cdots$ ideal IMC)

Step (i) The application of model reduction to $|G|$, G^{11} and G^{12} produces

$$|\hat{G}| = \frac{-0.1077s^2 - 4.239s - 0.3881}{s^2 + 0.1031s + 0.0031}e^{-6.3s},$$

$$\hat{G}^{11} = \frac{-1.349s - 42.81}{s^2 + 31.85s + 2.207}e^{-3s},$$

$$\hat{G}^{12} = \frac{-0.6076s - 17.66}{s^2 + 29.26s + 2.676}e^{-7s}.$$

It is clear that $\boldsymbol{\tau}(|\hat{G}|) = 6.3$, $\boldsymbol{\tau}(\hat{G}^{11}) = 3$, and $\boldsymbol{\tau}(\hat{G}^{12}) = 7$. Thus, by (7.10), one finds $\tau_1 = \min\{3,\ 7\} = 3$. Since $|\hat{G}|$ is of minimum phase, $\mathbf{Z}^+_{|\hat{G}|} = \emptyset$ and there is no need to calculate $\eta_1(z)$.
Step (ii) According to (7.21), we have $h_{11} = \frac{e^{-3.3s}}{s+1}$ with the filter chosen as $\frac{1}{s+1}$. k_{11} is then obtained as $k_{11} = \tilde{g}_{11}^{-1}h_{11}$ from (7.22). The application of model reduction to k_{11} yields

$$\hat{k}_{11} = \frac{0.3168s^2 + 0.0351s + 0.0012}{s^2 + 0.1678s + 0.0074}e^{-0.9s}.$$

Step (iii) k_{21} is calculated from (7.4). Applying model reduction to k_{21} yields

$$\hat{k}_{21} = \frac{0.1405s^2 + 0.0180s + 0.0007}{s^2 + 0.2013s + 0.0123}e^{-4.9s}.$$

Step (iv) The smallest time delay in $\hat{k}_{11}$ and $\hat{k}_{21}$ is 0.9 and then subtracted from both elements' time delays to generate the final $\hat{k}_{11}$ and $\hat{k}_{21}$.
Step (v) Loop two is treated similarly, where $h_{22} = \frac{e^{-5.3s}}{s+1}$. This results in the overall controller as

$$K = \begin{bmatrix} \dfrac{0.3168s^2 + 0.0351s + 0.0021}{s^2 + 0.1678s + 0.0074} & \dfrac{-0.2130s^2 - 0.0222s - 0.0008}{s^2 + 0.1399s + 0.0051}e^{-2s} \\ \dfrac{0.1405s^2 + 0.0180s + 0.0007}{s^2 + 0.2031s + 0.0123}e^{-4s} & \dfrac{-0.1798s^2 - 0.0206s - 0.0008}{s^2 + 0.1607s + 0.0073} \end{bmatrix}.$$

The set-point responses of the resulting IMC system (shortened as IMC) are shown in Figure 7.2 with solid line. The corresponding set-point response of the Jerome's multivariable Smith predictor scheme with the controller settings given in Jerome and Ray (1986) is depicted with dashed line in the figure. The corresponding responses of the ideal IMC system $H(s) = \mathrm{diag}\{h_{ii}\}$ are also shown with dotted line in the figure. The resulting ISE (Integral Square Error) with equal weighting of the two outputs is calculated as 7.16, 4.73 and 5.01 for the IMC design, Jerome's scheme and the ideal IMC control system, respectively. Although the ISE for the Jerome's scheme is smaller than that of the proposed method, the Jerome's control system is not robust and already exhibits very poor damping and severe oscillation even for the nominal case, while the responses of the IMC are smooth and the decoupling is almost perfect.

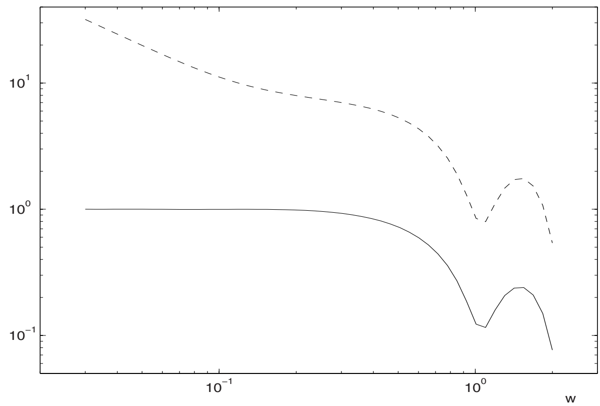


Fig. 7.3. Robustness Analysis for Example 7.1.1
(—— left hand side of (7.24); - - - right hand side of (7.24))

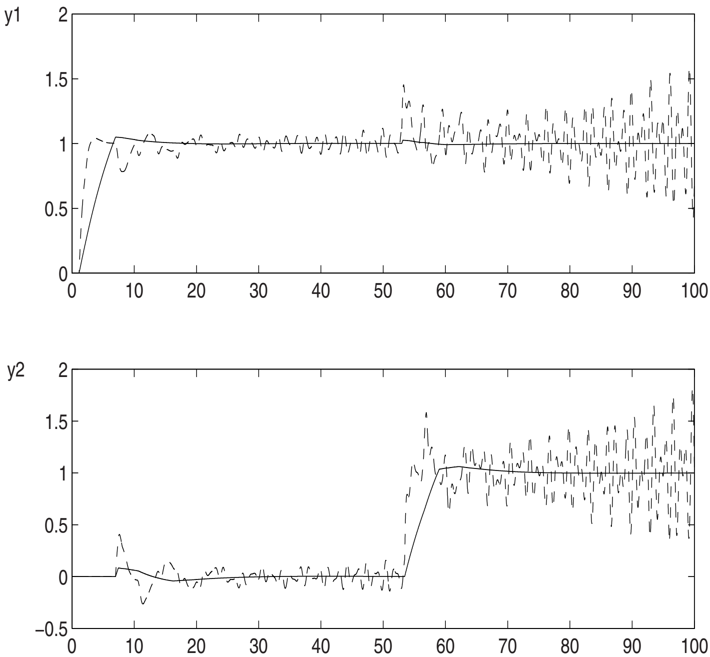


Fig. 7.4. Control System Robustness for Example 7.1.1
(—— IMC; - - - Jerome)

In order to see the robustness of the IMC design, we increase the dead times of the process' diagonal elements by 15%. In order to check the stability of the perturbed system, the left hand side and right hand side of (7.24) are plotted in Fig 7.3 with solid line and dashed line respectively. From the figure, we conclude that the IMC system remains stable under the given perturbations. The set-point responses are shown in Figure 7.4, exhibiting that the performance of the IMC design is much superior compared with the Jerome's multivariable Smith predictor scheme. The resulting ISE is calculated as 7.76 for the IMC method while it is infinite for Jerome's scheme due to instability. ◊

Example 7.1.2. Consider a variation of the above example:

$$G(s) = \begin{bmatrix} g_{110}(s)e^{-s} & g_{120}(s)e^{-9s} \\ g_{210}(s)e^{-2s} & g_{220}(s)e^{-15s} \end{bmatrix},$$

where the delay free parts $g_{ij0}(s)$ are identical to Example 7.1.1. It follows from the IMC procedure that

$$|\hat{G}| = \frac{0.0374s^2 + 3.72s - 0.5042}{s^2 + 0.1239s + 0.0041} e^{-13.6s},$$

$$\hat{G}^{11} = \frac{-1.3493s - 42.81}{s^2 + 31.85s + 2.207} e^{-15s},$$

$$\hat{G}^{12} = \frac{-0.6072s - 17.66}{s^2 + 29.26s + 2.676} e^{-2s}.$$

One readily sees that $\boldsymbol{\tau}(|\hat{G}|) = 13.6$, $\boldsymbol{\tau}(\hat{G}^{11}) = 15$, $\boldsymbol{\tau}(\hat{G}^{12}) = 2$ and thus $\tau_1 = \min\{15, 2\} = 2$ from (7.10). A difference from Example 7.1.1 is that $|\hat{G}|$ has a RHP zero at $z = 0.136$ with a multiplicity of one, i.e., $\mathbf{Z}^+_{|\hat{G}|} = \{0.136\}$ and $\eta_z(|\hat{G}|)|_{z=0.136} = 1$. It is easy to find $\eta_z(\hat{G}^{11})|_{z=0.136} = 0$ and $\eta_z(\hat{G}^{12})|_{z=0.136} = 0$ since $\hat{G}^{11}$ and $\hat{G}^{12}$ are both of minimum phase. We thus have $\eta_1(z)|_{z=0.136} = \min\{0,\ 0\} = 0$ from (7.14) and $h_{11} = \frac{0.136-s}{(0.136+s)(s+1)} e^{-11.6s}$ from (7.21) with the filter of $\frac{1}{s+1}$. For the second loop, with the filter chosen as $\frac{1}{s+1}$, h_{22} is obtained as $h_{22} = \frac{0.136-s}{(0.136+s)(s+1)} e^{-12.6s}$. The rest of design is straightforward and gives

$$K = \begin{bmatrix} \dfrac{0.3656s^2 + 0.0524s + 0.002}{s^2 + 0.2265s + 0.0124} e^{-13.1s} & \dfrac{-0.2375s^2 - 0.0294s - 0.0011}{s^2 + 0.1790s + 0.0070} e^{-8s} \\ \dfrac{0.1551s^2 + 0.0291s + 0.012}{s^2 + 0.2726s + 0.0226} & \dfrac{-0.2003s^2 - 0.0317s - 0.0014}{s^2 + 0.2215s + 0.0136} \end{bmatrix}.$$

For comparison, the Ogunnaike and Ray's multivariable Smith predictor is considered here, where the primary controller given in Ogunnaike and Ray (1979) can be still used since the delay-free part of the process is identical for our case and their case. The set-point responses shown in Figure 7.5 indicate that the performance of the proposed IMC design is significantly better and

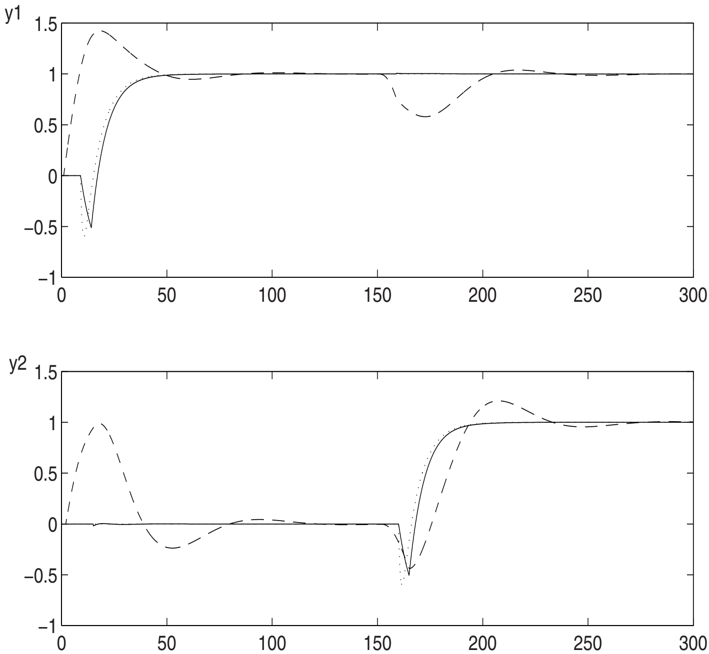


Fig. 7.5. Control Performance for Example 7.1.2
(—— IMC; - - - Ogunnaike and Ray; $\cdots$ ideal IMC)

its response is very close to that of the ideal IMC controller (dotted line). The resulting ISE is calculated as 52.5, 71.16 and 49.4 for the IMC, Ogunnaike and Ray's scheme, and the ideal IMC, respectively. ♢

Example 7.1.3. The process studied by Luyben (1986*a*) has the following transfer function matrix:

$$G(s) = \begin{bmatrix} \frac{0.126e^{-6s}}{60s+1} & \frac{-0.101e^{-12s}}{(45s+1)(48s+1)} \\ \frac{0.094e^{-8s}}{38s+1} & \frac{-0.12e^{-8s}}{35s+1} \end{bmatrix}.$$

It follows from the design procedure that with filters chosen as $f_1(s) = f_2(s) = \frac{1}{3s+1}$, h_{11} and h_{22} are obtained as $h_{11} = \frac{e^{-6.5s}}{3s+1}$ and $h_{22} = \frac{e^{-8.5s}}{3s+1}$, respectively. The resultant controller is

$$K = \begin{bmatrix} \frac{146.8s^2+6.355s+0.0733}{s^2+0.3465s+0.0034} & \frac{-6.027s^2-0.3416s-0.0053}{s^2+0.0395s+0.0003}e^{-9.3s} \\ \frac{104.5s^2+4.719s+0.0561}{s^2+0.34s+0.0034} & \frac{-68.2s^2-3.490s-0.0527}{s^2+0.2496s+0.0024} \end{bmatrix}.$$

Since the multivariable Smith predictor design for this process is not available in the literature, the BLT tuning method (Luyben, 1986*a*) is used for

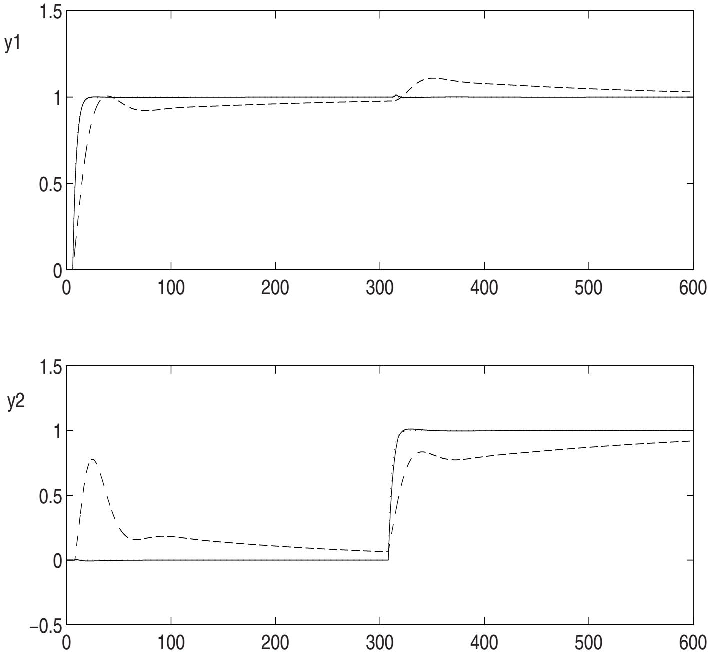


Fig. 7.6. Control Performance for Example 7.1.3
(—— IMC; - - - BLT; $\cdots$ ideal IMC)

comparison. The set-point responses are shown in Figure 7.6, exhibiting that significant performance improvement has been achieved with the IMC design and again its response is very close to that of the ideal IMC system. The resulting ISE is calculated as 17.7, 51.5 and 17.0 for the IMC, BLT, and the ideal IMC system, respectively. $\diamondsuit$

In this section, an approach to the decoupling and stable IMC analysis and design has been presented for multivariable processes with multiple time delays. All the stabilizing controllers which solve this decoupling problem and the resultant closed-loop systems are characterized in terms of their unavoidable time delays and non-minimum phase zeros. Such delays and zeros can be readily calculated from the process transfer function matrix and clearly quantify performance limitations for any multivariable IMC system which is decoupled and stable. It is interesting to note that a controller may necessarily include some time delays and non-minimum phase zeros to make itself a solution to the problem. A theoretical control design for the best achievable performance is carried out based on the characterizations. Model reduction is then exploited to simplify both analysis and design involved. Examples have been given to illustrate our approach with which significant performance improvement over the existing multivariable Smith predictor control methods is evident.

7.2 The Smith Scheme

In this section, a multivariable Smith predictor controller design is presented for decoupling and stabilizing multivariable processes with multiple time delays. A decoupler is first introduced in this new control scheme and it simplifies the multivariable Smith predictor controller design to multiple single-loop Smith predictor controller designs.

7.2.1 The Structure

Consider a multivariable process with the transfer matrix:

$$G(s) = \begin{bmatrix} g_{11}(s) & \dots & g_{1m}(s) \\ \vdots & & \vdots \\ g_{m1}(s) & \cdots & g_{mm}(s) \end{bmatrix}, \tag{7.25}$$

where

$$g_{ij}(s) = g_{ij0}(s)e^{-L_{ij}s},$$

and $g_{ij0}(s)$ are strictly proper, stable scalar rational functions, and non-negative L_{ij} are the time delay associated with $g_{ij}(s)$. Let the delay-free part of the process be denoted by $G_0 = [g_{ij0}]$.

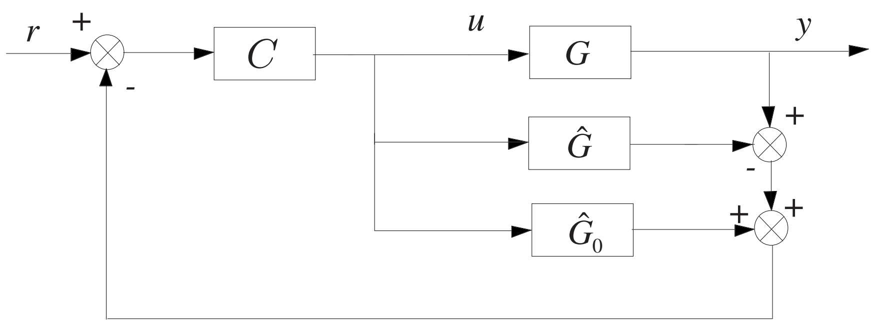


Fig. 7.7. Multivariable Smith Predictor Control

The multivariable Smith predictor control scheme is shown in Figure 7.7 where $G(s)$ and $\hat{G}(s)$ are the process and its model, respectively. $\hat{G}_0(s)$ is the same as $\hat{G}(s)$ except that all the delays have been removed. $C(s)$ is the primary controller. When the model is perfect, i.e., $\hat{G}(s) = G(s)$ and $\hat{G}_0(s) = G_0(s)$, the closed-loop transfer function from r to y becomes

$$H(s) = G(s)C(s)[I + G_0(s)C(s)]^{-1}.$$

It can be seen that $I + G_0(s)C(s)$ contains no delays provided that $C(s)$ is so and it suggests that the primary controller $C(s)$ can be designed with respect to the delay free part $G_0(s)$. This is the main attractiveness of the scheme. However, unlike SISO case, even though $C(s)$ is designed such that $H_0(s) = G_0(s)C(s)[I + G_0(s)C(s)]^{-1}$ has desired performance, the actual system performance can not be guaranteed. This can be seen from the closed-loop transfer function

$$H(s) = G(s)G_0^{-1}(s)G_0(s)C(s)[I + G_0(s)C(s)]^{-1} = G(s)G_0(s)^{-1}H_0(s).$$

The actual system performance could be quite poor due to the existence of $G(s)G_0^{-1}(s)$. For the special case where the delays of all the elements in each row of the transfer matrix are identical, the finite poles and zeros in $G(s)G_0^{-1}(s)$ will all be cancelled. In this case, $G(s)G_0^{-1}(s) = diag\{e^{-L_{ii}s}\}$ and the system output is the delayed output of $H_0(s)$. However, in general this desired property is not preserved.

In order to overcome this problem and improve the performance of the multivariable Smith predictor control system, we here propose a decoupling Smith predictor control scheme depicted in Figure 7.8, where $D(s)$ is a decoupler for G, $Q(s)$ the decoupled process $G(s)D(s)$, $Q_0(s)$ is the same as $Q(s)$ except that all the delays are removed. Suppose that $G(s)D(s)$ is decoupled, it is obvious that the $Q(s)$ and $Q_0(s)$ will be diagonal matrices. The multivariable smith predictor design is then simplified to multiple single-loop smith predictor designs for which various methods can be applied. This decoupling Smith scheme involves three parts, decoupler $D(s)$, the decoupled process Q and the primary controller $C(s)$. Their design will be discussed in the next subsection.

7.2.2 Design

For $G(s)D(s)$ to be decoupled, it is shown in Chapter 6 that the elements of the $i-th$ column of $D(s)$ should satisfy the conditions

$$GD = diag\{\frac{|G|}{G^{ii}}d_{ii}, \quad i = 1, 2, \cdots, m\}, \tag{7.26}$$

$$d_{ji} = \frac{G^{ij}}{G^{ii}}d_{ii} := \psi_{ji}d_{ii}, \quad \forall i, j \in \mathbf{m}, j \neq i. \tag{7.27}$$

where $\mathbf{m} = \{1, 2, \cdots, m\}$. We may adopt the procedure in Section 1 to design D. But the present case has a separate $C(s)$ which will take care of all control issues except decoupling which is the only duty of $D(s)$. Thus, we wish to get a simplest decoupling $D(s)$ with minimum calculations. Let θ_{ji} as the smallest non-negative number such that $\psi_{ji}e^{-\theta_{ji}s}$ does not have prediction for all $j \in \mathbf{m}$. Let $\theta_i = \max_{j \in \mathbf{m}, j \neq i} \theta_{ji}$. By choosing $d_{ii} = e^{-\theta_i s}$ in (7.27), it is obvious that d_{ji} will have no predictions.

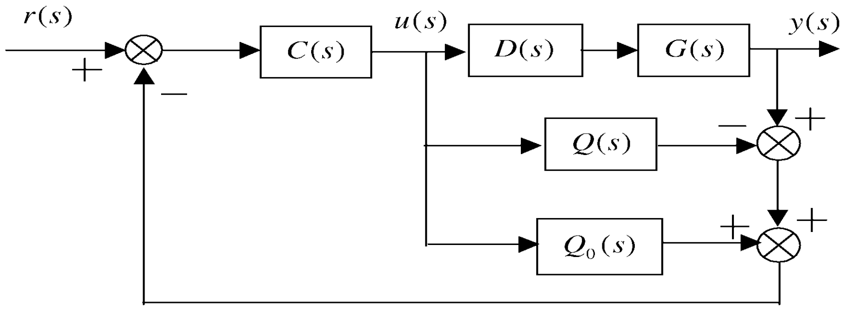


Fig. 7.8. Decoupling Smith Control Scheme

Generally, for a multivariable system in (7.25), the dynamics of the resultant d_{ji}, $j \neq i$, could be highly complicated. For ease of implementation, low-order transfer functions are usually preferred. Yet, good performance should be maintained. Thus, we apply model reduction to get a simpler, stable and realizable approximation to each of them and use it for implementation.

Consider now Q in Figure 7.8. In the case of perfect decoupling, Q should be

$$\begin{aligned} Q = GD &= diag\{\frac{|G|}{G^{ii}} d_{ii}, i = 1, 2, \cdots, m\} \\ &= diag\{q_{11}, q_{22}, \cdots, q_{mm}\}. \end{aligned}$$

Generally, q_{ii} is too complicated to implement. Thus, model reduction is also applied to q_{ii} to obtain a simpler yet good approximation for implementation. And Q_0 in Figure 7.8 is readily obtained as the delay free part of $Q(s)$.

With decoupling by D, the multivariable smith predictor design for Figure 7.8 is now simplified to multiple single-loop smith predictor control designs. Let the primary controller be

$$C(s) = diag\{ c_{11}(s) \cdots c_{mm}(s)\} .$$

Each individual $c_{ii}(s)$ is designed with respect to the delay free part q_{ii0} of q_{ii} such that closed-loop system formed by $c_{ii}(s)$ and q_{ii0} has the desired performance. This is a number of SISO problems and deserves no further discussions.

7.2.3 Stability Analysis

The multivariable Smith system in Figure 7.8 is referred to as the nominal case if $GD = Q$. As the process G in (7.25) is assumed to be stable and the decoupler D is designed to be stable, the control system in Figure 7.8 will be stable if and only if the primary controller C stabilizes Q_0.

The nominal condition $GD = Q$ can be violated in practice for several reasons. Firstly, due to the approximation of the decoupler by model reduction,

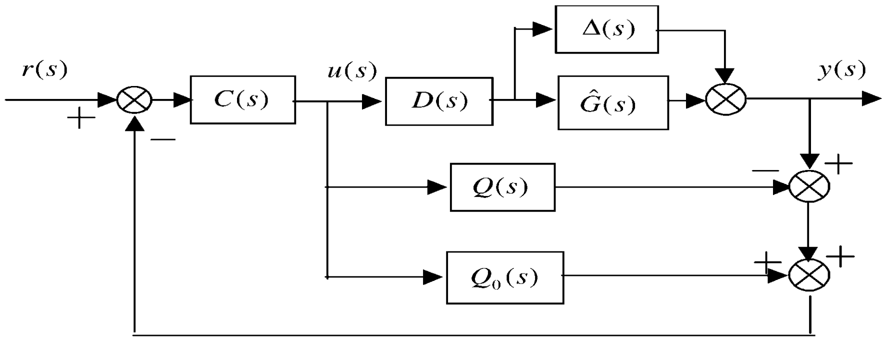


Fig. 7.9. Uncertain Smith System

the transfer matrix GD may not be diagonal whereas Q is implemented as diagonal form. Secondly, approximation of q_{ii} by model reduction also causes discrepancy between GD and Q even if the decoupling is perfect. Most importantly, in the real world, the model $G(s)$ may not represent the actual process exactly. For robustness analysis, the actual process is assumed to be any member of a family of possible processes. It is shown by an example in Palmor (1980) that the Smith predictor controller could be unstable for infinitesimal perturbation in the dead time, even though it may be nominally stable. Therefore, nominal stability is not sufficient. Robust stability of the closed-loop has to be ensured for the application of the multivariable Smith system.

Let the nominal stable process transfer function be $\hat{G}(s)$ and the real process be described by the family:

$$\Pi = \Big\{ G(s) : G(s) = \hat{G}(s) + \Delta(s), \bar{\sigma}(\Delta(j\omega)) \leq |\gamma(j\omega)| \Big\}, \tag{7.28}$$

where the perturbation $\Delta(s)$ is stable. The multivariable Smith predictor control system in Figure 7.8 is said to be *robustly stable* if the closed-loop system is stable for each member in the family Π. This uncertain Smith system is shown in Figure 7.9, and this can be redrawn into the standard form (Chapter 3), with $\Delta(s)$ appearing in a feedback path, as in Figure 7.10. Note that if we assume that the Smith system is *nominally stable*, then the dashed line encloses a stable system. Let $F(s)$ be the transfer function from $\tilde{z}$ to $\tilde{u}$ in Figure 7.10. It follows that the uncertain Smith system is *robustly stable*, if and only if

$$\bar{\sigma}[F(j\omega)]|\gamma(\mathrm{j}\omega)| < 1, \quad \omega \in [0, \infty), \tag{7.29}$$

or equivalently

$$\|F(s)\gamma(s)\|_\infty < 1, \tag{7.30}$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix, and $\|\cdot\|_\infty$ the H_∞-norm of a transfer matrix. To find F, by Figure 7.10, we have

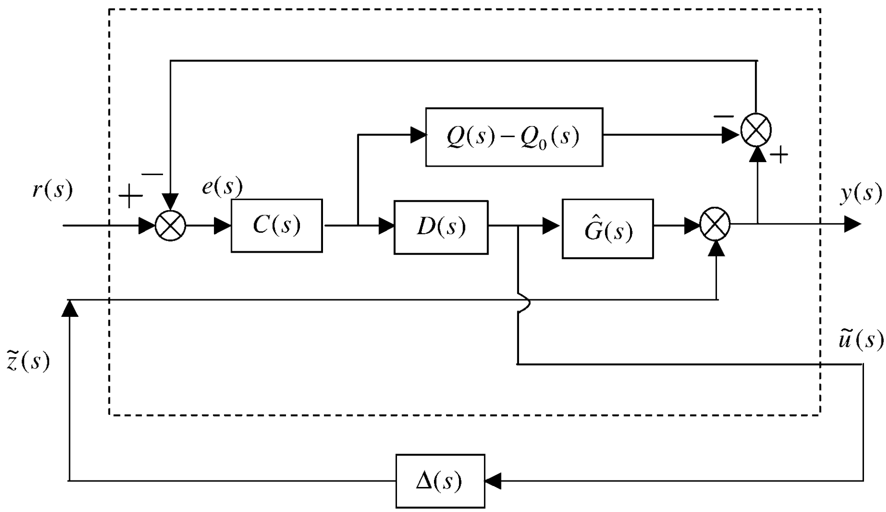


Fig. 7.10. Equivalent Uncertain Smith System

$$\tilde{u} = DCe, \tag{7.31}$$

$$e = r - y + (Q - Q_0)Ce, \tag{7.32}$$

$$y = \hat{G}DCe + \tilde{z}. \tag{7.33}$$

Collecting (7.31) to (7.33) yields

$$\tilde{u} = DC[I + \hat{G}DC - (Q - Q_0)C]^{-1}r - DC[I + \hat{G}DC - (Q - Q_0)C]^{-1}\tilde{z}.$$

The transfer function from $\tilde{z}$ to $\tilde{u}$ is then given by

$$F(s) = -DC[I + \hat{G}DC - (Q - Q_0)C]^{-1}.$$

Theorem 7.2.1. *Assume that the family of stable processes Π is described by (7.28) and that the nominal closed-loop system is stable. Then the multivariable Smith predictor control system in Figure 7.9 is robustly stable if and only if (7.29) or (7.30) holds true.*

7.2.4 Simulation

Several simulation examples are now given to show the effectiveness of the decoupling Smith control.

Example 7.2.1. Consider the well-known Wood/Berry binary distillation column process (Wood and Berry, 1973):

$$G(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}.$$

The diagonal elements of decoupler $D(s)$ are chosen as $d_{11}(s) = 1$ and $d_{22}(s) = 1$ since the $\psi_{12} = g_{12}/g_{11}$and $\psi_{21} = g_{21}/g_{22}$ in (7.27) have no predictions. Model reduction yields the off-diagonal elements of the decoupler as

$$d_{12} = \frac{(24.66s + 1.48)e^{-2s}}{21s + 1},$$

and

$$d_{21} = \frac{(4.90s + 0.34)e^{-4s}}{10.9s + 1},$$

and the decoupler is formed as

$$D(s) = \begin{bmatrix} 1 & \frac{(24.66s+1.48)e^{-2s}}{21s+1} \\ \frac{(4.90s+0.34)e^{-4s}}{10.9s+1} & 1 \end{bmatrix}.$$

The application of the model reduction to $g_{ii}(s)$, the diagonal elements of $G(s)D(s)$, produces

$$g_{11} = \frac{6.37e^{-0.92s}}{0.61s^2 + 5.45s + 1},$$

and

$$g_{22} = \frac{-9.65e^{-3.31s}}{4.59s + 1},$$

giving

$$Q(s) = \begin{bmatrix} \frac{6.37e^{-0.92s}}{0.61s^2+5.45s+1} & 0 \\ 0 & \frac{-9.65e^{-3.31s}}{4.59s+1}, \end{bmatrix}$$

and

$$Q_0(s) = \begin{bmatrix} \frac{6.37}{0.61s^2+5.45s+1} & 0 \\ 0 & \frac{-9.65}{4.59s+1} \end{bmatrix}.$$

respectively, The primary controller is designed as simple PID:

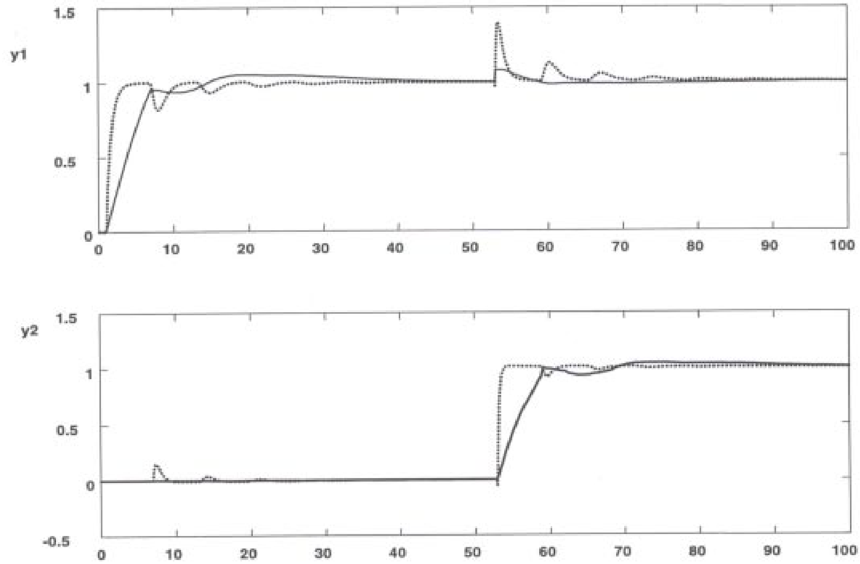


Fig. 7.11. Control Performance for Example 7.2.1 (— Proposed; - - - Jerome)

$$C(s) = \begin{bmatrix} 0.23 + \frac{0.066}{s} - 0.09s & 0 \\ 0 & -0.14 - \frac{0.048}{s} + 0.097s \end{bmatrix}.$$

The set-point response is shown in Figure 7.11. For comparison, the Jerome's multivariable Smith predictor scheme is used with the controller settings from Jerome and Ray (1986). It can be seen that the responses of the decoupling Smith control are much better and the decoupling is almost perfect. ♢

Example 7.2.2. Wardle and Wood (1969) give the following transfer function matrix model for an industrial distillation column:

$$G(s) = \begin{bmatrix} \frac{0.126e^{-6s}}{60s+1} & \frac{-0.101e^{-12s}}{(45s+1)(48s+1)} \\ \frac{0.094e^{-8s}}{38s+1} & \frac{-0.12e^{-8s}}{35s+1} \end{bmatrix}.$$

For this process, the diagonal elements of the decoupler $D(s)$ are chosen as $d_{11}(s) = 1$ and $d_{22}(s) = 1$ since the $\psi_{12} = g_{12}/g_{11}$ and $\psi_{21} = g_{21}/g_{22}$ have no predictions. Model reduction yields

$$D(s) = \begin{bmatrix} 1 & \frac{(48.09s+0.802)e^{-6s}}{2160s^2+93s+1} \\ \frac{47.72s^2+28.78s+0.783}{66.14s^2+39.74s+1} & 1 \end{bmatrix}.$$

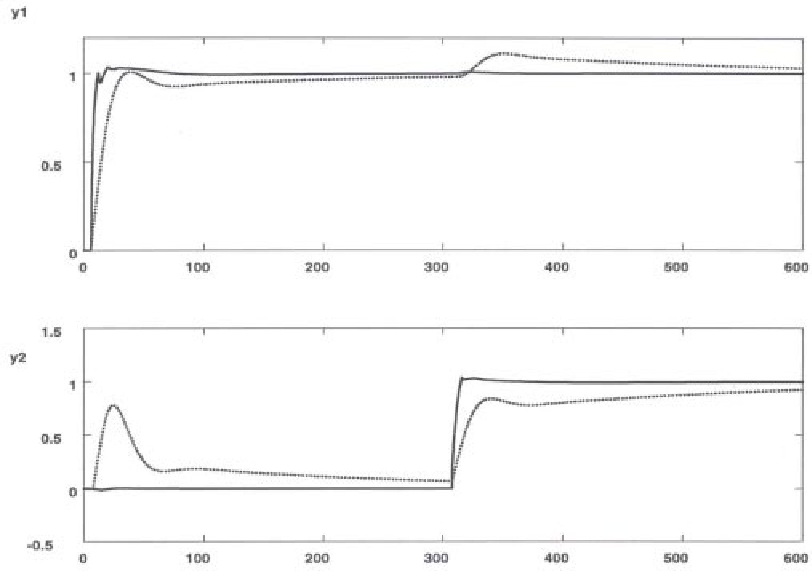


Fig. 7.12. Control Performance for Example 7.2.2 (— Proposed; - - - Jerome)

The reduced-order modelling of the diagonal elements of $G(s)D(s)$ yields

$$Q(s) = \begin{bmatrix} \frac{(19.54s+1)e^{-6.52s}}{8827.65s^2+712.54s+21.33} & 0 \\ 0 & \frac{(57.16s+1)e^{-8.6s}}{15578s^2+964.2s+22.4} \end{bmatrix},$$

and $Q_0(s)$ is readily obtained by taking the delay free part of $Q(s)$. The primary controller is then designed as

$$C(s) = \begin{bmatrix} 205.4 + \frac{11.1}{s} - 47.75s & 0 \\ 0 & -73.0 - \frac{7.02}{s} - 17.2s \end{bmatrix}.$$

For comparison, the BLT tuning method (Luyben, 1986) is used here. The set-point responses are shown in Figure 7.12, exhibiting that significant performance improvement has been achieved with the decoupling Smith design. ♢

Example 7.2.3. Consider the following system in Jerome and Ray (1986)

$$G(s) = \begin{bmatrix} \frac{(1-s)e^{-2s}}{s^2+1.5s+1} & \frac{0.5(1-s)e^{-4s}}{(2s+1)(3s+1)} \\ \frac{0.33(1-s)e^{-6s}}{(4s+1)(5s+1)} & \frac{(1-s)e^{-3s}}{4s^2+6s+1} \end{bmatrix}.$$

The design procedure yields

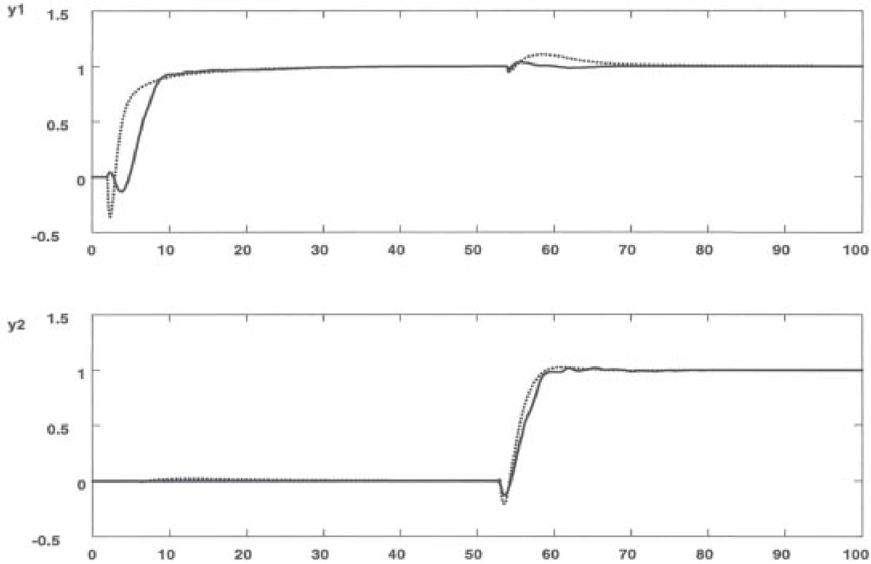


Fig. 7.13. Control Performance for Example 7.2.3

$$D(s) = \begin{bmatrix} 1 & -\frac{(0.5s^2+0.75s+0.5)e^{-2s}}{6s^2+5s+1} \\ -\frac{(1.32s^2+1.98s+0.33)e^{-3s}}{20s^2+9s+1} & 1 \end{bmatrix}.$$

$$Q(s) = \begin{bmatrix} \frac{(16.9s^2+10.7+1)e^{-4.54s}}{17.4s^2+9.77s+1.2} & 0 \\ 0 & \frac{(5.18s+1)e^{-4.83s}}{26.79s^2+9.72s+1.2} \end{bmatrix},$$

and

$$C(s) = \begin{bmatrix} -0.22 + \frac{0.37}{s} + 0.11s & 0 \\ 0 & 1.79 + \frac{0.16}{s} - 0.90s \end{bmatrix}.$$

The set-point change response is shown in Figure 7.13. For comparison the Jerome's multivariable Smith predictor (Jerome and Ray, 1986) is used, where an augmented compensator is required in their controller. The results indicate that the decoupling Smith method also gives improvement in performances.

In order to see the robustness of the decoupling Smith method, we increase the static gains and dead-times of the process' diagonal elements by 20% and 30%, respectively. The set-point response is depicted in Figure 7.14, showing that the performance of the decoupling Smith scheme is much superior compared with the Jerome's multivariable Smith predictor controller. ♢

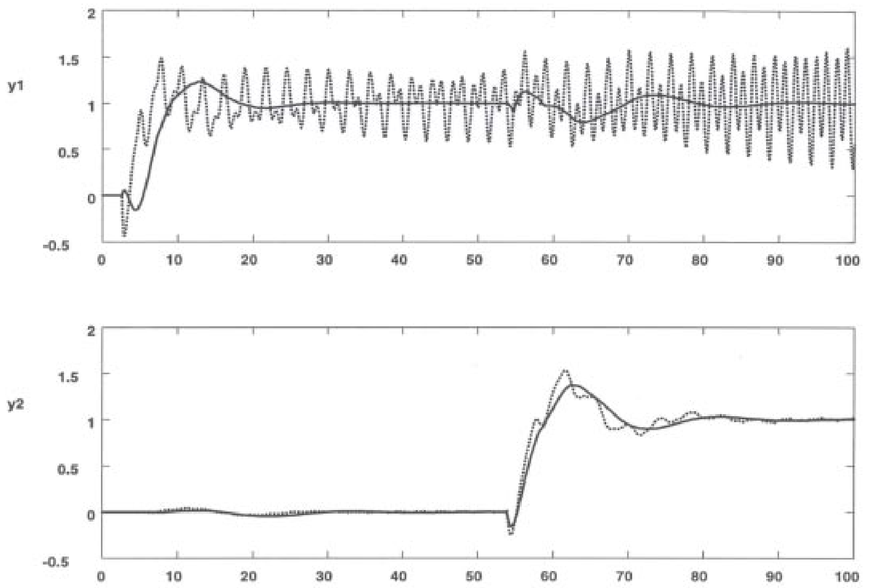


Fig. 7.14. Control System Robustness for Example 7.2.3 (— Proposed; - - - Jerome)

In this section, a stable and decoupling multivariable Smith predictor scheme is presented for multivariable processes with multiple time delay. Our approach to the decoupler design leads to a number of independent single-loop Smith predictor designs. Both nominal and robust stability analysis is given. Examples have been given to illustrate our approach with which significant performance improvement over the existing multivariable Smith predictor control methods is observed. This design is simpler, compared with the IMC control in Section 1.

7.3 Notes and References

The first time delay compensation scheme was proposed by Smith (1957) for SISO plants and is named after him. This scheme was extended to multivariable systems with single delay by Alevisakis and Seborg (1973),Alevisakis and Seborg (1974) and with multi-delays by Ogunnaike and Ray (1979),Ogunnaike *et al.* (1983). Due to multivariable interactions, the performance of such MIMO Smith schemes might sometimes be poor ((Garcia and Morari, 1985). Jerome and Ray (1986) proposed a different version of multivariable Smith predictor control to improve overall performance. However, their design of the primary controller is based on a transfer matrix with time delays and is difficult to carry out. This also contradicts the Smith's philosophy of delay removal from closed-loop characteristic equation for ease of stable design. The work presented in Section 2 is based on Wang *et al.* (2000) and can achieve better performance.

The IMC was introduced by Garcia and Morari (1982) and studied thoroughly in Morari and Zafiriou (1989) as a powerful control design strategy for linear systems. In principle, if a multivariable delay model, $\hat{G}(s)$, is factorized as $\hat{G}(s) = \hat{G}^+(s)\hat{G}^-(s)$ such that $\hat{G}^+(s)$ is diagonal and contains all the time delays and non-minimum phase zeros of $\hat{G}(s)$, then the IMC controller can be designed, like the scalar case, as $K(s) = \{\hat{G}^-(s)\}^{-1}$ with a possible filter appended to it, and the performance improvement of the resultant IMC controller over various versions of the multivariable Smith predictor schemes can be expected (Garcia and Morari, 1985). For a scalar transfer function, it is quite easy to obtain $\hat{G}^+(s)$. However, it becomes much more difficult for a transfer matrix with multi-delays. The factorization is affected not only by the time delays in individual elements but also by their distributions within the transfer function matrix, and a non-minimum phase zero is not related to that of elements of the transfer matrix at all (Holt and Morari, 1985*b*; Holt and Morari, 1985*a*). The systematic method in Section 1 can effectively deal with such a problem and it is from Zhang (1999).

8. Near-Decoupling

Several approaches to decoupling control have been presented in details in the previous chapters and plant uncertainties not addressed explicitly in design though robust stability is checked sometimes. It is noted that exact decoupling may not be necessary even in the nominal case for many applications and usually impossible in the uncertain case. What is really needed instead is to limit the loop interaction to an acceptable level. In this chapter, we propose the concept of near-decoupling and develop an approach to near-decoupling controller design for both nominal and uncertain systems.

This chapter is organized as follows. In Section 8.1, we provide an example to illustrate the necessity for proposing the new concept of near-decoupling. Its exact definition is then given in Section 8.2. Fundamental lemmas for solving near-decoupling control problem and the basic result on the near-decoupling controller design for the case of exact model and state feedback are also presented in this section. The design method is then extended to the cases of (i) uncertain models and state feedback, (ii) exact models and dynamic output feedback, and (iii) uncertain models and dynamic output feedback respectively in Sections 8.3, 8.4 and 8.5. A numerical illustrative example is presented in Section 8.6 to show the whole design procedure of our approach. Finally concluding remarks are drawn in Section 8.7.

Notations: R^n denotes $n-$dimensional Euclidean space, $R^{n\times m}$ is the set of all $n \times m$ real matrices, I is an identity matrix whose dimension is implied from context, $\|\cdot\|_2$ refers to either the Euclidean vector norm or the induced matrix 2-norm, and $\|H\|_\infty$ represents the ∞-norm of a transfer matrix $H(s)$ in $\mathcal{H}_\infty$ space. $\underline{\sigma}(M)$ and $\lambda_{\min}(M)$ denote the smallest singular value and minimum eigenvalue of a matrix M, respectively. M^* stands for the complex conjugate transpose of M. The notation $X > Y$ (or $X \geq Y$, respectively) means that $X - Y$ is positive definite (or positive semidefinite, respectively).

8.1 Motivation for the Concept of Near-Decoupling

Consider the following 2×2 system

$$\dot{x} = A_0 x + B_0 u, \quad y = C_0 x,$$

where

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 267-292, 2003.

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -3 & -3 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Its transfer matrix is

$$H_0(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{4s+6}{(s+1)(s+2)(s+3)} & \frac{3}{(s+2)(s+3)} \end{bmatrix}.$$

It is found in Chapter 4 that the above system can be decoupled by the following state feedback

$$u = Kx + Gv,$$

where

$$K = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{3} & -\frac{4}{3} & -3 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ -\frac{4}{3} & \frac{1}{3} \end{bmatrix}.$$

Indeed, the resulting closed-loop transfer matrix is

$$H_{0cl}(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}.$$

Now suppose that there are some parameter perturbations in matrix A_0, i.e., suppose

$$A_0 = \begin{bmatrix} -1+a_1 & 0 & 0 \\ 0 & -2+a_2 & 0 \\ 0 & 0 & -3+a_3 \end{bmatrix},$$

Under the same decoupling law, the closed-loop transfer matrix of the perturbed system is as follows

$$H_{cl}(s) = \begin{bmatrix} \frac{1}{s-a_1} & 0 \\ \frac{(a_3-2a_2+a_1)s+a_1a_2+a_2a_3-2a_1a_3-a_1+4a_2-3a_3}{(s-a_1)(s^2-(a_2+a_3)s+6a_2-6a_3+a_2a_3)} & \frac{a_2-a_3+1}{s^2-(a_2+a_3)s+6a_2-6a_3+a_2a_3} \end{bmatrix}.$$

Thus it is clear that the decoupling still holds only when

$$\begin{cases} a_3 - 2a_2 + a_1 = 0, \\ a_1a_2 + a_2a_3 - 2a_1a_3 - a_1 + 4a_2 - 3a_3 = 0. \end{cases}$$

This is the case if

$$\begin{cases} a_1 = a_3, \\ a_2 = a_3, \end{cases}$$

or

$$\begin{cases} a_1 = a_3 - 2, \\ a_2 = a_3 - 1. \end{cases} \tag{8.1}$$

The relationship (8.1) is inadmissable since it leads to $H_{cl}(2,2) \equiv 0$. Therefore only when $a_1 = a_2 = a_3$, does the perturbed system remain decoupled. If we regard $a := (a_1, a_2, a_3)$ as a point in R^3 and a_1, a_2 and a_3 vary independently in the space R^3, then the exact decoupling can be achieved only when a hits the line $\{(a_1, a_2, a_3) \in R^3 : a_1 = a_2 = a_3\}$. It is well known that the probability that a point randomly distributed in the space R^3 hits a line is zero in the sense of geometric probability measure.

It is clear that the above procedure is applicable to general multiinput multioutput (MIMO) linear systems with parameter perturbations and it can be shown that a (fixed) decoupling law can decouple a perturbed MIMO system only when the parameter perturbations are in a subspace of the space consisting of all free parameter perturbations. Therefore we can generally say that exact decoupling cannot be achieved almost for every perturbed system.

Based upon the aforementioned fact, we will propose a new concept for decoupling: *near-decoupling*. According to this concept, our aim of designing an near-decoupling law is to make the magnitude of off-diagonal elements in the transfer matrix of a system as small as possible, instead of zero, under the existence of parameter perturbations, while to maintain the magnitude of diagonal elements in the transfer matrix to be larger than a given number.

8.2 Near-Decoupling for Unperturbed Systems: State Feedback

In this section, we first define the concept of near-decoupling and then develop a method to find solutions for near-decoupling controllers using state feedback for systems with no perturbations, which will lay a basis for the developments of the sections to be followed. Let us consider a linear time-invariant system (A, B, C, D) described by

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{8.2}$$

where $x \in R^n$ denotes the state, $u \in R^m$ denotes the control input, $y \in R^l$ denotes the output, and A, B, C and D are constant matrices with appropriate dimensions. The output y is partitioned in p block outputs y_i, each of size l_i, with $\sum_{i=1}^{p} l_i = l$. Similarly, The input u is also partitioned in p block inputs u_i, each of size m_i, with $\sum_{i=1}^{p} m_i = m$. The block subsystems (A, B_j, C_i, D_{ij}) of (A, B, C, D) are introduced where C_i, B_j and D_{ij} are the ith block of C, jth block of B and ij-th block of D, respectively, in connection with the splitting of $y = [y_1^T, y_2^T, \cdots, y_p^T]^T$ and $u = [u_1^T, u_2^T, \cdots, u_p^T]^T$. Thus for such a splitting, we write

$$B = [B_1, B_2, \cdots, B_p], \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1p} \\ D_{21} & D_{22} & \cdots & D_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ D_{p1} & D_{p2} & \cdots & D_{pp} \end{bmatrix}.$$

For the convenience of later exposition, we write

$$D_i = \begin{bmatrix} D_{i1} & D_{i2} & \cdots & D_{ip} \end{bmatrix}.$$

Correspondingly, the transfer matrix of system (8.2) is also split as

$$H(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1p}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ H_{p1}(s) & H_{p2}(s) & \cdots & H_{pp}(s) \end{bmatrix}$$

with

$$H_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}, \quad i, j = 1, 2, \ldots, p.$$

Definition 8.2.1. *System (8.2) is said to be near-decoupled for the input-output pairs* (u_i, y_i), $i = 1, 2, \ldots, p$, *if*

(i) *system (8.2) is stable;*
(ii) *the following inequalities*

$$\underline{\sigma}(H_{ii}(j\omega)) \geq \beta_i, \quad \forall i \in \{1, 2, \ldots, p\}, \forall \omega \in (-\infty, +\infty), \tag{8.3}$$
$$\|H_{ij}\|_\infty < \gamma, \quad \forall i, j \in \{1, 2, \ldots, p\}, i \neq j. \tag{8.4}$$

hold for given numbers $\beta_i > 0$, $i = 1, 2, \ldots, p$, *and* $\gamma > 0$.

Definition 8.2.2. *System (8.2) is said to be nearly state-feedback decouplable if there exists a state feedback law*

$$u = Kx + v \tag{8.5}$$

such that the closed loop system (8.2)-(8.5) is near-decoupled for the new input-output pairs (v_i, y_i), $i = 1, 2, \ldots, p$, *in the sense of Definition 8.2.1.*

Notice that the parameter γ in the above definitions should be sufficiently small, whose value can be either specified by the designer or minimized in the controller design process. In the first case, we can generally specify, for example, $\gamma = 10^{-3} \times \min\{\beta_1, \beta_2, \ldots, \beta_p\}$.

To make the design of controller (8.5) be numerically tractable, it is highly desirable to relate the performance indices described in Definition 8.2.1 directly to the system parameters (A, B, C, D). For requirement (8.4), it can be dealt with by the following lemma, which has become a standard result in H_∞ control.

Lemma 8.2.1. *(Boyd* et al., *1994) Denote by $H(s)$ the transfer matrix of a system (A, B, C, D). Then the condition*

$$\|H\|_\infty < \gamma$$

is equivalent to $\exists P > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

To deal with the requirement (8.3), let us first establish the following result.

Lemma 8.2.2. *Consider system (8.2) with transfer matrix $H(s)$. Then the inequality*

$$\underline{\sigma}(H(j\omega)) \geq \beta, \quad \forall \omega \in (-\infty, +\infty), \tag{8.6}$$

holds if and only if

$$\int_{-\infty}^{+\infty} y^T(t)y(t)\mathrm{d}t \geq \beta^2 \int_{-\infty}^{+\infty} u^T(t)u(t)\mathrm{d}t, \quad \forall u \in \mathcal{L}_2(-\infty, +\infty) \tag{8.7}$$

holds.

Proof. The *only if* part. First notice, in terms of Parseval's theorem, the fact that inequality (8.7) is equivalent to the following inequality:

$$\int_{-\infty}^{+\infty} Y^*(j\omega)Y(j\omega)\mathrm{d}\omega \geq \beta^2 \int_{-\infty}^{+\infty} U^*(j\omega)U(j\omega)\mathrm{d}\omega \quad \forall U \in \mathcal{H}_2, \tag{8.8}$$

where $U(j\omega)$ and $Y(j\omega)$ denote the Fourier's transform of $u(t)$ and $y(t)$ respectively. For a brief introduction on the spaces $\mathcal{L}_2(-\infty, +\infty)$ and $\mathcal{H}_2$, readers is referred to (Zhou *et al.*, 1996). Inequality (8.6) means that

$$H^*(j\omega)H(j\omega) \geq \beta^2 I, \quad \forall \omega \in (-\infty, +\infty).$$

Thus we have

$$\begin{aligned} \int_{-\infty}^{+\infty} y^T(t)y(t)\mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y^*(j\omega)Y(j\omega)\mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} U^*(j\omega)H^*(j\omega)H(j\omega)U(j\omega)\mathrm{d}\omega \\ &\geq \frac{\beta^2}{2\pi} \int_{-\infty}^{+\infty} U^*(j\omega)U(j\omega)\mathrm{d}\omega \\ &= \beta^2 \int_{-\infty}^{+\infty} u^T(t)u(t)\mathrm{d}t. \end{aligned}$$

Now let us prove the *if* part. Suppose, by contradiction, that inequality (8.6) does not hold for all $\omega \in (-\infty, +\infty)$. Without loss of generality, we temporarily assume $m \le l$. According to singular value decomposition theorem, there exist unitary matrices $E_L(j\omega)$ and $E_R(j\omega)$ such that

$$H(j\omega) = E_L(j\omega)\Lambda(j\omega)E_R^*(j\omega)$$

where

$$\Lambda(j\omega) = \begin{bmatrix} \sigma_1(j\omega) & 0 & \cdots & 0 \\ 0 & \sigma_2(j\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m(j\omega) \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Notice that the last row of zeros actually includes $l - m$ rows of zeros. Now we have

$$H^*(j\omega)H(j\omega) = E_R(j\omega) \begin{bmatrix} |\sigma_1(j\omega)|^2 & 0 & \cdots & 0 \\ 0 & |\sigma_2(j\omega)|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\sigma_m(j\omega)|^2 \end{bmatrix} E_R^*(j\omega)$$

Suppose that there exists an ω_0 such that

$$|\sigma_m(j\omega_0)|^2 < \beta^2.$$

Since $\sigma_m(j\omega)$ is continuous with respect to ω, it follows that there exists a closed interval $\mathcal{I}$ with $\omega_0 \in \mathcal{I}$ and $|\mathcal{I}| \neq 0$, where $|\mathcal{I}|$ denotes the length of the set $\mathcal{I}$, such that

$$|\sigma_m(j\omega)|^2 < \beta^2, \quad \forall \omega \in \mathcal{I}.$$

Notice that there might be many such closed intervals and the length of the such interval might be infinite. In this case, we just choose one of such intervals and limit its length to be finite. Define $e_m = [0 \ \ \cdots \ \ 0 \ \ 1]^T \in R^m$. Now we choose the signal $U(j\omega)$ as follows

$$U(j\omega) = \begin{cases} E_R(j\omega)e_m & \text{when } \omega \in \mathcal{I}, \\ 0 & \text{otherwise.} \end{cases}$$

Such a choice on $U(j\omega)$ will lead to

$$\int_{-\infty}^{+\infty} U^*(j\omega)U(j\omega)\mathrm{d}\omega = |\mathcal{I}| \tag{8.9}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} Y^*(j\omega)Y(j\omega)\mathrm{d}\omega \\
&= \int_{\mathcal{I}} e_m^T E_R^*(j\omega)E_R(j\omega)\Lambda^*(j\omega)\Lambda(j\omega)E_R^*(j\omega)E_R(j\omega)e_m \mathrm{d}\omega \\
&= \int_{\mathcal{I}} |\sigma_m(j\omega)|^2 \mathrm{d}\omega \\
&< \beta^2 |\mathcal{I}|.
\end{aligned} \tag{8.10}$$

Combining (8.9) and (8.10) gives

$$\int_{-\infty}^{+\infty} Y^*(j\omega)Y(j\omega)\mathrm{d}\omega < \beta^2 \int_{-\infty}^{+\infty} U^*(j\omega)U(j\omega)\mathrm{d}\omega,$$

which contradicts the inequality (8.7). Thus the proof is completed. □

Lemma 8.2.3. *Denote by $H(s)$ the transfer matrix of system (8.2). If there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} -A^T P - PA + C^T C & -PB + C^T D \\ -B^T P + D^T C & -\beta^2 I + D^T D \end{bmatrix} \geq 0, \tag{8.11}$$

the following inequality

$$\underline{\sigma}(H(j\omega)) \geq \beta, \quad \forall \omega \in [-\infty, +\infty], \tag{8.12}$$

holds.

Proof. Let $V(x) = -x^T P x$. Consider the motion of system (8.2) with zero initial condition, i.e., $x(0) = 0$. If one can prove that

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t} + y^T(t)y(t) - \beta^2 u^T(t)u(t) \geq 0, \quad \forall u \in \mathcal{L}_2(-\infty, +\infty), \tag{8.13}$$

then one has

$$\begin{aligned}
&\int_0^{+\infty} [y^T(t)y(t) - \beta^2 u^T(t)u(t)]\mathrm{d}t \\
&\geq - \int_0^{+\infty} \frac{\mathrm{d}V(x)}{\mathrm{d}t}\mathrm{d}t = x^T(+\infty)Px(+\infty) - x^T(0)Px(0) \geq 0.
\end{aligned}$$

Thus

$$\int_0^{+\infty} y^T(t)y(t)\mathrm{d}t \geq \beta^2 \int_0^{+\infty} u^T(t)u(t)\mathrm{d}t, \quad \forall u \in \mathcal{L}_2(-\infty, +\infty).$$

From Lemma 8.2.2, we can therefore conclude that inequality (8.12) holds true.

Now simple algebra yields that (8.13) means

$$[x^T \quad u^T] \left\{ \begin{bmatrix} -(A^T P + PA) & -PB \\ -B^T P & -\beta^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C \quad D] \right\} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0.$$

Thus it is clear that if inequality (8.11) holds, inequalities (8.13) and then (8.12) hold too. □

Corollary 8.2.1. *For system (8.2), if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} -A^T P - PA & -PB + C^T D \\ -B^T P + D^T C & -\beta^2 I + D^T D \end{bmatrix} \geq 0,$$

then inequality (8.12) holds.

Remark 8.2.1. From Lemma 8.2.3 it can be seen that the parameter β is upper bounded by

$$\beta \leq \sqrt{\lambda_{\min}(D^T D)}.$$

Based upon the above development, we can establish the following theorem.

Theorem 8.2.1. *For system (8.2), if there are matrices $Q > 0$ and F such that the following LMIs*

$$\begin{bmatrix} -QA^T - AQ - F^T B^T - BF & -B_i + (QC_i^T + F^T D_i^T)D_{ii} \\ -B_i^T + D_{ii}^T (C_i Q + D_i F) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix} > 0, \quad \forall i \in \{1, 2, \dots, m\}; \tag{8.14}$$

$$\begin{bmatrix} QA^T + AQ + F^T B^T + BF & B_j & QC_i^T + F^T D_i^T \\ B_j^T & -\gamma I & D_{ij}^T \\ C_i Q + D_i F & D_{ij} & -\gamma I \end{bmatrix} < 0, \quad \forall i, j \in \{1, 2, \dots, m\}, i \neq j. \tag{8.15}$$

hold, then system (8.2) is nearly state-feedback decouplable. The state-feedback law is given by

$$u = Kx + v, \quad K = FQ^{-1} \tag{8.16}$$

and the closed-loop performance indices satisfy

$$\underline{\sigma}(H_{cl,ii}(j\omega)) \geq \beta_i, \quad \forall i \in \{1, 2, \dots, p\}, \forall \omega \in [-\infty, +\infty], \tag{8.17}$$

$$\|H_{cl,ij}\|_\infty < \gamma, \quad \forall i, j \in \{1, 2, \dots, p\}, i \neq j, \tag{8.18}$$

where $\beta_i < \sqrt{\lambda_{\min}(D_{ii}^T D_{ii})}$, $i = 1, 2, \dots, p$.

Proof. System (8.2) under the control (8.16) reads as

$$\dot{x} = (A + BK)x + Bv,$$
$$y = (C + DK)x + Dv.$$

Applying Lemma 8.2.1 and Corollary 8.2.1 to the block transfer matrix $H_{cl,ij}(s) = \mathtt{trans}(A + BK, B_j, (C + DK)_i, D_{ij}) = \mathtt{trans}(A + BK, B_j, C_i + D_iK, D_{ij})$, $i, j \in \{1, 2, \dots, p\}$ (where `trans` refers the transfer matrix of a state-space model) yields that performance requirements (8.17)-(8.18) are satisfied if there exists a matrix $P > 0$ such that the following matrix inequalities

$$\begin{bmatrix} -(A + BK)^T P - P(A + BK) & -PB_i + (C_i + D_iK)^T D_{ii} \\ -B_i^T P + D_{ii}^T (C_i + D_iK) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix} > 0,$$
$$\forall i \in \{1, 2, \dots, m\}; \tag{8.19}$$

$$\begin{bmatrix} (A + BK)^T P + P(A + BK) & PB_j & (C_i + D_iK)^T \\ B_j^T P & -\gamma I & D_{ij}^T \\ (C_i + D_iK) & D_{ij} & -\gamma I \end{bmatrix} < 0,$$
$$\forall i, j \in \{1, 2, \dots, m\}, i \neq j \tag{8.20}$$

hold. Let $Q = P^{-1}$ and $F = KQ$. Then we have

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -(A + BK)^T P - P(A + BK) & -PB_i + (C_i + D_iK)^T D_{ii} \\ -B_j^T P + D_{ii}^T (C_i + D_iK) & -\beta^2 I + D_{ii}^T D_{ii} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} -QA^T - AQ - F^T B^T - BF & -B_i + (QC_i^T + F^T D_i^T) D_{ii} \\ -B_i^T + D_{ii}^T (C_i Q + D_i F) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix},$$

$$\begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} (A + BK)^T P + P(A + BK) & PB_j & (C_i + D_iK)^T \\ B_j^T P & -\gamma I & D_{ij}^T \\ (C_i + D_iK) & D_{ij} & -\gamma I \end{bmatrix} \begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} QA^T + AQ + F^T B^T + BF & B_j & QC_i^T + F^T D_i^T \\ B_j^T & -\gamma I & D_{ij}^T \\ C_i Q + D_i F & D_{ij} & -\gamma I \end{bmatrix},$$

i.e., matrix inequalities (8.19) and (8.20) are equivalent to LMIs (8.14) and (8.15), respectively. In regard to the stableness of the closed-loop system, we notice that both (8.19) and (8.20) imply that $(A+BF)^T P+P(A+BK) < 0$, which further implies that the closed-loop system is stable. This completes the proof. □

Notice that the argument in the end of the above proof also applies to Theorems 8.3.1, 8.4.1, and 8.5.1 given later on. Hence we will not repeat it in the proof of these theorems.

In the decoupling controller design for practical plants, the magnitude of parameters β_i, $i = 1, 2, \dots, m$, can be readily estimated from Remark 8.2.1, but it is hard to infer the magnitude of parameter γ. Therefore we formulate the near-decoupling controller design (NDCD) in the case of state-feedback (SF) as the following optimization problem.

Problem NDCD-SF:

minimize γ
subject to LMIs $Q > 0$, (8.14), (8.15).

The above problem is a standard eigenvalue problem (EVP) in LMI language (Boyd *et al.*, 1994), hence its numerical solutions can be found through LMI toolbox of Matlab very efficiently (Gahinet *et al.*, 1995).

8.3 Robust Near-Decoupling: State Feedback

Now consider the case that there are some parameter perturbations in the system matrix A. We assume that the perturbations are structured as follows

$$A = A_0 + A_l \Psi A_r, \tag{8.21}$$
$$\Psi \in \breve{\Psi} := \{\Psi \in R^{n_1 \times n_2} : \|\Psi\|_2 \leq 1\},$$

where A_0, a constant matrix, is the nominal parameter of A, Ψ denotes the uncertainty, A_l and A_r, constant matrices with appropriate dimensions, are used to describe how the uncertainty Ψ enters the system matrix A. Notice that any uncertainties with bounded norm can be described by the form (8.21).

Theorem 8.3.1. *Consider uncertain system (8.2) with matrix A described by (8.21). If there are matrices $Q > 0$, F and numbers $\varepsilon_{ij} > 0$ $(i, j = 1, 2, \ldots, p)$ such that the following LMIs*

$$\begin{bmatrix} -QA_0^T - A_0 Q - F^T B^T - BF - \varepsilon_{ii} A_l A_l^T & -B_i + (QC_i^T + F^T D_i^T) D_{ii} & QA_r^T \\ -B_i^T + D_{ii}^T (C_i Q + D_i F) & -\beta_i^2 I + D_{ii}^T D_{ii} & 0 \\ A_r Q & 0 & \varepsilon_{ii} I \end{bmatrix} > 0,$$
$$\forall i \in \{1, 2, \ldots, m\}, \tag{8.22}$$

$$\begin{bmatrix} QA_0^T + A_0 Q + F^T B^T + BF + \varepsilon_{ij} A_l A_l^T & B_j & QC_i^T + F^T D_i^T & QA_r^T \\ B_j^T & -\gamma I & D_{ij}^T & 0 \\ C_i Q + D_i F & D_{ij} & -\gamma I & 0 \\ A_r Q & 0 & 0 & -\varepsilon_{ij} I \end{bmatrix} < 0,$$
$$\forall i, j \in \{1, 2, \ldots, m\}, i \neq j, \tag{8.23}$$

hold, then system (8.2) is nearly state-feedback decouplable. The state-feedback law and closed-loop performance indices are given by (8.16) and (8.17)-(8.18) respectively.

To prove this theorem, we need the following lemma.

Lemma 8.3.1. *Let Φ_l, Φ_r, Υ be real matrices of appropriate dimensions with Υ satisfying $\|\Upsilon\|_2 \leq 1$. Then for any real number $\varepsilon > 0$ we have*

$$\Phi_l \Upsilon \Phi_r + \Phi_r^T \Upsilon^T \Phi_l^T \leq \varepsilon \Phi_l \Phi_l^T + \varepsilon^{-1} \Phi_r^T \Phi_r. \tag{8.24}$$

Proof. The assumption $\|\Upsilon\|_2 \leq 1$ leads to $I - \Upsilon\Upsilon^T \geq 0$, which again yields the following inequality

$$\begin{bmatrix} I & \Upsilon \\ \Upsilon^T & I \end{bmatrix} \geq 0$$

by applying Schur complements (Boyd *et al.*, 1994). Then for any two real numbers ε_1 and ε_2 we will have

$$\begin{bmatrix} \varepsilon_1 \Phi_l & -\varepsilon_2 \Phi_r^T \end{bmatrix} \begin{bmatrix} I & \Upsilon \\ \Upsilon^T & I \end{bmatrix} \begin{bmatrix} \varepsilon_1 \Phi_l^T \\ -\varepsilon_2 \Phi_r \end{bmatrix} \geq 0. \tag{8.25}$$

If $\varepsilon_1 \varepsilon_2 = 1$, the inequality (8.25) will be of the following form

$$\varepsilon_1^2 \Phi_l \Phi_l^T + \varepsilon_2^2 \Phi_r^T \Phi_r - \Phi_l \Upsilon \Phi_r - \Phi_r^T \Upsilon^T \Phi_l^T \geq 0,$$

which is equivalent to inequality (8.24). □

Proof of theorem 8.3.1. It follows from Theorem 8.2.1 that if there are matrices $Q > 0$ and F such that the following inequalities

$$\begin{bmatrix} -QA^T - AQ - F^T B^T - BF & -B_i + (QC_i^T + F^T D_i^T) D_{ii} \\ -B_i^T + D_{ii}^T (C_i Q + D_i F) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix} > 0,$$
$$\forall i \in \{1, 2, \dots, m\}, \forall \Psi \in \breve{\Psi}; \tag{8.26}$$

$$\begin{bmatrix} QA^T + AQ + F^T B^T + BF & B_j & QC_i^T + F^T D_i^T \\ B_j^T & -\gamma I & D_{ij}^T \\ C_i Q + D_i F & D_{ij} & -\gamma I \end{bmatrix} < 0,$$
$$\forall i, j \in \{1, 2, \dots, m\}, i \neq j, \forall \Psi \in \breve{\Psi} \tag{8.27}$$

hold, then system (8.2) is nearly state-feedback decouplable with performance requirement (8.17)-(8.18) being satisfied. The difficulty lies in the fact that the above inequalities should be satisfied for all possible uncertainty Ψ. Now we try to remove the term Ψ in the above inequalities. In view of Lemma 8.3.1, we have

$$\begin{aligned} QA^T + AQ &= QA_0^T + A_0 Q + QA_r^T \Psi^T A_l^T + A_l \Psi A_r Q \\ &\leq QA_0^T + A_0 Q + \varepsilon^{-1} QA_r^T A_r Q + \varepsilon A_l A_l^T, \quad \forall \varepsilon \in (0, +\infty). \end{aligned}$$

Therefore inequalities (8.26) and (8.27) hold if the following inequalities

$$\begin{bmatrix} -(QA_0^T + A_0 Q + \varepsilon_{ii}^{-1} QA_r^T A_r Q + \varepsilon_{ii} A_l A_l^T) - F^T B^T - BF & -B_i + (QC_i^T + F^T D_i^T) D_{ii} \\ -B_i^T + D_{ii}^T (C_i Q + D_i F) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix} > 0,$$
$$\forall i \in \{1, 2, \dots, m\}; \tag{8.28}$$

$$\begin{bmatrix} QA_0^T + A_0 Q + \varepsilon_{ij}^{-1} QA_r^T A_r Q + \varepsilon_{ij} A_l A_l^T + F^T B^T + BF & B_j & QC_i^T + F^T D_i^T \\ B_j^T & -\gamma I & D_{ij}^T \\ C_i Q + D_i F & D_{ij} & -\gamma I \end{bmatrix} < 0,$$
$$\forall i, j \in \{1, 2, \dots, m\}, i \neq j \tag{8.29}$$

are satisfied for some positive numbers ε_{ij} $(i, j = 1, 2, \dots, m)$. It follows from Schur complements that inequalities (8.28) and (8.29) are equivalent to LMIs (8.22) and (8.23), respectively. Thus the proof is completed. $\square$

Similarly, we can formulate the robust near-decoupling controller design (RNDCD) in the case of state-feedback as the following optimization problem.

Problem RNDCD-SF:

minimize	γ
subject to	LMIs $Q > 0$, (8.22), (8.23).

In both problems NDCD-SF and RNDCD-SF, the minimization procedure will give a very large Q, making Q^{-1} ill-conditioned. Therefore for these two problems, we impose the following additional constraint

$$Q < \mu I$$

on the corresponding problems in the practical controller computation, where μ is a given positive number.

8.4 Near-Decoupling: Dynamic Output Feedback

In many cases, all state variables are not available. Thus it is necessary to develop a counterpart of the results of the preceding two sections based on output feedback. In the two sections to be followed, we will try to deal with this issue. Our controller is described by the following dynamical output feedback law:

$$\begin{cases} \dot{z} = A_K z + B_K(y - Du), \\ u = C_K z + D_K(y - Du) + v, \end{cases} \tag{8.30}$$

i.e.,

$$\begin{cases} \dot{z} = A_K z + B_K C x, \\ u = C_K z + D_K C x + v. \end{cases} \tag{8.31}$$

We suppose that

$$|D_K D + I| \neq 0.$$

Our basic idea is to use the results already obtained hereto to develop a method for the design of controller (8.30). To proceed, we write the closed-loop system consisting of (8.2) and (8.31) as follows

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v, \\ \quad y = \begin{bmatrix} C + DD_K C & DC_K \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + Dv. \end{cases} \tag{8.32}$$

Let

$$\bar{A} = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C + DD_K C & DC_K \end{bmatrix}, \quad \bar{D} = D.$$

Then it follows from Lemma 8.2.1 and Corollary 8.2.1 that system (8.32) is near-decoupled with performance requirement (8.17)-(8.18) being satisfied if there exists a matrix $\bar{P} > 0$ such that the following matrix inequalities

$$\begin{bmatrix} -\bar{A}^T\bar{P} - \bar{P}\bar{A} & -\bar{P}\bar{B}_i + \bar{C}_i^T\bar{D}_{ii} \\ -\bar{B}_i^T\bar{P} + \bar{D}_{ii}^T\bar{C}_i & -\beta_i^2 I + \bar{D}_{ii}^T\bar{D}_{ii} \end{bmatrix} > 0, \quad \forall i \in \{1, 2, \dots, m\}, \tag{8.33}$$

$$\begin{bmatrix} \bar{A}^T\bar{P} + \bar{P}\bar{A} & \bar{P}\bar{B}_j & \bar{C}_i^T \\ \bar{B}_j^T\bar{P} & -\gamma I & \bar{D}_{ij}^T \\ \bar{C}_i & \bar{D}_{ij} & -\gamma I \end{bmatrix} < 0, \quad \forall i, j \in \{1, 2, \dots, m\}, i \neq j \tag{8.34}$$

hold.

Due to the fact that the matrix $\bar{A}$ also includes the design parameters A_K, B_K, C_K and D_K, (8.33) and (8.34) are bilinear matrix inequalities in the unknowns $\bar{P}$, A_K, B_K, C_K and D_K, which is very difficulty to solve. Thanks to the pioneer work by Chilali, Gahinet and Scherer (Chilali and Gahinet, 1996; Scherer *et al.*, 1997), both (8.33) and (8.34) can be transformed into two LMIs by taking a change of variables. In the following, we will use the method developed in (Chilali and Gahinet, 1996; Scherer *et al.*, 1997) to solve our problem.

Let us partition $\bar{P}$ and $\bar{P}^{-1}$ as

$$\bar{P} = \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix}, \quad \bar{P}^{-1} = \begin{bmatrix} X & M \\ M^T & \star \end{bmatrix}, \tag{8.35}$$

where X and Y are $n \times n$ positive definite matrices, M and N are $n \times k$ matrices with k being the order of the controller (8.30), and $\star$ denotes the matrices in which we are not interested. Clearly M and N should satisfy the following relationship

$$MN^T = I - XY. \tag{8.36}$$

It will be clear later that X and Y, together with other matrices, are design variables which make the required performance indices be satisfied, while M and N are free parameters, the unique requirement for which is to satisfy equation (8.36). Define

$$\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}.$$

It is clear that

$$\bar{P}\Pi_1 = \Pi_2.$$

Now define the change of controller variables as follows:

$$\begin{cases} \hat{A} := Y(A + BD_K C)X + NB_K CX + YBC_K M^T + NA_K M^T \\ \hat{B} := YBD_K + NB_K \\ \hat{C} := D_K CX + C_K M^T \\ \hat{D} := D_K \end{cases} \tag{8.37}$$

Then a short calculation yields the following identities

$$\begin{cases} \Pi_1^T \bar{P} \bar{A} \Pi_1 = \Pi_2^T \bar{A} \Pi_1 = \begin{bmatrix} AX + B\hat{C} & A + B\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{bmatrix}, \\ \Pi_1^T \bar{P} \bar{B} = \Pi_2^T \bar{B} = \begin{bmatrix} B \\ YB \end{bmatrix}, \\ \bar{C}\Pi_1 = \begin{bmatrix} CX + D\hat{C} & C + D\hat{D}C \end{bmatrix}, \\ \Pi_1^T \bar{P} \Pi_1 = \Pi_1^T \Pi_2 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}. \end{cases} \tag{8.38}$$

The partition of $\bar{P}$ and $\bar{P}^{-1}$ defined in (8.35), the new controller variables identified in (8.37) and the relationship shown in (8.38) are developed in (Chilali and Gahinet, 1996; Scherer *et al.*, 1997), which make it easy to transform many problems of dynamic output feedback based controller design to LMI related problems. In the sequel, we make another try along this direction while solving our problems.

Multiplying the left- and right-hand-sides of inequality (8.33) by block-diag $\{\Pi_1^T,\ I\}$ and its transpose respectively yields

$$\begin{bmatrix} \Pi_1^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -\bar{A}^T\bar{P} - \bar{P}\bar{A} & -\bar{P}\bar{B}_i + \bar{C}_i^T\bar{D}_{ii} \\ -\bar{B}_i^T\bar{P} + \bar{D}_{ii}^T\bar{C}_i & -\beta_i^2 I + \bar{D}_{ii}^T\bar{D}_{ii} \end{bmatrix} \begin{bmatrix} \Pi_1 & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} -AX - XA^T - B\hat{C} - \hat{C}^T B^T & -A - B\hat{D}C - \hat{A}^T & -B_i + (C_iX + D_i\hat{C})^T D_{ii} \\ -(A + B\hat{D}C)^T - \hat{A} & -YA - A^TY - \hat{B}C - C^T\hat{B}^T & -YB_i + (C_i + D_i\hat{D}C)^T D_{ii} \\ -B_i^T + D_{ii}^T(C_iX + D_i\hat{C}) & -B_i^TY + D_{ii}^T(C_i + D_i\hat{D}C) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix}$$

$$> 0, \quad \forall i \in \{1, 2, \dots, m\}; \tag{8.39}$$

Multiplying the left- and right-hand-sides of inequality (8.34) by block-diag $\{\Pi_1^T,\ I,\ I\}$ and its transpose respectively yields

$$\begin{bmatrix} \Pi_1^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}^T\bar{P} + \bar{P}\bar{A} & \bar{P}\bar{B}_j & \bar{C}_i^T \\ \bar{B}_j^T\bar{P} & -\gamma I & \bar{D}_{ij}^T \\ \bar{C}_i & \bar{D}_{ij} & -\gamma I \end{bmatrix} \begin{bmatrix} \Pi_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} AX + XA^T + B\hat{C} + \hat{C}^T B^T & A + B\hat{D}C + \hat{A}^T & B_j & (C_iX + D_i\hat{C})^T \\ (A + B\hat{D}C)^T + \hat{A} & YA + A^TY + \hat{B}C + C^T\hat{B}^T & YB_j & (C_i + D_i\hat{D}C)^T \\ B_j^T & B_j^TY & -\gamma I & D_{ij}^T \\ C_iX + D_i\hat{C} & C_i + D_i\hat{D}C & D_{ij} & -\gamma I \end{bmatrix}$$

$$< 0, \quad \forall i, j \in \{1, 2, \dots, m\}, i \neq j. \tag{8.40}$$

In the above derivation, we have used the following convention:

$$\bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} C_i + D_iD_KC & D_iC_K \end{bmatrix},$$

$$\bar{D}_{ij} = D_{ij}, \quad \bar{D}_i = \begin{bmatrix} \bar{D}_{i1} & \bar{D}_{i2} & \cdots & \bar{D}_{ip} \end{bmatrix}.$$

Thus, similar to equation (8.38), one has the following identities

$$\begin{aligned} \Pi_1^T \bar{P} \bar{B}_i &= \Pi_2^T \bar{B}_i = \begin{bmatrix} B_i \\ Y B_i \end{bmatrix}, \\ \bar{C}_i \Pi_1 &= \begin{bmatrix} C_i X + D_i \hat{C} & C_i + D_i \hat{D} C \end{bmatrix}. \end{aligned}$$

Notice that inequalities (8.39) and (8.40) are linear in the unknown variables X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$. Once the solutions for these variables are found, the original controller parameters can be recovered from (Chilali and Gahinet, 1996; Scherer *et al.*, 1997)

$$\begin{cases} D_K = \hat{D} \\ C_K = (\hat{C} - D_K C X) M^{-T} \\ B_K = N^{-1}(\hat{B} - Y B D_K) \\ A_K = N^{-1}[\hat{A} - Y(A + B D_K C) X - N B_K C X - Y B C_K M^T] M^{-T} \end{cases} \tag{8.41}$$

with the assumption that both M and N are square. The nonsingularness of M and N comes from the fact that $X - Y^{-1} > 0$, which can be inferred from the last equation of equation (8.38).

We summarize the above results into the following theorem.

Theorem 8.4.1. *System (8.2) is near-decoupled under dynamic output feedback controller (8.30) with performance (8.17)-(8.18) if there exist matrices* X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$ *and* $\hat{D}$ *such that the following LMIs*

$$\begin{bmatrix} -AX - XA^T - B\hat{C} - \hat{C}^T B^T & -A - B\hat{D}C - \hat{A}^T & -B_i + (C_i X + D_i \hat{C})^T D_{ii} \\ -(A + B\hat{D}C)^T - \hat{A} & -YA - A^T Y - \hat{B}C - C^T \hat{B}^T & -YB_i + (C_i + D_i \hat{D} C)^T D_{ii} \\ -B_i^T + D_{ii}^T (C_i X + D_i \hat{C}) & -B_i^T Y + D_{ii}^T (C_i + D_i \hat{D} C) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix} > 0,$$
$$\forall i \in \{1, 2, \ldots, m\}, \tag{8.42}$$

$$\begin{bmatrix} AX + XA^T + B\hat{C} + \hat{C}^T B^T & A + B\hat{D}C + \hat{A}^T & B_j & (C_i X + D_i \hat{C})^T \\ (A + B\hat{D}C)^T + \hat{A} & YA + A^T Y + \hat{B}C + C^T \hat{B}^T & YB_j & (C_i + D_i \hat{D} C)^T \\ B_j^T & B_j^T Y & -\gamma I & D_{ij}^T \\ C_i X + D_i \hat{C} & C_i + D_i \hat{D} C & D_{ij} & -\gamma I \end{bmatrix} < 0,$$
$$\forall i, j \in \{1, 2, \ldots, m\}, i \neq j, \tag{8.43}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{8.44}$$

hold. In this case, the controller parameters are given by (8.41).

As before, the near-decoupling controller design in the case of output-feedback (OF) can be formulated as the following optimization problem.

Problem NDCD-OF:

minimize	γ
subject to	LMIs (8.42), (8.43), (8.44).

8.5 Robust Near-Decoupling: Dynamic Output Feedback

Now suppose that the system matrix A is uncertain and of the structure defined by equation (8.21). Again we try to find a dynamic output feedback controller of the form (8.30) to achieve the objective of near-decoupling. To this end, we have

Theorem 8.5.1. *Consider uncertain system (8.2) with matrix A described by (8.21). If there exist matrices X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and positive numbers ε_3 and ε_4 such that the following matrix inequalities*

$$\begin{bmatrix} -A_0X - XA_0^T - B\hat{C} - \hat{C}^TB^T & -A_0 - B\hat{D}C - \hat{A}_0^T & -B_i + (C_iX + D_i\hat{C})^TD_{ii} & A_l & XA_r^T \\ -(A_0 + B\hat{D}C)^T - \hat{A}_0 & -YA_0 - A_0^TY - \hat{B}C - C^T\hat{B}^T & -YB_i + (C_i + D_i\hat{D}C)^TD_{ii} & YA_l & A_r^T \\ -B_i^T + D_{ii}^T(C_iX + D_i\hat{C}) & -B_i^TY + D_{ii}^T(C_i + D_i\hat{D}C) & -\beta_i^2I + D_{ii}^TD_{ii} & 0 & 0 \\ A_l^T & A_l^TY & 0 & \varepsilon_3I & 0 \\ A_rX & A_r & 0 & 0 & \varepsilon_3^{-1}I \end{bmatrix} > 0,$$

$$\forall i \in \{1, 2, \ldots, m\}, \tag{8.45}$$

$$\begin{bmatrix} A_0X + XA_0^T + B\hat{C} + \hat{C}^TB^T & A_0 + B\hat{D}C + \hat{A}_0^T & B_j & (C_iX + D_i\hat{C})^T & A_l & XA_r^T \\ (A_0 + B\hat{D}C)^T + \hat{A}_0 & YA_0 + A_0^TY + \hat{B}C + C^T\hat{B}^T & YB_j & (C_i + D_i\hat{D}C)^T & YA_l & A_r^T \\ B_j^T & B_j^TY & -\gamma I & D_{ij}^T & 0 & 0 \\ C_iX + D_i\hat{C} & C_i + D_i\hat{D}C & D_{ij} & -\gamma I & 0 & 0 \\ A_l^T & A_l^TY & 0 & 0 & -\varepsilon_4I & 0 \\ A_rX & A_r & 0 & 0 & 0 & -\varepsilon_4^{-1}I \end{bmatrix} < 0$$

$$\forall i, j \in \{1, 2, \ldots, m\}, i \neq j, \tag{8.46}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{8.47}$$

hold, then system (8.2) is nearly output-feedback decouplable. The output-feedback law and closed-loop performance indices are given by (8.30)-(8.41) and (8.17)-(8.18) respectively.

Proof. Define

$$\begin{aligned} \hat{A}_0 &:= Y(A_0 + BD_KC)X + NB_KCX + YBC_KM^T + NA_KM^T, \\ \hat{A} &:= Y(A + BD_KC)X + NB_KCX + YBC_KM^T + NA_KM^T \\ &= \hat{A}_0 + YA_l\Psi A_rX, \\ \hat{B}, \hat{C}, \hat{D} &:= \text{ as in equation (8.37).} \end{aligned}$$

Following the same lines of the argument as in Theorem 8.4.1, we can conclude that if the following inequalities

$$\begin{bmatrix} -AX - XA^T - B\hat{C} - \hat{C}^TB^T & -A - B\hat{D}C - \hat{A}^T & -B_i + (C_iX + D_i\hat{C})^TD_{ii} \\ -(A + B\hat{D}C)^T - \hat{A} & -YA - A^TY - \hat{B}C - C^T\hat{B}^T & -YB_i + (C_i + D_i\hat{D}C)^TD_{ii} \\ -B_i^T + D_{ii}^T(C_iX + D_i\hat{C}) & -B_i^TY + D_{ii}^T(C_i + D_i\hat{D}C) & -\beta_i^2I + D_{ii}^TD_{ii} \end{bmatrix}$$

$$= \begin{bmatrix} -A_0X - XA_0^T - B\hat{C} - \hat{C}^TB^T & -A_0 - B\hat{D}C - \hat{A}_0^T & -B_i + (C_iX + D_i\hat{C})^TD_{ii} \\ -(A_0 + B\hat{D}C)^T - \hat{A}_0 & -YA_0 - A_0^TY - \hat{B}C - C^T\hat{B}^T & -YB_i + (C_i + D_i\hat{D}C)^TD_{ii} \\ -B_i^T + D_{ii}^T(C_iX + D_i\hat{C}) & -B_i^TY + D_{ii}^T(C_i + D_i\hat{D}C) & -\beta_i^2I + D_{ii}^TD_{ii} \end{bmatrix}$$

$$- \begin{bmatrix} A_l\Psi A_rX + X(A_l\Psi A_r)^T & A_l\Psi A_r + (YA_l\Psi A_rX)^T & 0 \\ (A_l\Psi A_r)^T + YA_l\Psi A_rX & YA_l\Psi A_r + (A_l\Psi A_r)^TY & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$> 0, \quad \forall i \in \{1, 2, \ldots, m\}, \forall \Psi \in \breve{\Psi}, \tag{8.48}$$

$$\begin{bmatrix} AX + XA^T + B\hat{C} + \hat{C}^T B^T & A + B\hat{D}C + \hat{A}^T & B_j & (C_iX + D_i\hat{C})^T \\ (A + B\hat{D}C)^T + \hat{A} & YA + A^TY + \hat{B}C + C^T\hat{B}^T & YB_j & (C_i + D_i\hat{D}C)^T \\ B_j^T & B_j^TY & -\gamma I & D_{ij}^T \\ C_iX + D_i\hat{C} & C_i + D_i\hat{D}C & D_{ij} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} A_0X + XA_0^T + B\hat{C} + \hat{C}^T B^T & A_0 + B\hat{D}C + \hat{A}_0^T & B_j & (C_iX + D_i\hat{C})^T \\ (A_0 + B\hat{D}C)^T + \hat{A}_0 & YA_0 + A_0^TY + \hat{B}C + C^T\hat{B}^T & YB_j & (C_i + D_i\hat{D}C)^T \\ B_j^T & B_j^TY & -\gamma I & D_{ij}^T \\ C_iX + D_i\hat{C} & C_i + D_i\hat{D}C & D_{ij} & -\gamma I \end{bmatrix}$$

$$+ \begin{bmatrix} A_l\Psi A_rX + X(A_l\Psi A_r)^T & A_l\Psi A_r + (YA_l\Psi A_rX)^T & 0 & 0 \\ (A_l\Psi A_r)^T + YA_l\Psi A_rX & YA_l\Psi A_r + (A_l\Psi A_r)^TY & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$< 0, \quad \forall i, j \in \{1, 2, \dots, m\}, i \neq j, \forall \Psi \in \breve{\Psi} \tag{8.49}$$

together with (8.47) hold, system (8.2) is near-decoupled and the closed-loop performance indices are given by (8.17)-(8.18). In view of Lemma 8.3.1, we have

$$\begin{aligned} &\begin{bmatrix} A_l\Psi A_rX + X(A_l\Psi A_r)^T & A_l\Psi A_r + (YA_l\Psi A_rX)^T & 0 \\ (A_l\Psi A_r)^T + YA_l\Psi A_rX & YA_l\Psi A_r + (A_l\Psi A_r)^TY & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_l\Psi A_rX & A_l\Psi A_r & 0 \\ YA_l\Psi A_rX & YA_l\Psi A_r & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_l\Psi A_rX & A_l\Psi A_r & 0 \\ YA_l\Psi A_rX & YA_l\Psi A_r & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} A_l \\ YA_l \\ 0 \end{bmatrix} \Psi \begin{bmatrix} A_rX & A_r & 0 \end{bmatrix} + \left(\begin{bmatrix} A_l \\ YA_l \\ 0 \end{bmatrix} \Psi \begin{bmatrix} A_rX & A_r & 0 \end{bmatrix} \right)^T \\ &\leq \varepsilon_3^{-1} \begin{bmatrix} A_l \\ YA_l \\ 0 \end{bmatrix} \begin{bmatrix} A_l^T & A_l^TY & 0 \end{bmatrix} + \varepsilon_3 \begin{bmatrix} XA_r^T \\ A_r^T \\ 0 \end{bmatrix} \begin{bmatrix} A_rX & A_r & 0 \end{bmatrix}, \quad \forall \varepsilon_3 \in (0, +\infty) \end{aligned}$$

and

$$\begin{aligned} &\begin{bmatrix} A_l\Psi A_rX + X(A_l\Psi A_r)^T & A_l\Psi A_r + (YA_l\Psi A_rX)^T & 0 & 0 \\ (A_l\Psi A_r)^T + YA_l\Psi A_rX & YA_l\Psi A_r + (A_l\Psi A_r)^TY & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\leq \varepsilon_4^{-1} \begin{bmatrix} A_l \\ YA_l \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_l^T & A_l^TY & 0 & 0 \end{bmatrix} + \varepsilon_4 \begin{bmatrix} XA_r^T \\ A_r^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_rX & A_r & 0 & 0 \end{bmatrix}, \ \forall \varepsilon_4 \in (0, +\infty). \end{aligned}$$

Therefore, if the following inequalities

$$\begin{bmatrix} -A_0X - XA_0^T - B\hat{C} - \hat{C}^T B^T & -A_0 - B\hat{D}C - \hat{A}_0^T & -B_i + (C_iX + D_i\hat{C})^T D_{ii} \\ -(A_0 + B\hat{D}C)^T - \hat{A}_0 & -YA_0 - A_0^TY - \hat{B}C - C^T\hat{B}^T & -YB_i + (C_i + D_i\hat{D}C)^T D_{ii} \\ -B_i^T + D_{ii}^T(C_iX + D_i\hat{C}) & -B_i^TY + D_{ii}^T(C_i + D_i\hat{D}C) & -\beta_i^2 I + D_{ii}^T D_{ii} \end{bmatrix}$$

$$-\varepsilon_3^{-1} \begin{bmatrix} A_l \\ YA_l \\ 0 \end{bmatrix} \begin{bmatrix} A_l^T & A_l^TY & 0 \end{bmatrix} - \varepsilon_3 \begin{bmatrix} XA_r^T \\ A_r^T \\ 0 \end{bmatrix} \begin{bmatrix} A_rX & A_r & 0 \end{bmatrix}$$

$$> 0, \quad \forall i \in \{1, 2, \dots, m\}, \tag{8.50}$$

$$\begin{bmatrix} A_0X + XA_0^T + B\hat{C} + \hat{C}^TB^T & A_0 + B\hat{D}C + \hat{A}_0^T & B_j & (C_iX + D_i\hat{C})^T \\ (A_0 + B\hat{D}C)^T + \hat{A}_0 & YA_0 + A_0^TY + \hat{B}C + C^T\hat{B}^T & YB_j & (C_i + D_i\hat{D}C)^T \\ B_j^T & B_j^TY & -\gamma I & D_{ij}^T \\ C_iX + D_i\hat{C} & C_i + D_i\hat{D}C & D_{ij} & -\gamma I \end{bmatrix}$$
$$+\varepsilon_4^{-1}\begin{bmatrix} A_l \\ YA_l \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} A_l^T & A_l^TY & 0 & 0 \end{bmatrix} + \varepsilon_4\begin{bmatrix} XA_r^T \\ A_r^T \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} A_rX & A_r & 0 & 0 \end{bmatrix}$$
$$< 0, \quad \forall i, j \in \{1, 2, \ldots, m\}, i \neq j \tag{8.51}$$

hold, so do inequalities (8.48) and (8.49). By Schur complements, inequalities (8.50) and (8.51) is equivalent to inequalities (8.45) and (8.46), respectively. This completes the proof. □

Different from the problems dealt with in the preceding sections, inequalities (8.45) and (8.46) are not linear in the unknowns ε_3 and ε_4, respectively. Thus we cannot use standard LMI toolbox of Matlab to solve these two inequalities. An important characteristic of these two inequalities is that they reduce to LMIs once upon ε_3 and ε_4 are fixed. Therefore one possible way is to solve inequalities (8.45), (8.46) and (8.47) by sweeping all possible ε_3 and ε_4, but the difficulty is where one should start his sweeping. From inequalities (8.50) and (8.51), we can see that ε_3 and ε_4 in inequalities (8.45) and (8.46) are of the similar rule and structure as the "communicating variables" in the problems studied in (Zheng *et al.*, 2002; Zheng *et al.*, to appear). In (Zheng *et al.*, 2002; Zheng *et al.*, to appear), an optimal estimate for the communicating variables is given, which facilitates us to solve the relevant problems. In the sequel, we will give an estimate for the initial values of ε_3 and ε_4 based on the method in (Zheng *et al.*, 2002; Zheng *et al.*, to appear).

From inequalities (8.50) and (8.51) and following the same argument as that in (Zheng *et al.*, 2002; Zheng *et al.*, to appear), one can see that the most likely values for ε_3 and ε_4 which make inequalities (8.45) and (8.46) hold are given by

$$\begin{aligned} \varepsilon_3^* &= \sqrt{\frac{\mathtt{tr}\left(\begin{bmatrix} A_l \\ YA_l \\ 0 \end{bmatrix}\begin{bmatrix} A_l^T & A_l^TY & 0 \end{bmatrix}\right)}{\mathtt{tr}\left(\begin{bmatrix} XA_r^T \\ A_r^T \\ 0 \end{bmatrix}\begin{bmatrix} A_rX & A_r & 0 \end{bmatrix}\right)}} = \sqrt{\frac{\mathtt{tr}(A_lA_l^T) + \mathtt{tr}(YA_lA_l^TY)}{\mathtt{tr}(A_r^TA_r) + \mathtt{tr}(XA_r^TA_rX)}}. \\ &:= \epsilon(X, Y) \qquad\qquad (8.52) \\ \varepsilon_4^* &= \epsilon(X, Y). \end{aligned}$$

Since both ε_3^* and ε_4^* are functions of the unknowns X and Y, it is still difficult to solve our problem. Because matrix inequalities (8.45), (8.46) and (8.47) should have solutions when there are no parameter perturbations, we can reasonably use the matrices X and Y that solve problem NDCD-OF to calculate ε_3^* and ε_4^*, which are then used as the initial values for ε_3 and ε_4 respectively in the sweeping procedure.

Based upon the above discussion, we can adopt the following procedure to solve robust near-decoupling controller design (RNDCD) problem for output feedback case.

Design procedure for RNDCD-OF problem:

Step 0. Initial data: System's state space realization (A_0, B, C, D, A_l, A_r), performance indices β_i $(i = 1, 2, \dots, p)$.
Step 1. Solve problem NDCD-OF. Denote by X_0 and Y_0 its solution for the variables X and Y, respectively.
Step 2. Calculate $\varepsilon_0 = \epsilon(X_0, Y_0)$ according to equation (8.52).
Step 3. `for` $\varepsilon = \varepsilon_0$ `down to` ε_{f1} or `up to` ε_{f2}, where ε_{f1} and ε_{f2} are two positive numbers, solve the following optimization problem:

minimize	γ
subject to	LMIs (8.45), (8.46), (8.47), in which ε_3 and ε_4 are set to be ε.

Notice that in the above procedure we set $\varepsilon_3 = \varepsilon_4$ to avoid a two dimensional sweeping, which would introduce additional conservativeness for the original problem.

As pointed out in (Chilali and Gahinet, 1996; Scherer *et al.*, 1997), a minimization procedure often make the matrix $I - XY$ nearly singular. This phenomenon is also evidenced while we numerically solve problems NDCD-OF and RNDCD-OF. Therefore, similar to (Chilali and Gahinet, 1996; Scherer *et al.*, 1997), we use the following matrix inequality

$$\begin{bmatrix} X & \eta I \\ \eta I & Y \end{bmatrix} > 0, \tag{8.53}$$

to substitute matrix inequalities (8.44) and (8.47) in the corresponding problems, where η is a given number satisfying $\eta \gg 1$.

8.6 A Numerical Example

Now let us consider the following 2×2 system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

with

$$A = \begin{bmatrix} -1+\Delta & 0 & 0 \\ 0 & -2+2\Delta & 0 \\ 0 & 0 & -3+3\Delta \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -3 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where Δ is an uncertain parameter varying in the closed interval $[-0.1, 0.1]$. The nominal value for matrix A is

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and the uncertainty can be expressed in the form of (8.21) with

$$A_l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

Its nominal transfer matrix is

$$H_0(s) = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ \frac{4s+6}{(s+1)(s+2)(s+3)} & \frac{s^2+5s+9}{(s+2)(s+3)} \end{bmatrix}.$$

The Bode diagrams for this system (nominal part) are illustrated in Fig. 8.1.

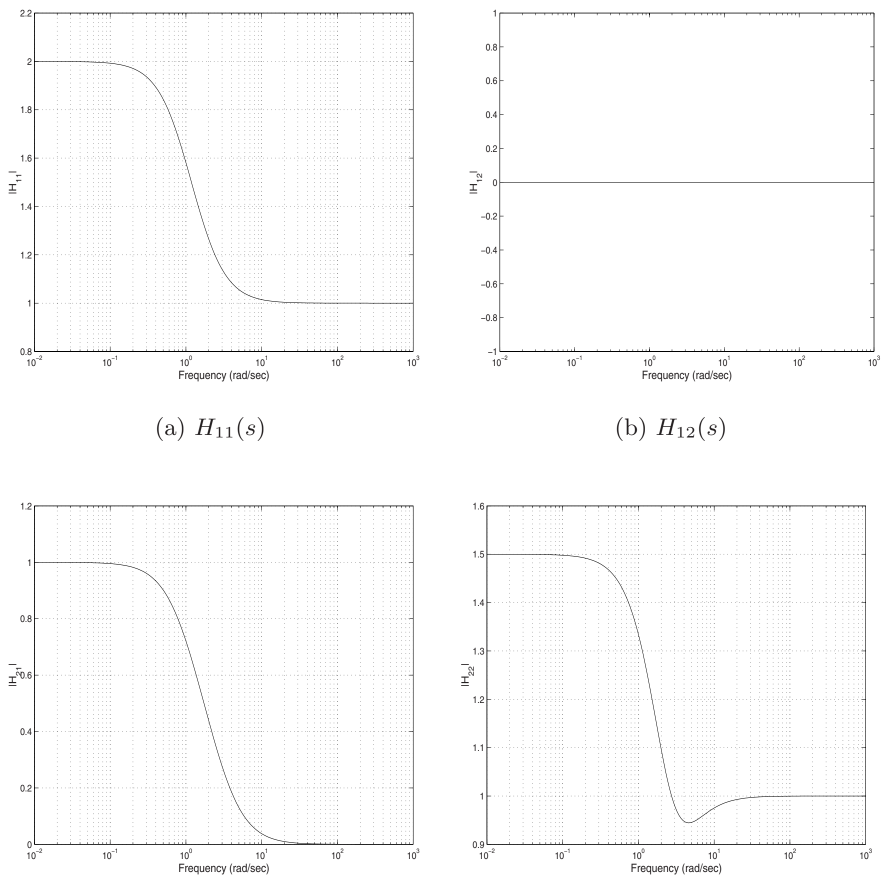
Using the methods developed in the preceding sections, we can obtain the near-decoupling controllers for SF and OF cases and achievable performance, as summarized in Table 8.1.

Table 8.1. Near-decoupling Controllers and Their Performance

Problem	Design param.	Perfor. guaranteed	Feedback matrices
NDCD-SF	$\beta_1 = \beta_2 = 0.95$ $\mu = 10^4$	$\gamma = 3.61 \times 10^{-4}$	$K = \begin{bmatrix} -0.9998 & 0.0018 & 0.0023 \\ -0.9993 & -1.0001 & -0.9997 \end{bmatrix}$
RNDCD-SF	$\beta_1 = \beta_2 = 0.95,$ $\mu = 10^4$	$\gamma = 4.43 \times 10^{-4}$	$K = \begin{bmatrix} -1.0009 & 0.0006 & 0.0009 \\ -0.9993 & -1.0001 & -0.9998 \end{bmatrix}$
NDCD-OF	$\beta_1 = \beta_2 = 0.95$ $\eta = 10^4$	$\gamma = 6.7166 \times 10^{-4}$	$A_K = \begin{bmatrix} 0.0103 & 6.6284 & -6.6386 \\ 0.0159 & 6.6257 & -6.6416 \\ 0.0159 & 6.6257 & -6.6416 \end{bmatrix} \times 10^8$ $B_K = \begin{bmatrix} -1.0485 & -2.1362 \\ -0.5438 & -1.8575 \\ -0.5438 & -0.5748 \end{bmatrix} \times 10^8$ $C_K = \begin{bmatrix} 0.0223 & -0.0105 & -0.0118 \\ 7.2e\{-9\} & 2.0e\{-5\} & -2.0e\{-5\} \end{bmatrix}$ $D_K = D$
NDCD-ROF	$\beta_1 = \beta_2 = 0.95$ $\eta = 10$	$\gamma = 0.001$ at $\varepsilon_3 = \varepsilon_4 = 0.001$	$A_K = \begin{bmatrix} -2.2333 & -1.2025 & -0.5697 \\ -0.0930 & -2.1850 & -2.7781 \\ -0.0930 & -2.1850 & -2.7781 \end{bmatrix} \times 10^6$ $B_K = \begin{bmatrix} -5.9078 & -2.4216 \\ 1.7431 & -4.1658 \\ 1.7431 & -4.1658 \end{bmatrix} \times 10^8$ $C_K = \begin{bmatrix} 0.0063 & -0.0002 & -0.0030 \\ 0.0031 & 0.0114 & 0.0111 \end{bmatrix}$ $D_K = D$

The Bode diagrams for the closed-loop system (we choose $\Delta = 0.1$) are illustrated in Figs. 8.2, 8.3, 8.4 and 8.5 for the cases NDCD-SF, RNDCD-SF, NDCD-OF and RNDCD-OF respectively.

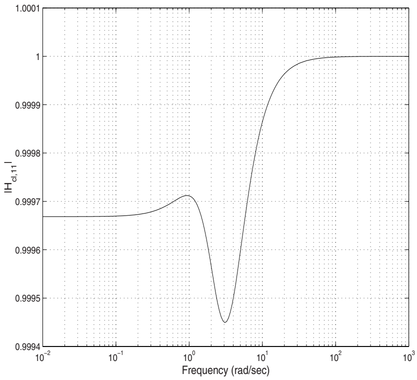
Comparing Table 8.1 and Figs. 8.2, 8.3, 8.4 and 8.5, we can see that the performance indices guaranteed by our methods are somewhat conservative, especially for the case where there are parameter perturbations. This is not strange since the performance index γ shown in Table 8.1 is guaranteed for all possible uncertain models.



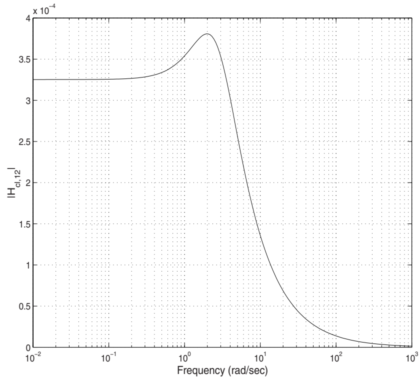
(a) $H_{11}(s)$ (b) $H_{12}(s)$

(c) $H_{21}(s)$ (d) $H_{22}(s)$

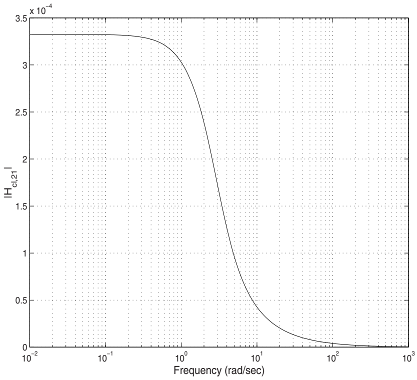
Fig. 8.1. Bode diagrams of the open-loop transfer matrices.



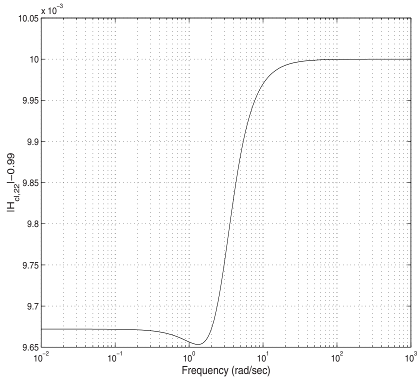
(a) $H_{cl,11}(s)$



(b) $H_{cl,12}(s)$



(c) $H_{cl,21}(s)$

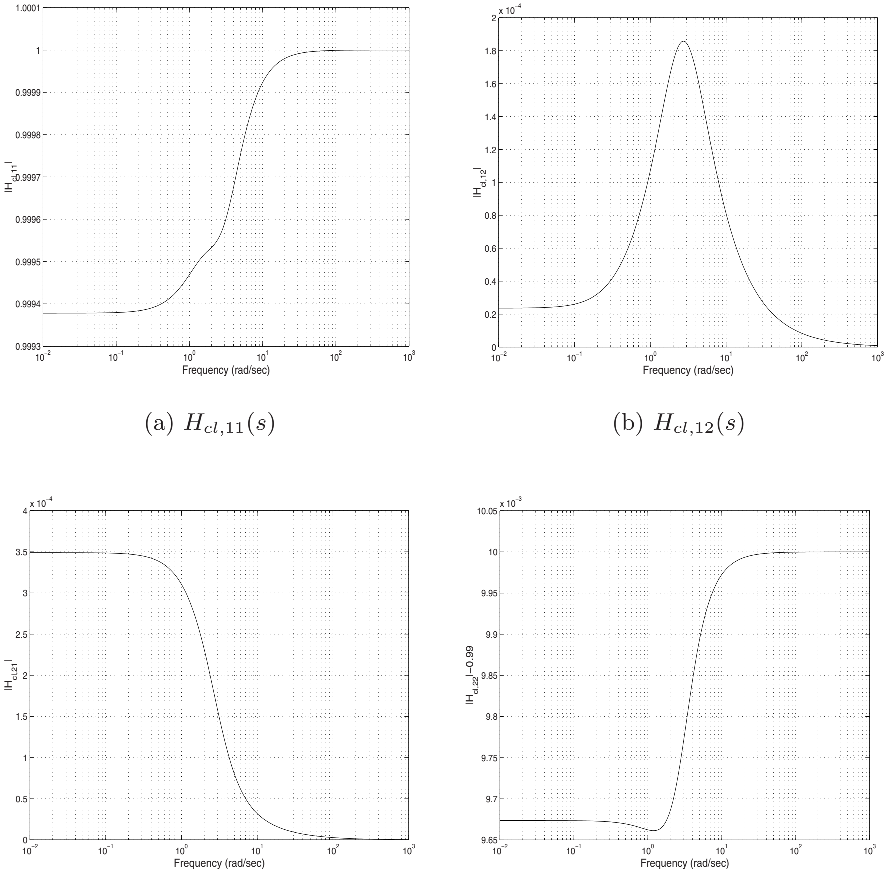


(d) $H_{cl,22}(s)$

Fig. 8.2. Bode diagrams of the closed-loop transfer matrices: ADCD-SF.

8.7 Concluding Remarks

In this chapter, we have proposed the new concept of near-decoupling control. Fundamental lemmas for solving near-decoupling control problem are established in the form of linear matrix inequalities. Design methods on the near-decoupling controllers have been presented for four general cases: (i) exact models and state feedback, (ii) uncertain models and state feedback, (iii) exact models and dynamic output feedback, and (iv) uncertain models and dynamic output feedback. In the case of uncertain models, we assume that the parameter uncertainties are norm-bounded and may be of some structure properties. All these results are described in LMI language, which



(a) $H_{cl,11}(s)$ (b) $H_{cl,12}(s)$

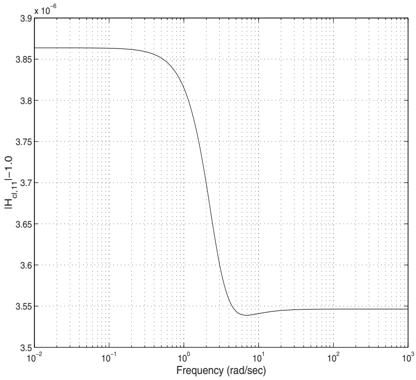
(c) $H_{cl,21}(s)$ (d) $H_{cl,22}(s)$

Fig. 8.3. Bode diagrams of the closed-loop transfer matrices: RADCD-SF.

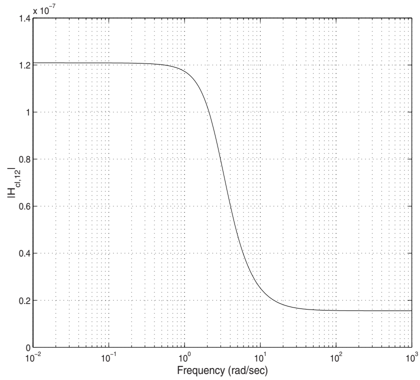
makes the numerical solutions of near-decoupling controller design problem easily tractable. The results given in this chapter provides a viable way for solving the problem of robust decoupling control, which is of conspicuous significance, but unfortunately, has long remained rarely studied in control theory.

While we have witnessed the effectiveness of the approach developed here through the illustrative example, we should admit that there exist two main limitations in our approach, which are

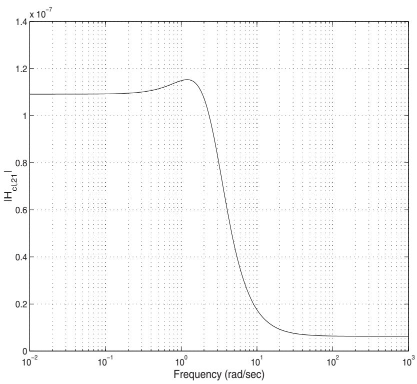
(i) The matrix D cannot become vanished. This is due to the specification defined in equation (8.3). If we change this equation as follows



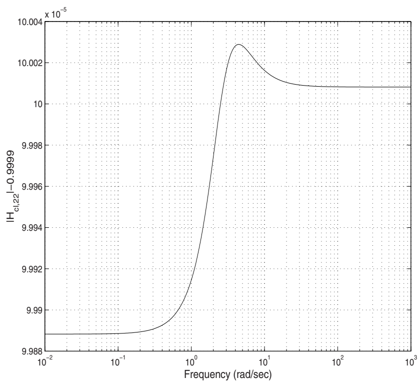
(a) $H_{cl,11}(s)$



(b) $H_{cl,12}(s)$



(c) $H_{cl,21}(s)$



(d) $H_{cl,22}(s)$

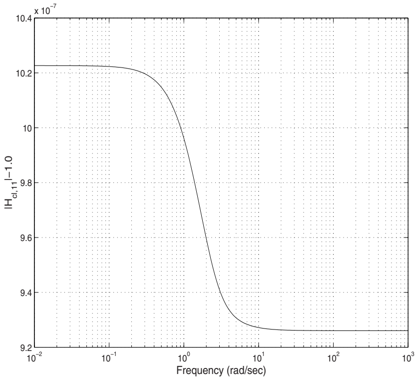
Fig. 8.4. Bode diagrams of the closed-loop transfer matrices: ADCD-OF.

$$\underline{\sigma}(H_{ii}(j\omega)) \geq \beta_i, \quad \forall i \in \{1, 2, \ldots, p\}, \forall \omega \in \mathcal{B}, \tag{8.54}$$

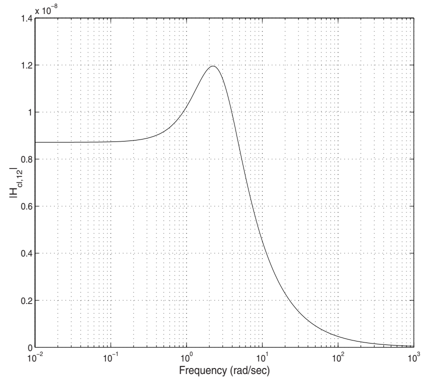
where $\mathcal{B}$ is a limited band of frequencies in which the designer is interested, the matrix D can then be allowed to be zero. The difficulty lies in the shortage of effective methods which convert specification (8.54) to a numerical tractable formula concerning system matrices.

(ii) The decoupling controller is limited to the class of (8.5), which decreases the freedom of controller design parameters. In practice, it is desirable that the controller is of the following general form

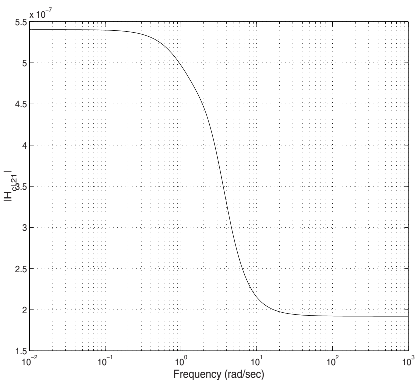
$$u = Kx + Gv.$$



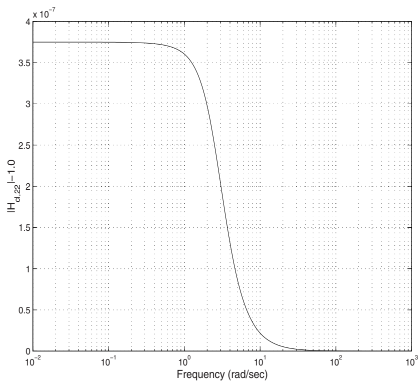
(a) $H_{cl,11}(s)$



(b) $H_{cl,12}(s)$



(c) $H_{cl,21}(s)$



(d) $H_{cl,22}(s)$

Fig. 8.5. Bode diagrams of the closed-loop transfer matrices: RADCD-OF.

It is observed that the main rule of matrix G is to statically change the transfer matrices of the closed-loop systems. We hope that this limitation can be partly overcome in the pre-design stage by considering the input-output pairs to be decoupled.

Finally we would like to point out that the concept of near-decoupling design proposed in (Cheng *et al.*, 1995) is of some similarity with the concept of robust near-decoupling proposed in this chapter, but the design procedures are completely different between this chapter and the references (Cheng *et al.*, 1995), where the "decoupling" controller is designed still based on the nominal model of the plants, but their objective is to nearly, instead of completely, decouple the system.

8.8 Notes and References

A literature survey will reveal that nominal decoupling problem has now been well studied. In sharp contrast to other control topics where robustness has been extensively investigated, robust decoupling problem for uncertain systems has rarely been reported in the literature. This phenomena is unsatisfactory since parameter perturbations or deviations from their nominal values are inevitable in real plants, especially in industrial process control. Intuitively, exact decoupling is very difficult to achieve owing to the presence of plant parameter perturbations. In practice, it is often the case that one first designs a decoupling controller (decoupler) for the nominal model of the real plant, put the decoupler as the inner loop controller and then on-line tune the outer loop controller to make the overall system performance be roughly satisfied. Reference (Wu *et al.*, 1998) describes such a procedure. In robot control systems, the calculated torque method combined with gain scheduling (Ito and Shiraishi, 1997) also belongs to this approach. It is believed that near or robust decoupling problem deserves more attention from researchers.

9. Dynamic Disturbance Decoupling

The attenuation of load disturbance is always of a primary concern for any control system design, and is even the ultimate objective for process control, where the set-point may be kept unchanged for years. As such, this and next chapters will deal with decoupling disturbances from the plant output in dynamic and asymptotic senses, respectively. The dynamic disturbance decoupling problem is to find a control scheme such that in the resulting control system the transfer matrix function from the disturbance to the controlled output is zero for all frequencies, i.e. there is no effect of the disturbance on the controlled output. If the disturbance is measurable, the feedforward compensation scheme can be employed to eliminate its effect on the system output, which will be the topic of Section 1 of this chapter. For unmeasurable disturbances which are more often encountered in industry, feedback control has to be adopted. However, there will be inevitably a design tradeoff between the set-point response and disturbance rejection performance. To alleviate this problem, a control scheme, called the disturbance observer, is introduced and it acts as an add-on mechanisms to the conventional feedback system. The disturbance observer estimates the equivalent disturbance as the difference between the actual process output and the output of the nominal model. The estimate is then fed to a process model inverse to produce an extra control effort which can compensate for the disturbance effect on the output. The disturbance observer will be addressed in Section 2 of this chapter. Stability is a crucial issue and will be considered in both schemes.

9.1 Measurable Disturbance: Feedforward Control

In this section, supposing measurable disturbances, we aim to solve disturbance decoupling problem with stability via disturbance feedforward control in connection with the normal output feedback configuration. Let a given system be described by a transfer matrix model

$$y(s) = G_u(s)u(s) + G_d(s)d(s), \tag{9.1}$$
$$y_m(s) = G_m(s)y(s), \tag{9.2}$$

where $y(s)$ is the m-dimensional controlled output, $u(s)$ the l-dimensional control input, $d(s)$ the p-dimensional disturbance, $y_m(s)$ q-dimensional measurement, and the proper rational matrices G_u, G_m and G_d with appropriate

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 293-325, 2003.

dimensions represent the plant model, sensor model and disturbance channel model, respectively. Suppose that both y_m and d are accessible. A general lin-

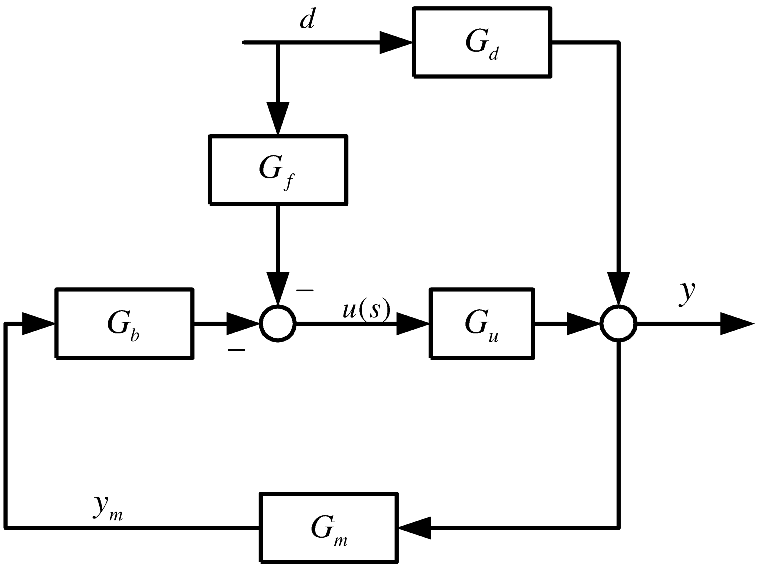


Fig. 9.1. Control Scheme for Decoupling Measurable Disturbances

ear compensator which makes use of all the available information is described by

$$u(s) = -G_f(s)d(s) - G_b(s)y_m(s), \tag{9.3}$$

where G_f and G_b are proper rational matrices to be designed for disturbance decoupling. This compensator can be thought as a combination of a disturbance feedforward part and a output feedback part. The overall control system is depicted in Figure 9.1. For the sake of simplicity the intermediate variable s will be dropped as long as the omission causes no confusion.

It is well known that (9.1)∼(9.3) can equivalently be described by polynomial matrix fraction representations:

$$A\,y = B\,u + C\,d, \tag{9.4}$$

$$F\,\xi = y, \tag{9.5}$$

$$y_m = E\,\xi, \tag{9.6}$$

$$P\,u = -Q\,d - R\,y_m, \tag{9.7}$$

where two pairs of polynomial matrices, A and $[B \quad C]$, and P and $[-Q \quad -R]$, are both left coprime such that $A^{-1}[B \quad C] = [G_u \quad G_d]$ and $P^{-1}[-Q \quad -R] = [-G_f \quad - G_b]$; and F and E are right coprime such that $EF^{-1} = G_m$. Combining (9.4)∼(9.7) yields

$$\begin{bmatrix} -B & 0 & A \\ 0 & F & -I \\ P & RE & 0 \end{bmatrix} \begin{bmatrix} u \\ \xi \\ y \end{bmatrix} = \begin{bmatrix} C \\ 0 \\ -Q \end{bmatrix} d. \tag{9.8}$$

Let

$$W = \begin{bmatrix} -B & 0 & A \\ 0 & F & -I \\ P & RE & 0 \end{bmatrix}, \tag{9.9}$$

then the transfer function matrix between y and d is

$$G_{yd} = [0 \;\; 0 \;\; I]W^{-1}[C^T \;\; 0 \;\; -Q^T]^T. \tag{9.10}$$

We can now formulate the disturbance decoupling problem with stability as follows. (DDP): given a controlled system described by (9.1) and (9.2) with G_u, G_d and G_m being proper rational matrices, find the conditions under which a compensator in the form of (9.3) with G_f and G_b being proper rational matrices will make (i) the resulting system stable, i.e., all the roots of the determinant $det(W)$ lie in $\mathbb{C}^-$, the open left half of complex plane; and (ii) the transfer matrix between y and d equals zero, i.e., $G_{yd} = 0$.

9.1.1 Solvability

Our strategy to resolve (DDP) is first to determine the set of proper compensators which stabilize a controlled system. Within these stabilizable systems and their compensators, the conditions for zeroing disturbance are then derived.

Let G_1 and G_2 be rational matrices with the same number of rows, we can factorize them as

$$G_1 = D_1^{-1}N_1, \quad G_2 = D_2^{-1}N_2, \tag{9.11}$$

$$[G_1 \quad G_2] = D_3^{-1}N_3, \tag{9.12}$$

where all fractions are coprime. Let

$$D_2 = D_{21}D_{22}, \tag{9.13}$$

and partition N_3 consistent with $[G_1 \;\; G_2]$ as

$$N_3 = [N_{31} \quad N_{32}]. \tag{9.14}$$

Then the following can be established.

Lemma 9.1.1. *With the above notations, $\tilde{D}_{22}$ is a right divisor of any greatest common left divisor of N_{31} and D_3, where $\tilde{D}_{22}^{-1}\tilde{D}_1$ is a left coprime fraction of $D_1D_{22}^{-1}$.*

Proof. Equations (9.11), (9.12) and (9.14) show that $G_1 = D_1^{-1}N_1 = D_3^{-1}N_{31}$. Furthermore, the coprimeness of N_1 and D_1 implies that $[D_3 \ \ N_{31}] = L_1[D_1 \ \ N_1]$ for some polynomial matrix L_1 which is a greatest common left divisor of N_{31} and D_3. Similarly, one can show that $D_3 = L_2D_2$ for some polynomial matrix L_2. Grouping all these facts, we obtain

$$L_2 = D_3D_2^{-1} = L_1D_1D_2^{-1} = L_1D_1D_{22}^{-1}D_{21}^{-1} = L_1\tilde{D}_{22}^{-1}\tilde{D}_1D_{21}^{-1}. \quad (9.15)$$

It follows from the coprimeness of $\tilde{D}_{22}$ and $\tilde{D}_1$ that $L_1\tilde{D}_{22}^{-1}$ must be a polynomial matrix, that is, $\tilde{D}_{22}$ is a right divisor of L_1. □

If $D(s)$ is nonsingular, we will denote by $\sigma(D)$ the set of all complex numbers c such that $det(D(c)) = 0$.

Corollary 9.1.1. *If $\sigma(\tilde{D}_{22})$ is not empty, $[D_3(c) \quad N_{31}(c)]$ is of rank defect for $c \in \sigma(\tilde{D}_{22})$.*

Proof. It follows from Lemma 9.1.1 that there is a polynomial matrix L_3 such that $L_1 = L_3\tilde{D}_{22}$ and $[D_3 \quad N_{31}] = L_3\tilde{D}_{22}[D_1 \quad N_1]$. For $c \in \sigma(\tilde{D}_{22})$, $[D_3(c) \quad N_{31}(c)]$ is clearly of rank defect, and the corollary thus follows. □

Corollary 9.1.2. *Let $\tilde{D}_2^{-1}\tilde{D}_1$ be a left coprime fraction of $D_1D_2^{-1}$, then D_3 and $\tilde{D}_2D_1$ are left equivalent, i.e. there is a unimodular polynomial matrix U such that $D_3 = U\tilde{D}_2D_1$.*

Proof. Set $D_{22} = D_2$ and $D_{21} = I$. It follows from Lemma 9.1 that $D_3 = L_4\tilde{D}_2D_1$ for some polynomial matrix L_4. One the other hand, we have

$$\begin{aligned}[G_1 \quad G_2] &= [D_1^{-1}N_1 \quad D_2^{-1}N_2] = D_1^{-1}[N_1 \quad D_1D_2^{-1}N_2]. \\ &= D_1^{-1}[N_1 \quad \tilde{D}_2^{-1}\tilde{D}_1N_2] = (\tilde{D}_2D_1)^{-1}[\tilde{D}_2N_1 \quad \tilde{D}_1N_2].\end{aligned}$$

The last representation is also a fraction of $[G_1 \quad G_2]$ and can be related to its coprime fraction $D_3^{-1}N_3$ by $\tilde{D}_2D_1 = L_5D_3$ for some polynomial matrix L_5. Then, both L_4 and $L_4^{-1} = L_5$ are polynomial and it is unimodular. □

For a nonsingular polynomial matrix D, it can be factorized into $D = D^-D^+$ such that $\sigma(D^{-1}) \subset \mathbb{C}^-$ and $\sigma(D^+) \subset \mathbb{C}^+$, the closed right half of the complex plane (RHP). Such a factorization can be done, for example, by determining the Smith form of D and then factorizing every invariant polynomial as the product of a stable polynomial and an anti-stable polynomial. For rational matrices G_1 and G_2 given above, this factorization is applied to D_2, i.e., $D_2 = D_2^-D_2^+$. G_1 is said to have the unstable model of G_2 if D_2^+ is a right divisor of D_1, i.e., $D_1(D_2^+)^{-1}$ is a polynomial matrix.

Theorem 9.1.1. *Given a controlled system described by (9.1) and (9.2) with G_u, G_d and G_m being proper, then a proper compensator described by (9.3) stabilizes the system if and only if the following hold.*

(i) G_u has the unstable model of G_d, and G_b the unstable model of G_f,

(ii) $G_m \cdot G_u$ has no RHP pole-zero cancellations, and

(iii) G_b stabilizes G_mG_u.

Proof. With the fraction representations given by (9.4)∼(9.7), we recall that the compensated system is stable if and only if $det(W)$ has no zero in $\mathbb{C}^+$. An equivalent statement is

$$W(c) \ is \ of \ full \ rank \ for \ any \ c \in \mathbb{C}^+. \tag{9.16}$$

Necessity: (i) Let $D_u^{-1}N_u$ and $D_d^{-1}N_d$ be left coprime fractions of G_u and G_d, respectively, and factorize D_d as $D_d = D_d^- D_d^+$ where $\sigma(D_d^-) \subset \mathbb{C}^-$ and $\sigma(D_d^+) \subset \mathbb{C}^+$. We now assume that G_u does not have the unstable model of G_d. This means that D_d^+ is not a right divisor of D_u and $\sigma(\tilde{D}_d^+)$ is not empty where $(\tilde{D}_d^+)^{-1}\tilde{D}_u$ is a left coprime fraction of $D_u(D_d^+)^{-1}$. Note $A^{-1}[B \quad C] = [G_u \quad G_d] = [D_u^{-1}N_u \quad D_d^{-1}N_d]$. It follows from Corollary 9.1 that $[A(c) \quad B(c)]$ is of rank defect for $c \in \sigma(\tilde{D}_d^+) \subset \sigma(D_d^+) \subset \mathbb{C}^+$. Then $W(c)$ must be of rank defect for these c's because

$$Rank[-B(c) \quad 0 \quad A(c)] = Rank[A(c) \quad B(c)].$$

In a similar way, using

$$[P \quad RE \quad 0] = [P \quad R \quad 0] \begin{bmatrix} I & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I \end{bmatrix},$$

and

$$Rank[P(c) \quad R(c)E(c) \quad 0] \le Rank[P(c) \quad R(c) \quad 0],$$

one can show that if G_b does not have the unstable model of G_f, then $W(c)$ is of rank defect for some $c \in \mathbb{C}^+$. These facts show that (i) is necessary for (9.16) to hold.

(ii) We perform the following elementary row operation on W

$$W_1 = U_1 W := \begin{bmatrix} I & A & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -B & 0 & A \\ 0 & F & -I \\ P & RE & 0 \end{bmatrix} = \begin{bmatrix} -B & AF & 0 \\ 0 & F & -I \\ P & RE & 0 \end{bmatrix}. \tag{9.17}$$

Since U_1 is unimodular, (9.16) is equivalent to the fact that $W_1(c)$ is of full rank for any $c \in \mathbb{C}^+$. Then $[-B \quad AF \quad 0]$ must thus be of full rank for any $c \in \mathbb{C}^+$, which implies

$$[A(c)F(c) \quad B(c)] \ is \ of \ full \ rank \ for \ any \ c \in \mathbb{C}^+. \tag{9.18}$$

In a similar way, postmultiplying W by the following unimodular matrix

$$U_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & F & I \end{bmatrix},$$

one can show that (9.16) implies

$$\begin{bmatrix} A(c)F(c) \\ E(c) \end{bmatrix} \ is \ of \ full \ rank \ for \ any \ c \in \mathbb{C}^+. \tag{9.19}$$

(9.18) and (9.19) together ensure that $G_m \cdot G_u = EF^{-1}A^{-1}B = E(AF)^{-1}B$ has no RHP pole-zero cancellations.

(*iii*) From (9.17), it follows that

$$det(W) = det(W_1) = a_1 det \begin{bmatrix} AF & -B \\ RE & P \end{bmatrix}$$

$$= a_1 det(AF) \cdot det(P + RE(AF)^{-1}B), \tag{9.20}$$

where $a_1 = +1$ or -1. Using Corollary 9.1.2,

$$det(A) = a_2 det(D_u) det(\tilde{D}_d), \tag{9.21}$$

where a_2 is a nonzero constant and $\tilde{D}_d^{-1}\tilde{D}_u$ is a left coprime fraction of $D_u D_d^{-1}$. Let

$$[P \quad R] = L[\bar{P} \quad \bar{R}] \tag{9.22}$$

for some polynomial matrix L such that $\bar{P}$ and $\bar{R}$ are coprime. Let

$$ND^{-1} = E(AF)^{-1}B, \tag{9.23}$$

where N and D are coprime. Substituting (9.21) $\sim$ (9.23) into (9.20) yields

$$det(W) = a \cdot det(\tilde{D}_d) \cdot det(L) \cdot det(\bar{P}D + \bar{R}N) \cdot g, \tag{9.24}$$

where a is a nonzero constant, and $g = det(D_u) \cdot det(F)/det(D)$. Since $ND^{-1} = G_m G_u = EF^{-1}D_u^{-1}N_u = E(D_u F)^{-1}N_u$, then g is a polynomial and $\sigma(g)$ is the set of all the cancelled poles when performing multiplication of G_m and G_u. It can be seen from (9.24) that for ensuring stability $det(\bar{P}D + \bar{R}N)$ must have no zeros in $\mathbb{C}^+$, which also means that G_b stabilizes $G_m G_u$. Necessity is thus proven.

Sufficiency: If G_u has the unstable model of G_d, then $D_u(D_d^+)^{-1}$ is a polynomial matrix and thus $\sigma(\tilde{D}_d) \subset \sigma(D_d^-) \subset \mathbb{C}^-$. Similarly, one has $\sigma(L) \subset \mathbb{C}^-$. Conditions (*ii*) and (*iii*) imply $\sigma(g) \subset \mathbb{C}^-$ and $\sigma(\bar{P}D + \bar{R}N) \subset \mathbb{C}^-$, respectively. It follows from (9.24) that $\sigma(W) \subset \mathbb{C}^-$. Proof is therefore completed. □

Remark 9.1.1. Theorem 9.1.1 is intuitively appealing. It indicates that in order to ensure stability, all the unstable modes of G_d must be common to G_u, i.e., $G_d = (D_d^+)^{-1}\tilde{G}_d$ and $G_u = (D_d^+)^{-1}\tilde{G}_u$ for some rational matrices $\tilde{G}_d$ and $\tilde{G}_u$, where $\tilde{G}_d$ is stable, and $(D_d^+)^{-1}$ and $\tilde{G}_u$ have no RHP pole-zero cancellations. There also hold $G_f = (D_f^+)^{-1}\tilde{G}_f$ and $G_b = (D_f^+)^{-1}\tilde{G}_b$ for some rational matrices $\tilde{G}_f$ and $\tilde{G}_b$, where $\tilde{G}_f$ is stable, and $(D_f^+)^{-1}$ and $\tilde{G}_b$ have no RHP pole-zero cancellations. These show that all the unstable modes $(D_d^+)^{-1}$ and $(D_f^+)^{-1}$ of G_d and G_f must be within the feedback loop, see Figure 9.2.

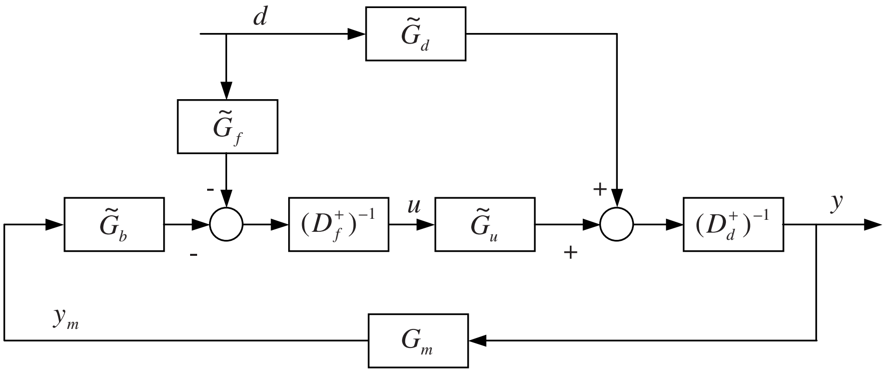


Fig. 9.2. Possible Unstable Modes in Stabilizable Disturbance Decoupling Control

Remark 9.1.2. To know if G_u has the unstable model of G_d, one only needs to check if $D_u D_d^{-1}$ is stable, which is much easier to perform without a need to factorize D_d into $D_d^- D_d^+$. It is also sufficient if $D_d G_u$ is stable.

Consider now (DDP) under the stability conditions given in Theorem 9.1.1. From Figure 9.1, we have

$$G_{yd} = [I + G_u G_b G_m]^{-1}[G_d - G_u G_f].$$

$G_{yd} = 0$ is equivalent to

$$G_u G_f = G_d. \tag{9.25}$$

Lemma 9.1.2. *If the conditions of Theorem 9.1.1 are satisfied and G_f meets (9.25), then G_f is stable.*

Proof. Assume that G_f satisfies (9.25) but is not stable. With the notations as in Theorem 9.1.1, premultiplying (9.25) by D_d^+ gives

$$(\bar{D}_u)^{-1} N_u G_f = \tilde{G}_d,$$

where $\tilde{G}_d = (D_d^-)^{-1} N_d$ and $\bar{D}_u = D_u (D_d^+)^{-1}$ with $\bar{D}_u$ being a polynomial matrix due to G_u having the unstable model of G_d. It follows from the stability of $\tilde{G}_d$ and coprimeness of N_u and $\bar{D}_u$ that $N_u \cdot G_f$ must has RHP pole-zero cancellations. Denote by $(D_f^+)^{-1}$ such a cancelled unstable model of G_f. Then $G_f = (D_f^+)^{-1} \tilde{G}_f$, $\sigma(D_f^+) \subset \mathbb{C}^+$, and $N_u (D_f^+)^{-1}$ is a polynomial matrix. Since G_b has the unstable model of G_f, it must have the form $G_b = (D_f^+)^{-1} \tilde{G}_b$. This results in the RHP pole-zero cancellations in $G_u \cdot G_b = D_u^{-1} N_u (D_f^+)^{-1} \tilde{G}_b$, which violates the condition (*ii*) of Theorem 9.1.1. Proof is therefore completed. □

Theorem 9.1.2. *(DDP) is solvable if and only if the following hold*
(i) G_u has the unstable model of G_d;
(ii) $G_m \cdot G_u$ has no RHP pole-zero cancellations; and
(iii) (9.25) has a proper stable solution G_f.

Proof. Assume that (DDP) is solvable, i.e., there are proper G_b and G_f such that the system is stable and $G_{yd} = 0$. In view of Theorem 9.1.1, conditions (i) and (ii) above must be satisfied in order to assure stability. In addition, $G_{yd} = 0$ implies that G_f satisfies (9.25). It follows from Lemma 9.2 that G_f must also be stable.

Conversely assume that the conditions (i) through (iii) are satisfied. Then $G_{yd} = 0$ for the solution G_f. And there is a proper G_b which stabilizes $G_m G_u$. Furthermore, such a G_b must have the unstable model of G_f due to G_f being itself stable. Hence, by Theorem 9.1, the system is stable. And the above G_f and G_b forms a solution of (DDP). □

The application of Theorem 9.1.2 to a special case yields the following corollary.

Corollary 9.1.3. *(DDP) using feedforward only (i.e., $G_b = 0$) is solvable if and only if the following hold*

(i) G_u, G_d and G_m are all stable; and

(ii) Equation (9.25) has a proper stable solution G_f.

9.1.2 Synthesis

Based on Theorem 9.1.2, a compensator which solves (DDP) can be constructed as follows.

Step 1. Check the solvability conditions. If they are satisfied, (DDP) is solvable and we can proceed to the next step; Otherwise, there is no solution.

Step 2. Construct a feedforward compensator G_f. Any solution of (9.25) can be taken as G_f provided that it is proper and stable. It is of course desirable to choose its order as low as possible.

Step 3. Construct a feedback compensator G_b. A proper G_b is constructed such that it stabilizes $G_m G_u$.

It is seen that G_b and G_f can be constructed independently. In order to check whether (i) and (ii) of Theorem 9.1.2 are satisfied, only matrix factorizations and divisions are needed and they have been well covered in Chapter 2. Stabilization problem in Step 3 has been discussed in Chapter 3. Therefore, our attention here will be paid only to (iii) of Theorem 9.1.2 and Step 2 above so as to find when (9.25) has a proper stable solution and how it can be calculated.

Given a $m \times l$ rational matrix $\bar{G}_1$ of rank r and a $m \times p$ rational matrix $\bar{G}_2$, an $l \times p$ proper rational matrix G needs to be found such that

$$\bar{G}_1 G = \bar{G}_2. \tag{9.26}$$

This is called the model-matching problem. We now present a unified approach, which can treat the properness, stability and minimality simultaneously, and give an explicit parameterization of all the solutions.

It is well known that there is a unimodular polynomial matrix U such that

$$U\bar{G}_1 = \begin{bmatrix} G_1 \\ 0 \end{bmatrix},$$

where G_1 has full row rank r. And the same operation is applied to $\bar{G}_2$:

$$U\bar{G}_2 = \begin{bmatrix} G_2 \\ G_2^{'} \end{bmatrix},$$

where G_2 has the same number of rows as G_1. (9.26) is clearly equivalent to

$$G_2^{'} = 0, \tag{9.27}$$

$$G_1 G = G_2. \tag{9.28}$$

Let a polynomial matrix X be such that $X[G_1 \quad -G_2] := P$ is a polynomial matrix. Elementary row operations are performed on P to make P row reduced. This means that there is an unimodular polynomial matrix U_1 such that $U_1 P := \tilde{P}$ is row reduced. $\tilde{P}$ is partitioned as

$$\tilde{P} = [\tilde{P}_1 \quad -\tilde{P}_2], \tag{9.29}$$

which is consistent with $[G_1 \quad -G_2]$. A greatest common left divisor of the columns of P_1 is denoted by L, i.e., $\tilde{P}_1 = L\bar{P}_1$, where columns of polynomial matrix $\bar{P}_1$ are left coprime. Let B be a minimal base of the right null space of $\tilde{P}_1$ and $\bar{P}_2 R^{-1}$ be a dual coprime fraction of $L^{-1}\tilde{P}_2$. Denote by $\partial_{ci}(A)$ the i th column degree of a polynomial matrix A, and by $\Gamma_r(A)$ the highest row-degree coefficient matrix of $\tilde{A}$. $\Gamma_r(\tilde{P})$ is partitioned as

$$\Gamma_r(\tilde{P}) = [\tilde{P}_{h1} \quad \tilde{P}_{h2}],$$

which is consistent with (9.29).

Theorem 9.1.3. *Given $\bar{G}_1$ and $\bar{G}_2$ as above, (9.26) has a solution for G if and only if (9.27) holds. If it is the case, all the solutions are given by the following polynomial matrix fraction:*

$$G = ND^{-1}, \tag{9.30}$$

$$N = BW + N_0 V, \tag{9.31}$$

$$D = RV, \tag{9.32}$$

where W and V are arbitrary polynomial matrices, and N_0 satisfies

$$\tilde{P}_1 N_0 = \tilde{P}_2 R. \tag{9.33}$$

Furthermore,

(i) a solution is of minimal order if and only if V is unimodular. Without loss of generality all the minimal solutions are given by (9.30) and (9.33) with $V = I$;

(ii) *There is a proper solution if and only if*

$$Rank[\tilde{P}_{h1}] = r. \tag{9.34}$$

If it holds, all the proper solutions are given by (9.30) through (9.33) with N_0R^{-1} being proper, D column reduced and $\partial_{ci}(BW) \leq \partial_{ci}(D)$; And

(iii) *there is a stable solution if and only if $det(R)$ is a stable polynomial. If it is, all the stable solutions are given by (9.30) through (9.33) with V^{-1} being stable.*

Proof. It is clear that (9.27) is necessary for the existence of solutions. Conversely assume that it holds. Then (9.26) will have a solution if (9.28) has a solution, which is guaranteed due to G_1 having full rank. Indeed, replace G in (9.28) by a right coprime polynomial matrix fraction,

$$G = ND^{-1}. \tag{9.35}$$

Equation (9.28) can be rewritten as

$$\tilde{P}_1 N = \tilde{P}_2 D. \tag{9.36}$$

Premultiplying (9.36) by L^{-1} yields

$$\bar{P}_1 N = \bar{P}_2 R^{-1} D. \tag{9.37}$$

Because the columns of $\bar{P}_1$ are coprime by the definition of L, then there is a polynomial matrix M such that $\bar{P}_1 M = I$. Setting $N = M\bar{P}_2$ and $D = R$, then such N and D satisfy (9.37) and thus ND^{-1} is a solution of (9.28).

In order to search for a general solution of (9.36), it will be first shown that if D is any solution of (9.36), it must have the form

$$D = RV, \tag{9.38}$$

for some polynomial matrix V. Indeed, assume that D is a solution of (9.36), it then satisfies (9.37). This implies that $R^{-1}D$ must be a polynomial matrix because $\bar{P}_2$ and R are coprime. In other words, $D = RV$ for some polynomial matrix V.

A general solution of (9.36) can be expressed as the sum of the general solution of the associated homogeneous equation,

$$\tilde{P}_1 N_1 = 0, \tag{9.39}$$

and a particular solution N_2 of the inhomogeneous equation (9.36). If B is a minimal base of the right null space of $\tilde{P}_1$, then it is well known (Forney 1975) that the general solution N_1 of (9.39) is $N_1 = BW$, where W is a arbitrary polynomial matrix. For the particular solution N_2 of (9.36), set $N_2 = N_0V$, where N_0 satisfies

$$\tilde{P}_1 N_0 = \tilde{P}_2 R, \tag{9.40}$$

then N_2 is clearly a solution of (9.36) due to $D = RV$ as shown above. Therefore, we have

$$N = N_1 + N_2 = BW + N_0 V. \tag{9.41}$$

Next, we prove (i)-(iii).

(i) The order of a solution is $deg\, det(D)$, where $deg\, det(D) = deg\, det(R) + deg\, det(V) \geq deg\, det(R)$. The minimal order is achieved if and only if $deg\, det(V) = 0$, i.e., V is unimodular. It is clear that V can be taken as I, which does not change the form of (9.30) if W is replaced by WV^{-1}. And WV^{-1} is equivalent to W since V is unimodular.

(ii) It has been shown (Kung and Kailath, 1980) that there is a proper minimal solution if and only if (9.34) hold. If it does, there is a proper minimal solution which is, of course, proper. Conversely if a proper solution exists, then in the set of all the proper solutions, there must be one with minimal order. Thus (9.34) is necessary and sufficient for the existence of a proper solution. Some constraints will be imposed on the general solution given by (9.30) to (9.33) in order to obtain all the proper solutions. For the simplicity, assuming D is column reduced. It can be easily seen that this is not a restriction since there is always a unimodular polynomial matrix U such that DU is column reduced. If N and D is replaced by NU and DU, respectively, no solution will be changed or lost. Therefore D can be assumed column reduced. As mentioned above, when (9.34) holds, there is a proper minimal solution of (9.36). But one sees from the proof of (i) that such a solution can be obtained by setting $V = I$. Now let N_0 be a solution of (9.33) such that $N_0 R^{-1}$ is proper. This can easily be done by any algorithm for the minimal design problem. For such a N_0, all the solutions of (9.28), as shown above, can be given by (9.30). Furthermore, since $ND^{-1} = (BW)D^{-1} + N_0 R^{-1}$ with $N_0 R^{-1}$ being proper, then the solutions will be proper if and only if $(BW)D^{-1}$ is proper, which is equivalent to $\partial_{ci}(BW) \leq \partial_{ci}(D)$ when D is column reduced.

(iii) Because R is common to all the solutions and V is arbitrary, there is a stable solution if and only if $det(R)$ is a stable polynomial. If it is the case, all the stable solutions are given by (9.30) to (9.33) with V^{-1} being stable. □

For illustration, consider

$$G_u = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s+2)} \\ \frac{1}{s-1} & \frac{1}{(s-1)(s+1)} \end{bmatrix},$$

$$G_d = \begin{bmatrix} \frac{1}{(s-1)(s+2)} & 0 \\ \frac{1}{(s-1)(s+1)} & \frac{1}{(s-1)(s+1)(s+2)} \end{bmatrix},$$

$$G_m = I.$$

Clearly G_d has an unstable pole at $s = 1$. We got left coprime fractions $G_u = D_u^{-1} N_u$ and $G_d = D_d^{-1} N_d$, where

$$D_u = \begin{bmatrix} (s-1)(s+2) & 0 \\ 0 & (s-1)(s+1) \end{bmatrix}, \quad N_u = \begin{bmatrix} (s+2) & 1 \\ (s+1) & 1 \end{bmatrix},$$
$$D_d = \begin{bmatrix} (s-1)(s+2) & 0 \\ 0 & (s-1)(s+1)(s+2) \end{bmatrix}, \quad N_d = \begin{bmatrix} 1 & 0 \\ (s+2) & 1 \end{bmatrix}.$$

Because

$$D_u D_d^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

is stable so that G_u has the unstable model of G_d. For $G_m = I$, $G_m \cdot G_u$ has no RHP pole-zero cancellations. It follows from Theorem 9.1.2 that there is a solution for (DDP) if and only if the equation,

$$G_u G_f = G_d,$$

has a proper stable solution. According to Theorem 9.1.3, it has a solution due to nonsingularity of G_u. Premultiplying $[G_u \quad -G_d]$ by D_d yields

$$P = D_d \begin{bmatrix} G_u & -G_d \end{bmatrix} = \begin{bmatrix} (s+2) & 1 & -1 & 0 \\ (s+1)(s+2) & (s+2) & -(s+2) & -1 \end{bmatrix}$$

which is left equivalent to

$$\tilde{P} = \begin{bmatrix} (s+2) & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

which is row reduced. One sees

$$\Gamma_r(\tilde{P}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix},$$

$$Rank(\tilde{P}_{h1}) = 2.$$

Thus a proper solution exists. Because $\tilde{P}_1 = \tilde{P}_1 \cdot I$, the greatest common left divisor of $\tilde{P}_1$ can be taken as itself, i.e., $L = \tilde{P}_1$. And the minimal base B of L is clearly zero. Hence

$$\begin{aligned} L^{-1}\tilde{P}_2 &= \begin{bmatrix} (s+2) & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{1}{s+2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} := \bar{P}_2 R^{-1}. \end{aligned} \tag{9.42}$$

$det(R) = (s+2)$ is a stable polynomial so that a stable solution exists. Therefore, (DDP) for this example is solvable. It is now possible to give an explicit parametric form of all the proper stable solutions. In order to search for a particular solution N_0 of the equation: $\tilde{P}_1 N = \tilde{P}_2 R$, for N such that $N_0 R^{-1}$ is proper. Set $N_0 = \bar{P}_2$ which is clearly a needed solution. A proper and stable solution is then given by $G_f = \bar{P}_2 R^{-1}$ as given by (9.42).

It can be noted that for this example, since G_u is nonsingular, G_f is in fact unique and can be obtained directly by $G_f = G_u^{-1} G_d$. In this case (DDP) is solvable if and only if this G_f is proper and stable. The computations in the example is thus not necessary, but is carried out to illustrate the procedure for a general case.

For the feedback compensator G_b which stabilizes G_u there are many well-known algorithms for its construction (Chapter 3). Therefore, specific computations are omitted here.

In this section the disturbance decoupling problem with stability by dynamic measurement feedback and disturbance feedforward has been considered. The problem is solved in a polynomial matrix setting. The stability conditions are established. The main observation is that for ensuring the stability, the transfer matrix of control channel and the feedback compensator must have the unstable models of disturbance channel and of the feed-forward compensator, respectively, and there should be no RHP pole-zero cancellations between the plant and sensor. They are intuitively appealing and have an explicit dynamic interpretation. Under such stability conditions it is then shown that solvability of disturbance decoupling is equivalent to the existence of a proper stable solution for some rational matrix equation. It is also shown that the construction of a compensator for (DDP) can be made into two independent parts. One is the feedback compensator synthesis, which is a well-known problem. Another is the feed-forward compensator synthesis, which can be incorporated into a more general problem, namely the model matching. A unified approach, which can simultaneously treat properness, stability and minimality, is presented. This approach also gives an explicit parametric form of all the solutions.

9.2 Unmeasurable Disturbances: Disturbance Observers

In this section, supposing unmeasurable disturbance, we present a general scheme for disturbance decoupling control via output feedback for processes with possibly time delay. The objective is to fully compensate for the disturbance while retaining the input-output servo response of the conventional feedback system. Internal stability and disturbance rejection performance will be analyzed, and the special case of periodic disturbances highlighted. SISO systems will be treated first and then extended to the MIMO systems.

Our development will involve both time and Laplace domain representations of signals. For clarity, the former is denoted by small case letters such as $d(t)$ while the latter by capitals such as $D(s)$. Throughout this section, we assume that the process is stable and has no zeros in the right half plane (RHP), unless otherwise stated. The diagram of the conventional feedback control system is depicted in Figure 9.3, where $P(s)$ and $C(s)$ denote the process and feedback controller respectively, and $d(t)$ is an unknown disturbance acting on the system through a stable dynamic element $G(s)$. It follows from Figure 9.3 that

$$Y^o = \frac{PC}{1+PC}R + \frac{G}{1+PC}D,$$

where the superscription o indicates the output response in the conventional feedback system.

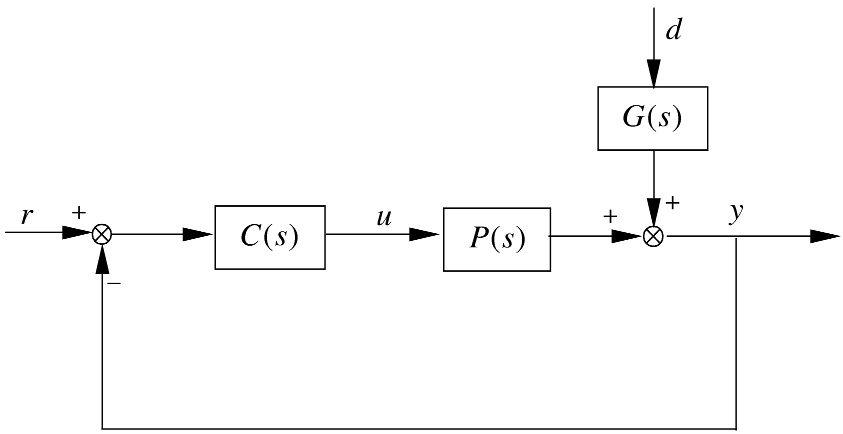


Fig. 9.3. Conventional Feedback System

9.2.1 Disturbance Observers

A control scheme, depicted in Figure 9.4, is exploited to compensate for the disturbance. The intention is to estimate the disturbance by the difference between the actual process output and its model output, and feed the estimate into another block to fully cancel the disturbance. From Figure 9.4, one has

$$Y = \frac{PC}{1-HF+PC+PH}R + \frac{1-HF}{1-HF+PC+PH}GD. \tag{9.43}$$

To fully compensate for the disturbance while retaining the input-output relation for servo response of the conventional feedback system requires

$$\frac{PC}{1-HF+PC+PH} = \frac{PC}{1+PC},$$

$$\frac{1-HF}{1-HF+PC+PH}GD = 0, \tag{9.44}$$

which are satisfied by

$$F(s) = P(s), \tag{9.45}$$

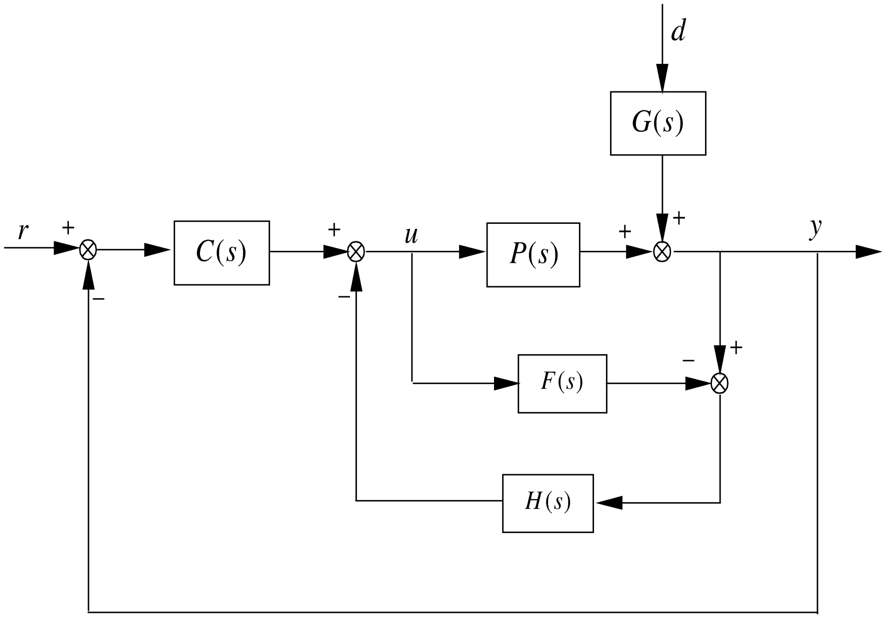


Fig. 9.4. Control Scheme with Disturbance Observer

and

$$H(s) = P^{-1}(s). \tag{9.46}$$

Let $P(s) = P_0(s)e^{-Ls}$ and its model $\hat{P}(s) = \hat{P}_0(s)e^{-\hat{L}s}$, where P_0 and $\hat{P}_0$ are delay-free rational transfer functions. Then, using the model, (9.45) and (9.46) are replaced by

$$F(s) = \hat{P}(s) = \hat{P}_0(s)e^{-\hat{L}s}, \tag{9.47}$$

$$H(s) = \frac{1}{F(s)} = \frac{1}{\hat{P}_0(s)}e^{\hat{L}s}, \tag{9.48}$$

respectively. For a strictly proper $\hat{P}_0(s)$, its inverse is not physically realizable, and nor is the pure predictor $e^{\hat{L}s}$. For the term $e^{\hat{L}s}$, the approximation presented in Huang *et al.* (1990) as shown in Figure 9.5 is adopted here, where the overall transfer function is

$$G_v(s) = \frac{1 + V(s)}{1 + V(s)e^{-\hat{L}s}}.$$

If $V(s)$ is of low-pass with high gain, then

$$G_v(s) \approx \frac{V(s)}{V(s)e^{-\hat{L}s}} = e^{\hat{L}s}$$

in the low-frequency range, while $G_v(s) \approx \frac{1+0}{1+0} = 1$ in the high-frequency range. The simplest choice for $V(s)$ is a first-order element:

$$V(s) = \frac{K_v}{\tau_v s + 1}, \qquad \tau_v > 0,$$

and then the corresponding G_v is given by

$$G_v(s) = \frac{1 + \frac{K_v}{\tau_v s+1}}{1 + \frac{K_v}{\tau_v s+1} e^{-\hat{L}s}}. \tag{9.49}$$

According to our study, the value of τ_v is suggested to be set as $0.1\hat{L} \sim \hat{L}$. The remaining parameter K_v is chosen as large as possible for high low-frequency gain of G_v while preserving stability of G_v. The critical value $K_{v_{max}}$ of K_v at which the Nyquist curve of $\frac{K_v}{\tau_v s+1} e^{-\hat{L}s}$ touches $-1 + j0$ can be found from

$$\begin{aligned} K_{v_{max}} &= \sqrt{1 + {\omega^*}^2 \tau_v^2}, \\ \tan^{-1}(\omega^* \tau_v) &+ \omega^* \hat{L} = \pi. \end{aligned} \tag{9.50}$$

For sufficient stability margin, K_v should be much smaller than $K_{v_{max}}$ and its default value is suggested to be $K_{v_{max}}/3$.

$\hat{P}_0(s)$ is usually strictly proper and $\frac{1}{\hat{P}_0(s)}$ is then improper. A filter $Q(s)$ is introduced to make $\frac{Q(s)}{\hat{P}_0(s)}$ proper and thus realizable. $Q(s)$ should have at least the same relative degree as that of $\hat{P}_0(s)$ to make $Q(s)/\hat{P}_0(s)$ proper, and should be approximately 1 in the disturbance frequency range for a satisfactory disturbance rejection performance. The following form is a possible choice:

$$Q(s) = \frac{1}{(\tau_q s + 1)^n}, \qquad \tau_q > 0, \tag{9.51}$$

where n is selected to make $\frac{Q(s)}{\hat{P}_0(s)}$ just proper. τ_q is a user specified parameter and should be smaller than the equivalent process time constant T_p. The

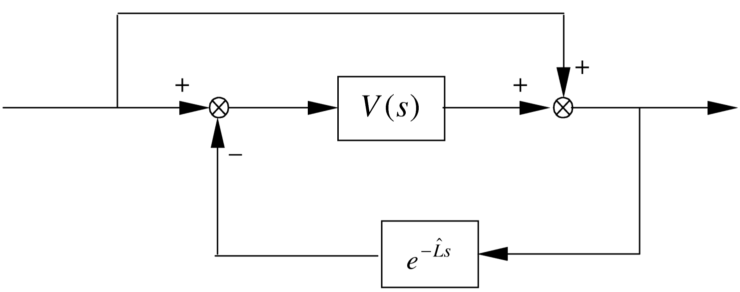


Fig. 9.5. Block Diagram of $G_v(s)$ Approximating $e^{\hat{L}s}$

initial value is set as $0.01T_p \sim 0.1T_p$ and its tuning will be discussed below to make the system stable. Other forms of $Q(s)$ have been discussed by Umeno and Hori (1991).

In view of the above development, $H(s)$ is approximated by

$$H(s) = \frac{Q(s)G_v(s)}{\hat{P}_0(s)} = \frac{Q(s)}{\hat{P}_0(s)} \frac{1 + V(s)}{1 + V(s)e^{-\hat{L}s}}. \tag{9.52}$$

The actual system to realize the proposed control scheme with such an approximation is depicted in Figure 9.6.

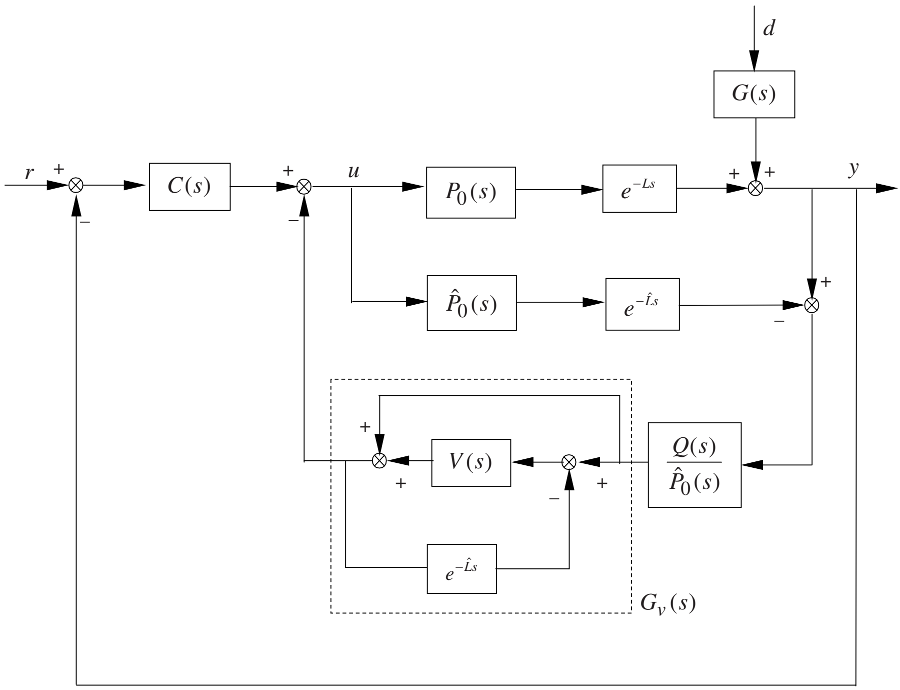


Fig. 9.6. Control Scheme with Approximate Disturbance Observer

If $\hat{P}(s)$ is bi-proper (the degrees of the numerator and denominator are the same), the filter $Q(s)$ is not required. For a dead-time free case, G_v is not required or $G_v(s) = 1$, (9.52) becomes

$$H(s) = \frac{Q(s)}{\hat{P}(s)},$$

which is the original disturbance observer proposed by Ohnishi (1987).

9.2.2 Analysis

Consider stability and performance of the control scheme in Figure 9.6. Let a system consist of n subsystems with scalar transfer functions $g_i(s)$, $i = 1, 2, \cdots, n$, and $p_i(s)$ be the characteristic polynomial of $g_i(s)$. Define

$$p_c(s) = \Delta(s) p_0(s), \tag{9.53}$$

where $\Delta(s)$ is the system determinant as given by the Mason's formula (Mason, 1956) and $p_0(s) = \prod_{i=1}^{n} p_i(s)$. It is shown (Chapter 3) that *the system is internally stable if $p_c(s)$ in (9.53) has all its roots in the open left-half of the complex plane.*

For the proposed scheme, write the transfer function of each block into a coprime polynomial fraction with possible time delay, i.e., $C(s) = \frac{n_c(s)}{d_c(s)}$, $P(s) = \frac{n_p(s)}{d_p(s)} e^{-Ls}$, $\hat{P}(s) = \frac{n_{\hat{p}}(s)}{d_{\hat{p}}(s)} e^{-\hat{L}s}$, $V(s) = \frac{n_v(s)}{d_v(s)}$, $Q(s) = \frac{n_q(s)}{d_q(s)}$ and $G(s) = \frac{n_g(s)}{d_g(s)}$, where $n_c(s)$, $d_c(s)$, $n_p(s)$, $d_p(s)$, $n_{\hat{p}}(s)$, $d_{\hat{p}}(s)$, $n_v(s)$, $d_v(s)$, $n_q(s)$ $d_q(s)$, $n_g(s)$ and $d_g(s)$ are all polynomials. It follows from Figure 9.6 that

$$\begin{aligned}
\Delta(s) &= 1 + PC + \frac{P_0}{\hat{P}_0} QVe^{-Ls} + \frac{P_0}{\hat{P}_0} Qe^{-Ls} \\
&\quad - QVe^{-\hat{L}s} - Qe^{-\hat{L}s} + Ve^{-\hat{L}s} + PCVe^{-\hat{L}s}, \\
&= (1 + PC)(1 + Ve^{-\hat{L}s}) + (1 + V)Q(\frac{P_0}{\hat{P}_0} e^{-Ls} - e^{-\hat{L}s}), \qquad (9.54)
\end{aligned}$$

$$p_0(s) = n_{\hat{p}}(s) d_p(s) d_{\hat{p}}(s) d_v(s) d_c(s) d_q(s) d_g(s). \tag{9.55}$$

Under the nominal case, i.e., $\hat{P}(s) = P(s)$, Δ becomes

$$\hat{\Delta} = (1 + PC)(1 + Ve^{-Ls}),$$

so that

$$\hat{p}_c = \hat{\Delta} p_0 = \{d_p d_c (1 + PC)\} \left\{d_v (1 + Ve^{-Ls})\right\} \{n_p d_p d_q d_g\}.$$

The assumed stability of the conventional feedback system in Figure 9.3 means that $d_p d_c (1 + PC)$ only has LHP roots. Under the assumptions that the process is stable and without RHP zero, and G is stable, d_p, d_g and n_p are then all stable. Therefore, it is concluded that the proposed scheme is nominally stable, provided that stable G_v (equivalently $d_v(1 + Ve^{-Ls})$) and Q (or d_q) are used.

Turn now to the case where there is a process/model mismatch:

$$\hat{P}(s) \neq P(s),$$

and the modelling error is described by

$$\left| P(j\omega) - \hat{P}(j\omega) \right| < \gamma(j\omega).$$

Then, $p_c(s)$ can be re-written as

$$\frac{p_c(s)}{p_0(s)} = \Delta(s) = \hat{\Delta}(s) + \delta(s),$$

where the perturbed term from the nominal case is $\delta(s) = (1+V)Q(\frac{P_0}{\hat{P}_0}e^{-Ls} - e^{-\hat{L}s})$ and

$$\begin{aligned} |\delta(j\omega)| &= \left| \frac{1+V(j\omega)}{\hat{P}_0(j\omega)} Q(j\omega)(P(j\omega) - \hat{P}(j\omega)) \right| \\ &< \left| \frac{1+V(j\omega)}{\hat{P}_0(j\omega)} Q(j\omega) \right| \gamma(j\omega) := \bar{\delta}(j\omega). \end{aligned}$$

Since the nominal stability has been proven, the Nyquist curve of $\hat{\Delta}(s)$ should have a correct number of encirclements of the origin, as required by the Nyquist stability criterion. One can superimpose the $\bar{\delta}(j\omega)$ onto $\hat{\Delta}(s)$'s Nyquist curve to form a Nyquist curve band, and check its stability.

Consider now the performance of the proposed method and its robustness. With a stabilizing and tracking controller C, the process output response $y_d(t)$ to a disturbance will settle down, i.e., $y_d(t) \to 0$ as $t \to \infty$. Thus, the widely used index in the time domain:

$$\|y_d(t)\|_2 = \sqrt{\int_0^\infty y_d^2(t)dt},$$

makes sense and can be used for performance evaluation. To have a better disturbance rejection than the conventional feedback system requires

$$\|y_d(t)\|_2 \leqslant \|y_d^o(t)\|_2, \tag{9.56}$$

where $y_d^o(t)$ is the output response to the same disturbance if the conventional feedback system is used. Applying the Parseval's Identity (Spiegel, 1965):

$$\int_0^\infty y_d^2(t)dt = \frac{1}{2\pi}\int_0^\infty |Y_d(j\omega)|^2 d\omega,$$

to (9.56) gives

$$\|Y_d\|_2 \leqslant \|Y_d^o\|_2, \tag{9.57}$$

where

$$\begin{aligned} \|Y_d\|_2 &\triangleq \sqrt{\int_0^\infty |Y_d(j\omega)|^2 d\omega}, \\ \|Y_d^o\|_2 &\triangleq \sqrt{\int_0^\infty |Y_d^o(j\omega)|^2 d\omega}. \end{aligned}$$

Substituting (9.43), (9.47) and (9.52) into (9.57) yields

$$\left\| \frac{1 - G_v Q e^{-\hat{L}s}}{1 + PC + G_v Q(\frac{P_0}{\hat{P}_0} e^{-Ls} - e^{-\hat{L}s})} GD \right\|_2 \leqslant \left\| \frac{GD}{1 + PC} \right\|_2 . \tag{9.58}$$

In case of $\hat{P}(s) = P(s)$, (9.58) becomes

$$\left\| \frac{GD}{1 + PC} (1 - G_v Q e^{-Ls}) \right\|_2 \leqslant \left\| \frac{GD}{1 + PC} \right\|_2 . \tag{9.59}$$

For illustration, let us consider the simple case with $G_v = e^{Ls}$, $P = \frac{e^{-Ls}}{T_p s + 1}$ and $C(s) = k_c + \frac{k_i}{s}$. The disturbance is supposed to be $D(s) = \frac{1}{s}$ through a dynamics $G(s) = \frac{1}{T_d s + 1}$, and the filter is supposed to be $Q(s) = \frac{1}{\tau_q s + 1}$. With simple algebra, we evaluate

$$\left\| \frac{GD}{1 + PC} \right\|_2^2 = \int_0^\infty A(\omega) d\omega, \tag{9.60}$$

where

$$\begin{aligned} &A(\omega) \\ &= \frac{T_p^2 \omega^2 + 1}{(T_d^2 \omega^2 + 1)[(-T_p \omega^2 + k_i \cos \omega L + k_c \omega \sin \omega L)^2 + (\omega + k_c \omega \cos \omega L - k_i \sin \omega L)^2]} \\ &\geqslant 0. \end{aligned}$$

Similarly, the left-hand side of (9.59) becomes

$$\left\| \frac{GD}{1 + PC} (1 - Q) \right\|_2^2 = \int_0^\infty A(\omega) \frac{\tau_q^2 \omega^2}{\tau_q^2 \omega^2 + 1} d\omega \tag{9.61}$$

$$\leqslant \int_0^\infty A(\omega) d\omega = \left\| \frac{GD}{1 + PC} \right\|_2^2 , \tag{9.62}$$

so that (9.59) holds. The performance enhancement of the proposed method over the conventional feedback system is ensured for this special case in the sense of (9.56).

9.2.3 Periodic Disturbances

In the discussion made so far, no specific characteristic is assumed for the disturbance. If the disturbance is of a special type and this information is utilized beneficially, disturbance rejection performance may be further enhanced. In this subsection, a special type of disturbances, periodic disturbances, will be investigated in details. Periodic disturbances are often encountered in mechanical systems such as industrial robots (Hara *et al.*, 1988). Disturbances acting on the track-following servo systems of an optical disk drive or a hard disk drive inherently contain significant periodic components that cause tracking errors of a periodic nature. Also, continuous periodic patterns/disturbances are abundant in chemical process industries and examples include pressure swing absorption systems extruders, hybrid reactors and simulated moving bed (SMB) chromatography systems (Natarajan and Lee, 2000).

We begin with some definitions and preliminary results. Let a periodic signal $x(t)$ have a period T and act as an input to the stable system. The output steady state response will also be periodic with the same period. Let the periodic steady state part of the output response $y(t)$ be $y^s(t)$, and the corresponding Laplace transform be $Y^s(s) = \mathcal{L}\{y^s(t)\}$.

Lemma 9.2.1. *Under a periodic input $x(t)$, the output steady state of the system $Y(s) = G_2(s)G_1(s)X(s)$ with stable $G_2(s)$ and $G_1(s)$ satisfies*

$$(g_2 * g_1 * x)^s(t) = [g_2 * (g_1 * x)^s]^s(t),$$

where $*$ is the standard convolution, and g_1 and g_2 are the impulse responses of $G_1(s)$ and $G_2(s)$ respectively.

Proof. Suppose that $x(t)$ has the period T and is expressed by its Fourier series (Cartwright, 1990):

$$\begin{aligned} x(t) &= x_0 + \sum_{i=1}^{\infty} C_i \cos(2\pi i t/T + \phi_i) \\ &= x_0 + \sum_{i=1}^{\infty} C_i \cos(\omega_i t + \phi_i) \end{aligned}$$

with $x_0 = \frac{1}{T}\int_0^T x dt$, $C_i = \sqrt{A_i^2 + B_i^2}$, $\omega_i = 2\pi i/T$, and $\phi_i = -\tan^{-1}(B_i/A_i)$, where $A_i = \frac{2}{T}\int_0^T \cos(2\pi i t/T) x dt$ and $B_i = \frac{2}{T}\int_0^T \sin(2\pi i t/T) x dt$. Having such a signal transmitted through a stable G_1, the output is given by $g_1(t) * x(t)$. The steady state of the output is a periodic signal with the same period T (Cartwright, 1990):

$$(g_1 * x)^s(t) = G_1(0)x_0 + \sum_{i=1}^{\infty} C_i |G_1(j\omega_i)| \cos(\omega_i t + \phi_i + \angle G_1(j\omega_i)).$$

Similarly, one gets

$$\begin{aligned} [g_2 * (g_1 * x)^s]^s(t) = G_2(0)G_1(0)x_0 + \sum_{i=1}^{\infty} C_i |G_1(j\omega_i)||G_2(j\omega_i)| \\ \cos(\omega_i t + \phi_i + \angle G_1(j\omega_i) + \angle G_2(j\omega_i)). \end{aligned}$$

On the other hand, it follows from

$$g_2(t) * g_1(t) * x(t) = [g_2(t) * g_1(t)] * x(t)$$

that $y^s(t)$ is directly given by

$$(g_2 * g_1 * x)^s(t) = G_2(0)G_1(0)x_0 + \sum_{i=1}^{\infty} C_i |G_2(j\omega_i)G_1(j\omega_i)| \cos(\omega_i t + \phi_i + \angle G_2(j\omega_i)G_1(j\omega_i)).$$

Noting that

$$|G_1(j\omega_i)G_2(j\omega_i)| = |G_2(j\omega_i)||G_1(j\omega_i)|,$$

and

$$\angle[G_2(j\omega_i)G_1(j\omega_i)] = \angle G_1(j\omega_i) + \angle G_2(j\omega_i),$$

we then have

$$y^s(t) = (g_2 * g_1 * x)^s(t) = [g_2 * (g_1 * x)^s]^s(t). \qquad \square$$

Turn back to the rejection of periodic disturbances, we use the same F as in (9.47) but for H replace (9.48) by

$$H(s) = \frac{1}{\hat{P}_0(s)} e^{-L_h s}, \tag{9.63}$$

where $L_h = k_0 T - \hat{L}$, and k_0 is the smallest no-negative integer that renders a non-negative $L_h \geqslant 0$. With the preceding discussions, Figure 9.6 is recasted as Figure 9.7. Substituting (9.63) into (9.43) yields

$$Y_d(s) = \frac{1 - e^{-k_0 T s}}{1 - HF + PC + PH} GD. \tag{9.64}$$

Let $\eta(t)$ the impulse response of $G/(1 - HF + PC + PH)$. One obtains from (9.64) that

$$\begin{aligned} y_d^s(t) &= [\eta(t) * \mathcal{L}^{-1}\{(D(s) - e^{-k_0 T s} D(s))\}]^s \\ &= (\eta(t) * (d(t) - d(t - k_0 T)))^s \\ &= (\eta(t) * (d^s(t) - d^s(t - k_0 T)))^s \\ &= (\eta(t) * 0)^s \\ &= 0. \end{aligned} \tag{9.65}$$

Thus, the periodic disturbance can be asymptotically rejected. For physical realization of $H(s)$ in (9.63), a filter Q, as discussed above, has to be introduced to make $Q(s)/\hat{P}_0(s)$ proper.

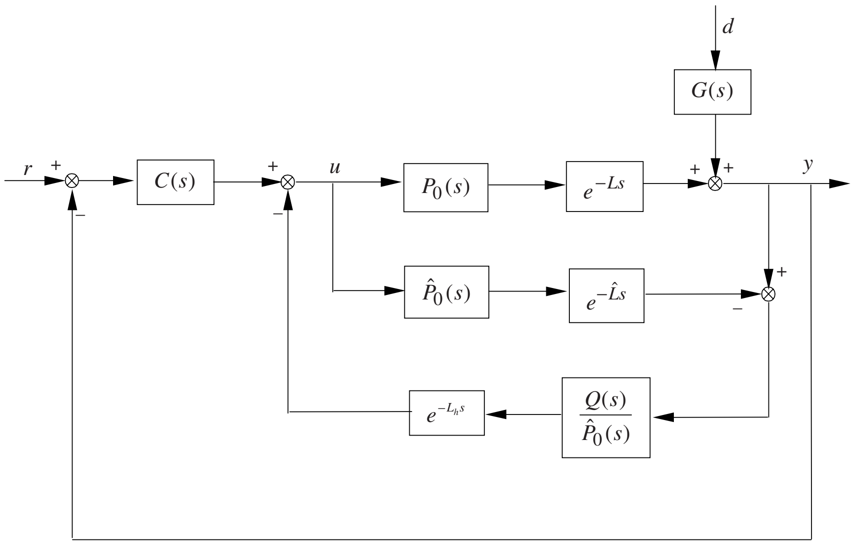


Fig. 9.7. Control Scheme for Periodic Disturbance Rejection

9.2.4 Simulation

We now present several examples to demonstrate the disturbance observer based control and compare with conventional feedback systems.

Example 9.2.1. Consider a first-order plus dead-time process (Astrom and Hagglund, 1995):

$$P(s) = \frac{e^{-5s}}{10s+1}.$$

The PI controller design method proposed in Zhuang and Atherton (1993) is applied to give

$$C(s) = 1.01 + \frac{1.01}{s}.$$

For the scheme in Figure 9.6 with perfect model assumption $\hat{P}(s) = P(s)$, take $\tau_v = 0.4L = 2$. We obtain $K_{v_{max}} = 1.38$ from (9.50). Setting $K_v = K_{v_{max}}/3 = 0.46$ yields $V(s) = \frac{0.46}{2s+1}$. The filter $Q(s)$ is in the form of $Q(s) = \frac{1}{\tau_q s+1}$, and τ_q is chosen as $0.1T_p = 1$. A unit step change in the set-point is made at $t = 0$, and a step disturbance of the size 1 is introduced through $G(s) = \frac{1}{20s+1}$ to the system at $t = 50$, the resulting responses are displayed in Figure 9.8. Clearly, the disturbance observer control exhibits better capability to reject load disturbance compared with the conventional controller.

To evaluate the scheme of Figure 9.7 for periodic disturbances, the following periodic disturbance is introduced at $t = 50$ as a separate test:

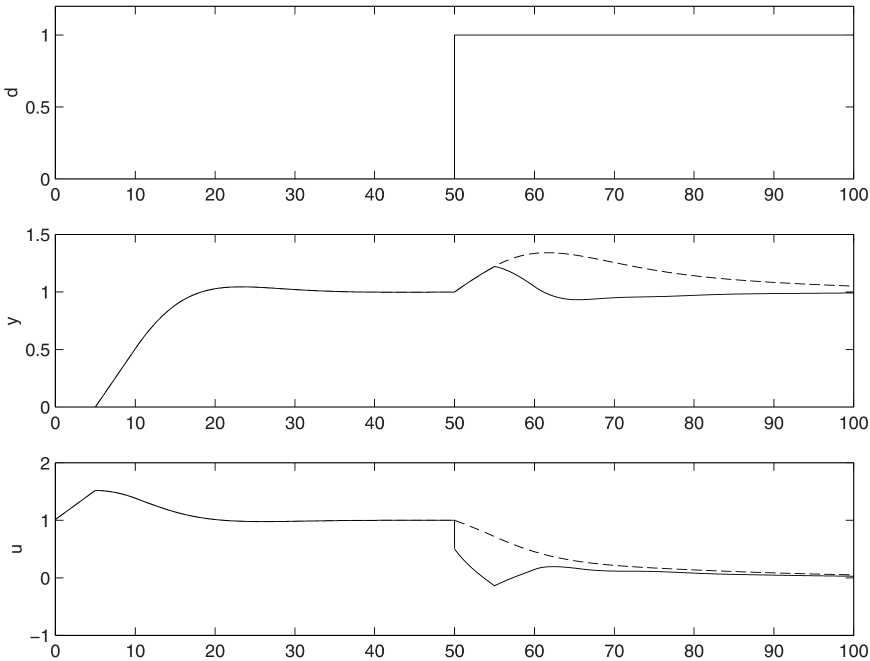


Fig. 9.8. Time Responses for Example 9.2.1
(—— Disturbance Observer; – – – Conventional Control)

$$d(t) = \sin(0.2\pi t) + 0.3 \sin(0.4\pi t) + 0.1 \sin(0.6\pi t) + 0.1 \sin(0.8\pi t)$$

with the period of $T = \frac{2\pi}{0.1\pi} = 10$. Let $G(s) = 1$. Calculate from (9.63) $L_h = T - L = 5$. In the case of no plant/model mismatch, the resultant responses are shown in Figure 9.9. The output response of the proposed method approaches zero asymptotically, as expected from (9.65). Comparing with the conventional control, the disturbance observer control produces great improvement in the load response. ♢

Example 9.2.2. Consider a process with multiple leg:

$$P(s) = \frac{1}{(s+1)^8},$$

with its model (Bi *et al.*, 1999):

$$\hat{P}(s) = \frac{1.06}{3.81s + 1} e^{-4.94s}.$$

A PI controller is designed (Zhuang and Atherton, 1993) as

$$C(s) = 0.3685 + \frac{0.0967}{s}.$$

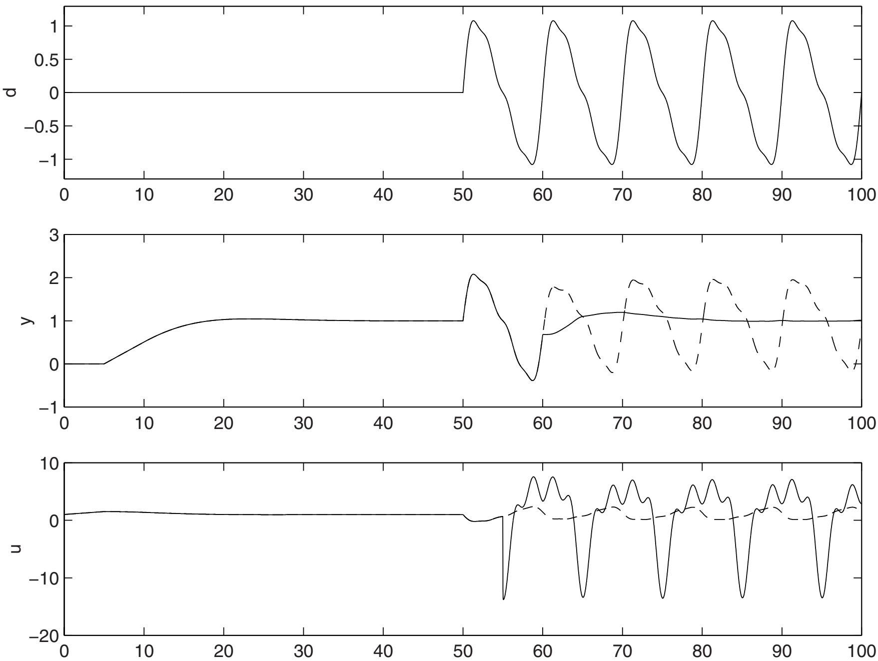


Fig. 9.9. Time Responses for Example 9.2.1 under Periodic Disturbance (——Disturbance Observer; − − −Conventional Control)

For the scheme of Figure 9.6, $V(s) = \frac{0.4625}{2s+1}$ and $Q(s) = \frac{1}{5s+1}$ are designed. Using P and $\hat{P}$ in the scheme, the system step responses are shown in Figure 9.10, and improvement in the load response is substantial with the disturbance observer control. To verify stability, we draw the Nyquist curve of $\hat{\Delta}(s)$ with the perturbation term $\delta(s)$ superimposed onto it, as displayed in Figure 9.11. The resulting banks neither enclose nor encircle the origin. Therefore, the system with the actual process P in place is indeed stable. $\Diamond$

In the development made so far, the process $P(s)$ has been assumed to be stable without any zeros in the RHP. However, the control scheme in Figure 9.6 can be modified to Figure 9.12 so that it is applicable to unstable processes and/or those with RHP zeros. In this case, the process $P(s)$ is factorized as

$$P(s) = P_0^+(s)P_0^-(s)e^{-Ls},$$

where $P_0^-(s)$ represents the stable non-minimum-phase part, and $P_0^+(s)$ includes all RHP zeros and poles. Subsequently, $P_0^-(s)$ is used in (9.47) and (9.52) to design $F(s)$ and $H(s)$, respectively.

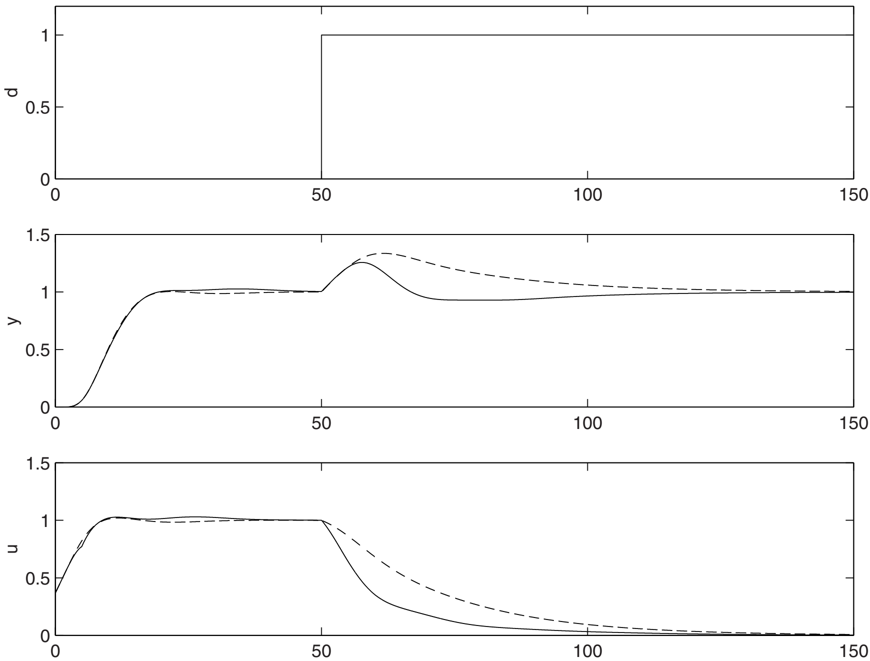


Fig. 9.10. Time Responses for Example 9.2.2
(—— Disturbance Observer; – – – Conventional Control)

Example 9.2.3. Consider an unstable process:

$$P(s) = \frac{e^{-0.4s}}{(s+1)(s-1)}.$$

A PID controller is designed (Ho and Xu, 1998) as

$$C(s) = 1.8075 + \frac{0.1175}{s} + 1.6900s.$$

For the scheme of Figure 9.12 with the modification mentioned above, $V(s) = \frac{0.3962}{0.1s+1}$ and $Q(s) = \frac{1}{10s+1}$ are designed. The resulting performance is shown in Figure 9.13, where the effectiveness of the disturbance observer control is clearly illustrated. $\Diamond$

9.2.5 Extension to the MIMO Case

We have so far considered the SISO case. The extension to the MIMO case is straightforward. Suppose that the process $P(s)$ is square, nonsingular, stable and of minimum phase. The same scheme as in Figure 9.4 is exploited. It then follows that

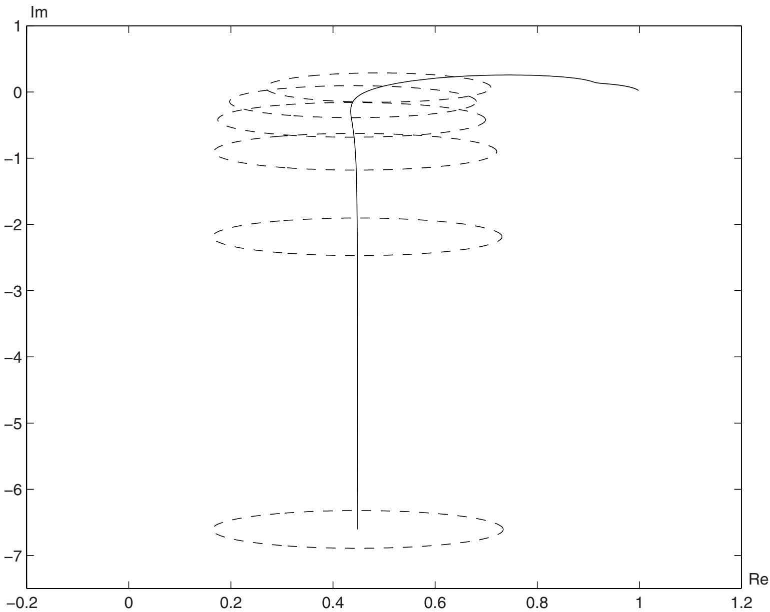


Fig. 9.11. Nyquist Curve Banks For Example 9.2.2 (—— $\hat{\Delta}$; − − − δ)

$$Y = GD + PU, \tag{9.66}$$

and

$$\begin{aligned} U &= C(R - Y) - H(Y - FU) \\ &= CR + HFU - (C + H)(GD + PU), \end{aligned}$$

which is then solved for U. Substituting this U into (9.66) gives

$$Y = P[I - HF + CP + HP]^{-1}CR + [I - P[I - HF + CP + HP]^{-1}(C + H)]GD. \tag{9.67}$$

Use F and H as in (9.45) and (9.46), respectively, that is,

$$F(s) = P(s), \quad H(s) = P^{-1}(s).$$

Equation (9.67) becomes

$$\begin{aligned} Y &= P[I + CP]^{-1}CR + [I - P[I + CP]^{-1}(C + P^{-1})]GD \\ &= [I + PC]^{-1}PCR + [I - [I + PC]^{-1}P(C + P^{-1})]GD \\ &= [I + PC]^{-1}PCR, \end{aligned} \tag{9.68}$$

which is independent of the disturbance D. For implementation of F and H, we use a process model $\hat{P}$ in place of P as usual. Now the inversion of $\hat{P}$ and its realization need our attention. Suppose also that $\hat{P}$ is nonsingular. Inverse

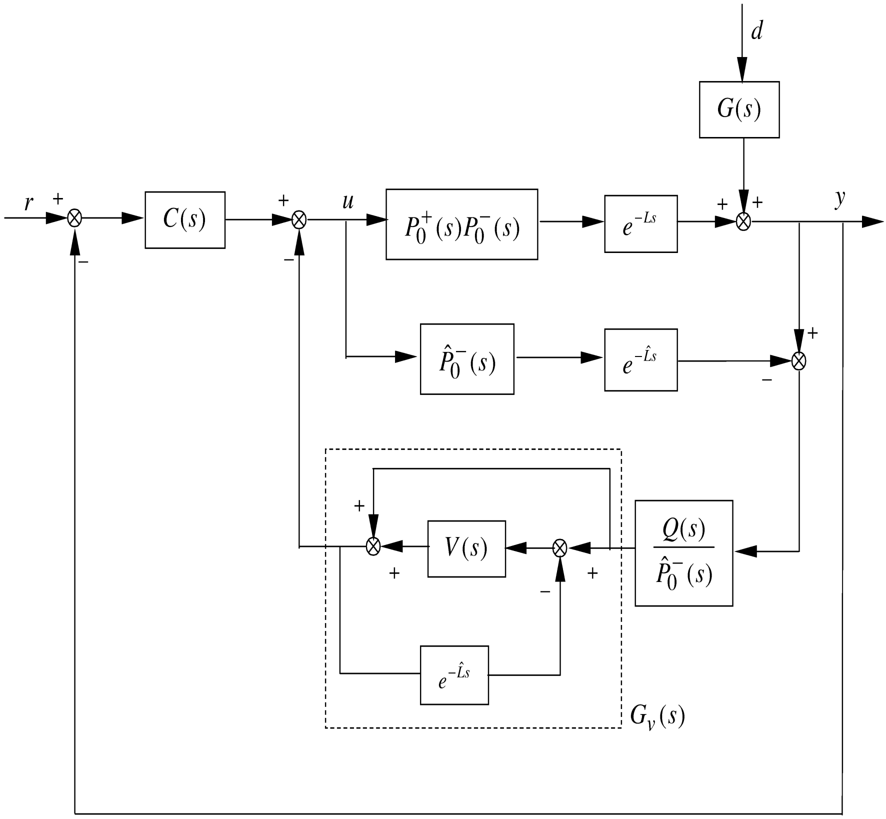


Fig. 9.12. Disturbance Observer Scheme for Unstable/non-minimum-phase Processes

$\hat{P}(j\omega)$ to get $\hat{P}^{-1}(j\omega)$. Apply the model reduction (Chapter 2) to $\hat{P}^{-1}(j\omega)$ to get its approximation $\tilde{P}(s)$ in form of

$$\tilde{p}_{ij}(s) = \tilde{p}_{ij0}(s)e^{-L_{ij}s}. \tag{9.69}$$

where $\tilde{p}_{ij0}(s)$ are proper rational functions. But terms $e^{-L_{ij}s}$ often happen to be a pure predictor which is not physically realizable, that is, some L_{ij} may be negative. They will require the approximation shown in Figure 9.5, each for one negative L_{ij}. It is costly. To reduce the number of such approximations, extract the largest possible pure prediction horizon L by

$$L = \begin{cases} 0, & \text{if all } -L_{ij} \le 0; \\ \max\{-L_{ij} | -L_{ij} > 0\}, & \text{otherwise.} \end{cases} \tag{9.70}$$

Write $-L_{ij} = L - (L + L_{ij})$ with $L^* := L + L_{ij} \ge 0$ due to $L \ge -L_{ij}$ so that $e^{-L^*_{ij}s}$ correspond to time delays. Equation (9.69) can be rewritten as

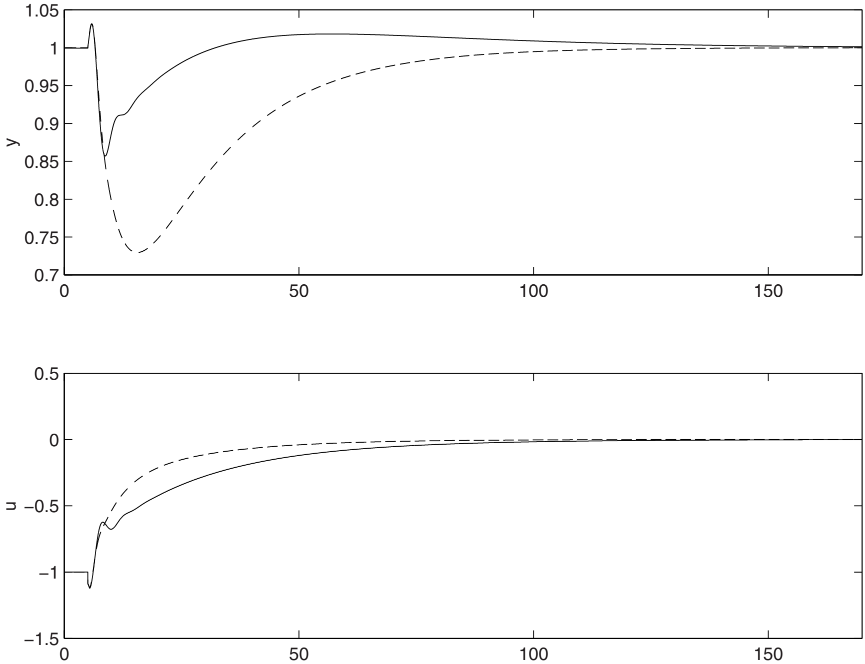


Fig. 9.13. Time responses for $P(s) = \frac{e^{-0.4s}}{(s+1)(s-1)}$
(—— Disturbance Observer; – – – Conventional Control)

$$\tilde{P} = diag\{L, L, \cdots, L\} \begin{bmatrix} \tilde{p}_{110}(s)e^{-L_{11}^* s} & \tilde{p}_{120}(s)e^{-L_{12}^* s} & \cdots & \tilde{p}_{1m0}(s)e^{-L_{1m}^* s} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{p}_{m10}(s)e^{-L_{m1}^* s} & \tilde{p}_{m20}(s)e^{-L_{m2}^* s} & \cdots & \tilde{p}_{mm0}(s)e^{-L_{mm}^* s} \end{bmatrix} \tag{9.71}$$

so that we only need to have m approximations to pure predictors instead of possible $m \times m$ ones, for $m \times m$ processes. Besides, the design involving the approximation in Figure 9.5 can be carried out only once.

In the case of periodic disturbance, we can avoid prediction in the loop, just like the SISO case. Use the same $F(s) = G(s)$ as before but change H to

$$H(s) = \tilde{P}(s)e^{-\tilde{\tau}_h s},$$

where $\tilde{\tau}_h = k_0 T$ and k_0 is the smallest nonnegative integer that renders $\tilde{\tau}_h + L_{ij} \geqslant 0$ for all i and j. Such a H then has no pure prediction.

Example 9.2.4. Consider the well-known Wood/Berry binary distillation column plant (Wood and Berry, 1973):

$$P(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.60e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}.$$

The PI controller given by Luyben (1986*b*) is

$$C(s) = \begin{bmatrix} 0.375(1+\frac{1}{8.29s}) & 0 \\ 0 & -0.075(1+\frac{1}{23.6s}) \end{bmatrix}.$$

With model reduction, we obtain

$$P^{-1} \approx \tilde{P} = \begin{bmatrix} \frac{0.41s^4+0.65s^3+0.41s^2+0.04s+0.001}{s^4+0.33s^3+1.17s^2+0.18s+0.008}e^{2s} & \frac{-0.27s^4-0.43s^3-0.27s^2-0.02s-0.001}{s^4+0.30s^3+1.16s^2+0.14s+0.005}e^{2s} \\ \frac{0.18s^4+0.29s^3+0.18s^2+0.02s+0.001}{s^4+0.35s^3+1.18s^2+0.20s+0.01}e^{-2s} & \frac{-0.23s^4-0.37s^3-0.23s^2-0.02s-0.001}{s^4+0.32s^3+1.17s^2+0.16s+0.006}e^{4s} \end{bmatrix},$$

where there are three pure predictors e^{2s}, e^{2s} and e^{4s} in $\tilde{P}(s)$. According to (9.70), we have

$$L = 4.$$

It then follows from (9.71) that $\tilde{P}(s)$ is rewritten as

$$\tilde{P}(s) = \mathrm{diag}\{e^{4s}, e^{4s}\} \cdot$$
$$\begin{bmatrix} \frac{0.41s^4+0.65s^3+0.41s^2+0.04s+0.001}{s^4+0.33s^3+1.17s^2+0.18s+0.008}e^{-2s} & \frac{-0.27s^4-0.43s^3-0.27s^2-0.02s-0.001}{s^4+0.30s^3+1.16s^2+0.14s+0.005}e^{-2s} \\ \frac{0.18s^4+0.29s^3+0.18s^2+0.02s+0.001}{s^4+0.35s^3+1.18s^2+0.20s+0.01}e^{-6s} & \frac{-0.23s^4-0.37s^3-0.23s^2-0.02s-0.001}{s^4+0.32s^3+1.17s^2+0.16s+0.006} \end{bmatrix}.$$

The Nyquist arrays of $P^{-1}(s)$ and $\tilde{P}(s)$ are depicted in Figure 9.14, exhibiting good matching of their frequency responses.

To approximate e^{4s}, we set $\tau_v = L/2 = 2$ and get $K_{v_{max}} = 1.52$. By setting $K_v = K_{v_{max}}/4 = 0.38$, $V(s) = \frac{0.38}{2s+1}$ is obtained. A unit step change is made at $t = 0$ for the 1st set-point and at $t = 200$ for 2nd set-point, respectively, and a step disturbance of size 1 is further introduced through $G_d = \frac{1}{20s+1}$ to the system at $t = 400$. The resulting output responses are shown in Figure 9.15. It can be seen that with the add-on control to the conventional feedback control, the system has achieved a better load disturbance rejection.

To evaluate the control scheme of Figure 9.7 for periodic disturbances, the following periodic disturbance instead of a step disturbance is introduced at t=400:

$$d(t) = \sin(0.1t) + 0.3\sin(0.2t) + 0.1\sin(0.3t) + 0.1\sin(0.4t)$$

with the period of $T = \frac{2\pi}{0.1} = 62.8$ and $G(s) = 1$. $\tilde{\tau}_h = k_0 T = 62.8$ is adopted with $k_0 = 1$. The system responses are shown in Figure 9.16. It is observed

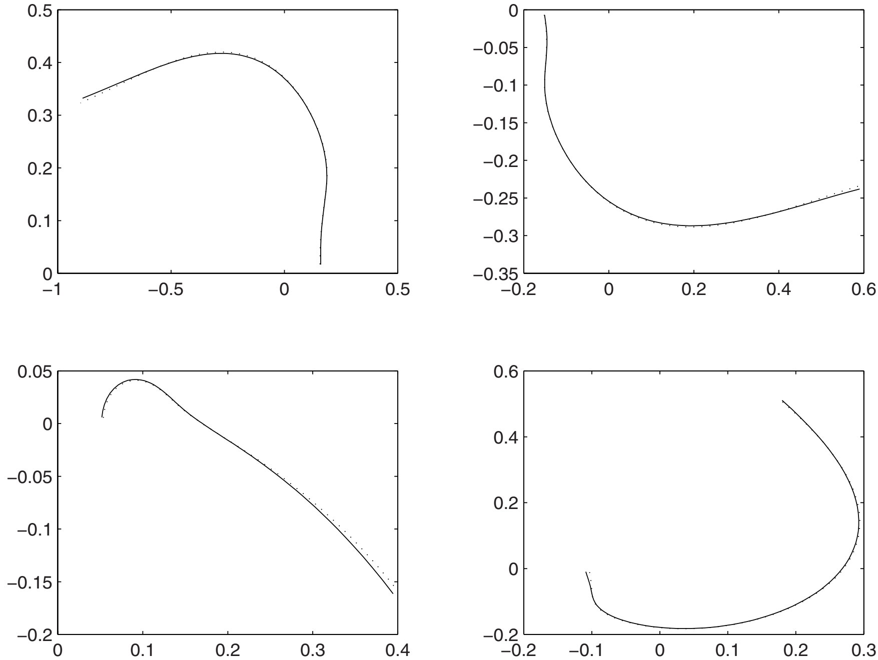


Fig. 9.14. Nyquist Array of $P^{-1}(s)$ and $\tilde{P}(s)$ for Example 9.2.4
(—— $P^{-1}(s)$, $\cdots$ $\tilde{P}(s)$)

that our output response approaches zero asymptotically, as expected, while the conventional feedback control has to sustain quite big steady state disturbance response. ♢

In this section, a control scheme for disturbance rejection is presented for time-delay processes. It can compensate for the disturbance while retaining the input-output servo response of the conventional feedback system. Stability and disturbance rejection performance are discussed.

9.3 Notes and References

The disturbance decoupling problem using a state feedback was considered in a geometric setting by Basile and Marro (1969) and Wonham (1979). Equivalent frequency domain solvability conditions were given by Bhattacharyya (1980; 1982). In the context of output feedback, the problem was solved by Akashi and Imai (1979) and Schumacher (1980). However, they did not consider the issue of stability. The disturbance decoupling by measurement feedback with stability or pole placement was solved for the first time by Willems and Commault (1981). This problem was also considered from an

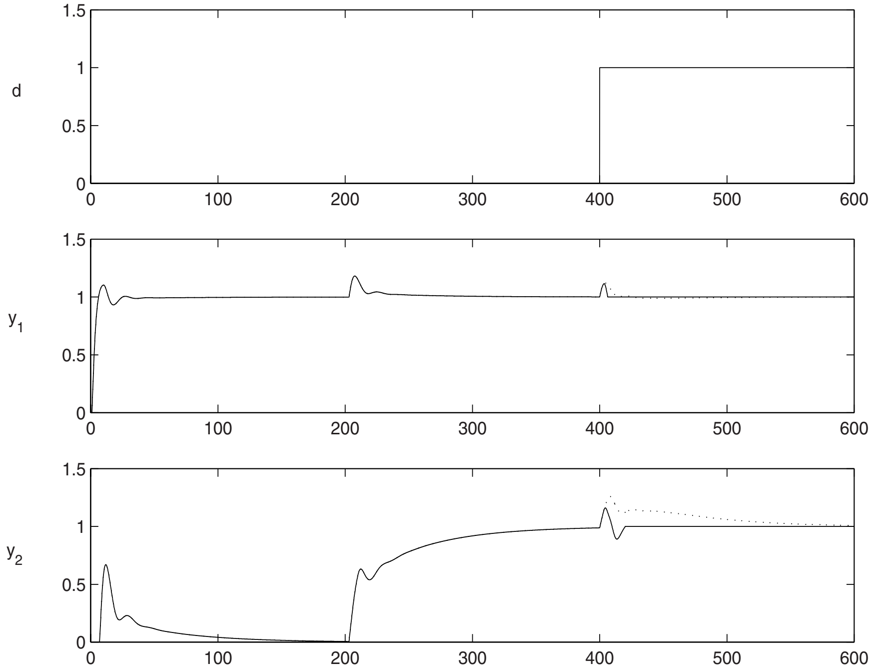


Fig. 9.15. Control Performance for Example 9.2.4
(—— with Disturbance Observer, · · · without Disturbance Observer)

input-output viewpoint by Kucera (1983) for single variable system and by Pernebo (1981) for multivariable systems. Section 1 of this chapter is based on the exiting results with some elaborations and modifications. These results on linear systems have recently been extended to nonlinear systems by Andiarti and Moog (1996) and Xia (1996).

The disturbance observer was introduced by Ohnishi (1987). It was further refined by Umeno and Hori (1991). But they considered delay-free systems only. Section 2 of this chapter extends them to the delay case and is from Zhang (2000).

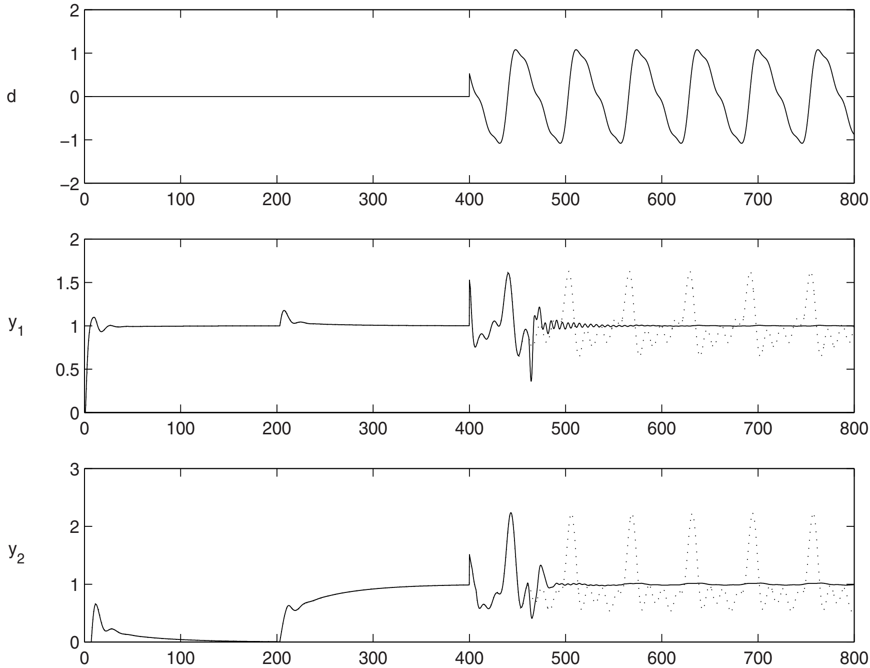


Fig. 9.16. Control performance for Example 9.2.4 under Periodic Disturbance (—— with Disturbance Observer, · · · without Disturbance Observer)

10. Asymptotic Disturbance Decoupling

Dynamic disturbance decoupling, as discussed in the preceding chapter, aims to decouple the disturbance from the output dynamic transient completely and thus inevitably results in some restrictive conditions on its solvability. For instance, the disturbance may not be measurable so that feedforward control is not applicable; And the plant inverse may not be stable or realizable, which may limit applicability or accuracy of the disturbance observer. A natural next target after dynamic disturbance decoupling would be static or asymptotic disturbance decoupling, that is, to decouple the disturbance from the plant output at the steady-state only. For measurable disturbances, asymptotic disturbance decoupling problem is straightforward. Rather, we will focus on the unmeasurable disturbance case in this chapter. A general case of disturbances will be considered in Section 1 whereas a specific yet common case of periodic disturbances is treated in Section 2.

10.1 General Disturbances: Internal Model Principle

Asymptotic disturbance decoupling is also called asymptotic disturbance rejection or asymptotic regulation. A closely related problem is asymptotic reference tracking since both the disturbance and reference are exogenous signals to the feedback system. Thus, in this section, we will develop the theory of asymptotic tracking and regulation for multivariable systems in a general setting.

Let the plant be described by a transfer matrix model:

$$y(s) = G_u(s)u(s) + G_d(s)d(s), \tag{10.1}$$

$$y_m(s) = G_m(s)y(s), \tag{10.2}$$

where $y(s)$ is the m-dimensional output to be controlled, $u(s)$ the l-dimensional control input, $d(s)$ the p-dimensional disturbance to be rejected, y_m(s) the m-dimensional measurement of the output y, to be used as the feedback signal for the compensator. Proper rational matrices G_u, G_d, and G_m with appropriate dimensions represent plant, disturbance channel and sensor models, respectively. We assume that the sensor G_m is stable, which is almost always the case in practice and therefore not restrictive. For closed-loop

Q.-G. Wang: Decoupling Control, LNCIS 285, pp. 327-356, 2003.

stabilizability, the series connection of the sensor and plant, $G_m \cdot G_u$, should have no unstable pole-zero cancellations. Let the compensator be

$$u(s) = G_c(s)e(s), \tag{10.3}$$

where $G_c(s)$ is a proper rational matrix, $e(s) = r(s) - y_m(s)$ is the error, and $r(s)$ is the m-dimensional reference input to be tracked by the control system. The overall system considered here is depicted in Figure 10.1.

Asymptotic regulation means $\lim_{t\to\infty} y(t) = 0$ in the case of $r(s) = 0$, for any $d(s)$ generated by

$$d(s) = \bar{G}_d(s)\tilde{d}, \tag{10.4}$$

where $\tilde{d}$ is a real vector and $\bar{G}_d$ a proper rational matrix. On the other hand, asymptotic tracking means $\lim_{t\to\infty} e(t) = 0$ in the case of $d(s) = 0$ for any $r(s)$ generated by

$$r(s) = \tilde{G}_r(s)\tilde{r}, \tag{10.5}$$

where $\tilde{r}$ is a real vector and $\tilde{G}_r$ a proper rational matrix. Stability of the system is obviously a prerequisite for asymptotic regulation and/or tracking to take place.

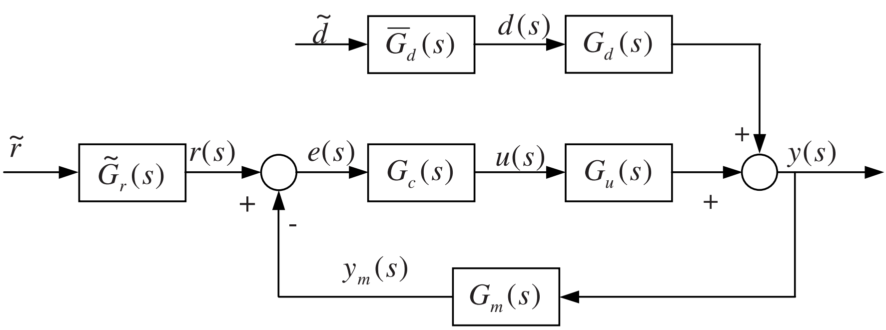


Fig. 10.1. Servo Control

For a (real) polynomial matrix $P(s)$ of full rank, we denote by σ(P(s)) the set of all c $\in \mathbb{C}$, the complex plane, such that $P(c)$ is of rank defect. By the polynomial division and partial fraction expansion, a rational matrix $G(s)$ can be written uniquely as the sum of a polynomial matrix $P_G(s)$ plus two strictly proper rational matrices $G^+(s)$ and $G^-(s)$:

$$G(s) = P_G(s) + G^+(s) + G^-(s),$$

where all the poles of anti-stable $G^+(s)$ and stable $G^-(s)$ lie, respectively, in $\mathbb{C}^+$, the closed right half of the complex plane (RHP), and $\mathbb{C}^-$, the open left half of the complex plane (LHP). Apply this decomposition to $\tilde{G}_r$ and

$$\tilde{G}_d := G_d\bar{G}_d. \tag{10.6}$$

By the final value theorem of the Laplace transform, one can conclude that for a stable system, the steady-state behavior of $y(t)$ and $e(t)$ is independent of the stable parts, $\tilde{G}_d^-$ and $\tilde{G}_r^-$, of $\tilde{G}_d$ and $\tilde{G}_r$. We can therefore assume without loss of generality that any stable modes, which may be initially associated with $\tilde{G}_d$ and $\tilde{G}_r$, have been removed when defining $\tilde{G}_d$ and $\tilde{G}_r$, in other words, all the poles of both $\tilde{G}_d$ and $\tilde{G}_r$ can be assumed to lie in $\mathbb{C}^+$.

We now formulate a general servocompensator problem with internal stability as follows. (SP): given a system in (10.1)-(10.3) and the disturbance and reference in (10.4)-(10.5), as depicted in Figure 10.1 with G_u, G_m, $\tilde{G}_d$, and $\tilde{G}_r$ all known and proper rational, G_u and G_m having no RHP pole-zero cancellations and $\tilde{G}_d$ and $\tilde{G}_r$ being anti-stable (i.e., having the RHP poles only), determine a proper compensator G_c such that the system of (10.1)-(10.3) is internally stable and $e(t)$ asymptotically tends to zero for any real vectors $\tilde{d}$ and $\tilde{r}$. In particular, the case of $\tilde{d} = 0$ is called tracking problem with internal stability (TP), and the case of $\tilde{r} = 0$ and e replaced by y is called regulation problem with internal stability (RP).

10.1.1 Preliminaries

Recall from Chapter 2 that two polynomial matrices, $D \in \mathbb{R}^{m\times m}[s]$ and $N \in \mathbb{R}^{m\times l}[s]$, are left coprime (implicitly in $\mathbb{C}$) if $Rank[D(s) \quad N(s)] = m, \quad for\ all\ s \in \mathbb{C}$. This can be generalized as follows.

Definition 10.1.1. *Two polynomial matrices, $D \in \mathbb{R}^{m\times m}[s]$ and $N \in \mathbb{R}^{m\times l}[s]$, are called left coprime in $\mathbb{C}^+$, if*

$$Rank[D(s) \quad N(s)] = m, \quad for\ all\ s \in \mathbb{C}^+.$$

Right coprimeness in $\mathbb{C}^+$ can be defined in a dual way.

Consider a cascade system $G_1(s)G_2(s)$ with proper $G_1(s)$ and $G_2(s)$. They can be expressed as coprime polynomial matrix fractions

$$G_1(s) = D_1^{-1}(s)N_1(s), \quad G_2(s) = N_2(s)D_2^{-1}(s), \tag{10.7}$$

respectively. For the sake of simplicity the indeterminate s may be dropped as long as it causes no confusion. By Chapters 2 and 3, it is trivial to establish the following lemma.

Lemma 10.1.1. *For G_1 and G_2 in (10.7), the following are equivalent:*

(i) $G_1 \cdot G_2$ has no RHP pole-zero cancellations;
(ii) Neither $D_1^{-1} \cdot N_1N_2$ nor $N_1N_2 \cdot D_2^{-1}$ has any RHP pole-zero cancellations;
(iii) Both fractions, $D_1^{-1}N_1N_2$ and $N_1N_2D_2^{-1}$, are coprime in $\mathbb{C}^+$;
(iv) There are stable rational matrices A, B, X and Y such that

$$D_1A + N_1N_2B = I,$$
$$XN_1N_2 + YD_2 = I;$$

(v) The system is internally stabilizable, i.e. there is a proper compensator which internally stabilizes G_1G_2.

Lemma 10.1.2. *Let $G = G_1G_2$ with left coprime fractions $G = D^{-1}N$, $G_1 = D_1^{-1}N_1$ and $G_2 = D_2^{-1}N_2$, let R be a polynomial matrix and R^{-1} anti-stable. i.e. $\sigma(R) \subset \mathbb{C}^+$ and G_2 be stable, i.e., $\sigma(D_2) \subset \mathbb{C}^-$. Then if R is a right divisor of D, it is also a right divisor of D_1.*

Proof. We prove by contradiction. Suppose that D_1R^{-1} is not a polynomial matrix and represented by a right coprime fraction $\bar{D}_1\bar{R}^{-1}$ with a non-empty $\sigma(\bar{R}) \subset \mathbb{C}^+$. Since $D = PR$ for some polynomial matrix P, then we have

$$\begin{aligned} G &= G_1G_2 \\ (PR)^{-1}N &= D_1^{-1}N_1D_2^{-1}N_2 \\ N_1D_2^{-1}N_2 &= D_1R^{-1}P^{-1}N = \bar{D}_1(P\bar{R})^{-1}N \end{aligned}$$

Since PR and N are coprime, so are $P\bar{R}$ and N. This together with coprimeness of $\bar{R}$ and $\bar{D}_1$ implies that the RHP set $\sigma(\bar{R}) \subset \sigma(D_2)$, which contradicts the assumed stability of G_2. The proof is completed. □

For a nonsingular polynomial matrix D, it can be factorized into $D = D^-D^+$ such that $\sigma(D^{-1}) \subset \mathbb{C}^-$ and $\sigma(D^+) \subset \mathbb{C}^+$. Such a factorization can be done, for example, by determining the Smith form of D and then factorizing every invariant polynomial as the product of a stable polynomial and an anti-stable polynomial. Let rational matrices G_1 and G_2 have the same number of rows with coprime fractions $G_1 = D_1^{-1}N_1$ and $G_2 = D_2^{-1}N_2$. The above factorization is applied to D_2, i.e., $D_2 = D_2^-D_2^+$. G_1 is said to have the unstable model of G_2 if D_2^+ is a right divisor of D_1, i.e., $D_1(D_2^+)^{-1}$ is a polynomial matrix. Applying Theorem 9.1 of Chapter 9 to our case of $G_f = 0$ (comparing Figure 9.1 with Figure 10.1) yields the following.

Lemma 10.1.3. *The system (10.1)-(10.3) is internally stable if and only if the following hold.*

(i) G_u has the unstable model of G_d; and

(ii) G_c internally stabilizes G_mG_u, which is equivalent to that $G_m \cdot G_u$ has no RHP pole-zero cancellations and G_c stabilizes G_mG_u.

The notion of skew primeness was introduced by Wolovich (1978). A full row rank polynomial matrix $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular polynomial matrix $D \in \mathbb{R}^{m\times m}[s]$ are said to be externally skew prime (implicitly in $\mathbb{C}$) if there are a full row rank $\bar{N} \in \mathbb{R}^{m\times l}[s]$ and a nonsingular $\bar{D} \in \mathbb{R}^{l\times l}[s]$ such that $DN = \bar{N}\bar{D}$, with D and $\bar{N}$ left coprime (in $\mathbb{C}$) and N and $\bar{D}$ right coprime (in $\mathbb{C}$). We will need its generalization as follows.

Definition 10.1.2. *Let a full row rank polynomial matrix $N \in \mathbb{R}^{m\times l}[s]$ and a nonsingular polynomial matrix $D \in \mathbb{R}^{m\times m}[s]$ be given. N and D are said to be externally skew prime in $\mathbb{C}^+$ if there are a full row rank polynomial matrix $\bar{N} \in \mathbb{R}^{m\times l}[s]$ and a nonsingular polynomial matrix $\bar{D} \in \mathbb{R}^{l\times l}[s]$ such that*

$$DN = \bar{N}\bar{D}, \tag{10.8}$$

where D and $\bar{N}$ are left coprime in $\mathbb{C}^+$, and N and $\bar{D}$ right coprime in $\mathbb{C}^+$. Such $(\bar{N}, \bar{D})$ is refereed to as an internally skew prime pair of (N, D) in $\mathbb{C}^+$.

In our proofs below, we will frequently make use of Lemma 5.2 of Chapter 5, and thus repeat here for ease of reference.

Lemma 10.1.4. *Let A, B, C, D be all polynomial matrices such that $AB^{-1}C = D$. If $B^{-1}C$ (resp. AB^{-1}) is coprime, then AB^{-1} (resp. $B^{-1}C$) is a polynomial matrix.*

10.1.2 Solvability

We will solve (RP) first, and subsequently apply its solution to (TP) and (SP). Given G_u, $G_m$and $\tilde{G}_d$, we can obtain polynomial fraction representations as follows:

$$G_u = D_u^{-1}N_u, \tag{10.9}$$

$$G_m = D_m^{-1}N_m, \tag{10.10}$$

$$D_u\tilde{G}_d = D_d^{-1}N_d, \tag{10.11}$$

$$N_m D_u^{-1} = D_1^{-1}N_1, \tag{10.12}$$

where all fractions on the right-hand sides are coprime and D_d^{-1} is anti-stable.

Theorem 10.1.1. *(RP) is solvable if and only if*
(i) G_u has the unstable model of G_d;
(ii) N_1 and D_d are right coprime in $\mathbb{C}^+$; and
(iii) N_u and D_d are externally skew-prime in $\mathbb{C}^+$.

Proof. Sufficiency: The external skew-primeness of N_u and D_d in $\mathbb{C}^+$ means that there are $\bar{D}_d$ and $\bar{N}_u$ such that $D_d^{-1}\bar{N}_u = N_u\bar{D}_d^{-1}$ and both fractions are coprime in $\mathbb{C}^+$. Take

$$G_c = \bar{D}_d^{-1}\bar{G}_c.$$

Write

$$G_m G_u \bar{D}_d^{-1} = [(D_1 D_m)^{-1} N_1 N_u][\bar{D}_d^{-1}].$$

Our problem formulation assumes that $G_m \cdot G_u$ has no RHP pole-zero cancellations. Internal stability will follow from Lemma 10.1.3 if $G_m G_u \bar{D}_d^{-1}$ is internally stabilizable, or equivalently, by Lemma 10.1.1, if $(N_1 N_u)\bar{D}_d^{-1}$ is right coprime in $\mathbb{C}^+$. Since both (N_1, D_d) and $(N_u, \bar{D}_d)$ are right coprime in $\mathbb{C}^+$, there exist stable rational matrices X_1, Y_1, X_2, Y_2 such that

$$X_1 N_1 + Y_1 D_d = I,$$
$$X_2 N_u + Y_2 \bar{D}_d = I.$$

Post-multiplying the first by N_u and substituting to the second with replacement of $D_d N_u = \bar{N}_u \bar{D}_d$ give

$$X_2(X_1 N_1 N_u + Y_1 \bar{N}_u \bar{D}_d) + Y_2 \bar{D}_d = I,$$

or

$$(X_2 X_1) N_1 N_u + (Y_2 + X_2 Y_1 \bar{N}_u) \bar{D}_d = I.$$

The last equation implies that $N_1 N_u$ and $\bar{D}_d$ are right coprime in $\mathbb{C}^+$. Let $\bar{G}_c$ be such that it stabilizes $G_m G_u \bar{D}_d^{-1}$, which is always possible (Chapter 3).

Consider now the steady state error. The transfer matrix between y and $\tilde{d}$ is obtained from Figure 10.1 as

$$H = (I + G_u G_c G_m)^{-1} \tilde{G}_d.$$

Let $\bar{G}_c G_m = N_2 D_2^{-1}$ be a right coprime fraction. It follows that

$$\begin{aligned} H &= (I + D_u^{-1} N_u \bar{D}_d^{-1} N_2 D_2^{-1})^{-1} \tilde{G}_d \\ &= D_2 (D_d D_u D_2 + \bar{N}_u N_2)^{-1} N_d \\ &:= D_2 P^{-1} N_d. \end{aligned}$$

Stabilization of $G_m G_u \bar{D}_d^{-1}$ by $\bar{G}_c$ ensures $\sigma(P) \subset \mathbb{C}^-$. Hence, H is stable and by the final-value theorem of the Laplace transform, $\lim_{t\to\infty} y(t) = 0$ for any $\tilde{d}$. (RP) is thus solvable.

Necessity: If (RP) is solvable, then the system is internally stable and H is stable. Condition (i) is obviously necessary, in view of Lemma 10.1.3. Write $N_u G_c G_m = D_3^{-1} N_3$ where D_3 and N_3 are left coprime. It follows that

$$H(s) = [D_3 D_u + N_3]^{-1} D_3 D_d^{-1} N_d.$$

Internal stability of the system implies that $[D_3 D_u + N_3]^{-1}$ is stable. Note that D_d^{-1} is anti-stable and D_d and N_d are right coprime. As a result, stability of H implies that $D_3 D_d^{-1}$ must be a polynomial matrix, or D_d is a right divisor of D_3. It follows from Lemma 10.1.2 that for stable G_m as assumed in the formulation, D_d is also a right divisor of D_4, where $N_u G_c = D_4^{-1} N_4$ is a left coprime fraction.

Look now at $N_u G_c$. Let $G_c = D_c^{-1} N_c$ be coprime and $\tilde{D}_c^{-1} \tilde{N}_u$ be a coprime fraction of $N_u D_c^{-1}$. Note that closed-loop stability implies that $N_u D_c^{-1}$ has no RHP pole-zero cancellations and

$$\sigma^+(\tilde{D}_c) = \sigma^+(D_c), \tag{10.13}$$

where $\sigma^+(D)$ denotes the set of all c in $\mathbb{C}^+$ such that $D(c)$ is singular. By our construction, $(\tilde{D}_c^{-1})(\tilde{N}_u N_c)$ is also a coprime fraction of $N_u G_c$ so that

$$(\tilde{D}_c^{-1})(\tilde{N}_u N_c) = D_4^{-1} N_4,$$

$$\tilde{D}_c = U D_4 = \tilde{D}_{c1} D_d \tag{10.14}$$

for a unimodular U and a polynomial $\tilde{D}_{c1}$. It then follows from $N_u D_c^{-1} = \tilde{D}_c^{-1}\tilde{N}_u$ that

$$\tilde{D}_{c1} D_d N_u = \tilde{N}_u D_c. \tag{10.15}$$

$\tilde{D}_{c1}^{-1}\tilde{N}_u$ must be coprime because otherwise $D_d^{-1}\tilde{D}_{c1}^{-1}\tilde{N}_u = \tilde{D}_c^{-1}\tilde{N}_u$ (coprime by construction) could not be coprime. Let $\bar{N}_u\bar{D}_{c1}^{-1}$ be a dual right coprime fraction of left coprime $\tilde{D}_{c1}^{-1}\tilde{N}_u$ so that

$$\sigma^+(\tilde{D}_{c1}) = \sigma^+(\bar{D}_{c1}). \tag{10.16}$$

Equation (10.15) can be rewritten as

$$D_d N_u = \bar{N}_u \bar{D}_d,$$

where

$$\bar{D}_d := \bar{D}_{c1}^{-1} D_c \tag{10.17}$$

must be, by Lemma 10.1.4, a polynomial matrix, due to right coprimeness of $\bar{N}_u\bar{D}_{c1}^{-1}$. Then, N_u and D_d are externally skew-prime in $\mathbb{C}^+$ if both $(D_d, \bar{N}_u)$ and $(N_u, \bar{D}_d)$ are coprime in $\mathbb{C}^+$. D_d and $\bar{N}_u$ must be coprime (then coprime in $\mathbb{C}^+$ as well) because otherwise in

$$D_d^{-1}\bar{N}_u\bar{D}_{c1}^{-1} = D_d^{-1}\tilde{D}_{c1}^{-1}\tilde{N}_u = (\tilde{D}_{c1}D_d)^{-1}\tilde{N}_u = (\tilde{D}_c)^{-1}\tilde{N}_u,$$

the last fraction which is coprime by construction could not be coprime. It follows from (10.17), (10.16), (10.13) and (10.14) that

$$\begin{aligned}
\sigma^+(\bar{D}_d) &= \sigma^+(D_c) - \sigma^+(\bar{D}_{c1}) \\
&= \sigma^+(D_c) - \sigma^+(\tilde{D}_{c1}) \\
&= \sigma^+(\tilde{D}_c) - \sigma^+(\tilde{D}_{c1}) \\
&= \sigma^+(D_d).
\end{aligned}$$

This implies that N_u and $\bar{D}_d$ are right coprime in $\mathbb{C}^+$, because otherwise $D_d^{-1}\bar{N}_u$ being equal to $N_u\bar{D}_d^{-1}$ could not be coprime.

One notices

$$\begin{aligned}
G_m G_u G_c &= D_m^{-1}(N_m D_u^{-1})(N_u D_c^{-1})N_c \\
&= D_m^{-1}D_1^{-1}N_1\tilde{D}_c^{-1}\tilde{N}_u N_c \\
&= D_m^{-1}D_1^{-1}(N_1 D_d^{-1})\tilde{D}_{c1}^{-1}\tilde{N}_u N_c.
\end{aligned}$$

It follows from Lemma 10.1.1 that D_d and N_1 must be right coprime in $\mathbb{C}^+$ in order to preserve internal stabilizability. The proof of Theorem 10.1.1 is therefore completed. □

A physical interpretation can now be given to Theorem 10.1.1. Let $\tilde{G}_d = \tilde{D}_d^{-1}\tilde{N}_d$ be a coprime fraction. Alternatively, $\tilde{G}_d = D_u^{-1}D_u\tilde{G}_d =$

$D_u^{-1}D_d^{-1}N_d = (D_dD_u)^{-1}N_d$, and the last representation is also a fraction of $\tilde{G}_d$. So there is a polynomial matrix L such that $D_dD_u = L\tilde{D}_d$. In the light of Theorem 10.1, in order to solve (RP) we must set $G_c = \bar{D}_d^{-1}\bar{G}_c$. One sees

$$\begin{aligned} G_uG_c &= D_u^{-1}N_u\bar{D}_d^{-1}\bar{G}_c \\ &= D_u^{-1}D_d^{-1}\bar{N}_u\bar{G}_c \\ &= (D_dD_u)^{-1}\bar{N}_u\bar{G}_c \\ &= (\tilde{D}_d^{-1})(L^{-1}\bar{N}_u\bar{G}_c), \end{aligned}$$

as shown in Figure 10.2. It is clear that when (RP) is solved, an exact duplication of the inverse of the denominator matrix of the exogenous system must be present in the internal loop at the same summation junction as the exogenous signals enter the loop. This is known as "internal model principle".

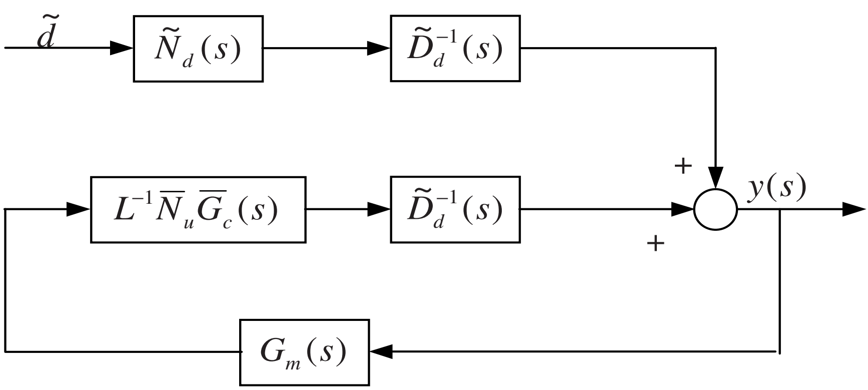


Fig. 10.2. Internal Model Principle

To apply Theorem 10.1.1 to (TP). Figure 10.1 is redrawn into Figure 10.3. Then, (TP) is obviously equivalent to (RP) with G_u, G_m and G_d replaced by G_mG_u, I and I respectively. Let

$$\begin{aligned} G_mG_u &= D_{mu}^{-1}N_{mu}, \\ D_{mu}\tilde{G}_r &= D_r^{-1}N_r, \end{aligned}$$

where all fractions on the right hand sides are coprime. Noting that G_mG_u and I cannot have any pole-zero cancellations and I has no unstable modes at all so that they trivially meet condition (i) of Theorem 10.1, we obtain from Theorem 10.1.1 the following corollary for solvability of asymptotic tracking problem.

Corollary 10.1.1. *(TP) is solvable if and only if N_{mu} and D_r are externally skew prime in $\mathbb{C}^+$.*

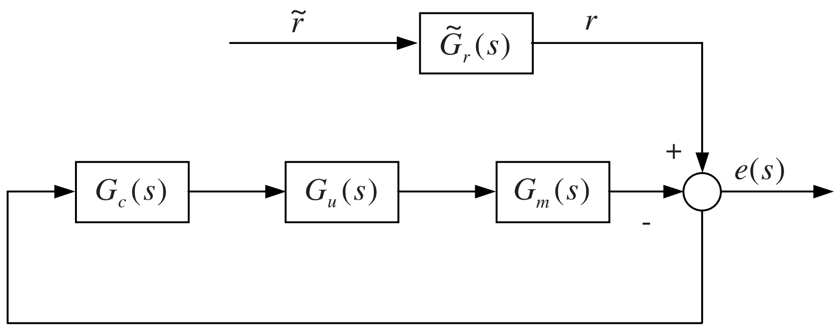


Fig. 10.3. Tracking Problem

For the general (SP), we combine Theorem 10.1 and Corollary 10.1 as follows.

Corollary 10.1.2. *(SP) is solvable if and only if*
(i) G_u *has the unstable model of* G_d*;*
(ii) N_1 *and* D_d *are right coprime in* $\mathbb{C}^+$*;*
(iii) N_u *and* D_d *are externally skew-prime in* $\mathbb{C}^+$*; And*
(iv) N_{mu} *and* D_r *are externally skew-prime in* $\mathbb{C}^+$*.*

10.1.3 Algorithms

Based on Theorem 10.1.1, a compensator which solves (RP) can be constructed as follows. Given transfer matrices G_u, G_m and $\tilde{G}_d$,

(i) determine left coprime fractions in (10.9)–(10.12);
(ii) check the solvability conditions of Theorem 10.1. If they are met, construct $\bar{D}_d$; and
(iii) design a compensator $\bar{G}_c$ that stabilizes $G_m G_u \bar{D}_d^{-1}$ and meets other control requirements if any.

Coprime Fractions. We wish to factorize a $m \times l$ proper transfer matrix $G(s)$ in the form of

$$G = D^{-1}N, \tag{10.18}$$

where D and N are left coprime polynomial matrices. Rewrite (10.18) as

$$\begin{bmatrix} D \ N \end{bmatrix} \begin{bmatrix} G \\ -I \end{bmatrix} = 0.$$

It can be seen that a left coprime fraction of G can be determined by finding a minimal basis for the left null space of $[G^T \ \ -I]^T$. This is a standard minimal design problem and its algorithm is available (Kung and Kailath,1980).

Right Coprimeness. One of the solvability conditions for (PR) is the right coprimeness of some matrix pair. A generalized Sylvester resultant s_k

(Bitmead et al 1978) can be used to check the coprimeness, where s_k of two polynomial matrices $D(s)$ and $N(s)$ of order k consists of the coefficient matrices of $D(s)$ and $N(s)$. The coprimeness of $D(s)$ and $N(s)$ is determined by the rank information of s_k.

Stabilizing Compensator. If $\bar{D}_d$ has been constructed somehow (see below for its construction), the feedback loop can be stabilized by an appropriate choice of the remaining unspecified part of compensator, namely $\bar{G}_c$. The actual selection of an appropriate stabilizing compensator is now a standard problem in the linear system theory, which has a variety of constructive solutions. No further discussion is needed.

Skew-Primeness. The external skew-primeness of two polynomials matrices N and D is crucial for the solvability of (RP), and it is also involved in construction of $\bar{D}$ for the compensator design. We now present a constructive procedure for direct computation of $\bar{D}$ from N and D for a case which is encountered very often.

Let an $m \times m$ nonsingular polynomial matrix D and an $m \times l$ full row rank polynomial matrix N be given. The case which we want to consider is that $\sigma(D)$ and $\sigma(N)$ are distinct, i.e., $\sigma(D) \cap \sigma(N) = \emptyset$. It is well known (Chapter 2) that a polynomial matrix can be transformed to its Smith form by elementary operations. In other words, there are unimodular L_i and $R_i, i = 1, 2, 3$, such that

$$D = L_1 S_1 R_1, \tag{10.19}$$
$$N = L_2 S_2 R_2, \tag{10.20}$$
$$DN = L_3 S_3 R_3, \tag{10.21}$$

where $S_i, i = 1, 2, 3$ are in the Smith form:

$$S_1 = \Lambda_1, \quad S_2 = [\Lambda_2 \quad 0], \quad S_3 = [\Lambda_3 \quad 0]$$

for diagonal Λ_i. Recall that a determinantal divisor of order k of a polynomial matrix is the monic greatest common divisor of all its minors of order k. Let d_{1k}, d_{2k} and d_{3k} be the determinantal divisors of order k of D, N and DN, respectively. Then Λ_i are given by

$$\Lambda_i = diag\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{im}\}, \quad i = 1, 2, 3,$$

where $\lambda_{ik} = d_{ik} d_{i,k-1}^{-1}, k = 1, 2, \ldots, m$, with $d_{i0} = 1$ by convention.

Lemma 10.1.5. *For nonsingular D and full row rank N with $\sigma(D) \cap \sigma(N) = \emptyset$, the Smith form of DN is the product of those of D and N, that is, $S_3 = S_1 S_2$.*

Proof. From (10.19) and (10.20), $DN = L_1(S_1 R_1 L_2 S_2) R_2$, the Smith form of DN is the same as that of $S_1 R_1 L_2 S_2$ because L_1 and R_2 are unimodular. It is therefore sufficient to prove that the Smith form of $S_1 M S_2$ is $S_1 S_2$, where

M is unimodular. Let $Q = S_1 M S_2$. Note that the determinantal divisors of order k for S_1, S_2 and Q are, respectively, the same as those for D, N and DN, and they are still denoted by d_{1k}, d_{2k} and d_{3k},respectively.

Set $Q = S_1 Q_1$ for $Q_1 = M S_2$. By the Binet-Cauchy formula, we can express a minor of Q of order k as a linear combination of minors of S_1 of the same order. It follows that

$$d_{3k} d_{1k}^{-1} = p_{1k}, \quad k = 1, 2, \ldots, m, \tag{10.22}$$

for some polynomial p_{1k}. Similarly, by setting $Q = Q_2 S_2$ where $Q_2 = S_1 M$, one gets

$$d_{3k} d_{2k}^{-1} = p_{2k}, \quad k = 1, 2, \ldots, m, \tag{10.23}$$

for some polynomial p_{2k}. For a fixed k, partition M into $[M_k \;\; M_{m-k}]$, where M_k is the first k columns of M. It can be seen from $Q = S_1 M S_2$ that $g_k m_k d_{2k}$ is a minor of order k of Q, where m_k is any minor of order k of M_k and g_k is the product of the diagonal elements of a submatrix of S_1 corresponding to m_k and is a polynomial. The greatest common divisor of all m_k must be a non-zero real constant, because otherwise a Laplace expansion of the $det(M)$ would show that M were not unimodular. Therefore the greatest common divisor of all $g_k m_k d_{2k}$ can be expressed as $f_{1k} d_{2k}$ where f_{1k} is a polynomial and $\sigma(f_{1k}) \subset \sigma(S_1) = \sigma(D)$. By definition of d_{3k}, we have

$$f_{1k} d_{2k} d_{3k}^{-1} = \bar{p}_{2k}, \quad k = 1, 2, \ldots, m, \tag{10.24}$$

where $\bar{p}_{2k}$ is a polynomial. Similarly, one can get

$$f_{2k} d_{1k} d_{3k}^{-1} = \bar{p}_{1k}, \quad k = 1, 2, \ldots, m, \tag{10.25}$$

where f_{2k} and $\bar{p}_{1k}$ are polynomials and $\sigma(f_{2k}) \subset \sigma(S_2) = \sigma(N)$. Factorize d_{3k} into $d_{3k} = d_{3k}^D d_{3k}^N$ where $\sigma(d_{3k}^D) \subset \sigma(D)$ and $\sigma(d_{3k}^N) \subset \sigma(N)$. Then we see that (10.22) and (10.25) are satisfied simultaneously if and only if both $d_{3k}^D d_{1k}^{-1}$ and $d_{1k}(d_{3k}^D)^{-1}$ are polynomial. This is possible only if $d_{3k}^D = d_{1k}$. Similarly, we have $d_{3k}^N = d_{2k}$. In view of these relations, we obtain

$$\begin{aligned}
\lambda_{3k} = d_{3k}(d_{3,k-1})^{-1} &= d_{3k}^D (d_{3,k-1}^D)^{-1} d_{3k}^N (d_{3,k-1}^N)^{-1} \\
&= (d_{1k} d_{1,k-1}^{-1})(d_{2k} d_{2,k-1}^{-1}) \\
&= \lambda_{1k} \cdot \lambda_{2k},
\end{aligned}$$

which implies that the Smith form of $Q = S_1 M S_2$ is $S_1 S_2$, The proof is thus complete. □

Theorem 10.1.2. *Let a full row rank $N \in \mathbb{R}^{m \times l}[s]$ and a nonsingular $D \in \mathbb{R}^{m \times m}[s]$ be given with $\sigma(D) \cap \sigma(N) = \emptyset$. Then N and D are externally skew prime and thus externally skew prime in $\mathbb{C}^+$ as well. Furthermore, $\bar{D}$ and $\bar{N}$, defined by*

$$\bar{D} = \begin{bmatrix} S_1 & 0 \\ 0 & I \end{bmatrix} R_3, \tag{10.26}$$

$$\bar{N} = L_3 S_2, \tag{10.27}$$

with I being an identity matrix of order $(l-m)$, constitute an internally skew prime pair of (N, D) in the normal sense as well as in $\mathbb{C}^+$.

Proof. From Lemma 10.1.5, one can write

$$\begin{aligned} DN &= L_3 S_3 R_3 = L_3 S_1 S_2 R_3 \\ &= L_3 [\Lambda_1 \Lambda_2 \quad 0] R_3 \\ &= L_3 [\Lambda_2 \quad 0] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I \end{bmatrix} R_3 \\ &= L_3 S_2 \begin{bmatrix} S_1 & 0 \\ 0 & I \end{bmatrix} R_3 \\ &:= \bar{N} \bar{D}, \end{aligned}$$

where $\bar{N} = L_3 S_2$ and $\bar{D} = Blockdiag\{S_1, I\} R_3$ are both polynomial matrices. Furthermore, D and $\bar{N}$ must be a left coprime because otherwise there is a non-unimodular common divisor P such that $\sigma(P) \subset \sigma(D)$ and $\sigma(P) \subset \sigma(\bar{N}) = \sigma(S_2) = \sigma(N)$, which contradicts the fact that $\sigma(D) \cap \sigma(N) = \emptyset$. Similarly, N and $\bar{D}$ must be right coprime. Hence, N and D are externally skew prime. Note the coprimeness in the normal sense, i.e., in $\mathbb{C}$, of course, implies coprimeness in $\mathbb{C}^+$, a subset of $\mathbb{C}$. N and D are thus externally skew prime in $\mathbb{C}^+$ as well. □

Example 10.1.1. For illustration of Theorems 10.2 and 10.1, consider the headbox model of a paper machine as established in Wang (1984):

$$y(s) = G_u(s) u(s) + G_d(s) d(s),$$

where $y = [y_1 \quad y_2]^T$, y_1 and y_2 are the liquid level and pulp concentration in the headbox, respectively; $u = [u_1 \quad u_2]^T$, u_1 and u_2 are the flow rates of thick pulp and white water, respectively; $d = [d_1 \quad d_2]^T$, d_1 and d_2 are the pulp concentrations of thick pulp and white water respectively; and

$$G_u = \begin{bmatrix} \frac{1.5}{2.6s+1} & \frac{1.5}{2.6s+1} \\ \frac{5.1}{2s+1} & \frac{-1.4}{2s+1} \end{bmatrix},$$

$$G_d = \begin{bmatrix} 0 & 0 \\ \frac{0.2}{2s+1} & \frac{0.8}{2s+1} \end{bmatrix}.$$

The sensor model is

$$G_m = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2s+1} \end{bmatrix}.$$

Suppose that both disturbances to be rejected are step signals:

$$d(s) = \begin{bmatrix} \frac{\tilde{d}_1}{s} \\ \frac{\tilde{d}_1}{s} \end{bmatrix} = \frac{1}{s} \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} := \bar{G}_d \tilde{d}.$$

We construct

$$\tilde{G}_{d:} = (G_d\bar{G}_d)^+ = \begin{bmatrix} 0 & 0 \\ \frac{0.2}{s(2s+1)} & \frac{0.8}{s(2s+1)} \end{bmatrix}^+ = \begin{bmatrix} 0 & 0 \\ \frac{0.2}{s} & \frac{0.8}{s} \end{bmatrix}.$$

Appropriate coprime fractions are obtained as

$$\begin{aligned} G_u &= \begin{bmatrix} 2.6s+1 & 0 \\ 0 & 2s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1.5 & 1.5 \\ 5.1 & -1.4 \end{bmatrix} := D_u^{-1}N_u, \\ G_m &= \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} := D_m^{-1}N_m, \\ D_u\tilde{G}_d &= \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0.2(2s+1) & 0.8(2s+1) \end{bmatrix} := D_d^{-1}N_d, \\ N_mD_u^{-1} &= \begin{bmatrix} 2.6s+1 & 0 \\ 0 & 2s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} := D_1^{-1}N_1. \end{aligned}$$

Since $\sigma(N_u) = \emptyset$, then $\sigma(D_d) \cap \sigma(N_u) = \emptyset$. In view of Theorem 10.1.2, D_d and N_u are externally skew-prime. It follows that

$$D_d = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} := S_1,$$

which is already in the Smith form, the Smith form for N_{11} is $S_2 = I$ and

$$\begin{aligned} D_dN_u &= \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1.5 & 1.5 \\ 5.1 & -1.4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1.5 & 1.5 \\ 5.1 & -1.4 \end{bmatrix} := L_3S_3R_3. \end{aligned}$$

By Theorem 10.1.2, we have $\bar{N}_u = L_3S_2 = I$ and

$$\bar{D}_d = S_1R_3 = \begin{bmatrix} 1.5 & 1.5 \\ 5.1s & -1.4s \end{bmatrix}.$$

Obviously, $G_m \cdot G_u$ has no RHP pole-zero cancellations. G_d is stable so that condition (i) of Theorem 10.1 is trivially met. It is readily seen that D_d and N_1 are right coprime. Hence, (RP) for this example is solvable. Any stabilizer for $G_mG_u\bar{D}_d^{-1}$ will solve (RP). $\Diamond$

In this section, a servocompensator problem with internal stability has been resolved in a polynomial matrix setting for general multivariable systems. It has been shown that the primary condition of solvability is that the plant numerator matrix and the denominator matrix of the product of the plant denominator matrix and exogenous system transfer matrix are externally skew-prime in the RHP. This important observation naturally leads to an internal model principle in an explicit way. We have also presented a procedure for designing a compensator and discussed relevant computational algorithms.

10.2 Periodic Disturbances: Virtual Feedforward Control

The internal model principle (IMP) presented in the preceding section is an elegant theory and has wide applications in industry. In particular, it is easy to apply if the disturbance is of simple type such as step, ramp and sinusoidal where the unstable modes of the disturbance Laplace transform are clear and simple to users, and stabilization of the generalized plant consisting of the original plant and the internal model is easy to achieve. However, there do exist other cases for which the IMP is difficult or even impossible to apply. For instance, some disturbances may be irregular. Others may not have analytical expressions for its Laplace transforms, so one is unable to extract the unable modes of the disturbance under consideration. Even when such unstable modes can be obtained, it may be impossible, too difficult or expensive to stabilize the resulting generalized plant. A class of disturbances which falls into this last case is general periodic signals other than sinusoidal.

In practice, we often encounter the situation where the reference commands to be tracked and/or disturbance to be rejected are periodic signals, e.g., repetitive commands, operations for mechanical systems such as industrial robots, or disturbances depending on the frequency of the power supply. Disturbances acting on the track-following servo system of an optical disk drive inherently contain significant periodic components that cause tracking errors of a periodic nature. For such disturbances, the controller designed for step type reference tracking and/or disturbance rejection will inevitably give an uncompensated error.

Theoretically, a signal of period T will contribute an unstable factor, $(1-e^{Ls})$, into the denominator of its Laplace transform. By the IMP, the reciprocal of this factor which has an infinite unstable modes has to be inserted into the feedback loop. But how to stabilize it remains an open problem in the literature. This motivates us to look for some alternative solution to the IMP. In this section, a new control scheme, called the virtual feedforward control, is presented for asymptotic rejection of periodic disturbance. The periodic disturbance is estimated when a periodic steady state error is detected, and the virtual feedforward control is then activated to compensate for such a disturbance.

10.2.1 The Scheme

Consider first the case of SISO linear time-invariant (LTI) continuous processes. Let $G(s)$ be the plant and $K(s)$ the feedback controller. The proposed scheme, called the virtual feedforward control scheme (VFC scheme for short), is depicted in Figure 10.4, where d is supposed to be an unknown periodic disturbance. Without the VFC controller, the proposed structure reduces to an ordinary feedback control system and in face of non-zero periodic d, the system will have non-zero steady state error. The intention of the proposed method is to activate the VFC controller timely and to give an extra control signal v to compensate for the disturbance. Without loss of generality, the reference r is assumed to be zero when the disturbance response in output is addressed.

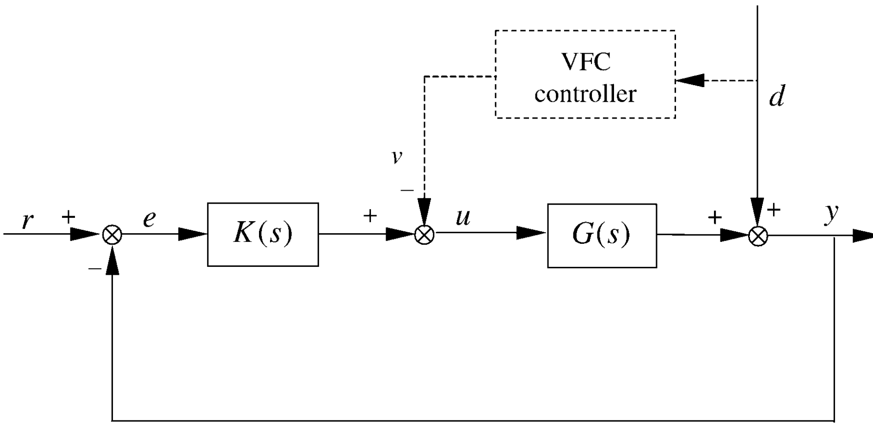


Fig. 10.4. Feedback System with VFC

It is well known that the steady state of the output $y(t)$ of a stable system G in response to a periodic input $x(t)$ is periodic with the same period as that of the input. With the standard convolution operation $*$, the response of a stable system $Y = G_2G_1X$ can be written as $y(t) = g_2(t) * g_1(t) * x(t)$, where $g_1(t)$ and $g_2(t)$ are impulse responses of G_1 and G_2, respectively. For simplicity, denote $g_2(t) * g_1(t) * x(t)$ by $(g_2 * g_1 * x)(t)$. For a signal $f(t)$ with a periodic steady state, let its steady state be $f^s(t)$, and the corresponding Laplace transform be $F^s(s) = \mathcal{L}\{f^s(t)\}$. Lemma 9.3 in Chapter 9 is needed in our development here and repeated as follows (readers are referred to Chapter 9 for its proof).

Lemma 10.2.1. *Under a periodic input $x(t)$, the output steady state of the system $Y = G_2G_1X$ with stable G_2 and G_1 satisfies*

$$(g_2 * g_1 * x)^s(t) = [g_2 * (g_1 * x)^s]^s(t),$$

equivalently, in the s-domain, there holds

$$(G_2G_1X)^s = [G_2(G_1X)^s]^s.$$

With this lemma in hand, we are now ready to develop the new method for periodic disturbance rejection. Suppose that the process is in a normal and stable feedback control ($v = 0$ in Figure 10.4) and the output has been in a constant steady state before the disturbance acts on the system. Without loss of generality, such a steady state is assumed to be zero. When a periodic disturbance d comes to the system at $t = 0$, the process output y will become periodic with the same period after some transient. The periodic output can be detected and its period be measured with high accuracy by monitoring the process output.

At $t = 0$, the VFC controller has not been activated yet ($v = 0$ in Figure 10.4), and it follows from Figure 10.4 that $Y_d = D + GU_d$, or

$$D = Y_d - GU_d, \tag{10.28}$$

where the subscription d indicates the signals' response to d only. Note that $y_d(t)$ and $u_d(t)$ as well as their steady states $y_d^s(t)$ and $u_d^s(t)$ are available. In the time domain, (10.28) becomes

$$d(t) = y_d(t) - (g * u_d)(t). \tag{10.29}$$

Since the actual process transfer function G is not available, its model $\hat{G}$ has to be used to estimate $d(t)$, i.e.,

$$\hat{d}(t) = y_d(t) - (\hat{g} * u_d)(t), \tag{10.30}$$

or in the s-domain,

$$\hat{D} = Y_d - \hat{G}U_d. \tag{10.31}$$

Note that $d(t)$ and $\hat{d}(t)$ have the periodic steady state $d^s(t)$ and $\hat{d}^s(t)$ respectively, since $y_d(t)$ and $u_d(t)$ are so and the system is stable. Now suppose that the VFC controller $v(t)$ is activated at $t = T_v$. Since the initial conditions of the system at $t = T_v$ are non-zero, we have to apply the Laplace transform at $t = 0$ to get

$$Y_v = \frac{1}{1+GK}D - \frac{G}{1+GK}V, \tag{10.32}$$

where the subscription v indicates the signal's response to both d and v. Let $\mathbf{1}(t)$ be the unit step function. We set the virtual feedforward control signal as

$$V(s) = [\hat{G}^{-1}(s)\hat{D}(s)]^s e^{-sT_v}. \tag{10.33}$$

Let

$$Q_{vd} = \hat{G}^{-1}(s)\hat{D}(s), \quad H_{yd} = \frac{1}{1+GK},$$

Equation (10.33) can be written in the time domain as

$$v(t) = [q_{vd}(t - T_v)]^s \mathbf{1}(t - T_v). \tag{10.34}$$

Substituting (10.33) into (10.32) gives

$$Y_v = H_{yd}D - H_{yd}G(\hat{G}^{-1}\hat{D})^s e^{-sT_v}.$$

It follows from Lemma 10.2.1, (10.28) and (10.31) that

$$
\begin{aligned}
& Y_v^s \\
&= [H_{yd}D - H_{yd}G(\hat{G}^{-1}\hat{D})^s e^{-sT_v}]^s \\
&= [H_{yd}D^s]^s - [H_{yd}G\hat{G}^{-1}\hat{D}^s e^{-sT_v}]^s \\
&= [H_{yd}Y_d^s - H_{yd}(GU_d)^s]^s - [H_{yd}G\hat{G}^{-1}Y_d^s e^{-sT_v} - H_{yd}G\hat{G}^{-1}(\hat{G}U_d)^s e^{-sT_v}]^s \\
&= [H_{yd}(1 - G\hat{G}^{-1}e^{-sT_v})Y_d^s]^s - [H_{yd}G(1 - e^{-sT_v})U_d^s]^s. \qquad (10.35)
\end{aligned}
$$

Choose

$$T_v = kT, \qquad (10.36)$$

where T is the period of $d(t)$, $y_d^s(t)$ and $u_d^s(t)$, and k is an integer, so that there hold

$$y_d^s(t - T_v) = y_d^s(t), \quad u_d^s(t - T_v) = u_d^s(t). \qquad (10.37)$$

Thus, (10.35) reduces to

$$y_v^s(t) = \{h_{yd}(t) * [1 - g(t) * \hat{g}^{-1}(t)] * y_d^s(t)\}^s. \qquad (10.38)$$

which becomes

$$y_v^s(t) = 0, \quad if \quad g(t) = \hat{g}(t),$$

i.e., in the case of perfect model-process match.

Now, if there is a model mismatch, i.e., $g(t) \neq \hat{g}(t)$, define

$$F \triangleq \frac{1 - G\hat{G}^{-1}}{1 + GK},$$

and f as its impulse response. (10.38) is written as

$$y_v^s(t) = \{f(t) * y_d^s(t)\}^s. \qquad (10.39)$$

Introduce the following norm for a periodic signal $x(t)$ with the period of T:

$$\|x(t)\| = \sqrt{\int_0^T x^2 dt},$$

and express $y_d^s(t)$ by its Fourier series:

$$y_d^s(t) = y_{d0} + \sum_{i=1}^{\infty} C_i \cos(\omega_i t + \phi_i)$$

with $\omega_i = 2\pi i/T$. We have

$$\|y_d^s(t)\|^2 = \int_0^T [y_{d0} + \sum_{i=1}^{\infty} C_i \cos(\omega_i t + \phi_i)]^2 dt. \tag{10.40}$$

Noting the orthogonality of $\cos(\omega_i t)$ and $\cos(\omega_i t+\phi_i)$ on the interval $t \in [0, T]$, (10.40) gives

$$\|y_d^s(t)\|^2 = Ty_{d0}^2 + \frac{T}{2}\sum_{i=1}^{\infty} C_i^2. \tag{10.41}$$

Similarly, we have

$$\begin{aligned}\|y_v^s(t)\|^2 &= \int_0^T [F(0)y_{d0} + \sum_{i=1}^{\infty} C_i|F(j\omega_i)|\cos(\omega_i t\phi_i + \angle F(j\omega_i))]^2 dt \\ &= F^2(0)Ty_{d0}^2 + \frac{T}{2}\sum_{i=1}^{\infty} C_i^2|F(j\omega_i)|^2. \end{aligned} \tag{10.42}$$

If

$$|F(j\omega_i)| < 1, \qquad i = 1, 2, \cdots, \tag{10.43}$$

it follows from (10.42) that

$$\|y_v^s(t)\| < \|y_d^s(t)\|. \tag{10.44}$$

In the VFC scheme, the extra control v is activated according to (10.34), (10.30) and (10.36). This v is independent of the closed-loop system once activated and acts as a "feedforward" control with the estimated disturbance $\hat{d}$. Therefore, stability of the closed-loop system can be guaranteed with the proposed VFC strategy, provided that the stability is taken into consideration at the stage of controller design. In view of the above development, we obtain the following theorem:

Theorem 10.2.1. *A stable feedback system remains stable when the virtual feedforward control signal v described by* (10.34), (10.30) *and* (10.36) *is introduced, and the resultant output steady state in response to a periodic disturbance satisfies*

$$\begin{cases} y_v^s(t) = 0, & \text{if } \hat{G} = G, \\ \|y_v^s(t)\| < \|y_d^s(t)\|, & \text{if } |\frac{1-G(j\omega_i)\hat{G}(j\omega_i)^{-1}}{1+G(j\omega_i)K(j\omega_i)}| < 1, \quad i = 1, 2, \cdots. \end{cases}$$

10.2.2 Implementation and Simulation

Let us discuss several practical issues in implementation of the VFC scheme.

Algorithm Execution. To implement the proposed VFC scheme shown in Figure 10.4, the process output is monitored all the time. When a periodic behavior is observed, the VFC controller is activated: the steady state of the disturbance d is estimated from (10.30) and v is computed from (10.34) and applied to the system at T_v according to (10.36).

Non-minimum-phase system. In (10.34), the impulse response of the model inverse is used to get the required $\hat{d}$ and thus v for disturbance compensation. For non-minimum-phase system, $\hat{G}^{-1}$ will give a divergent response. However, note that only its steady state of $\hat{d}$ is employed in the proposed method to compute v. It is possible to extract this steady state part from such a divergent response. For simplicity, consider a non-minimum-phase process $Y = GX$ with a single right-half-plant (RHP) zero at $s = a > 0$. Using the partial fraction expansion, we have

$$X(s) = G^{-1}(s)Y(s) = \frac{\beta}{a-s} + X^t + X^s, \tag{10.45}$$

where X^t denotes the stable part. The Laplace transform of the periodic signal $y(t)$ is given (Kuhfitting, 1978) by

$$Y(s) = \frac{1}{1-e^{-sT}} \int_0^T y(t)e^{-st}dt,$$

where T is the period. Applying the residue theorem yields

$$\beta = \left(\lim_{s\to a}(a-s)G^{-1}(s)\right) \cdot \left(\frac{1}{1-e^{-aT}} \int_0^T y(t)e^{-at}dt\right).$$

It can be seen from (10.45) that once the transient $x^t(t)$ dies out, the steady state part $x^s(t)$ can be found by

$$x^s(t) = [g^{-1}(t) * y(t) - \beta e^{at}]^s. \tag{10.46}$$

The extension of the method to non-minimum-phase systems with multiple RHP zeros is straightforward. In implementation, some numerical integration technique (Sinha and Rao, 1991) has to be used to compute β, and this inevitably gives a numerical error, which will accumulate and make $\eta(t) \triangleq g^{-1}(t) * y(t) - \beta e^{at}$ diverge. Thus, we only take one early period of $\eta(t)$, when $\eta(t)$ becomes approximately periodic, and duplicate it by $\eta^s(t) = \eta^s(t+T)$ to generate $x^s(t) = \eta^s(t)$ for all $t \geqslant 0$.

For example, consider a non-minimum-phase process $G(s) = \frac{-4s+1}{5s+1}$, and suppose that a sinusoidal signal $x(t)$ passes through $\frac{1}{9s^2+4s+1}$ and then acts on the process. With the suggested method, the steady state periodic signal $\hat{x}^s(t)$ is constructed and shown in Figure 10.5, where the solid line is $\hat{x}^s(t)$ and the dashed line is the actual $x(t)$. It is clear that $\hat{x}^s(t)$ is almost identical to the steady state part of $x(t)$.

Variation of v. The VFC signal v in (10.34) is equivalent to

$$v(t) = [\hat{g}^{-1}(t-T_v) * \hat{d}^s(t-T_v)]^s \mathbf{1}(t-T_v), \tag{10.47}$$

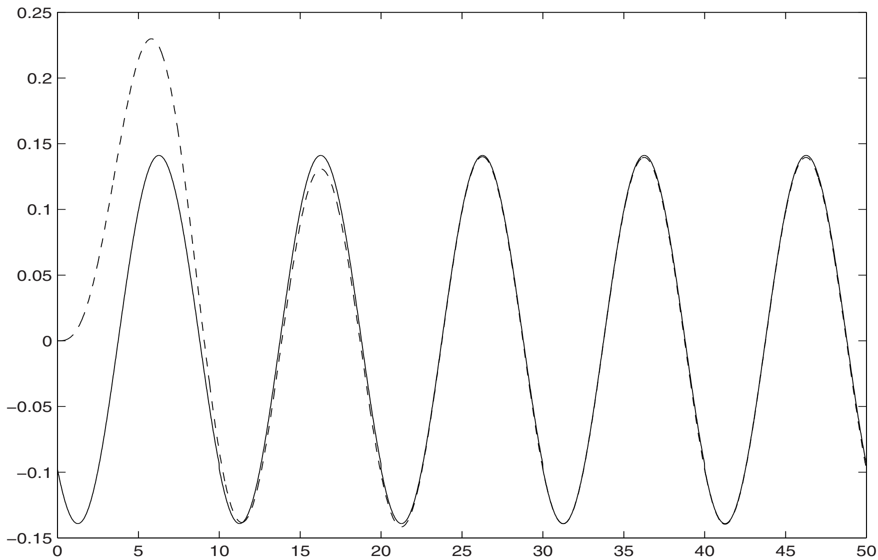


Fig. 10.5. Non-minimum-phase System $G(s) = \frac{-4s+1}{5s+1}$
(—— constructed $\hat{x}^s(t)$, $---$ actual $x(t)$)

which is the periodic steady state of

$$v(t) = [\hat{g}^{-1}(t - T_v) * \hat{d}^s(t - T_v)]\mathbf{1}(t - T_v). \tag{10.48}$$

Thus, (10.47) and (10.48) may be called static and dynamic VFC, respectively. One may guess that the dynamic control (10.48) can improve disturbance rejection performance over the static one (10.47). Dynamic VFC and static one have the same steady state, which causes the same steady state output response. The difference between the signals generated by these two schemes is its transient which depends on the plant and disturbance which vary from the case to the case, and is hard to predict. This difference will cause the transient difference in the output response and the latter is even harder to predict. Thus, there is no guarantee that the dynamic VFC will give an improved (dynamic) performance over the static VFC. What we see from extensive simulation is that for some examples, dynamic VFC does give a little improvement over static VFC, while for most examples, almost no difference can be observed. Furthermore, the dynamic VFC is not applicable to non-minimum-phase models. Therefore, we recommend (10.34) for practical implementation for all stable models of both minimum-phase and non-minimum-phase.

Adaptation. The proposed method can be extended to adaptively compensate for disturbances with variable periods. The process output is monitored all the time. When the periodic output is observed, v is applied and a new steady state with a suppressed error is expected after some transient. Thereafter, one waits for a different output waveform to occur, and then the VFC controller is de-activated (or removed) from the system. The procedure is then repeated.

We now present some different scenarios to demonstrate the effectiveness of the VFC scheme.

Example 10.2.1. Consider a process:

$$G(s) = \frac{e^{-4s}}{(10s+1)(2s+1)}$$

under a PID controller (Zhuang and Atherton, 1993):

$$K(s) = 2.0260 + \frac{0.2631}{s} + 4.9727s. \tag{10.49}$$

The external disturbance signal is introduced at $t = 0$:

$$d(t) = sin(0.1t) + 0.5sin(0.2t) + 0.3sin(0.3t) + 0.1sin(0.4t), \tag{10.50}$$

which has the period of $T = \frac{2\pi}{0.1} \approx 62.83$. The process output y is monitored until the periodic response is detected at $t = 150.0$. The disturbance is estimated via (10.30) and $v(t)$ is computed from (10.34). $v(t)$ is then applied at $T_v = 3T \approx 188.5$ to meet (10.36).

In the case of no model mismatch, i.e., $\hat{G} = G$, the resultant responses are shown in Figure 10.6. In the first plot, the disturbance is exhibited, where the dashed line is d and the solid line is $\hat{d}$. From this plot, it is clear that the stationary part $\hat{d}^s$ is identical to d^s. In the second plot, the process output is shown, where the solid line is from the proposed VFC scheme, while the dashed line is from the normal feedback control. The effectiveness of the VFC scheme is clear. Further, as stated in Theorem 10.3, the output from the proposed control approaches zero asymptotically. Such a property is guaranteed only when v is activated at T_v meeting (10.36). Suppose that v is applied with an extra delay of $t = 1$, and there results a steady state error as shown by the dotted line in the second plot of Figure 10.6.

The VFC scheme is independent of the feedback controller design, and the output steady state approaches to zero provided that the condition of Theorem 10.3 is satisfied. However, the output transient behavior inevitably relies on the controller parameters. To see the controller effects, different controller settings are used. Figure 10.7 plots the output responses, where the solid line is from the controller (10.49), while the dashed line is from the controller with the derivative part de-activated, i.e.,

$$K(s) = 2.0260 + \frac{0.2631}{s}.$$

As expected, such a controller gives a slower response compared with that in (10.49), though both produce a zero output asymptotically.

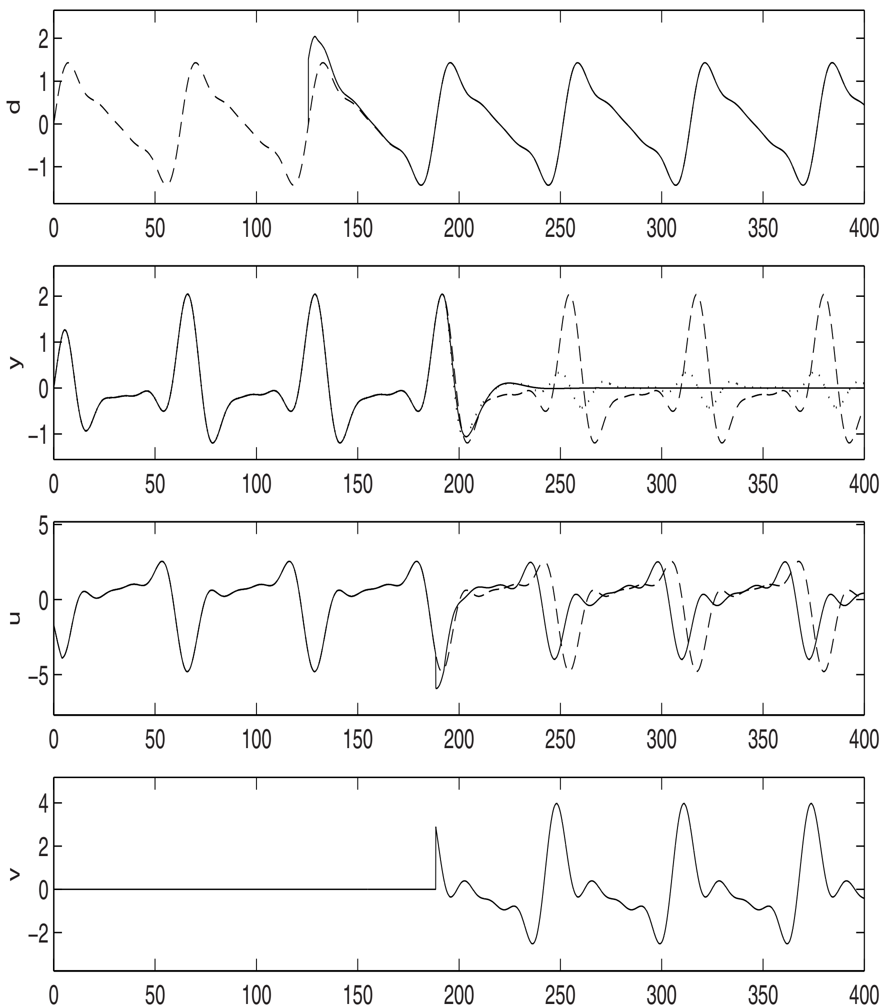


Fig. 10.6. VFC for Example 10.2.1 without Model Mismatch

In practice, the process is not fully known and it has to be described by a model. Suppose that the process is identified (Bi *et al.*, 1999) as

$$\hat{G}(s) = \frac{1.03e^{-5.47s}}{11.41s+1}.$$

From the recorded $y(t)$, $v(t)$ is computed with $\hat{G}(s)$ and applied with $T_v = 3T \approx 188.5$. The performance is shown in Figure 10.8 over $t \in [0, 377.0]$, where the solid line is with model mismatch, while the dashed line is for the case without model mismatch.

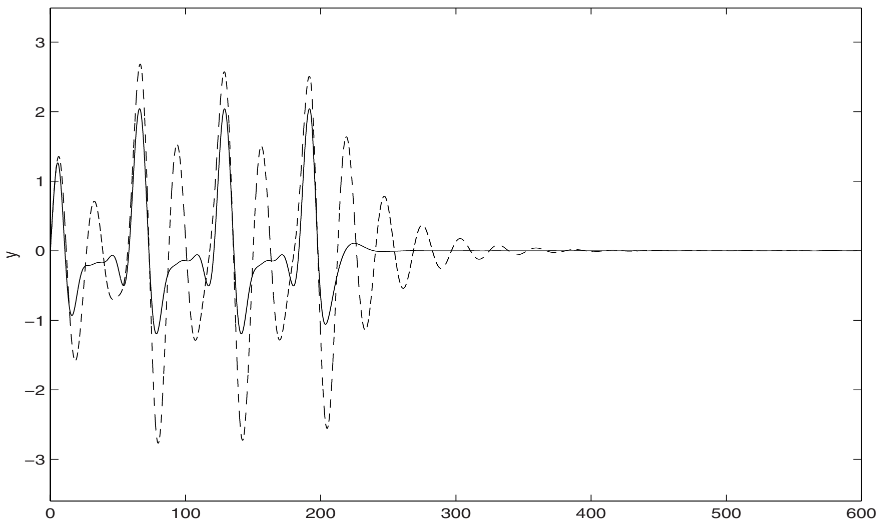


Fig. 10.7. VFC for Example 10.2.1 with Different $K(s)$
(—— $K(s) = 2.0260 + \frac{0.2631}{s} + 4.9727s$, $- - -$ $K(s) = 2.0260 + \frac{0.2631}{s}$)

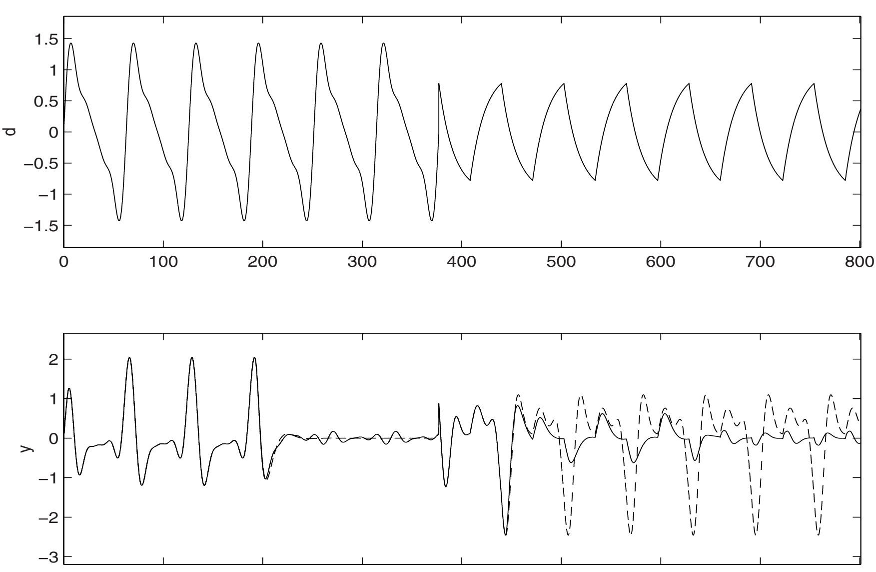


Fig. 10.8. VFC for Example 10.2.1 with Model Mismatch

Suppose that at $t = 6T \approx 377.0$, the disturbance is changed to $D(s) = \frac{1}{15s+1}W(s)$, where

$$w(t) = \begin{cases} -1, & \text{if } -T < t < 0, \\ 1, & \text{if } 0 < t < T. \end{cases}$$

Such a disturbance will change the waveform of the monitored $y(t)$, and a steady state periodic $y(t)$ with enlarged error is observed. The adaptive scheme is thus activated to de-activate $v(t)$ at $t = 7T \approx 439.8$. The virtual feedforward block (with model mismatch) is activated again at $t = 10T \approx 628.3$ to compensate for the new disturbance, as shown in the solid line in Figure 10.8. The response without the adaptive scheme, i.e., with the same $v(t)$ for the different $d(t)$, is shown by the dashed line over $t \in [377.0, 800.0]$ in Figure 10.8.

In practice, measurement noise is generally present, and its level may be measured by the noise-to-signal ratio (Haykin, 1989):

$$NSR = \frac{\text{mean(abs(noise))}}{\text{mean(abs(signal))}}.$$

To demonstrate the robustness of the proposed method, the system is tested again under a noise level with $NSR \approx 20\%$ for the above scenario, and the responses are shown in Figure 10.9. It is confirmed that the proposed method is robust to measurement noise. ♢

Example 10.2.2. The proposed method can be applied to servo control problems for better disturbance rejection over the existing methods, say the repetitive control scheme. For demonstration, consider a servo system in Moon *et al.* (1998) with the uncertain plant:

$$G(s) = \frac{3.2 \times 10^7 \beta_0}{s^2 + \alpha_1 s + \alpha_0},$$

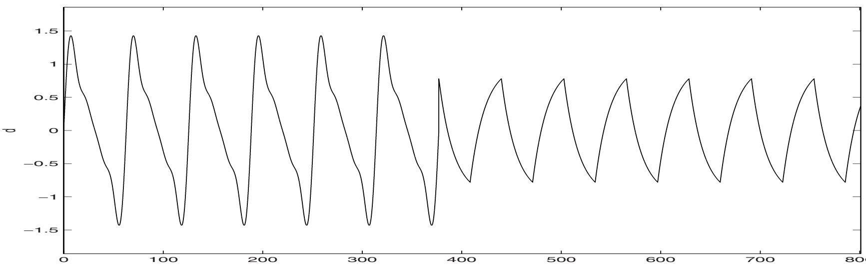
where $\alpha_1 \in [20, 27]$, $\alpha_0 \in [45000, 69000]$, $\beta_0 \in [60, 100]$; and the controller:

$$K(s) = \frac{0.26(s + 1900)(s + 4100)(s + 11300)^2}{(s + 100)(s + 19000)(s + 25100)(s + 31000)}.$$

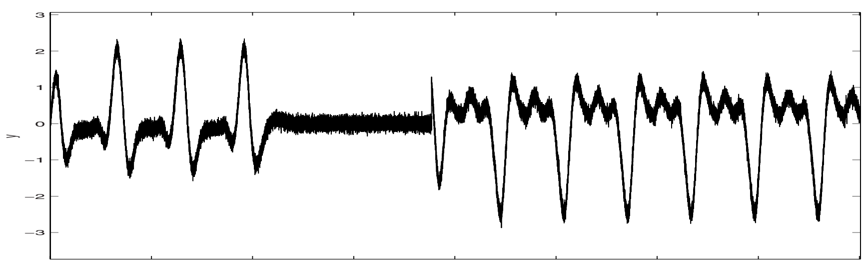
We compare our method with the repetitive control scheme discussed in Moon *et al.* (1998). Suppose that the system is in the steady-state before a disturbance is introduced at $t = 0$, the disturbance signal comprises a constant signal, eight out-of-phase sinusoidal signals from the first to the eighth harmonics of the spindle frequency, 30 Hz (Moon *et al.*, 1998).

The coefficients of the model are supposed to be center values of each interval, i.e.,

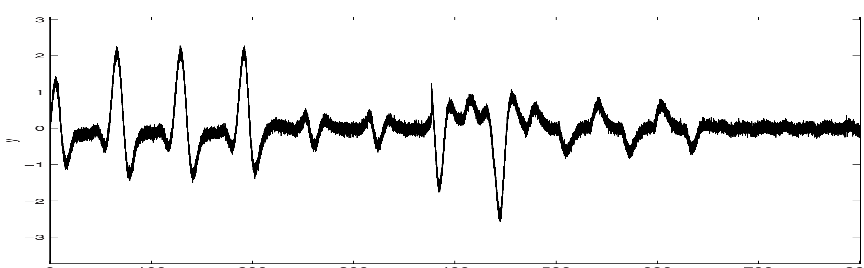
$$\alpha_1 = 23.5, \quad \alpha_0 = 57000, \quad \beta_0 = 80. \tag{10.51}$$



(a) disturbance



(b) $t = 0 \sim 377.0$: without model mismatch; $t = 377.0 \sim 800.0$: with model mismatch but without adaptation



(c) with model mismatch and adaptation

Fig. 10.9. VFC for Example 10.2.1 with Noise $NSR = 20\%$

For the case without model mismatch, i.e., the actual process is described exactly by the model, the responses are shown in Figure 10.10, where the solid line is from the proposed method, and the dashed line is from the repetitive control scheme. Since the control signals from these two methods are very close to each other, their difference Δu is displayed in Figure 10.10,

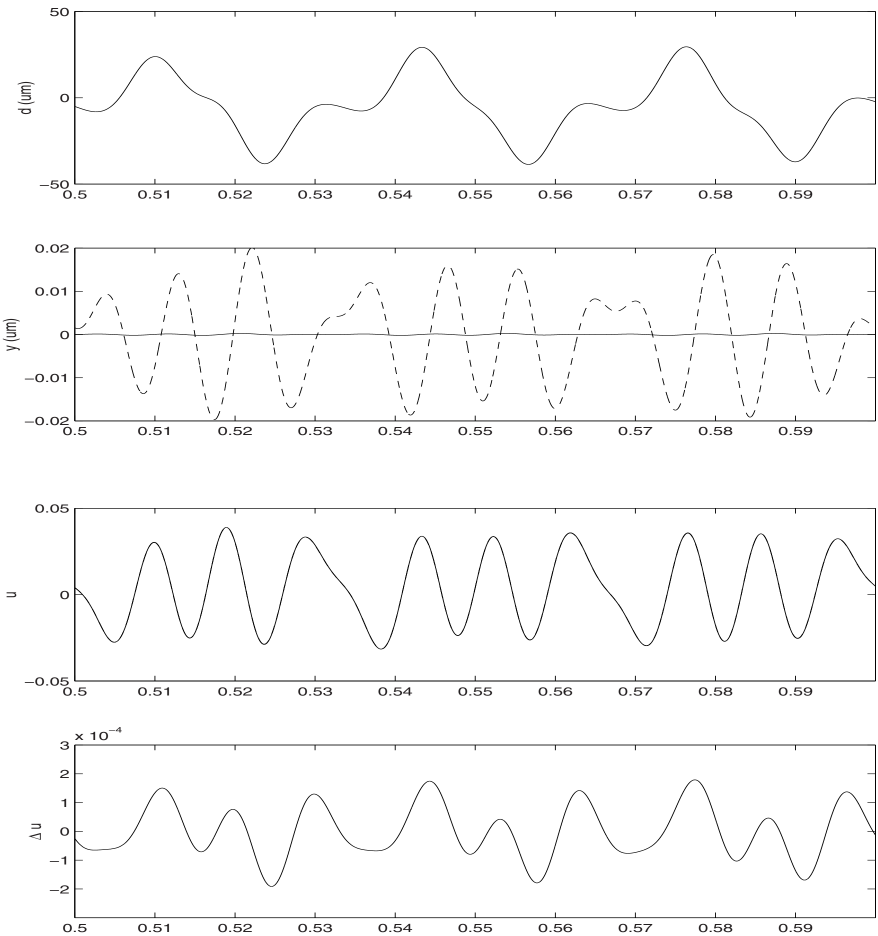


Fig. 10.10. Comparison of VFC and Repetitive Control (—— VFC, − − − Repetitive)

where only the steady state parts are shown for illustration. The effectiveness of the proposed method is obvious. Now we vary the plant coefficients within the possible intervals:

$$\alpha_1 = 25.5436, \quad \alpha_0 = 59770, \quad \beta_0 = 77.7881.$$

The responses are shown in Figure 10.11. The robustness of the proposed method to the model uncertainty is noticed. $\Diamond$

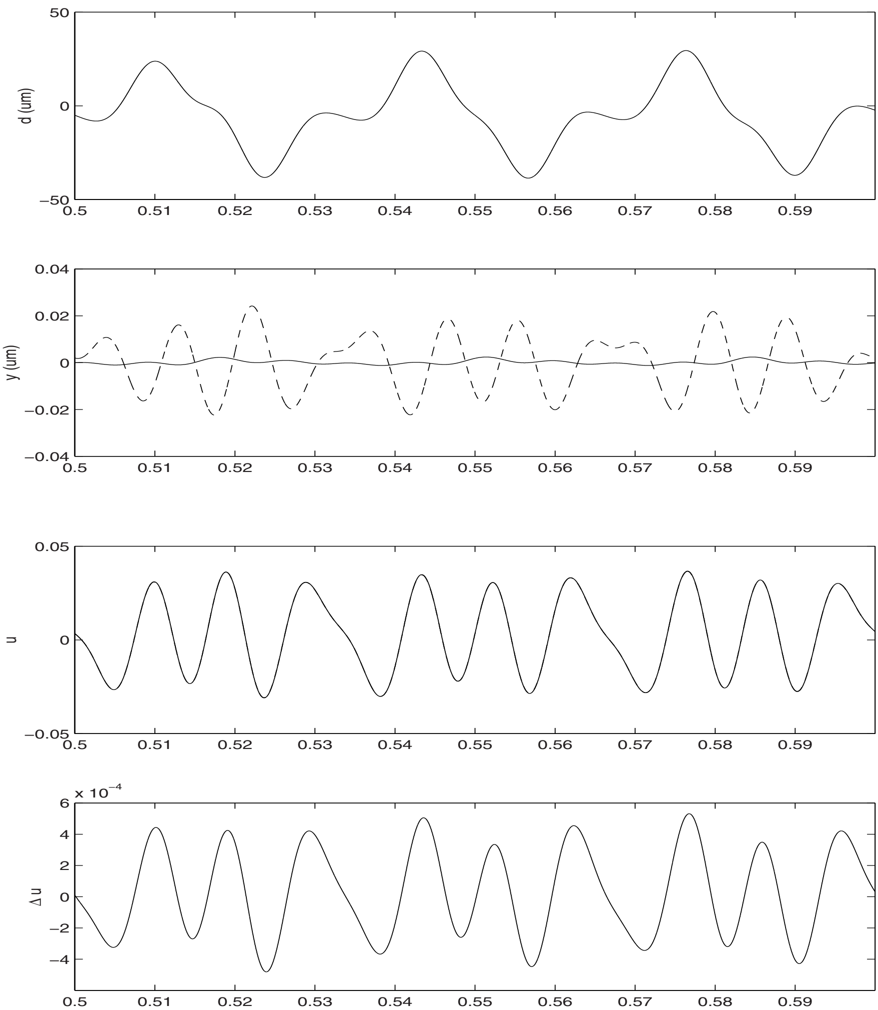


Fig. 10.11. VFC and Repetitive Control under Model Mismatch
(—— VFC, − − − repetitive)

10.2.3 Extension to the MIMO Case

We have so far considered the SISO case. The extension to the MIMO case is straightforward. Suppose that the process $P(s)$ is invertible and has been stabilized by the controller K in a conventional feedback system. The same scheme as in Figure 10.4 is exploited. The same derivation and implementation techniques as in the SISO case before applies with obvious modifications to suit the square matrix case. The only thing to take note is that the inverse of the process G is needed in (10.34) to compute the VFC vector signal $v(t)$ and is usually highly complicated and not realizable. We can apply the model reduction (Chapter 2) to $G^{-1}(j\omega)$ to get its approximation $\tilde{G}(s)$ in form of

$$\tilde{g}_{ij}(s) = \tilde{g}_{ij0}(s)e^{-\tau_{ij}s}. \tag{10.52}$$

where $\tilde{g}_{ij0}(s)$ are proper rational functions. Some terms $e^{-\tau_{ij}s}$ may happen to be a pure predictor but by choosing $T_v = kT$ with a suitable k (see (10.36)), we can always make all τ_{ij} negative, leading to time delays terms $e^{-\tau_{ij}s}$ which are realizable.

Example 10.2.3. Consider the process:

$$G = \begin{bmatrix} \frac{12.8}{16.7s+1}e^{-s} & \frac{-18.9}{21.0s+1}e^{-3s} \\ \frac{6.6}{10.9s+1}e^{-7s} & \frac{-19.4}{14.4s+1}e^{-3s} \end{bmatrix},$$

which is controlled by

$$K = \begin{bmatrix} 0.949 + 0.1173/s + 0.4289s & 0 \\ 0 & -0.131 - 0.0145/s - 0.1978s \end{bmatrix}.$$

With model reduction, we obtain

$$G^{-1} \approx \tilde{G}(s)$$
$$= \begin{bmatrix} \frac{0.31s^4+0.62s^3+0.42s^2+0.05s+0.002}{s^4+0.25s^3+1.16s^2+0.21s+0.013}e^{2s} & \frac{0.089s^4+0.04s^3-0.19s^2-0.03s-0.0015}{s^4+0.36s^3+1.03s^2+0.17s+0.01}e^{4s} \\ \frac{0.14s^4+0.278s^3+0.18s^2+0.02s+0.0007}{s^4+0.26s^3+1.16s^2+0.22s+0.01}e^{-2s} & \frac{0.08s^4+0.03s^3-0.16s^2-0.03s-0.001}{s^4+0.38s^3+1.03s^2+0.19s+0.01}e^{6s} \end{bmatrix}.$$

The following disturbance:

$$d(t) = sin(0.1t) + 0.5sin(0.2t) + 0.3sin(0.3t) + 0.1sin(0.4t)$$

is added to both outputs of the system at $t = 0$. The VFC controller is activated at $t = 3T$ (T is the period of the disturbance). The system time responses with and without VFC are shown in Figure 10.12 with solid line and dash line, respectively. Performance enhancement with VFC is obvious. Its non-zero steady state error is caused by the approximation of the process inverse. ♢

In this section, a control scheme, called the virtual feedforward control (VFC), is presented for asymptotic disturbance rejection. The VFC is an add-on function on top of normal feedback control, and it is activated when a periodic disturbance is found to be present in the system, based on output measurements and detection of output periodic behavior. It is shown that the VFC can reject the periodic disturbances asymptotically under the perfect plant modelling. Furthermore, the closed-loop stability is not affected by the VFC and thus there is no design trade-off between disturbance rejection and stability. The robustness of the control scheme to model mismatch is analyzed. The effectiveness of the VFC method is sustained by simulation and comparison.

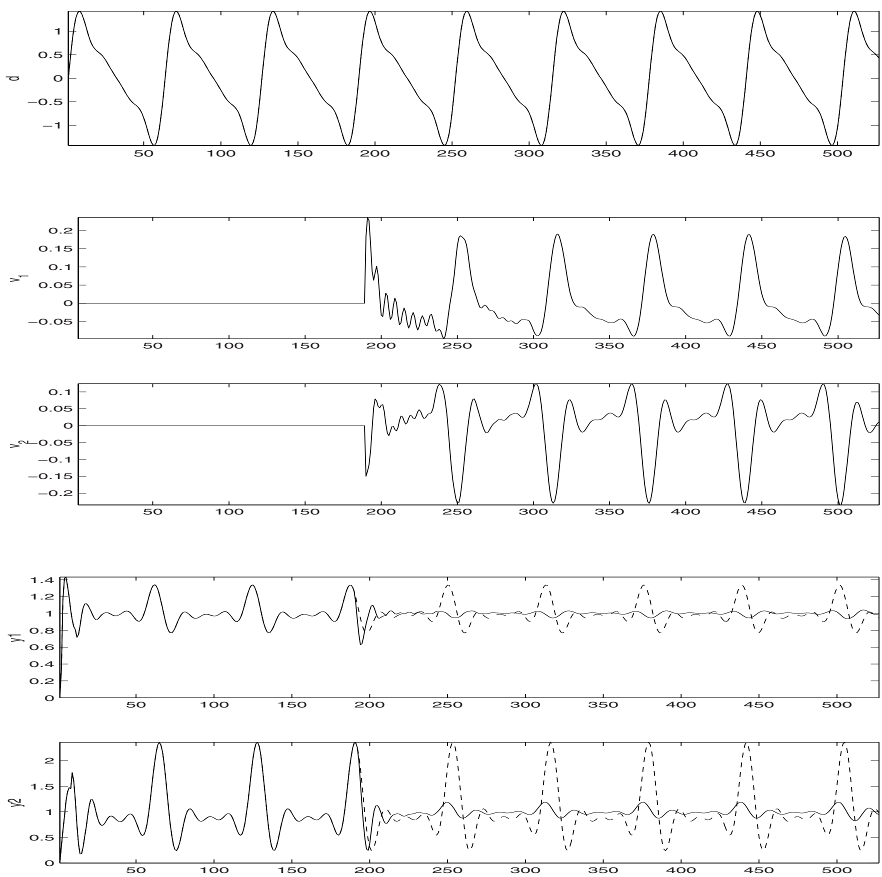


Fig. 10.12. Disturbance Response for Example 10.2.3
(—— With VFC – – –Without VFC)

10.3 Notes and References

A basic requirement of a closed -loop control system is its ability to regulate against disturbances, and to track references. It is thus not surprising that numerous investigators have considered these fundamental control system problems, employing a variety of mathematical formulations and techniques. The regulation problem without internal stability was considered by Wonham (1973) and that with internal stability by Wonham and Pearson (1974) and Francis and Wonham (1975) in a geometric setting. Wonham (1979) gives a complete exposition, where solutions together with (necessary and sufficient) solvability conditions, and a procedure for the construction of a compensator have been obtained. This problem was alternatively studied by Davison (1975) in an algebraic setting. Both works are carried out in a state-space viewpoint. The problem was also addressed from an input-output viewpoint by various researchers. Among these are Bengtsson (1977), Francis (1977), Seaks and Murray (1981) who considered the case where feedback signals coincide with the outputs; Chen and Pearson (1978), Wolovich and Ferreira (1979) who considered the case where feedback signals are certain functions of outputs; and finally Pernebo (1981), Cheng and Pearson (1981) and Khargonekar and Ozguler (1984) who considered the general case where feedback signals may be different from outputs and not related to each other. The crucial condition for the solvability in frequency domain setting is the existence condition for the solution of Diophantine equation (Kucera, 1979), which, in a special case, is simplified to skew-primeness (Wolovich, 1978). The case considered in this chapter is more general than that considered by Bengtsson (1977) who assume $G_m = I$ and by Wolovich and Ferreira (1979) who assume $G_d(s) = P^{-1}(s)D^{-1}(s)$ where both $P(s)$ and $D(s)$ are polynomial matrices and $P(s)$ is the denominator of left coprime fraction of plant $G_p(s)$. Section 1 of this chapter summarizes these elegant results from a polynomial matrix viewpoint.

For periodic disturbance rejection, Hara *et al.* (1988) pioneered a method called repetitive control. However, this method potentially makes the closed-loop system prone to instability, because the internal positive feedback loop that generates a periodic signal reduces the stability margin. Moon *et al.* (1998) proposed a repetitive controller design method for periodic disturbance rejection with uncertain plant coefficients. It is noted that such a method is robust at the expense of performance deterioration compared with the nominal case, and further, the disturbance cannot be fully compensated for even under the ideal case. Another way to solve the periodic disturbance problem is to use the double controller scheme (Tian and Gao, 1998). However, the complexity and the lack of tuning rules hinder its application. A plug-in adaptive controller (Hu and Tomizuka, 1993; Miyamoto *et al.*, 1999) is proposed to reject periodic disturbances. An appealing feature is that turning on or off the plug module will not affect the original structure. However, the shortcoming of this method is that the analysis and implementations are somewhat more complex than the conventional model based algorithm. Section 2 of this chapter presents the virtual feedforward control scheme, which is from Zhang (2000) and has several advantages over others mentioned above.