T H R E E

Modeling in the Time Domain

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: State-Space Representation

For the power amplifier, $\frac{E_a(s)}{V_p(s)} = \frac{150}{s+150}$. Taking the inverse Laplace transform, $\dot{e}_a + 150e_a = 150v_p$. Thus, the state equation is

$$\dot{e}_a = -150e_a + 150v_p$$

For the motor and load, define the state variables as $x_1 = \theta_m$ and $x_2 = \dot{\theta}_m$. Therefore,

$$\dot{x}_1 = x_2 \tag{1}$$

Using the transfer function of the motor, cross multiplying, taking the inverse Laplace transform, and using the definitions for the state variables,

$$\dot{x}_2 = -\frac{1}{J_m}\left(D_m + \frac{K_t K_a}{R_a}\right)x_2 + \frac{K_t}{R_a J_m}e_a \tag{2}$$

Using the gear ratio, the output equation is

$$y = 0.2x_1 \tag{3}$$

Also, $J_m = J_a + 5\left(\frac{1}{5}\right)^2 = 0.05+0.2 = 0.25$, $D_m = D_a + 3\left(\frac{1}{5}\right)^2 = 0.01+0.12 = 0.13$, $\frac{K_t}{R_a J_m} = \frac{1}{(5)(0.25)} = 0.8$, and $\frac{1}{J_m}\left(D_m + \frac{K_t K_a}{R_a}\right) = 1.32$. Using Eqs. (1), (2), and (3) along with the previous values, the state and output equations are,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1.32 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}e_a \; ; \; y = \begin{bmatrix} 0.2 & 0 \end{bmatrix}\mathbf{x}$$

Aquifer: State-Space Representation

$$C_1\frac{dh_1}{dt} = q_{i1}-q_{o1}+q_2-q_1+q_{21} = q_{i1}-0+G_2(h_2-h_1)-G_1h_1+G_{21}(H_1-h_1) =$$

$$-(G_2+G_1+G_{21})h_1+G_2h_2+q_{i1}+G_{21}H_1$$

$$C_2\frac{dh_2}{dt} = q_{i2}-q_{02}+q_3-q_2+q_{32} = q_{i2}-q_{o2}+G_3(h_3-h_2)-G_2(h_2-h_1)+0 = G_2h_1-[G_2+G_3]h_2+G_3h_3+q_{i2}-q_{o2}$$

$$C_3\frac{dh_3}{dt} = q_{i3}-q_{o3}+q_4-q_3+q_{43} = q_{i3}-q_{o3}+0-G_3(h_3-h_2)+0 = G_3h_2-G_3h_3+q_{i3}-q_{o3}$$

Dividing each equation by C_i and defining the state vector as $\mathbf{x} = [h_1 \;\; h_2 \;\; h_3]^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{-(G_1+G_2+G_3)}{C_1} & \frac{G_2}{C_1} & 0 \\ \frac{G_2}{C_2} & \frac{-(G_2+G_3)}{C_2} & \frac{G_3}{C_2} \\ 0 & \frac{G_3}{C_3} & \frac{-G_3}{C_3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{q_{i1}+G_{21}H_1}{C_1} \\ \frac{q_{i2}-q_{o2}}{C_2} \\ \frac{q_{i3}-q_{o3}}{C_3} \end{bmatrix}u(t)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

where u(t) = unit step function.

ANSWERS TO REVIEW QUESTIONS

1. (1) Can model systems other than linear, constant coefficients; (2) Used for digital simulation

2. Yields qualitative insight

3. That smallest set of variables that completely describe the system

4. The value of the state variables

5. The vector whose components are the state variables

6. The n-dimensional space whose bases are the state variables

7. State equations, an output equation, and an initial state vector (initial conditions)

8. Eight

9. Forms linear combinations of the state variables and the input to form the desired output

10. No variable in the set can be written as a linear sum of the other variables in the set.

11. (1) They must be linearly independent; (2) The number of state variables must agree with the order of the differential equation describing the system; (3) The degree of difficulty in obtaining the state equations for a given set of state variables.

12. The variables that are being differentiated in each of the linearly independent energy storage elements

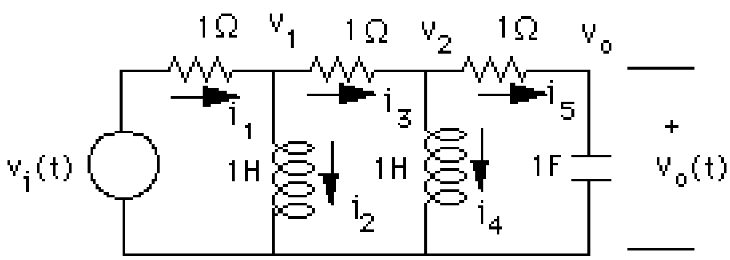
13. Yes, depending upon the choice of circuit variables and technique used to write the system equations. For example, a three -loop problem with three energy storage elements could yield three simultaneous second-order differential equations which would then be described by six, first-order differential equations. This exact situation arose when we wrote the differential equations for mechanical systems and then proceeded to find the state equations.

14. The state variables are successive derivatives.

SOLUTIONS TO PROBLEMS

1.

Add the branch currents and node voltages to the network.



Write the differential equation for each energy storage element.

$$\frac{di_2}{dt} = v_1$$
$$\frac{di_4}{dt} = v_2$$
$$\frac{dv_o}{dt} = i_5$$

Therefore, the state vector is $\mathbf{x} = \begin{bmatrix} i_2 \\ i_4 \\ v_o \end{bmatrix}$

Now obtain v_1, v_2, and i_5 in terms of the state variables. First find i_1 in terms of the state variables.

$-v_i + i_1 + i_3 + i_5 + v_o = 0$

But $i_3 = i_1 - i_2$ and $i_5 = i_3 - i_4$. Thus,

$-v_i + i_1 + (i_1 - i_2) + (i_3 - i_4) + v_o = 0$

Making the substitution for i_3 yields

$-v_i + i_1 + (i_1 - i_2) + ((i_1 - i_2) - i_4) + v_o = 0$

Solving for i_1

$$i_1 = \frac{2}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Thus,

$$v_1 = v_i - i_1 = -\frac{2}{3}i_2 - \frac{1}{3}i_4 + \frac{1}{3}v_o + \frac{2}{3}v_i$$

Also,

$$i_3 = i_1 - i_2 = -\frac{1}{3}i_2 + \frac{1}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

and

$$i_5 = i_3 - i_4 = -\frac{1}{3}i_2 - \frac{2}{3}i_4 - \frac{1}{3}v_o + \frac{1}{3}v_i$$

Finally,

$$v_2 = i_5 + v_o = -\frac{1}{3}i_2 - \frac{2}{3}i_4 + \frac{2}{3}v_o + \frac{1}{3}v_i$$

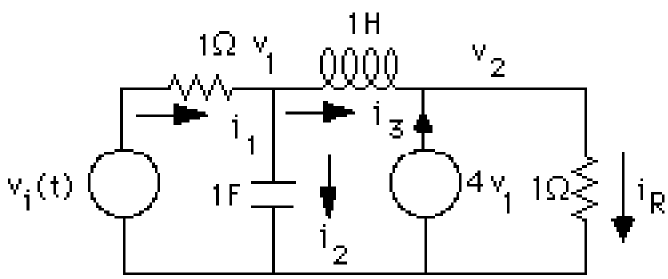
Using v_1, v_2, and i_5, the state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

2.

Add branch currents and node voltages to the schematic and obtain,



Write the differential equation for each energy storage element.

$$\frac{dv_1}{dt} = i_2$$

$$\frac{di_3}{dt} = v_L$$

Therefore the state vector is $\mathbf{x} = \begin{bmatrix} v_1 \\ i_3 \end{bmatrix}$

Now obtain v_L and i_2, in terms of the state variables,

$$v_L = v_1 - v_2 = v_1 - i_R = v_1 - (i_3 + 4v_1) = -3v_1 - i_3$$

$$i_2 = i_1 - i_3 = (v_i - v_1) - i_3 = -v_1 - i_3 + v_i$$

Also, the output is

$$y = i_R = 4v_1 + i_3$$

Hence,

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -1 \\ -3 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix}\mathbf{x}$$

3.

Let C_1 be the grounded capacitor and C_2 be the other. Now, writing the equations for the energy storage components yields,

$$\frac{di_L}{dt} = v_i - v_{C_1}$$

$$\frac{dv_{C_1}}{dt} = i_1 - i_2 \qquad (1)$$

$$\frac{dv_{C_2}}{dt} = i_2 - i_3$$

Thus the state vector is $\mathbf{x} = \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix}$. Now, find the three loop currents in terms of the state variables and the input.

Writing KVL around Loop 2 yields $v_{C_1} = v_{C_2} + i_2$ .Or,

$$i_2 = v_{C_1} - v_{C_2}$$

Writing KVL around the outer loop yields $i_3 + i_2 = v_i$ Or,

$$i_3 = v_i - i_2 = v_i - v_{C_1} + v_{C_2}$$

Also, $i_1 - i_3 = i_L$. Hence,

$$i_1 = i_L + i_3 = i_L + v_i - v_{C_1} + v_{C_2}$$

Substituting the loop currents in equations (1) yields the results in vector-matrix form,

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_{C_1}}{dt} \\ \frac{dv_{C_2}}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v_i$$

Since $v_o = i_2 = v_{C_1} - v_{C_2}$, the output equation is

$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix}$$

4.

Equations of motion in Laplace:

$$(s^2 + 3s + 1)x_1(s) - (s+1)x_2(s) - sx_3(s) = 0$$
$$-(s+1)x_1(s) + (s^2 + 2s + 1)x_2(s) - sx_3(s) = F(s)$$
$$-sx_1(s) - sx_2(s) + (s^2 + 3s)x_3(s) = 0$$

Equations of motion in the time domain:

$$\frac{d^2x_1}{dt^2} + 3\frac{dx_1}{dt} + x_1 - \frac{dx_2}{dt} - x_2 - \frac{dx_3}{dt} = 0$$
$$\frac{d^2x_2}{dt^2} + 2\frac{dx_2}{dt} + x_2 - \frac{dx_1}{dt} - x_1 - \frac{dx_3}{dt} = f(t)$$
$$\frac{d^2x_3}{dt^2} + 3\frac{dx_3}{dt} + x_1 - \frac{dx_2}{dt} - \frac{dx_1}{dt} = 0$$

Define state variables:

$$z_1 = x_1 \quad \text{or} \quad x_1 = z_1 \tag{1}$$
$$z_2 = \frac{dx_1}{dt} \quad \text{or} \quad \frac{dx_1}{dt} = z_2 \tag{2}$$
$$z_3 = x_2 \quad \text{or} \quad x_2 = z_3 \tag{3}$$
$$z_4 = \frac{dx_2}{dt} \quad \text{or} \quad \frac{dx_2}{dt} = z_4 \tag{4}$$
$$z_5 = x_3 \quad \text{or} \quad x_3 = z_5 \tag{5}$$
$$z_6 = \frac{dx_3}{dt} \quad \text{or} \quad \frac{dx_3}{dt} = z_5 \tag{6}$$

Substituting Eq. (1) in (2), (3) in (4), and (5) in (6), we obtain, respectively:

$$\frac{dz_1}{dt} = z_2 \tag{7}$$

$$\frac{dz_3}{dt} = z_4 \tag{8}$$

$$\frac{dz_5}{dt} = z_6 \tag{9}$$

Substituting Eqs. (1) through (6) into the equations of motion in the time domain and solving for the derivatives of the state variables and using Eqs. (7) through (9) yields the state equations:

$$\frac{dz_1}{dt} = z_2$$

$$\frac{dz_2}{dt} = -z_1 - 3z_2 + z_3 + z_4 + z_6$$

$$\frac{dz_3}{dt} = z_4$$

$$\frac{dz_4}{dt} = z_1 + z_2 - z_3 - 2z_4 + z_6 + f(t)$$

$$\frac{dz_5}{dt} = z_6$$

$$\frac{dz_6}{dt} = z_2 + z_4 - 3z_6$$

The output is $x_3 = z_5$.

In vector-matrix form:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{z}$$

5.

Writing the equations of motion,

$$(s^2+2s+1)X_1(s)-sX_2(s)-(s+1)X_3(s)=0$$
$$-sX_1(s)+(s^2+2s+1)X_2(s)-(s+1)X_3(s)=0$$
$$-(s+1)X_1(s)-(s+1)X_2(s)+(s^2+2s+2)X_3(s)=F(s)$$

Taking the inverse Laplace transform,

$$\ddot{x}_1+2\dot{x}_1+x_1-\dot{x}_2-\dot{x}_3-x_3=0$$
$$-\dot{x}_1+\ddot{x}_2+2\dot{x}_2+x_2-\dot{x}_3-x_3=0$$
$$-\dot{x}_1-x_1-\dot{x}_2-x_2+\ddot{x}_3+2\dot{x}_3+2x_3=f(t)$$

Simplifying,

$$\ddot{x}_1=-2\dot{x}_1-x_1+\dot{x}_2+\dot{x}_3+x_3$$
$$\ddot{x}_2=\dot{x}_1-2\dot{x}_2-x_2+\dot{x}_3+x_3$$
$$\ddot{x}_3=\dot{x}_1+x_1+\dot{x}_2+x_2-2\dot{x}_3-2x_3+f(t)$$

Defining the state variables,

$$z_1=x_1;\ z_2=\dot{x}_1;\ z_3=x_2;\ z_4=\dot{x}_2;\ z_5=x_3;\ z_6=\dot{x}_3$$

Writing the state equations using the simplified equations above yields,

$$\dot{z}_1=\dot{x}_1=z_2$$
$$\dot{z}_2=\ddot{x}_1=-2z_2-z_1+z_4+z_6+z_5$$
$$\dot{z}_3=\dot{x}_2=z_4$$
$$\dot{z}_4=\ddot{x}_2=z_2-2z_4-z_3+z_6+z_5$$
$$\dot{z}_5=\dot{x}_3=z_6$$
$$\dot{z}_6=\ddot{x}_3=z_2+z_1+z_4+z_3-2z_6-2z_5+f(t)$$

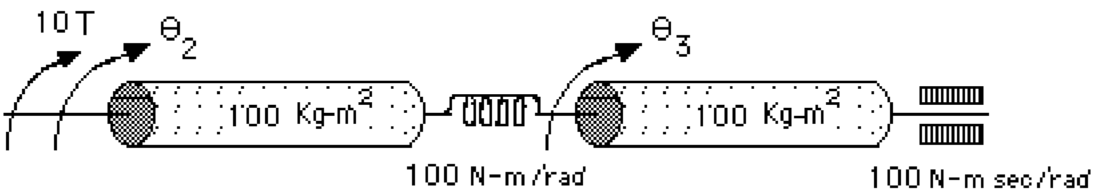
Converting to vector-matrix form,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & -2 & -2 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{z}$$

6.

Drawing the equivalent network,



Writing the equations of motion,

$$(100s^2 + 100)\theta_2 - 100\theta_3 = 10T$$
$$-100\theta_2 + (100s^2 + 100s + 100)\theta_3 = 0$$

Taking the inverse Laplace transform and simplifying,

$$\ddot{\theta}_2 + \theta_2 - \theta_3 = \frac{1}{10} T$$
$$-\theta_2 + \ddot{\theta}_3 + \dot{\theta}_3 + \theta_3 = 0$$

Defining the state variables as

$$x_1 = \theta_2,\ x_2 = \dot{\theta}_2,\ x_3 = \theta_3,\ x_4 = \dot{\theta}_3$$

Writing the state equations using the equations of motion and the definitions of the state variables

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ddot{\theta}_2 = -\theta_2 + \theta_3 + \frac{1}{10}T = -x_1 + x_3 + \frac{1}{10}T \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \ddot{\theta}_3 = \theta_2 - \theta_3 - \dot{\theta}_3 = x_1 - x_3 - x_4 \\
y &= 10\theta_2 = 10x_1
\end{aligned},$$

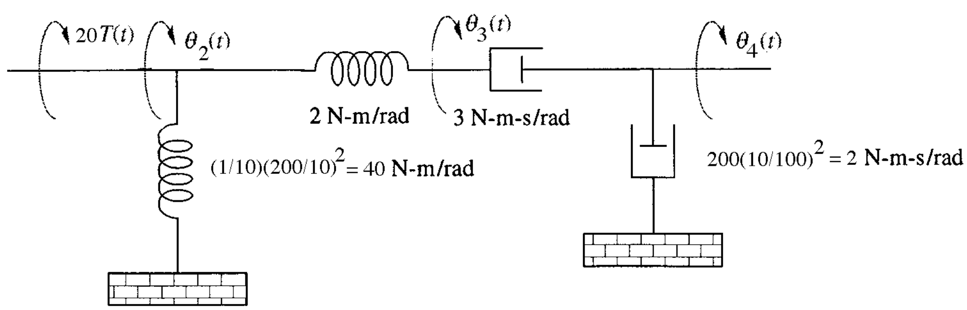
In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{10} \\ 0 \\ 0 \end{bmatrix}T$$

$$y = \begin{bmatrix} 10 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

7.

Drawing the equivalent circuit,



Writing the equations of motion,

$$\begin{aligned}
&42\theta_2(s) - 2\theta_3(s) = 20T(s) \\
&-2\theta_2(s) + (3s+2)\theta_3(s) - 3s\theta_4(s) = 0 \\
&-3s\theta_3(s) + 5s\theta_4(s) = 0
\end{aligned}$$

Taking the inverse Laplace transform,

$$42\,\theta_2(t) - 2\theta_3(t) = 20T(t) \qquad (1)$$

$$-2\theta_2(t) + 3\,\dot{\theta}_3(t) + 2\theta_3 - 3\,\dot{\theta}_4(t) = 0 \qquad (2)$$

$$-3\,\dot{\theta}_3(t) + 5\,\dot{\theta}_4(t) = 0 \qquad (3)$$

From (3),

$$\dot{\theta}_3(t) = \frac{5}{3}\dot{\theta}_4(t) \text{ and } \theta_3(t) = \frac{5}{3}\theta_4(t) \tag{4}$$

assuming zero initial conditions.

From (1)

$$\theta_2(t) = \frac{1}{21}\theta_3(t) + \frac{10}{21}T(t) = \frac{5}{63}\theta_4(t) + \frac{10}{21}T(t) \tag{5}$$

Substituting (4) and (5) into (2) yields the state equation (notice there is only one equation),

$$\dot{\theta}_4(t) = -\frac{100}{63}\theta_4(t) + \frac{10}{21}T(t)$$

The output equation is given by,

$$\theta_L(t) = \frac{1}{10}\theta_4(t)$$

8.

Solving Eqs. (3.44) and (3.45) in the text for the transfer functions $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$:

$$X_1(s) = \frac{\left|\begin{pmatrix} 0 & -K \\ F & M_2 s^2 + K \end{pmatrix}\right|}{\left|\begin{pmatrix} M_1 s^2 + D s + K & -K \\ -K & M_2 s^2 + K \end{pmatrix}\right|} \text{ and } X_2(s) = \frac{\left|\begin{pmatrix} M_1 s^2 + D s + K & 0 \\ -K & F \end{pmatrix}\right|}{\left|\begin{pmatrix} M_1 s^2 + D s + K & -K \\ -K & M_2 s^2 + K \end{pmatrix}\right|}$$

Thus,

$$\frac{X_1(s)}{F(s)} = \frac{K}{M_2 M_1 s^4 + D M_2 s^3 + K M_2 s^2 + K M_1 s^2 + D K s}$$

and

$$\frac{X_2(s)}{F(s)} = \frac{M_1 s^2 + D s + K}{M_2 M_1 s^4 + D M_2 s^3 + K M_2 s^2 + K M_1 s^2 + D K s}$$

Multiplying each of the above transfer functions by s to find velocity yields pole/zero cancellation at the origin and a resulting transfer function that is third order.

9.

a. . Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -7 & -10 & -20 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 100 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

b. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -30 & -1 & -6 & -9 & -8 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

10.

Program:

```
'a'
num=100;
den=[1 20 10 7 100];
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
'b'
num=30;
den=[1 8 9 6 1 30];
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
```

Computer response:

```
ans =

a

Transfer function:
               100
---------------------------------
s^4 + 20 s^3 + 10 s^2 + 7 s + 100




A =

     0     1     0     0
     0     0     1     0
     0     0     0     1
  -100    -7   -10   -20

B =

     0
     0
     0
     1

C =

   100     0     0     0

ans =

b
```

```
Transfer function:
                 30
------------------------------------
s^5 + 8 s^4 + 9 s^3 + 6 s^2 + s + 30

A =

     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
   -30    -1    -6    -9    -8

B =

     0
     0
     0
     0
     1

C =

    30     0     0     0     0
```

11.

a. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -5 & -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 10 & 5 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

b. Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -10 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 3 & 7 & 12 & 2 & 1 \end{bmatrix} \mathbf{x}$$

12.

Program:

```
'a'
num=[5 10];
den=[1 2 1 5 10]
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
```

```
C=fliplr(Ccc)
'b'
num=[1 2 12 7 3];
den=[1 9 10 8 0 0]
G=tf(num,den)
[Acc,Bcc,Ccc,Dcc]=tf2ss(num,den);
Af=flipud(Acc);
A=fliplr(Af)
B=flipud(Bcc)
C=fliplr(Ccc)
```

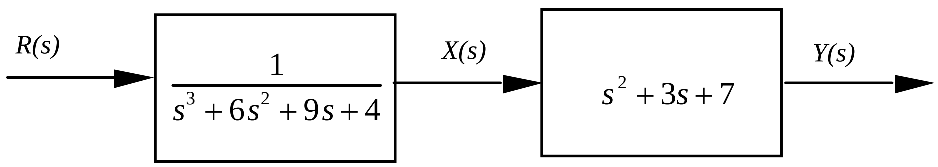
Computer response:

```
ans =

a

den =

     1     2     1     5    10


Transfer function:
          5 s + 10
----------------------------
s^4 + 2 s^3 + s^2 + 5 s + 10


A =

     0     1     0     0
     0     0     1     0
     0     0     0     1
   -10    -5    -1    -2

B =

     0
     0
     0
     1

C =

    10     5     0     0

ans =

b

den =

     1     9    10     8     0     0


Transfer function:
s^4 + 2 s^3 + 12 s^2 + 7 s + 3
------------------------------
 s^5 + 9 s^4 + 10 s^3 + 8 s^2


A =

     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
     0     0    -8   -10    -9
```

```
B =

     0
     0
     0
     0
     1

C =

     3     7    12     2     1
```

13.

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box:

$$\dddot{x}+6\ddot{x}+9\dot{x}+4x=r(t)$$

Defining the state variables:

$$x_1=x$$
$$x_2=\dot{x}$$
$$x_3=\ddot{x}$$

Thus,

$$\dot{x}_1=x_2$$
$$\dot{x}_2=x_3$$
$$\dot{x}_3=-4x-9\dot{x}-6\ddot{x}+r(t)=-4x_1-9x_2-6x_3+r(t)$$

From the second box,

$$y=\ddot{x}+3\dot{x}+7x=7x_1+3x_2+x_3$$

In vector-matrix form:

$$\dot{\mathbf{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -9 & -6\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}r(t)$$

$$y=\begin{bmatrix}7 & 3 & 1\end{bmatrix}\mathbf{x}$$

14.

a. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -2 & -5\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}0\\ 0\\ 10\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3+5s^2+2s+3}\begin{bmatrix}s^2+5s+2 & s+5 & 1\\ -3 & s(s+5) & s\\ -3s & -2s-3 & s^2\end{bmatrix}$$

Therefore, $G(s)=\dfrac{10}{s^3+5s^2+2s+3}$.

b. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{pmatrix}2 & 3 & -8\\ 0 & 5 & 3\\ -3 & -5 & -4\end{pmatrix};\ \mathbf{B}=\begin{pmatrix}1\\ 4\\ 6\end{pmatrix};\ \mathbf{C}=(1,3,6)$$

$$(s\,\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3-3s^2-27s+157}\begin{pmatrix}s^2-s-5 & 3s+52 & -8s+49\\ -9 & s^2+2s-32 & 3s-6\\ -3s+15 & -5s+1 & s^2-7s+10\end{pmatrix}$$

Therefore, $G(s)=\dfrac{49s^2-349s+452}{s^3-3s^2-27s+157}$.

c. $G(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{A}=\begin{bmatrix}3 & -5 & 2\\ 1 & -8 & 7\\ -3 & -6 & 2\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}5\\ -3\\ 2\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}1 & -4 & 3\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3+3s^2+19s-133}\begin{bmatrix}(s^2+6s+26) & -(5s+2) & (2s-19)\\ (s-23) & (s^2-5s+12) & (7s-19)\\ -(3s+30) & -(6s-33) & (s^2+5s-19)\end{bmatrix}$$

Therefore, $G(s)=\dfrac{23s^2-48s-7}{s^3+3s^2+19s-133}$.

15.

Program:

```
'a'
A=[0 1 3 0;0 0 1 0;0 0 0 1;-7 -9 -2 -3];
B=[0;5;8;2];
C=[1 3 4 6];
D=0;
statespace=ss(A,B,C,D)
```

```
[num,den]=ss2tf(A,B,C,D);
G=tf(num,den)
'b'
A=[3 1 0 4 -2;-3 5 -5 2 -1;0 1 -1 2 8;-7 6 -3 -4 0;-6 0 4 -3 1];
B=[2;7;6;5;4];
C=[1 -2 -9 7 6];
D=0;
statespace=ss(A,B,C,D)
[num,den]=ss2tf(A,B,C,D);
G=tf(num,den)
```

Computer response:

```
ans =

a


a =
       x1  x2  x3  x4
   x1   0   1   3   0
   x2   0   0   1   0
   x3   0   0   0   1
   x4  -7  -9  -2  -3


b =
       u1
   x1   0
   x2   5
   x3   8
   x4   2


c =
       x1  x2  x3  x4
   y1   1   3   4   6


d =
       u1
   y1   0

Continuous-time model.

Transfer function:
59 s^3 - 164 s^2 - 1621 s - 260
-------------------------------
s^4 + 3 s^3 + 2 s^2 + 30 s + 7


ans =

b


a =
       x1  x2  x3  x4  x5
   x1   3   1   0   4  -2
   x2  -3   5  -5   2  -1
   x3   0   1  -1   2   8
   x4  -7   6  -3  -4   0
   x5  -6   0   4  -3   1


b =
       u1
   x1   2
```

```
   x2   7
   x3   6
   x4   5
   x5   4


c =
       x1  x2  x3  x4  x5
   y1   1  -2  -9   7   6


d =
       u1
   y1   0

Continuous-time model.

Transfer function:
-7 s^4 - 408 s^3 + 1708 s^2 + 1.458e004 s + 2.766e004
-----------------------------------------------------
   s^5 - 4 s^4 - 32 s^3 + 148 s^2 - 1153 s - 4480
```

16.

Program:

```
syms s
'a'
A=[0 1 3 0
0 0 1 0
0 0 0 1
-7 -9 -2 -3];
B=[0;5;8;2];
C=[1 3 4 6];
D=0;
I=[1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1];
'T(s)'
T=C*((s*I-A)^-1)*B+D;
T=simple(T);
pretty(T)
'b'
A=[3 1 0 4 -2
-3 5 -5 2 -1
0 1 -1 2 8
-7 6 -3 -4 0
-6 0 4 -3 1];
B=[2;7;6;5;4];
C=[1 -2 -9 7 6];
D=0;
I=[1 0 0 0 0
0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1];
'T(s)'
T=C*((s*I-A)^-1)*B+D;
T=simple(T);
pretty(T)
```

Computer response:

```
ans =

a
```

```
ans =

T(s)

                    2                           3
         -164 s  - 1621 s - 260 + 59 s
         -----------------------------
          4        3        2
          s  + 3 s  + 2 s  + 30 s + 7

ans =

b

ans =
T(s)

                          2          3        4
      14582 s + 1708 s  - 408 s  - 7 s  + 27665
      -----------------------------------------
        5       4         3          2
      s  - 4 s  - 32 s  + 148 s  - 1153 s - 4480
```

17.

Let the input be $\frac{d\theta_z}{dt} = \omega_z$, $x_1 = \theta_x$, $x_2 = \dot{\theta}_x$. Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K_x}{J_x} x_1 - \frac{D_x}{J_x} x_2 + J\omega\omega_z$$

The output is θ_x.

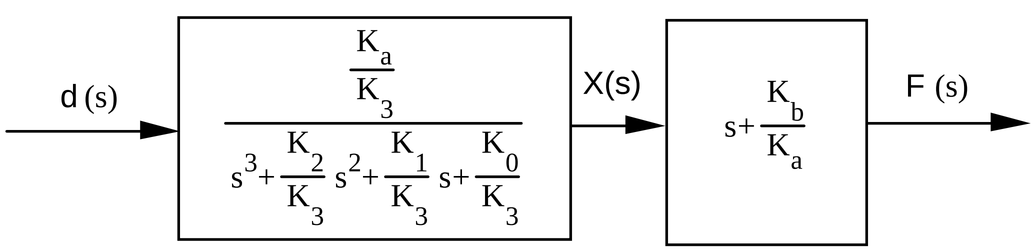
In vector-matrix form, $\theta_x = x_1$. Therefore, $y = x_1$.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{K_x}{J_x} & -\frac{D_x}{J_x} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ J\omega \end{bmatrix} \omega_z$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

18.

The equivalent cascade transfer function is as shown below.



For the first box, $\dddot{x}+\frac{K_2}{K_3}\ddot{x}+\frac{K_1}{K_3}\dot{x}+\frac{K_0}{K_3}x=\frac{K_a}{K_3}\delta(t)$.

Selecting the phase variables as the state variables: $x_1=x$, $x_2=\dot{x}$, $x_3=\ddot{x}$.

Writing the state and output equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -\frac{K_0}{K_3}x_1 - \frac{K_1}{K_3}x_2 - \frac{K_2}{K_3}x_3 + \frac{K_a}{K_3}\delta(t)$$

$$y = \phi(t) = \dot{x} + \frac{K_b}{K_a}x = \frac{K_b}{K_a}x_1 + x_2$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_0}{K_3} & -\frac{K_1}{K_3} & -\frac{K_2}{K_3} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_a}{K_3} \end{bmatrix}\delta(t)\ ;\ y = \begin{bmatrix} \frac{K_b}{K_a} & 1 & 0 \end{bmatrix}\mathbf{x}$$

19.

Since $T_m = J_{eq}\frac{d\omega_m}{dt} + D_{eq}\omega_m$, and $T_m = K_t\, i_a$,

$$J_{eq}\frac{d\omega_m}{dt} + D_{eq}\omega_m = K_t\, i_a \tag{1}$$

Or,

$$\frac{d\omega_m}{dt} = -\frac{D_{eq}}{J_{eq}}\omega_m + \frac{K_t}{J_{eq}}i_a$$

But, $\omega_m = \frac{N_2}{N_1}\omega_L$.

Substituting in (1) and simplifying yields the first state equation,

$$\frac{d\omega_L}{dt} = -\frac{D_{eq}}{J_{eq}}\omega_L + \frac{K_t}{J_{eq}}\frac{N_1}{N_2}i_a$$

The second state equation is:

$$\frac{d\theta_L}{dt} = \omega_L$$

Since

$$e_a = R_a i_a + L_a\frac{di_a}{dt} + K_b\omega_m = R_a i_a + L_a\frac{di_a}{dt} + K_b\frac{N_2}{N_1}\omega_L,$$

the third state equation is found by solving for $\frac{di_a}{dt}$. Hence,

$$\frac{di_a}{dt} = -\frac{K_b}{L_a}\frac{N_2}{N_1}\omega_L - \frac{R_a}{L_a}i_a + \frac{1}{L_a}e_a$$

Thus the state variables are: $x_1 = \omega_L$, $x_2 = \theta_L$, and $x_3 = i_a$.

Finally, the output is $y = \theta_m = \frac{N_2}{N_1}\theta_L$.

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{D_{eq}}{J_{eq}} & 0 & \frac{K_t}{J_{eq}}\frac{N_1}{N_2} \\ 1 & 0 & 0 \\ -\frac{K_b}{L_a}\frac{N_2}{N_1} & 0 & -\frac{R_a}{L_a} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix} e_a \;;\; y = \begin{bmatrix} 0 & \frac{N_2}{N_1} & 0 \end{bmatrix}\mathbf{x}$$

where,

$$\mathbf{x} = \begin{bmatrix} \omega_L \\ \theta_L \\ i_a \end{bmatrix}$$

20.

Writing the differential equations,

$$\frac{d^2x_1}{dt^2} + \frac{dx_1}{dt} + 2x_1^2 - \frac{dx_2}{dt} = 0$$

$$\frac{d^2x_2}{dt^2} + \frac{dx_2}{dt} - \frac{dx_1}{dt} = f(t)$$

Defining the state variables to be x_1, v_1, x_2, v_2, where v's are velocity,

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -v_1-2x_1^2+v_2$$

$$\dot{v}_2 = v_1-v_2+f(t)$$

Around $x_1 = 1$, $x_1 = 1+\delta x_1$, and $\dot{x}_1 = \delta\dot{x}_1$. Also,

$$x_1^2 = x_1^2\Big|_{x=1} + \frac{dx_1}{dt}\Big|_{x=1}\delta x_1 = 1+2x_1\Big|_{x=1}\delta x_1 = 1+2\delta x_1$$

Therefore, the state and output equations are,

$$\delta\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -v_1-2(1+2\delta x_1)+v_2$$

$$\dot{v}_2 = v_1 - v_2 + f(t)$$

$$y = x_2$$

In vector-matrix form,

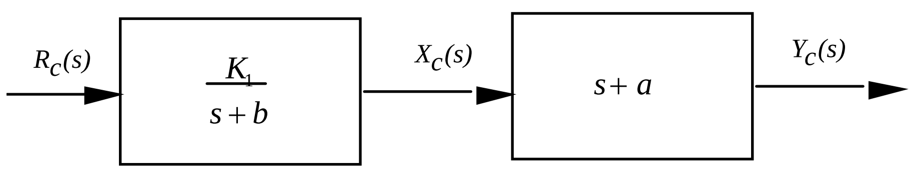
$$\begin{bmatrix} \delta\dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ f(t) \end{bmatrix}; \; y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}$$

where $f(t) = 2 + \delta f(t)$, since force in nonlinear spring is 2 N and must be balanced by 2 N force on damper.

21.

Controller:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box,

$$\frac{K_1}{s+b}$$

and solving for $\dot{x}_c$,

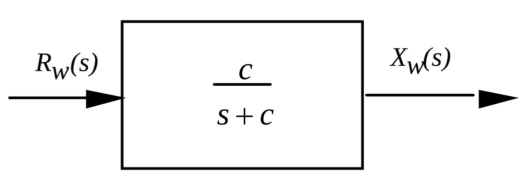
$$\dot{x}_c = -bx_c + K_1 r_c(t)$$

From the second box,

$$y_c = \dot{x}_c + ax_c = -bx_c + K_1 r_c(t) + ax_c$$
$$= (a-b)x_c + K_1 r_c(t)$$

Wheels:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the block of the form,

$$\frac{c}{s+c}$$

and solving for $\dot{x}_w$,

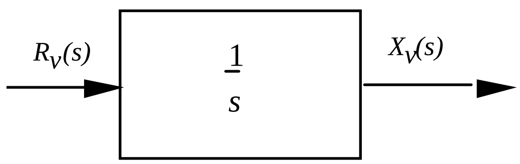
$$\dot{x}_w = -cx_w + cr_w(t)$$

The output equation is,

$$y_W = x_W$$

Vehicle:

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the block,

$$\frac{1}{s}$$

and solving for $\dot{x}_v$,

$$\dot{x}_v = r_v(t)$$

The output equation is

$$y_V = x_V$$

22.

$$\mathbf{A} = \begin{pmatrix} -1.702 & 50.72 & 263.38 \\ 0.22 & -1.418 & -31.99 \\ 0 & 0 & -14 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -272.06 \\ 0 \\ 14 \end{pmatrix}$$

For $G_1(s)$, $\mathbf{C_1} = (1, 0, 0)$, and

$$G_1(s) = \mathbf{C_1}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

Thus,

$$G_1(s) = \mathbf{C}_1\left(\frac{1}{s^3+17.12s^2+34.935s-122.43}\begin{bmatrix} s^2+15.418s+19.852 & 50.72s+710.08 & 263.38s-1249.1 \\ 0.22s+3.08 & s^2+15.702s+23.828 & -31.99s+3.4966 \\ 0 & 0 & s^2+3.12s-8.745 \end{bmatrix}\right)\mathbf{B}$$

Or

$$G_1(s) = \frac{-272.06s^2-507.3s-22888}{s^3+17.12s^2+34.935s-122.43} = \frac{-272.06(s^2+1.8647s+84.128)}{(s+14)(s-1.7834)(s+4.9034)}$$

For $G_2(s)$, $\mathbf{C_2} = (0,1,0)$, and

$$G_2(s) = \mathbf{C_2}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

Thus,

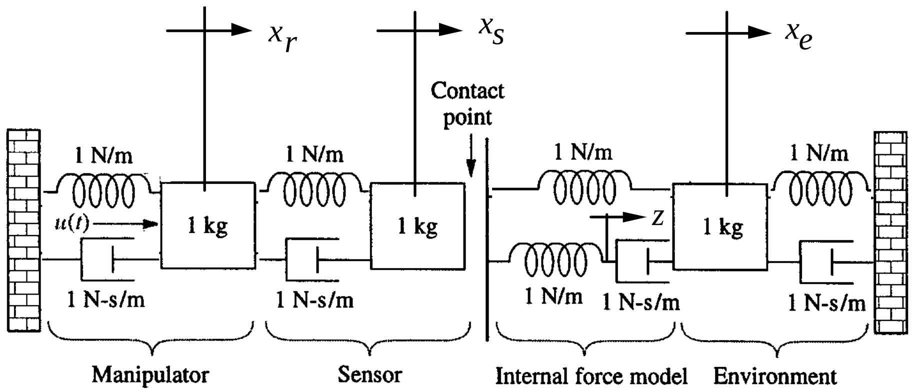
$$G_2(s) = \mathbf{C}_2 \left(\frac{1}{s^3+17.12s^2+34.935s-122.43} \begin{bmatrix} s^2+15.418s+19.852 & 50.72s+710.08 & 263.38s-1249.1 \\ 0.22s+3.08 & s^2+15.702s+23.828 & -31.99s+3.4966 \\ 0 & 0 & s^2+3.12s-8.745 \end{bmatrix} \right) \mathbf{B}$$

Or

$$G_2(s) = \frac{-507.71s-788.99}{s^3+17.12s^2+34.935s-122.43} = \frac{-507.71(s+1.554)}{(s+14)(s-1.7834)(s+4.9034)}$$

23.

Adding displacements to the figure,



Writing the differential equations for noncontact,

$$\frac{d^2x_r}{dt^2} + 2\frac{dx_r}{dt} + 2x_r - x_s - \frac{dx_s}{dt} = u(t)$$

$$-\frac{dx_r}{dt} - x_r + \frac{d^2x_s}{dt^2} + \frac{dx_s}{dt} + x_s = 0$$

Define the state variables as,

$$x_1 = x_r;\ x_2 = \dot{x}_r;\ x_3 = x_s;\ x_4 = \dot{x}_s$$

Writing the state equations, using the differential equations and the definition of the state variables, we get,

$$\dot{x}_1 = \dot{x}_r = x_2$$

$$\dot{x}_2 = \ddot{x}_r = -2x_1 - 2x_2 + x_3 + x_4 + u(t)$$

$$\dot{x}_3 = \dot{x}_s = x_4$$

$$\dot{x}_4 = \ddot{x}_s = x_1 + x_2 - x_3 - x_4$$

Assuming the output to be x_s, the output equation is,

$$y = x_3$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}u(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}$$

Writing the differential equations for contact,

$$\frac{d^2x_r}{dt^2} + 2\frac{dx_r}{dt} + 2x_r - x_s - \frac{dx_s}{dt} = u(t)$$

$$-\frac{dx_r}{dt} - x_r + \frac{d^2x_s}{dt^2} + \frac{dx_s}{dt} + x_s - z - x_e = 0$$

$$-x_s + \frac{dz}{dt} + z - \frac{dx_e}{dt} = 0$$

$$-x_s - \frac{dz}{dt} + \frac{d^2x_e}{dt^2} + 2\frac{dx_e}{dt} + 2x_e = 0$$

Defining the state variables,

$$x_1 = x_r;\ x_2 = \dot{x}_r;\ x_3 = x_s;\ x_4 = \dot{x}_s;\ x_5 = z;\ x_6 = \dot{z};\ x_7 = x_e;\ x_8 = \dot{x}_e$$

Using the differential equations and the definitions of the state variables, we write the state equations.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 2x_2 + x_3 + x_4 + u(t)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = x_1 + x_2 - x_3 - x_4 + x_5 + x_7$$

$$\dot{x}_5 = x_6$$

Differentiating the third differential equation and solving for d^2z/dt^2 we obtain,

$$\dot{x}_6 = \frac{d^2z}{dt^2} = \frac{dx_s}{dt} - \frac{dz}{dt} + \frac{d^2x_e}{dt^2}$$

But, from the fourth differential equation,

$$\frac{d^2x_e}{dt^2} = x_s + \frac{dz}{dt} - 2\frac{dx_e}{dt} - 2x_e = x_3 + x_6 - 2x_8 - 2x_7$$

Substituting this expression back into $\dot{x}_6$ along with the other definitions and then simplifying yields,

$$\dot{x}_6 = x_4 + x_3 - 2x_8 - 2x_7$$

Continuing,

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = x_3 + x_6 - 2x_7 - 2x_8$$

Assuming the output is x_s,

$$y = x_s$$

Hence, the solution in vector-matrix form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

24.

Writing the equations of motion,

$$M_f \frac{d^2 y_f}{dt^2} + (f_{vf} + f_{vh}) \frac{dy_f}{dt} + K_h y_f - f_{vh} \frac{dy_h}{dt} - K_h y_h = f_{up}(t)$$

$$-f_{vh} \frac{dy_f}{dt} - K_h y_f + M_h \frac{d^2 y_h}{dt^2} + f_{vh} \frac{dy_h}{dt} + (K_h + K_s) y_h - K_s y_{cat} = 0$$

$$-K_s y_h + (K_s + K_{ave}) y_{cat} = 0$$

The last equation says that

$$y_{cat} = \frac{K_s}{(K_s + K_{ave})} y_h$$

Defining state variables for the first two equations of motion,

$$x_1 = y_h;\ x_2 = \dot{y}_h;\ x_3 = y_f;\ x_4 = \dot{y}_f$$

Solving for the highest derivative terms in the first two equations of motion yields,

$$\frac{d^2y_f}{dt^2} = -\frac{(f_{vf} + f_{vh})}{M_f}\frac{dy_f}{dt} - \frac{K_h}{M_f}y_f + \frac{f_{vh}}{M_f}\frac{dy_h}{dt} + \frac{K_h}{M_f}y_h + \frac{1}{M_f}f_{up}(t)$$

$$\frac{d^2y_h}{dt^2} = \frac{f_{vh}}{M_h}\frac{dy_f}{dt} + \frac{K_h}{M_h}y_f - \frac{f_{vh}}{M_h}\frac{dy_h}{dt} - \frac{(K_h + K_s)}{M_h}y_h + \frac{K_s}{M_h}y_{cat}$$

Writing the state equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{f_{vh}}{M_h}x_4 + \frac{K_h}{M_h}x_3 - \frac{f_{vh}}{M_h}x_2 - \frac{(K_h + K_s)}{M_h}x_1 + \frac{K_s}{M_h}\frac{K_s}{(K_s + K_{ave})}x_1$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{(f_{vf} + f_{vh})}{M_f}x_4 - \frac{K_h}{M_f}x_3 + \frac{f_{vh}}{M_f}x_2 + \frac{K_h}{M_f}x_1 + \frac{1}{M_f}f_{up}(t)$$

The output is y_h - y_{cat}. Therefore,

$$y = y_h - y_{cat} = y_h - \frac{K_s}{(K_s + K_{ave})}y_h = \frac{K_{ave}}{(K_s + K_{ave})}x_1$$

Simplifying, rearranging, and putting the state equations in vector-matrix form yields,

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{M_h}\left(\frac{K_s^2}{(K_s + K_{ave})} - (K_h + K_s)\right) & -\frac{f_{vh}}{M_h} & \frac{K_h}{M_h} & \frac{f_{vh}}{M_h} \\ 0 & 0 & 0 & 1 \\ \frac{K_h}{M_f} & \frac{f_{vh}}{M_f} & -\frac{K_h}{M_f} & -\frac{(f_{vf} + f_{vh})}{M_f} \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_f} \end{bmatrix} f_{up}(t)$$

$$y = \begin{bmatrix} \frac{K_{ave}}{(K_s + K_{ave})} & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Substituting numerical values,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9353 & -14.29 & 769.2 & 14.29 \\ 0 & 0 & 0 & 1 \\ 406 & 7.558 & -406 & -9.302 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0581 \end{bmatrix} f_{up}(t)$$

$$y = \begin{bmatrix} 0.9491 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}$$