

F O U R

Time Response

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Open-Loop Response

The forward transfer function for angular velocity is,

$$G(s) = \frac{\omega_0(s)}{V_p(s)} = \frac{24}{(s+150)(s+1.32)}$$

a. $\omega_0(t) = A + Be^{-150t} + Ce^{-1.32t}$

b. $G(s) = \frac{24}{s^2 + 151.32s + 198}$. Therefore, $2\zeta\omega_n = 151.32$, $\omega_n = 14.07$, and $\zeta = 5.38$.

c. $\omega_0(s) = \frac{24}{s(s^2 + 151.32s + 198)} =$

$$\frac{24}{s(s+150)(s+1.32)} = 0.12121 \frac{1}{s} + 0.0010761 \frac{1}{s+150} - 0.12229 \frac{1}{s+1.32}$$

Therefore, $\omega_0(t) = 0.12121 + .0010761 e^{-150t} - 0.12229e^{-1.32t}$.

d. Using $G(s)$,

$$\ddot{\omega}_0 + 151.32 \dot{\omega}_0 + 198\omega_0 = 24v_p(t)$$

Defining,

$$x_1 = \omega_0$$

$$x_2 = \dot{\omega}_0$$

Thus, the state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -198x_1 - 151.32x_2 + 24v_p(t)$$

$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -198 & -151.32 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 24 \end{bmatrix} v_p(t); y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

e.

Program:

```
'Case Study 1 Challenge (e)'  
num=24;  
den=poly([-150 -1.32]);  
G=tf(num,den)  
step(G)
```

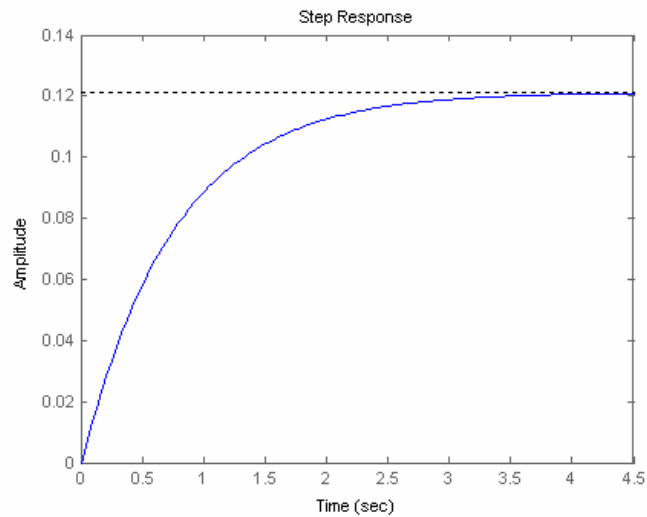
Computer response:

ans =

Case Study 1 Challenge (e)

Transfer function:

```
24  
-----  
s^2 + 151.3 s + 198
```



Ship at Sea: Open-Loop Response

a. Assuming a second-order approximation: $\omega_n^2 = 2.25$, $2\zeta\omega_n = 0.5$. Therefore $\zeta = 0.167$, $\omega_n = 1.5$.

$$T_s = \frac{4}{\zeta\omega_n} = 16; T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2.12;$$

$$\%OS = e^{-\zeta\pi} / \sqrt{1-\zeta^2} \times 100 = 58.8\%; \omega_n T_r = 1.169 \text{ therefore, } T_r = 0.77.$$

$$\begin{aligned} \text{b. } \theta(s) &= \frac{2.25}{s(s^2 + 0.5s + 2.25)} = \frac{1}{s} - \frac{s + 0.5}{s^2 + 0.5s + 2.25} \\ &= \frac{1}{s} - \frac{(s + 0.25) + \frac{0.25}{\sqrt{2.1875}}}{(s + 0.25)^2 + 2.1875} \end{aligned}$$

$$= \frac{1}{s} - \frac{(s + 0.25) + 0.16903 \cdot 1.479}{(s + 0.25)^2 + 2.1875}$$

Taking the inverse Laplace transform,

$$\theta(t) = 1 - e^{-0.25t} (\cos 1.479t + 0.16903 \sin 1.479t)$$

c.

Program:

```
'Case Study 2 Challenge (C)'
'(a)'
numg=2.25;
deng=[1 0.5 2.25];
G=tf(numg,deng)
omegan=sqrt(deng(3))
zeta=deng(2)/(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt(1-zeta^2))
pos=exp(-zeta*pi/sqrt(1-zeta^2))*100
t=0:.1:2;
[y,t]=step(G,t);
Tlow=interp1(y,t,.1);
Thi=interp1(y,t,.9);
Tr=Thi-Tlow
'(b)'
numc=2.25*[1 2];
denc=conv(poly([0 -3.57]),[1 2 2.25]);
[K,p,k]=residue(numc,denc)
'(c)'
[y,t]=step(G);
plot(t,y)
title('Roll Angle Response')
xlabel('Time(seconds)')
ylabel('Roll Angle(radians)')
```

Computer response:

ans =

Case Study 2 Challenge (C)

ans =

(a)

Transfer function:

```
      2.25
-----
s^2 + 0.5 s + 2.25
```

omegan =

1.5000

zeta =

0.1667

Ts =

16

$T_p =$

2.1241

$\rho s =$

58.8001

$T_r =$

0.7801

$\text{ans} =$

(b)

$K =$

0.1260
 $-0.3431 + 0.1058i$
 $-0.3431 - 0.1058i$
 0.5602

$p =$

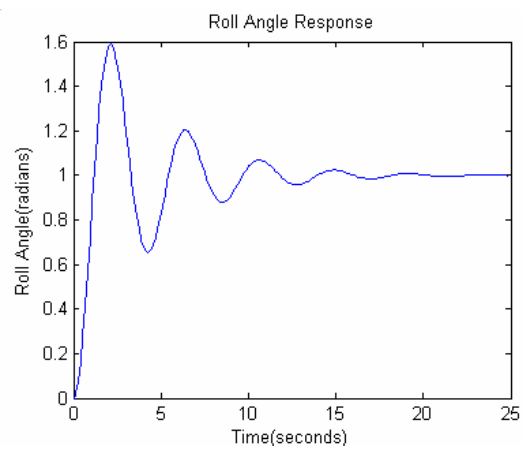
-3.5700
 $-1.0000 + 1.1180i$
 $-1.0000 - 1.1180i$
 0

$k =$

[]

$\text{ans} =$

(c)



ANSWERS TO REVIEW QUESTIONS

1. Time constant
2. The time for the step response to reach 67% of its final value
3. The input pole
4. The system poles
5. The radian frequency of a sinusoidal response
6. The time constant of an exponential response
7. Natural frequency is the frequency of the system with all damping removed; the damped frequency of oscillation is the frequency of oscillation with damping in the system.
8. Their damped frequency of oscillation will be the same.
9. They will all exist under the same exponential decay envelop.
10. They will all have the same percent overshoot and the same shape although differently scaled in time.
11. ζ , ω_n , T_P , %OS, T_S
12. Only two since a second-order system is completely defined by two component parameters
13. (1) Complex, (2) Real, (3) Multiple real
14. Pole's real part is large compared to the dominant poles, (2) Pole is near a zero
15. If the residue at that pole is much smaller than the residues at other poles
16. No; one must then use the output equation
17. The Laplace transform of the state transition matrix is $(s\mathbf{I} - \mathbf{A})^{-1}$
18. Computer simulation
19. Pole-zero concepts give one an intuitive feel for the problem.
20. State equations, output equations, and initial value for the state-vector
21. $\text{Det}(s\mathbf{I} - \mathbf{A}) = 0$

SOLUTIONS TO PROBLEMS

1.
 - a. Overdamped Case:

$$C(s) = \frac{9}{s(s^2 + 9s + 9)}$$

Expanding into partial fractions,

$$C(s) = \frac{9}{s(s + 7.854)(s + 1.146)} = \frac{1}{s} + \frac{0.171}{(s + 7.854)} - \frac{1.171}{(s + 1.146)}$$

Taking the inverse Laplace transform,

$$c(t) = 1 + 0.171 e^{-7.854t} - 1.171 e^{-1.146t}$$

b. Underdamped Case:

$$C(s) = \frac{9}{s(s^2 + 3s + 9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2 + 3s + 9)}$$

$$K_1 = \left. \frac{9}{(s^2 + 3s + 9)} \right|_{s \rightarrow 0} = 1$$

K_2 and K_3 can be found by clearing fractions with K_1 replaced by its value. Thus,

$$9 = (s^2 + 3s + 9) + (K_2 s + K_3)s$$

or

$$9 = s^2 + 3s + 9 + K_2 s^2 + K_3 s$$

Hence $K_2 = -1$ and $K_3 = -3$. Thus,

$$C(s) = \frac{1}{s} - \frac{s + 3}{(s^2 + 3s + 9)}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{2}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{\sqrt{27}} \sqrt{\frac{27}{4}}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$c(t) = 1 - e^{-\frac{3}{2}t} \cos \sqrt{\frac{27}{4}} t - \frac{3}{\sqrt{27}} e^{-\frac{3}{2}t} \sin \sqrt{\frac{27}{4}} t$$

$$c(t) = 1 - \frac{2}{\sqrt{3}} e^{-3t/2} \cos(\sqrt{\frac{27}{4}} t - \phi)$$

$$= 1 - 1.155 e^{-1.5t} \cos(2.598t - \phi)$$

where

$$\phi = \arctan\left(\frac{3}{\sqrt{27}}\right) = 30^\circ$$

c. Oscillatory Case:

$$C(s) = \frac{9}{s(s^2 + 9)}$$

$$C(s) = \frac{9}{s(s^2 + 9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2 + 9)}$$

The evaluation of the constants in the numerator are found the same way as they were for the underdamped case. The results are $K_2 = -1$ and $K_3 = 0$. Hence,

$$C(s) = \frac{1}{s} - \frac{s}{(s^2 + 9)}$$

Therefore,

$$c(t) = 1 - \cos 3t$$

d. Critically Damped

$$C(s) = \frac{9}{s(s^2 + 6s + 9)}$$

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{K_1}{s} + \frac{K_2}{(s + 3)^2} + \frac{K_3}{(s + 3)}$$

The constants are then evaluated as

$$K_1 = \left. \frac{9}{(s^2 + 6s + 9)} \right|_{s \rightarrow 0} = 1; \quad K_2 = \left. \frac{9}{s} \right|_{s \rightarrow -3} = -3; \quad K_3 = \left. \frac{d}{ds} \left(\frac{9}{s} \right) \right|_{s \rightarrow -3} = -1$$

Now, the transform of the response is

$$C(s) = \frac{9}{s(s^2 + 6s + 9)} = \frac{1}{s} - \frac{3}{(s + 3)^2} - \frac{1}{(s + 3)}$$

$$c(t) = 1 - 3t e^{-3t} - e^{-3t}$$

2.

a. $C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5}$. Therefore, $c(t) = 1 - e^{-5t}$.

Also, $T = \frac{1}{5}$, $T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44$, $T_s = \frac{4}{a} = \frac{4}{5} = 0.8$.

b. $C(s) = \frac{20}{s(s+20)} = \frac{1}{s} - \frac{1}{s+20}$. Therefore, $c(t) = 1 - e^{-20t}$. Also, $T = \frac{1}{20}$,

$T_r = \frac{2.2}{a} = \frac{2.2}{20} = 0.11$, $T_s = \frac{4}{a} = \frac{4}{20} = 0.2$.

3.

Program:

```

'(a)'
num=5;
den=[1 5];
Ga=tf(num,den)
subplot(1,2,1)
step(Ga)
title('(a)')
'(b)'
num=20;
den=[1 20];
Gb=tf(num,den)
subplot(1,2,2)
step(Gb)
title('(b)')

```

Computer response:

```
ans =
```

```
(a)
```

```
Transfer function:
```

```
5
```

```
-----
s + 5
```

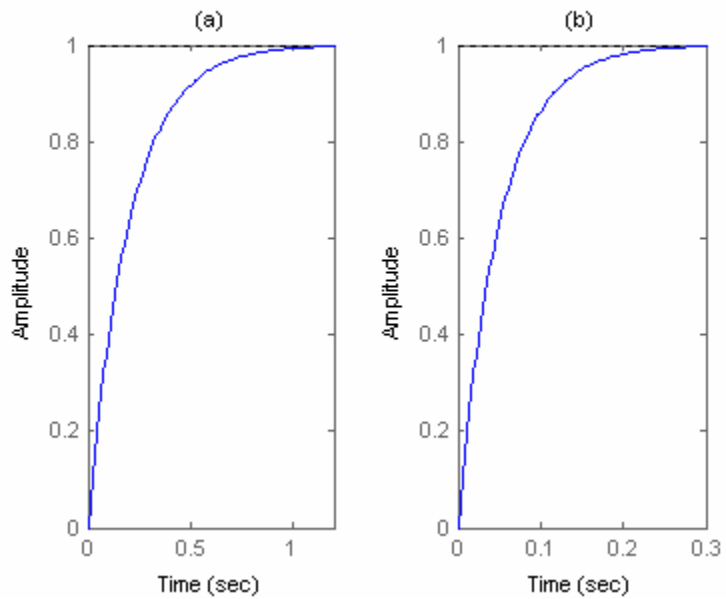
```
ans =
```

```
(b)
```

```
Transfer function:
```

```
20
```

```
-----
s + 20
```



4.

Using voltage division, $\frac{V_C(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{(R + \frac{1}{Cs})} = \frac{2}{(s+2)}$. Since $V_i(s) = \frac{5}{s}$, $V_C(s) = \frac{10}{s(s+2)} = \frac{5}{s} - \frac{5}{s+2}$.

Therefore $v_C(t) = 5 - 5e^{-2t}$. Also, $T = \frac{1}{2}$, $T_r = \frac{2.2}{a} = \frac{2.2}{2} = 1.1$, $T_s = \frac{4}{a} = \frac{4}{2} = 2$.

5.

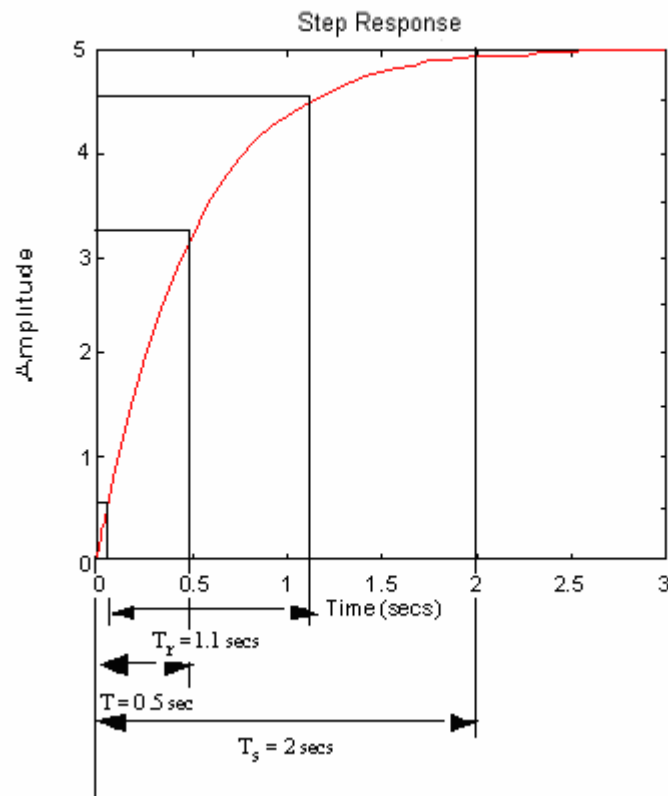
Program:

```
clf
num=2;
den=[1 2];
G=tf(num,den)
step(5*G)
```

Computer response:

Transfer function:

```
2
-----
s + 2
```



6.

Writing the equation of motion,

$$(Ms^2 + 8s)X(s) = F(s)$$

Thus, the transfer function is,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + 8s}$$

Differentiating to yield the transfer function in terms of velocity,

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms + 8} = \frac{1/M}{s + \frac{8}{M}}$$

Thus, the settling time, T_s , and the rise time, T_r , are given by

$$T_s = \frac{4}{8/M} = \frac{1}{2}M; \quad T_r = \frac{2.2}{8/M} = 0.275M$$

7.

Program:

```
Clf
M=1
num=1/M;
den=[1 8/M];
G=tf(num,den)
step(G)
pause
M=2
num=1/M;
den=[1 8/M];
G=tf(num,den)
step(G)
```

Computer response:

Transfer function:
M =

1

Transfer function:

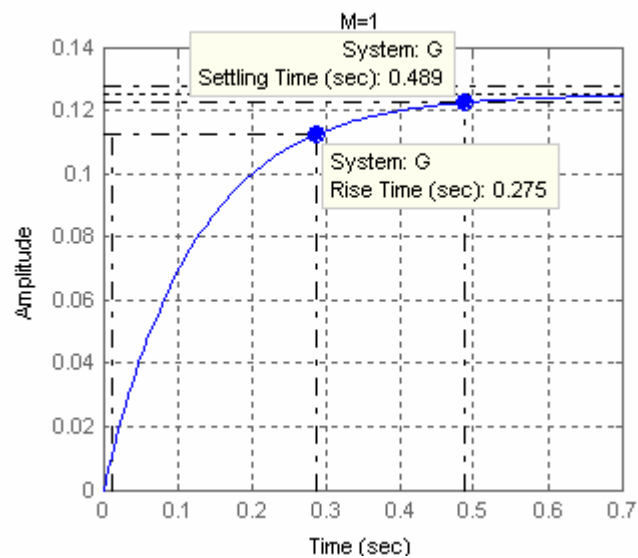
```
1
-----
s + 8
```

M =

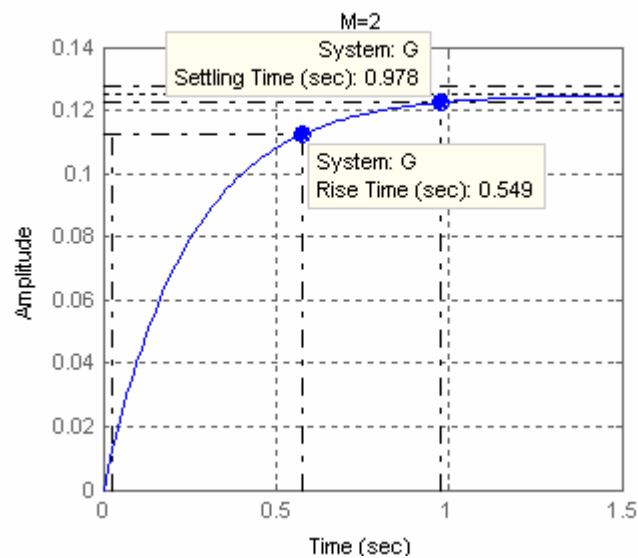
2

Transfer function:

```
0.5
-----
s + 4
```



From plot, time constant = 0.125 s.



From plot, time constant = 0.25 s.

8.

a. Pole: -2; $c(t) = A + Be^{-2t}$; first-order response.

b. Poles: -3, -6; $c(t) = A + Be^{-3t} + Ce^{-6t}$; overdamped response.

c. Poles: -10, -20; Zero: -7; $c(t) = A + Be^{-10t} + Ce^{-20t}$; overdamped response.

d. Poles: $(-3+j3\sqrt{15})$, $(-3-j3\sqrt{15})$; $c(t) = A + Be^{-3t} \cos(3\sqrt{15}t + \phi)$; underdamped.

e. Poles: $j3$, $-j3$; Zero: -2 ; $c(t) = A + B \cos(3t + \phi)$; undamped.

f. Poles: -10 , -10 ; Zero: -5 ; $c(t) = A + Be^{-10t} + Cte^{-10t}$; critically damped.

9.

Program:

```
p=roots([1 6 4 7 2])
```

Computer response:

```
p =
    -5.4917
   -0.0955 + 1.0671i
   -0.0955 - 1.0671i
    -0.3173
```

10.

$$G(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}$$

$$\mathbf{A} = \begin{bmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}; \mathbf{C} = [2 \quad 8 \quad -3]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3 - s^2 - 91s + 67} \begin{bmatrix} (s^2 + 7s - 18) & -(4s + 29) & (s - 2) \\ -(3s + 27) & (s^2 + s - 77) & -3 \\ 5s - 31 & 7s - 76 & (s^2 - 10s + 4) \end{bmatrix}$$

$$\text{Therefore, } G(s) = \frac{5s^2 + 136s - 1777}{s^3 - s^2 - 91s + 67}.$$

Factoring the denominator, or using $\det(s\mathbf{I} - \mathbf{A})$, we find the poles to be 9.683, 0.7347, -9.4179.

11.

Program:

```
A=[8 -4 1;-3 2 0;5 7 -9]
B=[1;3;7]
C=[2 8 -3]
D=0
[numg,deng]=ss2tf(A,B,C,D,1);
G=tf(numg,deng)
poles=roots(deng)
```

Computer response:

```
A =
     8     -4     1
    -3      2      0
     5      7     -9
B =
     1
```

```

      3
      7
C =   2      8      -3
D =   0

Transfer function:
5 s^2 + 136 s - 1777
-----
s^3 - s^2 - 91 s + 67

poles =

-9.4179
 9.6832
 0.7347

```

12.

Writing the node equation at the capacitor, $V_C(s) \left(\frac{1}{R_2} + \frac{1}{Ls} + Cs \right) + \frac{V_C(s) - V(s)}{R_1} = 0$.

Hence, $\frac{V_C(s)}{V(s)} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{Ls} + Cs} = \frac{10s}{s^2 + 20s + 500}$. The step response is $\frac{10}{s^2 + 20s + 500}$. The poles

are at

$-10 \pm j20$. Therefore, $v_C(t) = Ae^{-10t} \cos(20t + \phi)$.

13.

Program:

```

num=[10 0];
den=[1 20 500];
G=tf(num,den)
step(G)

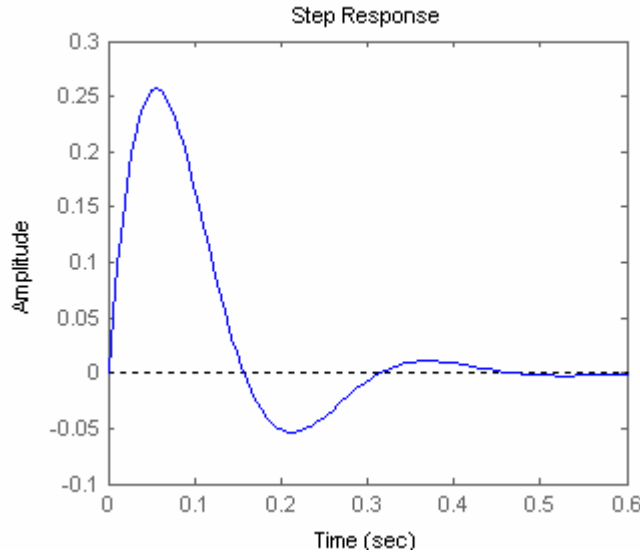
```

Computer response:

```

Transfer function:
10 s
-----
s^2 + 20 s + 500

```



14.

The equation of motion is: $(Ms^2 + f_v s + K_s)X(s) = F(s)$. Hence, $\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K_s} = \frac{1}{s^2 + s + 5}$.

The step response is now evaluated: $X(s) = \frac{1}{s(s^2 + s + 5)} = \frac{1/5}{s} - \frac{\frac{1}{5}s + \frac{1}{5}}{(s + \frac{1}{2})^2 + \frac{19}{4}} =$

$$\frac{\frac{1}{5}(s + \frac{1}{2}) + \frac{1}{5\sqrt{19}} \frac{\sqrt{19}}{2}}{(s + \frac{1}{2})^2 + \frac{19}{4}}.$$

Taking the inverse Laplace transform, $x(t) = \frac{1}{5} - \frac{1}{5} e^{-0.5t} \left(\cos \frac{\sqrt{19}}{2} t + \frac{1}{\sqrt{19}} \sin \frac{\sqrt{19}}{2} t \right)$

$$= \frac{1}{5} \left[1 - 2\sqrt{\frac{5}{19}} e^{-0.5t} \cos \left(\frac{\sqrt{19}}{2} t - 12.92^\circ \right) \right].$$

15.

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + \zeta\omega_n) + \zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2} = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}} \omega_n\sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2}$$

$$\text{Hence, } c(t) = 1 - e^{-\zeta\omega_n t} \left(\cos \omega_n\sqrt{1 - \zeta^2} t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_n\sqrt{1 - \zeta^2} t \right)$$

$$= 1 - e^{-\zeta \omega_n t} \sqrt{1 + \frac{\zeta^2}{1 - \zeta^2}} \cos(\omega_n \sqrt{1 - \zeta^2} t - \phi) = 1 - e^{-\zeta \omega_n t} \frac{1}{\sqrt{1 - \zeta^2}} \cos(\omega_n \sqrt{1 - \zeta^2} t - \phi),$$

$$\text{where } \phi = \tan^{-1} \frac{\zeta}{\sqrt{1 - \zeta^2}}$$

16.

$$\%OS = e^{-\zeta \pi} / \sqrt{1 - \zeta^2} \times 100. \text{ Dividing by 100 and taking the natural log of both sides,}$$

$$\ln\left(\frac{\%OS}{100}\right) = -\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}. \text{ Squaring both sides and solving for } \zeta^2, \zeta^2 = \frac{\ln^2\left(\frac{\%OS}{100}\right)}{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}. \text{ Taking the}$$

$$\text{negative square root, } \zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}.$$

17.

a.

$$C(s) = \frac{2}{s(s+2)}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+2}$$

$$c(t) = 1 - e^{-2t}$$

b.

$$C(s) = \frac{5}{s(s+3)(s+6)}$$

$$C(s) = \frac{5}{18} \frac{1}{s} - \frac{5}{9} \frac{1}{s+3} + \frac{5}{18} \frac{1}{s+6}$$

$$c(t) = \frac{5}{18} - \frac{5}{9} e^{-3t} + \frac{5}{18} e^{-6t}$$

c.

$$C(s) = \frac{10(s+7)}{s(s+10)(s+20)}$$

$$C(s) = \frac{7}{20} \frac{1}{s} + \frac{3}{10} \frac{1}{s+10} - \frac{13}{20} \frac{1}{s+20}$$

$$c(t) = \frac{7}{20} + \frac{3}{10} e^{-10t} - \frac{13}{20} e^{-20t}$$

d.

$$C(s) = \frac{20}{s(s^2 + 6s + 144)}$$

$$C(s) = \frac{5}{36} \frac{1}{s} - \frac{5}{36} \frac{s+6}{s^2 + 6s + 144}$$

$$C(s) = \frac{5}{36} \frac{1}{s} - \frac{5}{36} \frac{(s+3) + \frac{3}{\sqrt{135}} \sqrt{135}}{(s+3)^2 + 135}$$

$$c(t) = \frac{5}{36} - \frac{5}{36} e^{-3t} \left(\cos[\sqrt{135}]t + \frac{3}{\sqrt{135}} \sin[\sqrt{135}]t \right)$$

e.

$$C(s) = \frac{s+2}{s(s^2+9)}$$

$$C(s) = \frac{2}{9} \frac{1}{s} + \frac{1}{9} \frac{-2s+9}{s^2+9}$$

$$C(s) = \frac{2}{9} \frac{1}{s} + \frac{1}{9} \frac{-2s+3 \cdot 3}{s^2+9}$$

$$c(t) = \frac{2}{9} - \left(\frac{2}{9} \cos 3t - \frac{1}{3} \sin 3t \right)$$

f.

$$C(s) = \frac{s+5}{s(s+10)^2}$$

$$C(s) = \frac{1}{20} \frac{1}{s} - \frac{1}{20} \frac{1}{s+10} + \frac{1}{2} \frac{1}{(s+10)^2}$$

$$c(t) = \frac{1}{20} - \frac{1}{20} e^{-10t} + \frac{1}{2} t e^{-10t}$$

18.

a. N/A

b. $s^2+9s+18$, $\omega_n^2 = 18$, $2\zeta\omega_n = 9$, Therefore $\zeta = 1.06$, $\omega_n = 4.24$, overdamped.c. $s^2+30s+200$, $\omega_n^2 = 200$, $2\zeta\omega_n = 30$, Therefore $\zeta = 1.06$, $\omega_n = 14.14$, overdamped.d. $s^2+6s+144$, $\omega_n^2 = 144$, $2\zeta\omega_n = 6$, Therefore $\zeta = 0.25$, $\omega_n = 12$, underdamped.e. s^2+9 , $\omega_n^2 = 9$, $2\zeta\omega_n = 0$, Therefore $\zeta = 0$, $\omega_n = 3$, undamped.f. $s^2+20s+100$, $\omega_n^2 = 100$, $2\zeta\omega_n = 20$, Therefore $\zeta = 1$, $\omega_n = 10$, critically damped.

19.

$$X(s) = \frac{100^2}{s(s^2+100s+100^2)} = \frac{1}{s} - \frac{s+100}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50)+50}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50) + \frac{50}{\sqrt{7500}}\sqrt{7500}}{(s+50)^2+7500}$$

$$\text{Therefore, } x(t) = 1 - e^{-50t} \left(\cos \sqrt{7500} t + \frac{50}{\sqrt{7500}} \sin \sqrt{7500} t \right)$$

$$= 1 - \frac{2}{\sqrt{3}} e^{-50t} \cos \left(50\sqrt{3} t - \tan^{-1} \frac{1}{\sqrt{3}} \right)$$

20.

a. $\omega_n^2 = 16$ r/s, $2\zeta\omega_n = 3$. Therefore $\zeta = 0.375$, $\omega_n = 4$. $T_s = \frac{4}{\zeta\omega_n} = 2.667$ s; $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$

$$0.8472 \text{ s; } \%OS = e^{-\zeta\pi} / \sqrt{1-\zeta^2} \times 100 = 28.06 \% ; \omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1) = 1.4238;$$

therefore, $T_r = 0.356$ s.

b. $\omega_n^2 = 0.04$ r/s, $2\zeta\omega_n = 0.02$. Therefore $\zeta = 0.05$, $\omega_n = 0.2$. $T_s = \frac{4}{\zeta\omega_n} = 400$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$

15.73 s; %OS = $e^{-\zeta\pi} / \sqrt{1-\zeta^2} \times 100 = 85.45$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore,

$T_r = 5.26$ s.

c. $\omega_n^2 = 1.05 \times 10^7$ r/s, $2\zeta\omega_n = 1.6 \times 10^3$. Therefore $\zeta = 0.247$, $\omega_n = 3240$. $T_s = \frac{4}{\zeta\omega_n} = 0.005$ s; $T_P =$

$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001$ s; %OS = $e^{-\zeta\pi} / \sqrt{1-\zeta^2} \times 100 = 44.92$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta +$

1); therefore, $T_r = 3.88 \times 10^{-4}$ s.

21.

Program:

```
'(a)'  
clf  
numa=16;  
dena=[1 3 16];  
Ta=tf(numa,dena)  
omegana=sqrt(dena(3))  
zetaa=dena(2)/(2*omegana)  
Tsa=4/(zetaa*omegana)  
Tpa=pi/(omegana*sqrt(1-zetaa^2))  
Tra=(1.76*zetaa^3 - 0.417*zetaa^2 + 1.039*zetaa + 1)/omegana  
percenta=exp(-zetaa*pi/sqrt(1-zetaa^2))*100  
subplot(221)  
step(Ta)  
title('(a)')  
'(b)'  
numb=0.04;  
denb=[1 0.02 0.04];  
Tb=tf(numb,denb)  
omeganb=sqrt(denb(3))  
zetab=denb(2)/(2*omeganb)  
Tsb=4/(zetab*omeganb)  
Tpb=pi/(omeganb*sqrt(1-zetab^2))  
Trb=(1.76*zetab^3 - 0.417*zetab^2 + 1.039*zetab + 1)/omeganb  
percentb=exp(-zetab*pi/sqrt(1-zetab^2))*100  
subplot(222)  
step(Tb)  
title('(b)')  
'(c)'  
numc=1.05E7;  
denc=[1 1.6E3 1.05E7];  
Tc=tf(numc,denc)  
omeganc=sqrt(denc(3))  
zetac=denc(2)/(2*omeganc)  
Tsc=4/(zetac*omeganc)  
Tpc=pi/(omeganc*sqrt(1-zetac^2))  
Trc=(1.76*zetac^3 - 0.417*zetac^2 + 1.039*zetac + 1)/omeganc  
percentc=exp(-zetac*pi/sqrt(1-zetac^2))*100  
subplot(223)  
step(Tc)  
title('(c)')
```

Computer response:

ans =

(a)

Transfer function:

$$\frac{16}{s^2 + 3s + 16}$$

omegana =

4

zetaa =

0.3750

Tsa =

2.6667

Tpa =

0.8472

Tra =

0.3559

percenta =

28.0597

ans =

(b)

Transfer function:

$$\frac{0.04}{s^2 + 0.02s + 0.04}$$

omeganb =

0.2000

zetab =

0.0500

Tsb =

400

Tpb =

15.7276

Trb =

5.2556

percentb =

85.4468

ans =

(c)

Transfer function:

1.05e007

s^2 + 1600 s + 1.05e007

omeganc =

3.2404e+003

zetac =

0.2469

Tsc =

0.0050

Tpc =

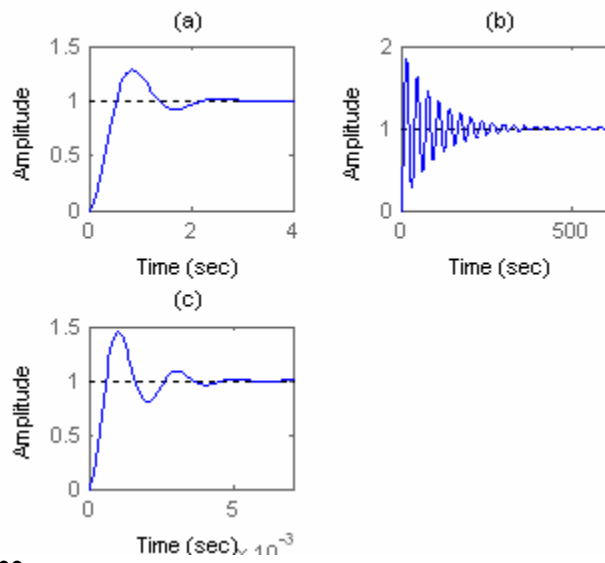
0.0010

Trc =

3.8810e-004

percentc =

44.9154



22.

Program:

```
T1=tf(16,[1 3 16])
T2=tf(0.04,[1 0.02 0.04])
T3=tf(1.05e7,[1 1.6e3 1.05e7])
ltiview
```

Computer response:

Transfer function:

16

s² + 3 s + 16

Transfer function:

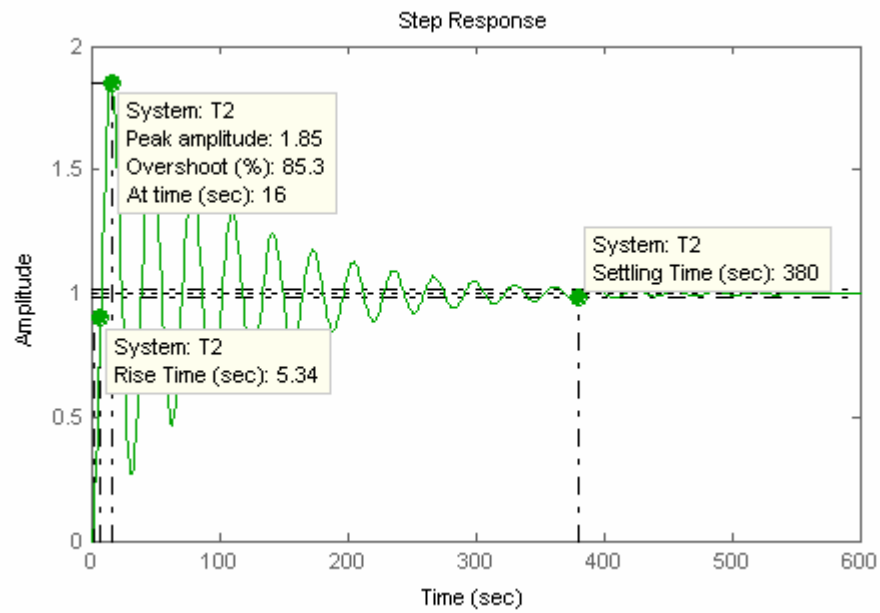
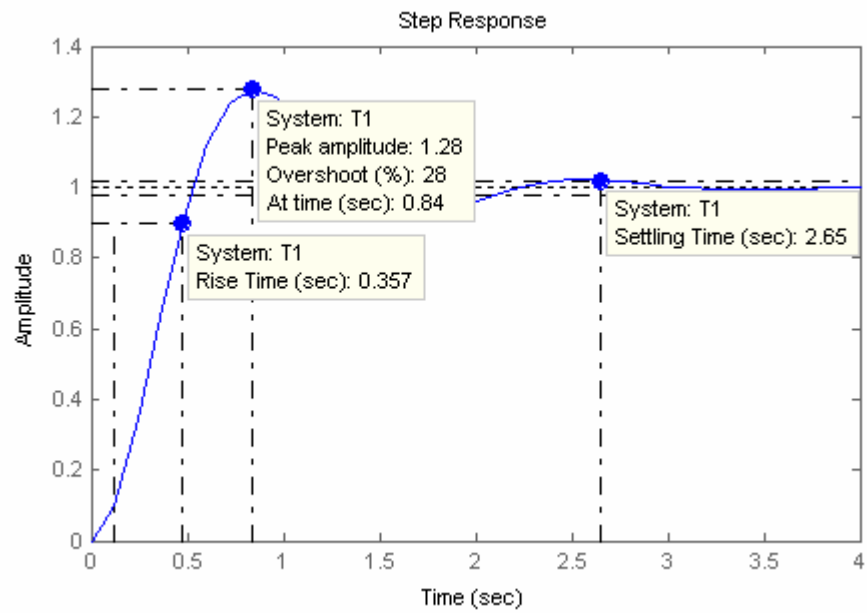
0.04

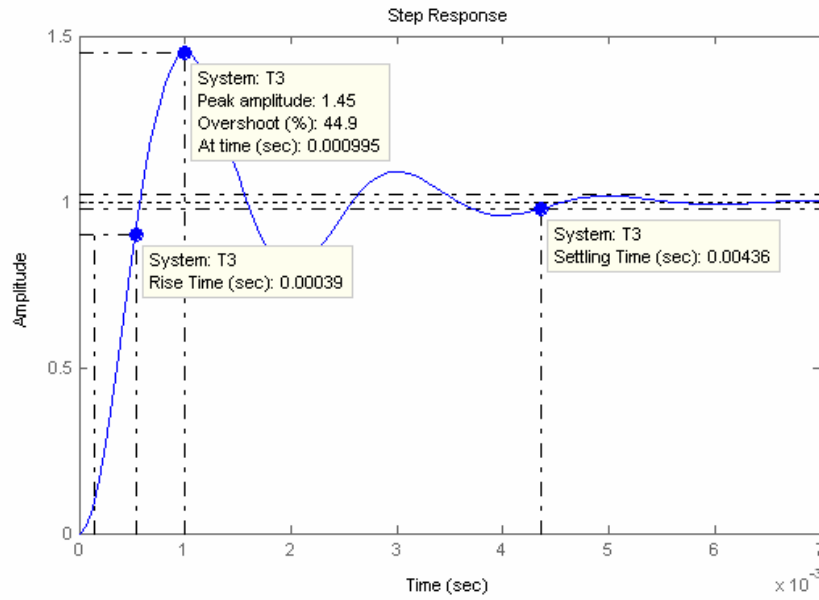
s² + 0.02 s + 0.04

Transfer function:

1.05e007

s² + 1600 s + 1.05e007





23.

$$\text{a. } \zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.56, \omega_n = \frac{4}{\zeta T_s} = 11.92. \text{ Therefore, poles} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -6.67 \pm j9.88.$$

$$\text{b. } \zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.591, \omega_n = \frac{\pi}{T_P\sqrt{1-\zeta^2}} = 0.779.$$

$$\text{Therefore, poles} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.4605 \pm j0.6283.$$

$$\text{c. } \zeta\omega_n = \frac{4}{T_s} = 0.571, \omega_n\sqrt{1-\zeta^2} = \frac{\pi}{T_p} = 1.047. \text{ Therefore, poles} = -0.571 \pm j1.047.$$

24.

$$\text{Re} = \frac{4}{T_s} = 4; \quad \zeta = \frac{-\ln(12.3/100)}{\sqrt{\pi^2 + \ln^2(12.3/100)}} = 0.5549$$

$$\text{Re} = \zeta\omega_n = 0.5549\omega_n = 4; \quad \therefore \omega_n = 7.21$$

$$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 6$$

$$\therefore G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{51.96}{s^2 + 8s + 51.96}$$

25.

$$\text{a. Writing the equation of motion yields, } (3s^2 + 15s + 33)X(s) = F(s)$$

Solving for the transfer function,

$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 5s + 11}$$

b. $\omega_n^2 = 11$ r/s, $2\zeta\omega_n = 5$. Therefore $\zeta = 0.754$, $\omega_n = 3.32$. $T_s = \frac{4}{\zeta\omega_n} = 1.6$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.44$ s; %OS = $e^{-\zeta\pi} / \sqrt{1-\zeta^2} \times 100 = 2.7$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 0.69$ s.

26.

Writing the loop equations,

$$(s^2 + s) \theta_1(s) - s \theta_2(s) = T(s)$$

$$-s \theta_1(s) + (s+1) \theta_2(s) = 0$$

Solving for $\theta_2(s)$,

$$\theta_2(s) = \frac{\begin{vmatrix} s^2 + s & T(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} s^2 + s & -s \\ -s & s+1 \end{vmatrix}} = \frac{T(s)}{s^2 + s + 1}$$

Forming the transfer function,

$$\frac{\theta_2(s)}{T(s)} = \frac{1}{s^2 + s + 1}$$

Thus $\omega_n = 1$, $2\zeta\omega_n = 1$. Thus, $\zeta = 0.5$. From Eq. (4.38), %OS = 16.3%. From Eq. (4.42), $T_s = 8$ seconds. From Eq. (4.34), $T_P = 3.63$ seconds.

27.

$$\text{a. } \frac{24.542}{s(s^2 + 4s + 24.542)} = \frac{1}{s} - \frac{s+4}{(s+2)^2 + 20.542} = \frac{1}{s} - \frac{(s+2) + \frac{2}{4.532} \cdot 4.532}{(s+2)^2 + 20.542}.$$

Thus $c(t) = 1 - e^{-2t} (\cos 4.532t + 0.441 \sin 4.532t) = 1 - 1.09e^{-2t} \cos(4.532t - 23.8^\circ)$.

b.

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029 \frac{1}{s+10} - \frac{0.70971s + 5.7418}{s^2 + 4s + 24.542}$$

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029 \frac{1}{s+10} - \frac{0.70971s + 5.7418}{(s+2)^2 + 20.542}$$

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029 \frac{1}{s+10} - \frac{0.70971(s+2) + \frac{4.3223}{\sqrt{20.542}} \sqrt{20.542}}{(s+2)^2 + 20.542}$$

$$\frac{245.42}{s(s+10)(s^2+4s+24.542)} = \frac{1}{s} - 0.29029 \frac{1}{s+10} - \frac{0.70971(s+2) + 0.95367\sqrt{20.542}}{(s+2)^2 + 20.542}$$

Therefore, $c(t) = 1 - 0.29e^{-10t} - e^{-2t}(0.71 \cos 4.532t + 0.954 \sin 4.532t)$

$$= 1 - 0.29e^{-10t} - 1.189 \cos(4.532t - 53.34^\circ).$$

c.

$$\begin{aligned} \frac{73.626}{s(s+3)(s^2+4s+24.542)} &= \frac{1}{s} - 1.1393 \frac{1}{s+3} + \frac{0.13926s - 2.8607}{s^2+4s+24.542} \\ &= \frac{1}{s} - 1.1393 \frac{1}{s+3} + \frac{0.13926(s+2) - \frac{3.1393}{\sqrt{20.542}}}{(s+2)^2 + 20.542} \\ \frac{73.626}{s(s+3)(s^2+4s+24.542)} &= \frac{1}{s} - 1.1393 \frac{1}{s+3} + \frac{0.13926(s+2) - 0.69264\sqrt{20.542}}{(s+2)^2 + 20.542} \end{aligned}$$

Therefore, $c(t) = 1 - 1.14e^{-3t} + e^{-2t}(0.14 \cos 4.532t - 0.69 \sin 4.532t)$

$$= 1 - 1.14e^{-3t} + 0.704 \cos(4.532t + 78.53^\circ).$$

28.

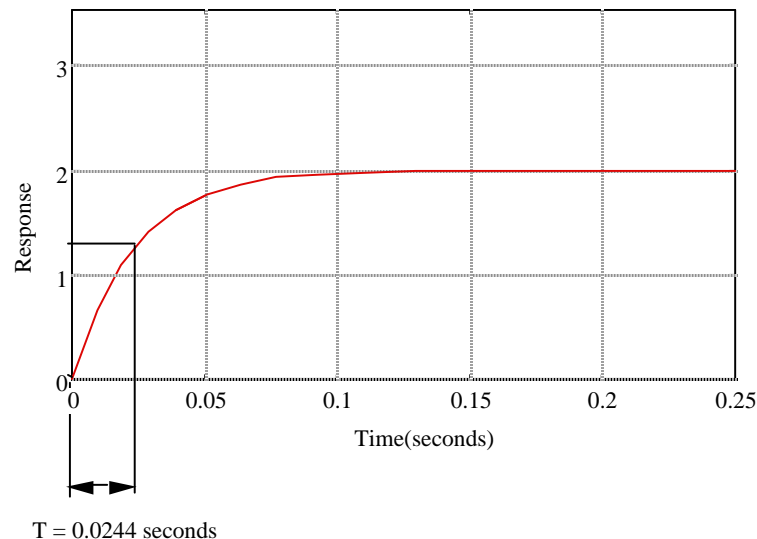
Since the third pole is more than five times the real part of the dominant pole, $s^2+1.204s+2.829$ determines the transient response. Since $2\zeta\omega_n = 1.204$, and $\omega_n = \sqrt{2.829} = \omega_n = 1.682$, $\zeta = 0.358$,

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 30\%, T_s = \frac{4}{\zeta\omega_n} = 6.64 \text{ sec}, T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2 \text{ sec}; \omega_n T_r = 1.4,$$

therefore, $T_r = 0.832$.

29.

a. Measuring the time constant from the graph, $T = 0.0244$ seconds.

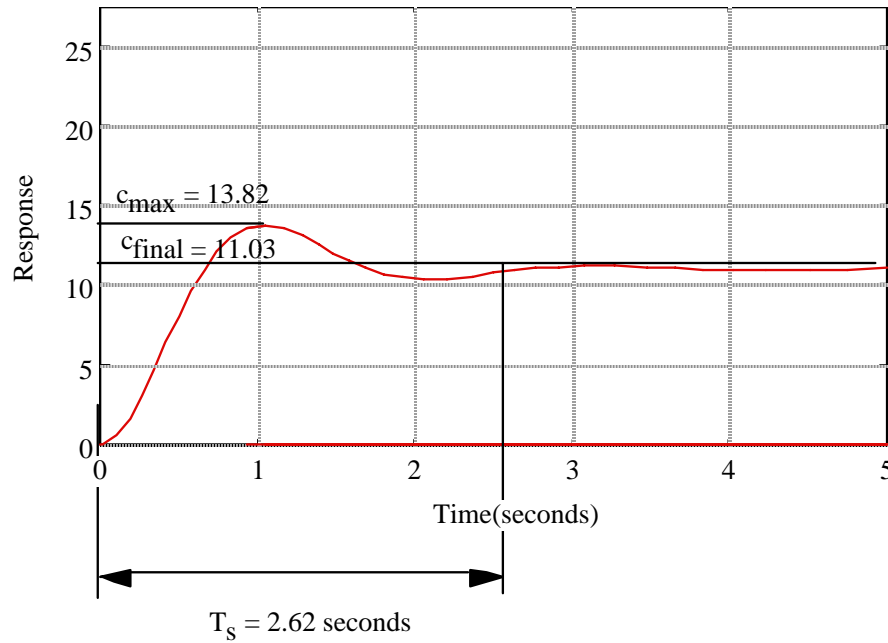


Estimating a first-order system, $G(s) = \frac{K}{s+a}$. But, $a = 1/T = 40.984$, and $\frac{K}{a} = 2$. Hence, $K = 81.967$.

Thus,

$$G(s) = \frac{81.967}{s+40.984}$$

b. Measuring the percent overshoot and settling time from the graph: $\%OS = (13.82-11.03)/11.03 = 25.3\%$,



and $T_s = 2.62$ seconds. Estimating a second-order system, we use Eq. (4.39) to find $\zeta = 0.4$, and Eq. (4.42) to find $\omega_n = 3.82$. Thus, $G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$. Since $C_{final} = 11.03$, $\frac{K}{\omega_n^2} = 11.03$. Hence,

$K = 160.95$. Substituting all values,

$$G(s) = \frac{160.95}{s^2 + 3.056s + 14.59}$$

c. From the graph, $\%OS = 40\%$. Using Eq. (4.39), $\zeta = 0.28$. Also from the graph,

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 4. \text{ Substituting } \zeta = 0.28, \text{ we find } \omega_n = 0.818.$$

Thus,

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.669}{s^2 + 0.458s + 0.669}.$$

30.

a.

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20} \frac{1}{s} - \frac{1}{16} \frac{1}{s+2} - \frac{1}{80} \frac{7s+31}{\left(s+\frac{3}{2}\right)^2 + \frac{31}{4}}$$

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20} \frac{1}{s} - \frac{1}{16} \frac{1}{s+2} - \frac{1}{80} \frac{7\left(s+\frac{3}{2}\right) + 7.3638\sqrt{\frac{31}{4}}}{\left(s+\frac{3}{2}\right)^2 + \frac{31}{4}}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -

2, pole-zero cancellation cannot be assumed.

b.

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16} \frac{1}{s} - \frac{1}{64} \frac{1}{s+2} - \frac{1}{64} \frac{3s+14}{s^2+4s+20}$$

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16} \frac{1}{s} - \frac{1}{64} \frac{1}{s+2} - \frac{1}{64} \frac{3(s+2) + 2\sqrt{16}}{(s+2)^2 + 16}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -

2, pole-zero cancellation cannot be assumed.

c.

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21 \frac{1}{s} - 0.0071429 \frac{1}{s+2} - \frac{0.20286s+0.21714}{s^2+s+5}$$

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21 \frac{1}{s} - 0.0071429 \frac{1}{s+2} - \frac{0.20286\left(s+\frac{1}{2}\right) + 0.053093\sqrt{\frac{19}{4}}}{\left(s+\frac{1}{2}\right)^2 + \frac{19}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole

at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 1$, and $\omega_n = \sqrt{5} = 2.236$, $\zeta = 0.224$,

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 48.64\%, T_s = \frac{4}{\zeta\omega_n} = 8 \text{ sec}, T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.44 \text{ sec}; \omega_n T_r = 1.23,$$

therefore, $T_r = 0.55$.

d.

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025 \frac{1}{s} - 0.00035714 \frac{1}{s+2} - \frac{0.049893s+0.25018}{\left(s+\frac{5}{2}\right)^2 + \frac{55}{4}}$$

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025 \frac{1}{s} - 0.00035714 \frac{1}{s+2} - \frac{0.049893\left(s+\frac{5}{2}\right) + 0.03383\sqrt{\frac{55}{4}}}{\left(s+\frac{5}{2}\right)^2 + \frac{55}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole

at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 5$, and $\omega_n = \sqrt{20} = 4.472$, $\zeta = 0.559$,

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 12.03\%, T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ sec}, T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.847 \text{ sec}; \omega_n T_r =$$

1.852, therefore, $T_r = 0.414$.

31.

Program:

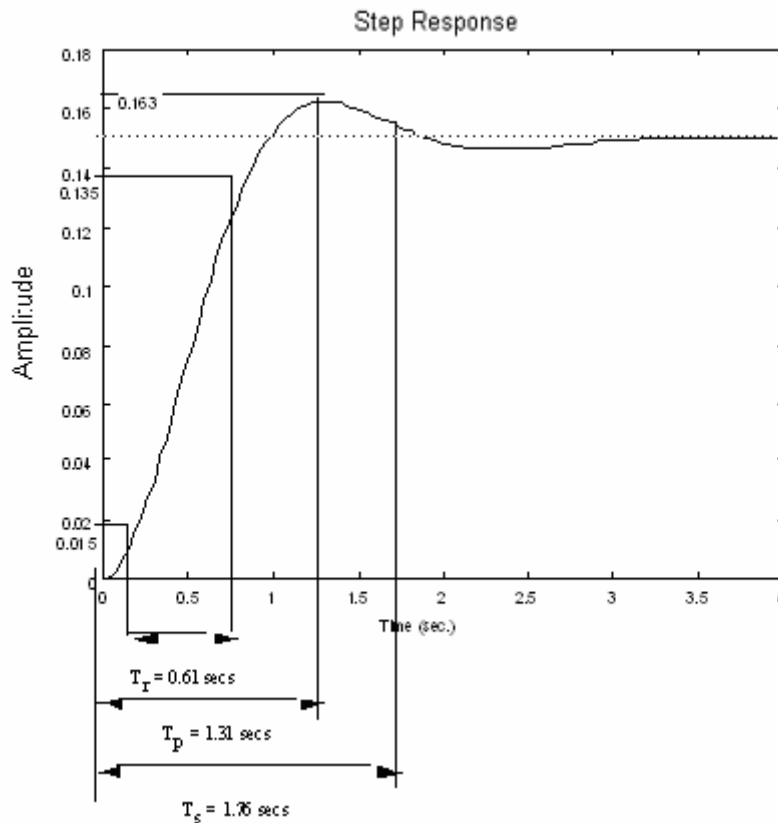
```
%Form sC(s) to get transfer function
clf
num=[1 3];
den=conv([1 3 10],[1 2]);
T=tf(num,den)
step(T)
```

Computer response:

Transfer function:

$$s + 3$$

$$s^3 + 5s^2 + 16s + 20$$



$$\%OS = \frac{(0.163 - 0.15)}{0.15} = 8.67\%$$

32.

Only part c can be approximated as a second-order system. From the exponentially decaying cosine the poles are located at $s_{1,2} = -2 \pm j9.796$. Thus,

$$T_s = \frac{4}{|\text{Re}|} = \frac{4}{2} = 2 \text{ s}; T_p = \frac{\pi}{|\text{Im}|} = \frac{\pi}{9.796} = 0.3207 \text{ s}$$

Also, $\omega_n = \sqrt{2^2 + 9.796^2} = 10$ and $\zeta\omega_n = |\text{Re}| = 2$. Hence, $\zeta = 0.2$, yielding 52.66 percent overshoot.

33.

a.

$$(1) \quad C_{a1}(s) = \frac{1}{s^2 + 3s + 36} = \frac{\frac{1}{\sqrt{33.75}} \sqrt{33.75}}{(s + 1.5)^2 + 33.75} = \frac{0.17213 \sqrt{33.75}}{(s + 1.5)^2 + 33.75} = \frac{0.17213 \cdot 5.8095}{(s + 1.5)^2 + 33.75}$$

Taking the inverse Laplace transform

$$C_{a1}(t) = 0.17213 e^{-1.5t} \sin 5.8095t$$

$$(2) \quad C_{a2}(s) = \frac{2}{s(s^2 + 3s + 36)} = \frac{1}{18} \frac{1}{s} - \frac{\frac{1}{18}s + \frac{1}{6}}{s^2 + 3s + 36} =$$

$$\frac{1}{18} \frac{1}{s} - \frac{\frac{1}{18}\left(s + \frac{3}{2}\right) + \frac{0.083333}{\sqrt{33.75}} \sqrt{33.75}}{\left(s + \frac{3}{2}\right)^2 + 33.75}$$

$$= 0.055556 \frac{1}{s} - \frac{0.055556\left(s + \frac{3}{2}\right) + 0.014344 \sqrt{33.75}}{\left(s + \frac{3}{2}\right)^2 + 33.75}$$

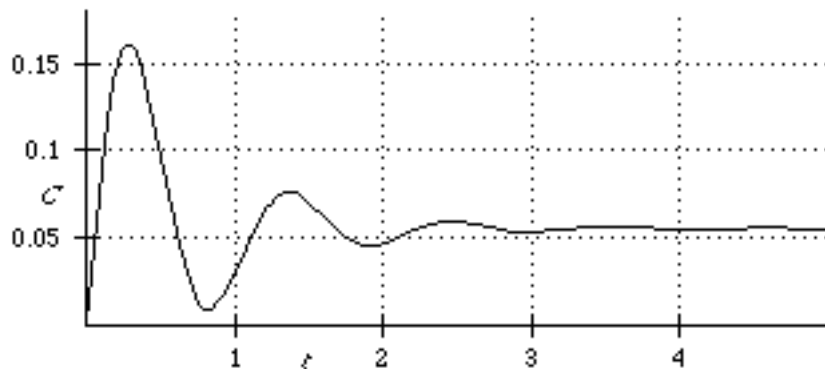
Taking the inverse Laplace transform

$$C_{a2}(t) = 0.055556 - e^{-1.5t} (0.055556 \cos 5.809t + 0.014344 \sin 5.809t)$$

The total response is found as follows:

$$C_{at}(t) = C_{a1}(t) + C_{a2}(t) = 0.055556 - e^{-1.5t} (0.055556 \cos 5.809t - 0.157786 \sin 5.809t)$$

Plotting the total response:



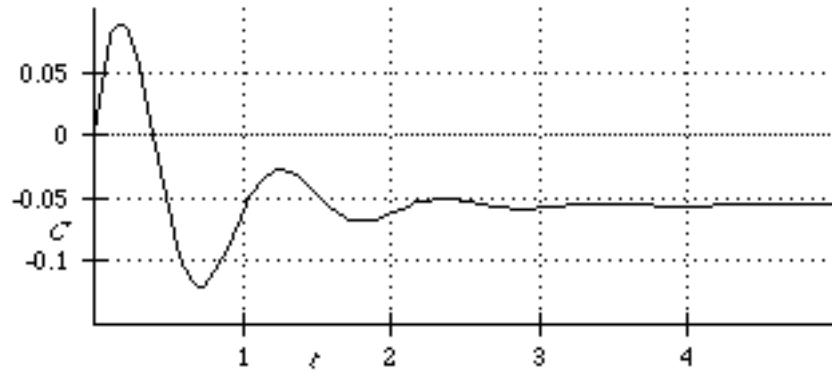
b.

(1) Same as (1) from part (a), or $C_{b1}(t) = C_{a1}(t)$

(2) Same as the negative of (2) of part (a), or $C_{b2}(t) = -C_{a2}(t)$

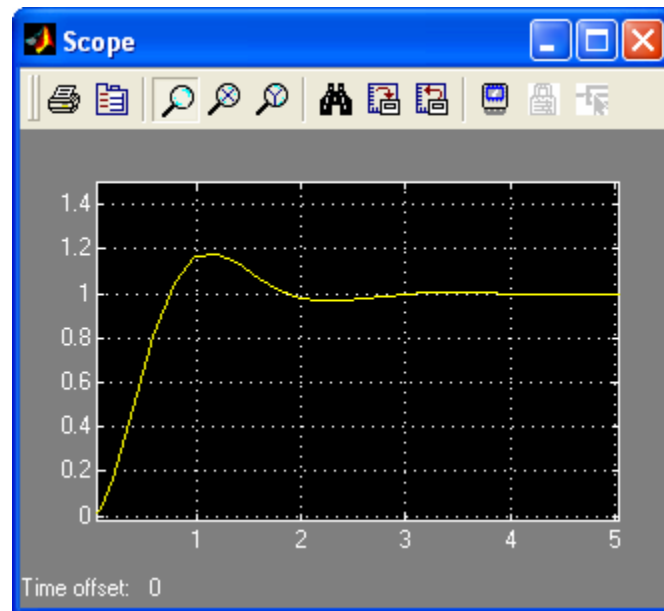
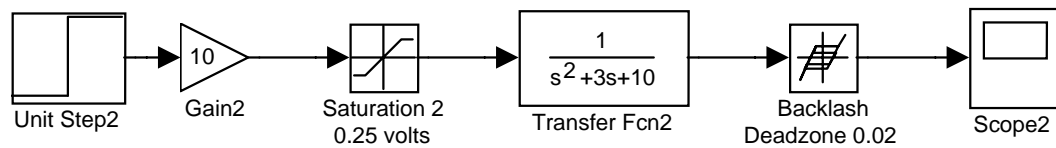
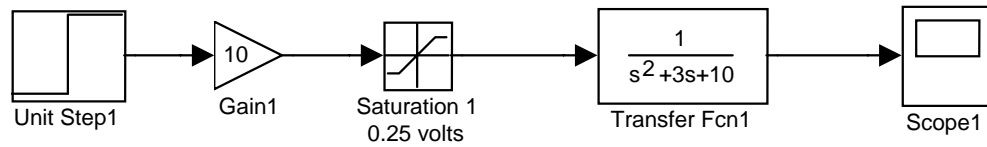
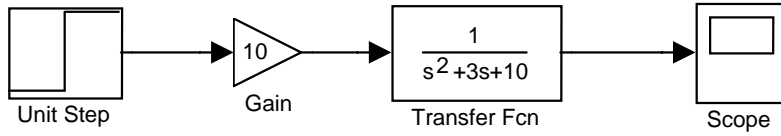
The total response is

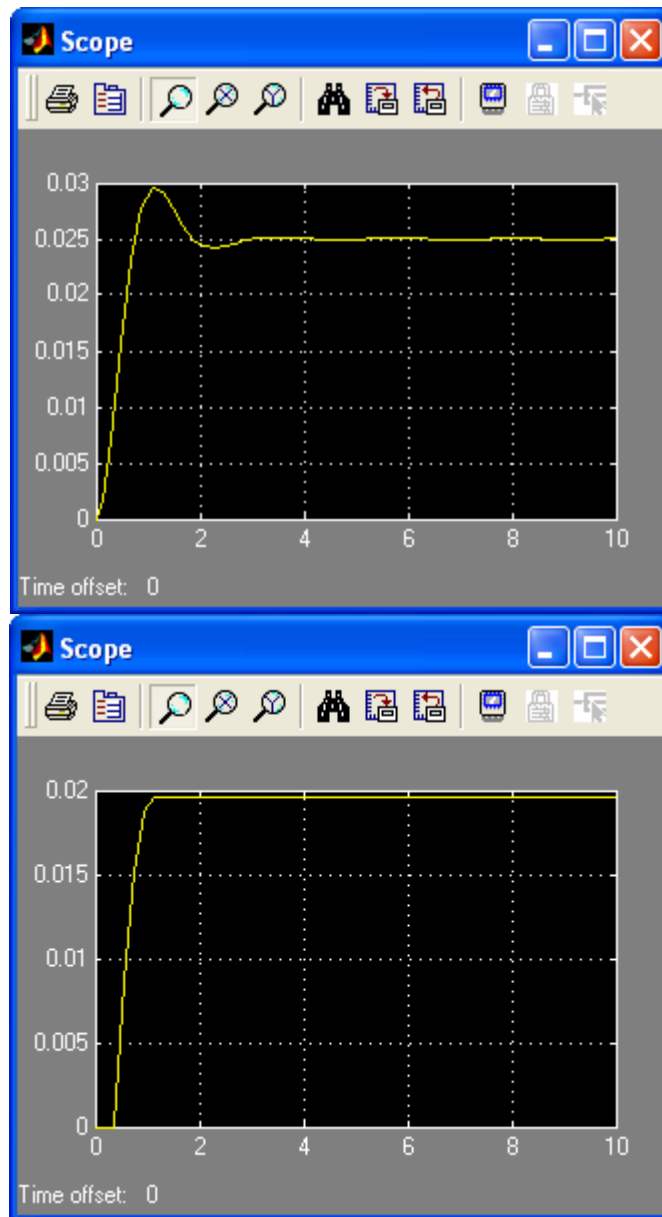
$$C_{bt}(t) = C_{b1}(t) + C_{b2}(t) = C_{a1}(t) - C_{a2}(t) = -0.055556 + e^{-1.5t} (0.055556 \cos 5.809t + 0.186474 \sin 5.809t)$$



Notice the nonminimum phase behavior for $C_{bt}(t)$.

34.





35.

$$s\mathbf{I} - \mathbf{A} = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} (s+2) & 1 \\ 3 & (s+5) \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^2 + 7s + 7$$

Factoring yields poles at -5.7913 and -1.2087 .

36.

a.

$$s\mathbf{I} - \mathbf{A} = s \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 3 \\ 0 & 6 & 5 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} s & -2 & -3 \\ 0 & (s-6) & -5 \\ -1 & -4 & (s-2) \end{vmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^3 - 8s^2 - 11s + 8$$

b. Factoring yields poles at 9.111, 0.5338, and -1.6448.

37.

$$\begin{aligned} \mathbf{x} &= (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{B}u) \\ \mathbf{x} &= \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 3 \frac{1}{s^2 + 9} \right) \\ \mathbf{x} &= \begin{pmatrix} \frac{2s^3 + 4s^2 + 21s + 45}{[s^2 + 9][s^2 + 5]} \\ \frac{s^3 - 7s^2 + 12s - 75}{[s^2 + 9][s^2 + 5]} \end{pmatrix} \\ y &= (1, 2)\mathbf{x} \\ y &= \left(\frac{4s^3 - 10s^2 + 45s - 105}{[s^2 + 9][s^2 + 5]} \right) \end{aligned}$$

38.

$$\begin{aligned} \mathbf{x} &= (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{B}u) \\ \mathbf{x} &= \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -2 & -4 & 1 \\ 0 & 0 & -6 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1}{s+1} \right) \\ \mathbf{x} &= \begin{pmatrix} \frac{1}{[s+6][s+1][s+0.58579][s+3.4142]} \\ \frac{s}{[s+6][s+1][s+0.58579][s+3.4142]} \\ \frac{s^2 + 4s + 2}{[s+6][s+1][s+0.58579][s+3.4142]} \end{pmatrix} \\ y &= (1, 0, 0)\mathbf{x} \\ y &= \left(\frac{1}{[s+6][s+1][s+0.58579][s+3.4142]} \right) \end{aligned}$$

39.

$$\begin{aligned} \mathbf{x} &= (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{B}u) \\ \mathbf{x} &= \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s} \right) \\ \mathbf{x} &= \begin{pmatrix} \frac{s+1}{s[s+2]} \\ \frac{1}{s[s+1][s+2]} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 y(s) &= (0, 1)x \\
 y(s) &= \left(\frac{1}{s[s+1][s+2]} \right) \\
 y(s) &= \left(\frac{1}{2} \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \frac{1}{s+2} \right) \\
 y(t) &= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}
 \end{aligned}$$

40.

$$\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_0 + \mathbf{B}u)$$

$$\mathbf{x} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 0 \\ s & 0 & 1 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 & 0 & -5 \end{array} \right)^{-1} \left(\begin{array}{c|c} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{array} \right) \frac{1}{s}$$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{s(s+3)(s+5)} \\ \frac{1}{s(s+5)} \\ \frac{1}{s(s+5)} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{15} - \frac{1}{6}e^{-3t} + \frac{1}{10}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} \end{bmatrix}$$

$$y(t) = [0 \quad 1 \quad 1]\mathbf{x} = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

41.

Program:

```

A=[-3 1 0;0 -6 1;0 0 -5];
B=[0;1;1];
C=[0 1 1];
D=0;
S=ss(A,B,C,D)
step(S)

```

Computer response:

```

a =
      x1      x2      x3
x1    -3       1       0
x2      0      -6       1
x3      0       0      -5

```

b =

```
      u1
x1      0
x2      1
x3      1
```

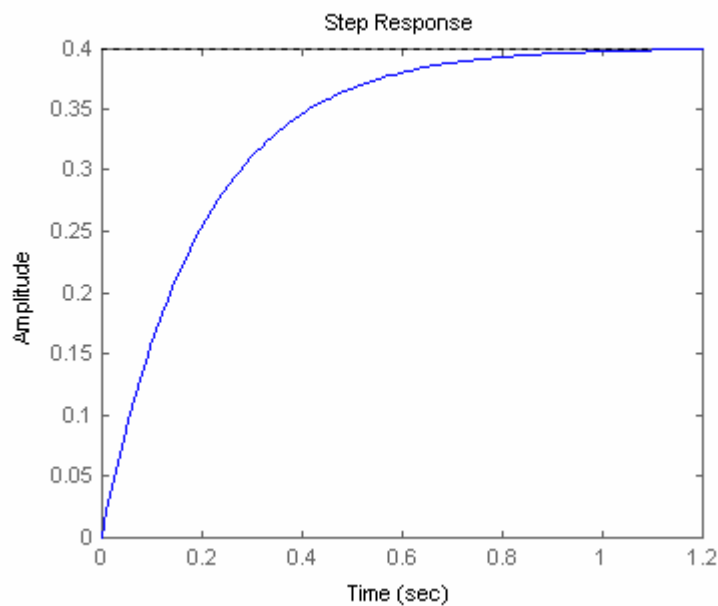
c =

```
      x1  x2  x3
y1      0   1   1
```

d =

```
      u1
y1      0
```

Continuous-time model.



42.

Program:

```
syms s                                %Construct symbolic object for
                                     %frequency variable 's'.
'a'                                   %Display label
A=[-3 1 0;0 -6 1;0 0 -5]             %Create matrix A.
B=[0;1;1];                           %Create vector B.
C=[0 1 1];                           %Create C vector
X0=[1;1;0]                           %Create initial condition vector,X(0).
U=1/s;                               %Create U(s).
I=[1 0 0;0 1 0;0 0 1];               %Create identity matrix.
X=((s*I-A)^-1)*(X0+B*U);              %Find Laplace transform of state vector.
x1=ilaplace(X(1))                    %Solve for X1(t).
x2=ilaplace(X(2))                    %Solve for X2(t).
x3=ilaplace(X(3))                    %Solve for X3(t).
y=C*[x1;x2;x3]                       %Solve for output, y(t).
y=simplify(y)                        %Simplify y(t).
'y(t)'                               %Display label.
pretty(y)                            %Pretty print y(t).
```

Computer response:

ans =

a

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 =$$

$$7/6 \exp(-3t) - 1/3 \exp(-6t) + 1/15 + 1/10 \exp(-5t)$$

$$x_2 =$$

$$\exp(-6t) + 1/5 - 1/5 \exp(-5t)$$

$$x_3 =$$

$$1/5 - 1/5 \exp(-5t)$$

$$y =$$

$$2/5 + \exp(-6t) - 2/5 \exp(-5t)$$

$$y =$$

$$2/5 + \exp(-6t) - 2/5 \exp(-5t)$$

$$\text{ans} =$$

$$y(t)$$

$$2/5 + \exp(-6t) - 2/5 \exp(-5t)$$

43.

$$|\lambda I - A| = \lambda^2 + 5\lambda + 1$$

$$|\lambda I - A| = (\lambda + 0.20871)(\lambda + 4.7913)$$

$$P = -0.20871$$

$$Q = -4.7913$$

$$\Phi = \begin{pmatrix} A_1 e^{-0.20871t} + A_2 e^{-4.7913t} & A_5 e^{-0.20871t} + A_6 e^{-4.7913t} \\ A_3 e^{-0.20871t} + A_4 e^{-4.7913t} & A_7 e^{-0.20871t} + A_8 e^{-4.7913t} \end{pmatrix}$$

$$\Phi_0 = \begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix}$$

$$\frac{\partial}{\partial t} \Phi = \begin{pmatrix} -0.20871 A_1 e^{-0.20871t} - 4.7913 A_2 e^{-4.7913t} & -0.20871 A_5 e^{-0.20871t} - 4.7913 A_6 e^{-4.7913t} \\ -0.20871 A_3 e^{-0.20871t} - 4.7913 A_4 e^{-4.7913t} & -0.20871 A_7 e^{-0.20871t} - 4.7913 A_8 e^{-4.7913t} \end{pmatrix}$$

$$\frac{d}{dt} \Phi_0 = \begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}$$

Solving for A_i 's two at a time, and substituting into the state-transition matrix

$$\Phi = \begin{pmatrix} 1.0455 e^{-0.20871 t} - 0.045545 e^{-4.7913 t} & 0.21822 e^{-0.20871 t} - 0.21822 e^{-4.7913 t} \\ -0.21822 e^{-0.20871 t} + 0.21822 e^{-4.7913 t} & -0.045545 e^{-0.20871 t} + 1.0455 e^{-4.7913 t} \end{pmatrix}$$

To find $x(t)$,

$$\begin{aligned} x &= \Phi x_0 \\ x &= \begin{pmatrix} 1.0455 e^{-0.20871 t} - 0.045545 e^{-4.7913 t} & 0.21822 e^{-0.20871 t} - 0.21822 e^{-4.7913 t} \\ -0.21822 e^{-0.20871 t} + 0.21822 e^{-4.7913 t} & -0.045545 e^{-0.20871 t} + 1.0455 e^{-4.7913 t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ x &= \begin{pmatrix} 1.0455 e^{-0.20871 t} - 0.045545 e^{-4.7913 t} \\ -0.21822 e^{-0.20871 t} + 0.21822 e^{-4.7913 t} \end{pmatrix} \end{aligned}$$

To find the output,

$$\begin{aligned} y &= (1, 2) x \\ y &= (1, 2) \begin{pmatrix} 1.0455 e^{-0.20871 t} - 0.045545 e^{-4.7913 t} \\ -0.21822 e^{-0.20871 t} + 0.21822 e^{-4.7913 t} \end{pmatrix} \\ y &= (0.60911 e^{-0.20871 t} + 0.39089 e^{-4.7913 t}) \end{aligned}$$

44.

$$\lambda I - A = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

$$|\lambda I - A| = \lambda^2 + 1$$

$$\begin{aligned} \Phi &= \begin{pmatrix} A_1 \cos[t] + A_2 \sin[t] & A_5 \cos[t] + A_6 \sin[t] \\ A_3 \cos[t] + A_4 \sin[t] & A_7 \cos[t] + A_8 \sin[t] \end{pmatrix} \\ \frac{d}{dt} \Phi &= \begin{pmatrix} A_2 \cos[t] - A_1 \sin[t] & A_6 \cos[t] - A_5 \sin[t] \\ A_4 \cos[t] - A_3 \sin[t] & A_8 \cos[t] - A_7 \sin[t] \end{pmatrix} \end{aligned}$$

$$\Phi_0 = \begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix}$$

$$\frac{d}{dt} \Phi_0 = \begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solving for the A_i 's and substituting into the state-transition matrix,

$$\Phi = \begin{pmatrix} \cos[t] & \sin[t] \\ -\sin[t] & \cos[t] \end{pmatrix}$$

To find the state vector,

$$x = \int_0^t (\Phi[t-\tau] B u[\tau]) d\tau$$

$$x = \int_0^t \begin{pmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$x = \int_0^t \begin{pmatrix} \sin[t-\tau] \\ \cos[t-\tau] \end{pmatrix} d\tau$$

$$t-\tau = \theta$$

$$x = \int_t^0 \begin{pmatrix} -\sin[\theta] \\ -\cos[\theta] \end{pmatrix} d\theta$$

$$x = \begin{pmatrix} 1 - \cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (3, 2) x$$

$$\Delta y = (3, 2) \begin{pmatrix} 1 - \cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (-3 \cos[t] + 2 \sin[t] + 3)$$

45.

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda + 2)(\lambda + 0.5 - 2.3979i)(\lambda + 0.5 + 2.3979i)$$

Let the state-transition matrix be

$$\phi = \begin{bmatrix} A_1 e^{-.5t} \cos(2.3979t) + A_2 e^{-.5t} \sin(2.3979t) + A_3 e^{-2t} & A_{10} e^{-.5t} \cos(2.3979t) + A_{11} e^{-.5t} \sin(2.3979t) + A_{12} & \bullet \\ A_4 e^{-.5t} \cos(2.3979t) + A_5 e^{-.5t} \sin(2.3979t) + A_6 e^{-2t} & A_{13} e^{-.5t} \cos(2.3979t) + A_{14} e^{-.5t} \sin(2.3979t) + A_{15} & \bullet \\ A_7 e^{-.5t} \cos(2.3979t) + A_8 e^{-.5t} \sin(2.3979t) + A_9 e^{-2t} & A_{16} e^{-.5t} \cos(2.3979t) + A_{17} e^{-.5t} \sin(2.3979t) + A_{18} & \bullet \end{bmatrix}$$

Since $\phi(0) = \mathbf{I}$, $\dot{\phi}(0) = \mathbf{A}$, and $\ddot{\phi}(0) = \mathbf{A}^2$, we can evaluate the coefficients, A_i 's. Thus,

$$\begin{pmatrix} A_3 + A_1 & A_{12} + A_{10} & A_{21} + A_{19} \\ A_6 + A_4 & A_{15} + A_{13} & A_{24} + A_{22} \\ A_9 + A_7 & A_{18} + A_{16} & A_{27} + A_{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -2A_3 + 2.3979A_2 - 0.5A_1 & -2A_{12} + 2.3979A_{11} - 0.5A_{10} & -2A_{21} + 2.3979A_{20} - 0.5A_{19} \\ -2A_6 + 2.3979A_5 - 0.5A_4 & -2A_{15} + 2.3979A_{14} - 0.5A_{13} & -2A_{24} + 2.3979A_{23} - 0.5A_{22} \\ -2A_9 + 2.3979A_8 - 0.5A_7 & -2A_{18} + 2.3979A_{17} - 0.5A_{16} & -2A_{27} + 2.3979A_{26} - 0.5A_{25} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 4A_3 - 2.3979A_2 - 5.4999A_1 & 4A_{12} - 2.3979A_{11} - 5.4999A_{10} & 4A_{21} - 2.3979A_{20} - 5.4999A_{19} \\ 4A_6 - 2.3979A_5 - 5.4999A_4 & 4A_{15} - 2.3979A_{14} - 5.4999A_{13} & 4A_{24} - 2.3979A_{23} - 5.4999A_{22} \\ 4A_9 - 2.3979A_8 - 5.4999A_7 & 4A_{18} - 2.3979A_{17} - 5.4999A_{16} & 4A_{27} - 2.3979A_{26} - 5.4999A_{25} \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & -6 & -1 \\ 0 & 6 & -5 \end{pmatrix}$$

Solving for the A_i 's taking three equations at a time,

Computer response:

ans =

a

A =

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -1 \end{bmatrix}$$

B =

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

C =

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

X0 =

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

ans =

E=(s*I-A)^-1

E =

$$\begin{bmatrix} 1/(s+2), & (s+1)/(s+2)/(s^2+s+6), & 1/(s+2)/(s^2+s+6) \\ 0, & (s+1)/(s^2+s+6), & 1/(s^2+s+6) \\ 0, & -6/(s^2+s+6), & s/(s^2+s+6) \end{bmatrix}$$

ans =

Fi(t)

$$\begin{bmatrix} \exp(-2t), & -1/8 \exp(-2t) + 1/8 \%1 + \frac{13}{184} \%2, \\ 1/8 \exp(-2t) - 1/8 \%1 + 3/184 \%2 \end{bmatrix}$$

$$\begin{bmatrix} 0, & 1/23 \%2 + \%1, & -1/23 \\ (-23)^{1/2} (\exp((-1/2 + 1/2 (-23)^{1/2})t) - \exp((-1/2 - 1/2 (-23)^{1/2})t)) \end{bmatrix}$$

$$\begin{bmatrix} 0, & 6/23 \\ (-23)^{1/2} (\exp((-1/2 + 1/2 (-23)^{1/2})t) - \exp((-1/2 - 1/2 (-23)^{1/2})t)) \end{bmatrix}$$

```

      ]
      , - 1/23 %2 + %1]

%1 := exp(- 1/2 t) cos(1/2 231/2 t)

%2 := exp(- 1/2 t) 231/2 sin(1/2 231/2 t)

ans =
Fi(t-tau)

[
[exp(-2 t + 2 tau) ,
[
- 1/8 exp(-2 t + 2 tau) + 1/8 %2 cos(%1) + --- %2 231/2 sin(%1) ,
184
1/8 exp(-2 t + 2 tau) - 1/8 %2 cos(%1) + 3/184 %2 231/2 sin(%1)]
]

[
0 , 1/23 %2 231/2 sin(%1) + %2 cos(%1) , - 1/23 (-23)1/2 (
exp((-1/2 + 1/2 (-23)1/2) (t - tau))
- exp((-1/2 - 1/2 (-23)1/2) (t - tau)))
]

[
0 , 6/23 (-23)1/2 (exp((-1/2 + 1/2 (-23)1/2) (t - tau))
- exp((-1/2 - 1/2 (-23)1/2) (t - tau))) ,
- 1/23 %2 231/2 sin(%1) + %2 cos(%1)]

%1 := 1/2 231/2 (t - tau)

%2 := exp(- 1/2 t + 1/2 tau)

ans =
x(t)

[.375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)
+ .339 exp(-.500 t) sin(2.40 t) + .500]

[.209 exp(-.500 t) sin(2.40 t) + exp(-.500 t) cos(2.40 t)]

[1.25 i (exp((- .500 + 2.40 i) t) - 1. exp((- .500 - 2.40 i) t))]

ans =
y(t)

.375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)

```


$$+ .339 \exp(-.500 t) \sin(2.40 t) + .500$$

47.

The state-space representation used to obtain the plot is,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -0.8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Using the Step Response software,

Step Response - Data Entry

About

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u(t)$
 $y = \mathbf{C}\mathbf{x}$

Order: 2

A =

	1	2
1	0	1
2	-1	-0.8

B =

1	0
2	1

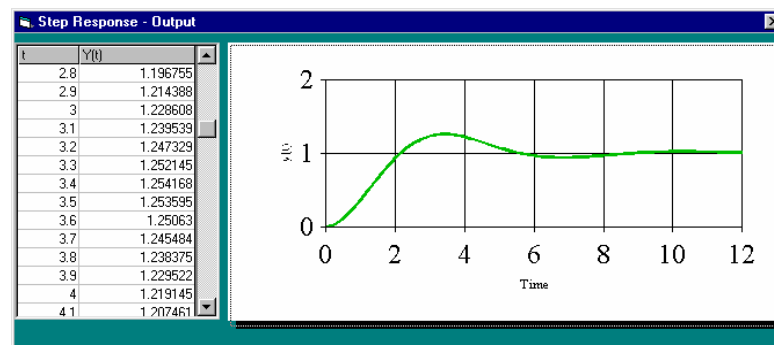
C =

1	2
1	0

x(0) =

1	0
2	0

Iteration Interval: 0.001 Print Interval: 0.1 Total Time: 12 **Calculate**



Calculating % overshoot, settling time, and peak time,

$$2\zeta\omega_n = 0.8, \omega_n = 1, \zeta = 0.4. \text{ Therefore, } \%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 25.38\%, T_s = \frac{4}{\zeta\omega_n} = 10 \text{ sec,}$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 3.43 \text{ sec.}$$

48.

Step Response - Data Entry

About

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u(t)$
 $y = \mathbf{C}\mathbf{x}$

Order: 3

A =

	1	2
1	0	1
2	-10	-7
3	0	0

B =

1	0
2	0
3	1

C =

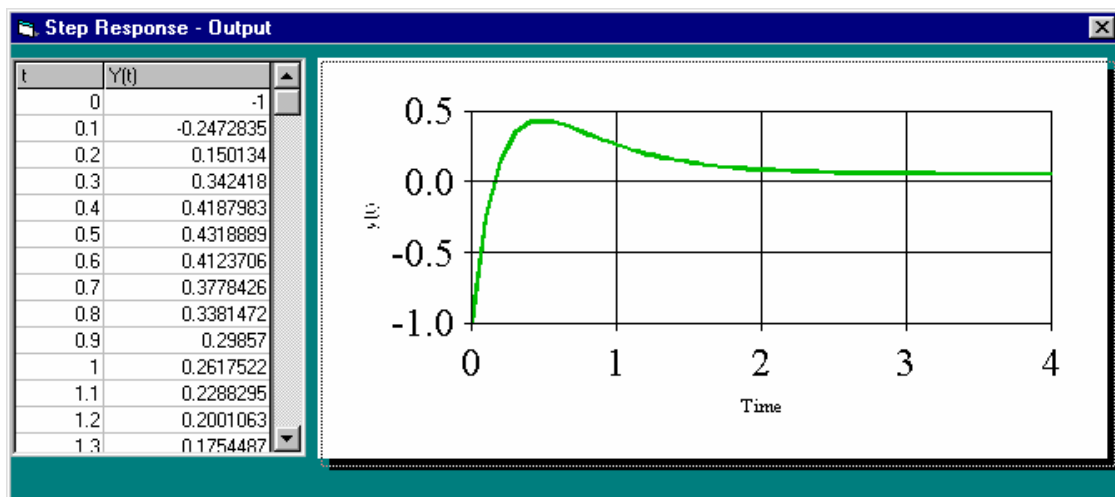
1	2
1	1

x(0) =

1	-1
2	0
3	0

Iteration Interval: .001 Print Interval: .1 Total Time: 4

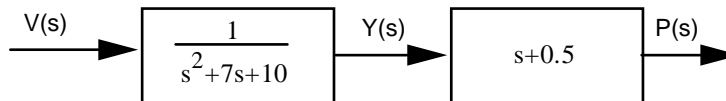
Calculate



49.

a. $P(s) = \frac{s+0.5}{s(s+2)(s+5)} = \frac{1/20}{s} + \frac{1/4}{s+2} - \frac{3/10}{s+5}$. Therefore, $p(t) = \frac{1}{20} + \frac{1}{4} e^{-2t} - \frac{3}{10} e^{-5t}$.

b. To represent the system in state space, draw the following block diagram.



For the first block,

$$\ddot{y} + 7\dot{y} + 10y = v(t)$$

Let $x_1 = y$, and $x_2 = \dot{y}$. Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10x_1 - 7x_2 + v(t)$$

Also,

$$p(t) = 0.5y + \dot{y} = 0.5x_1 + x_2$$

Thus,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad p(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \mathbf{x}$$

c.

Program:

```
A=[0 1;-10 -7];
```

```
B=[0;1];
```

```
C=[.5 1];
```

```
D=0;
```

```
S=ss(A,B,C,D)
```

```
step(S)
```

Computer response:

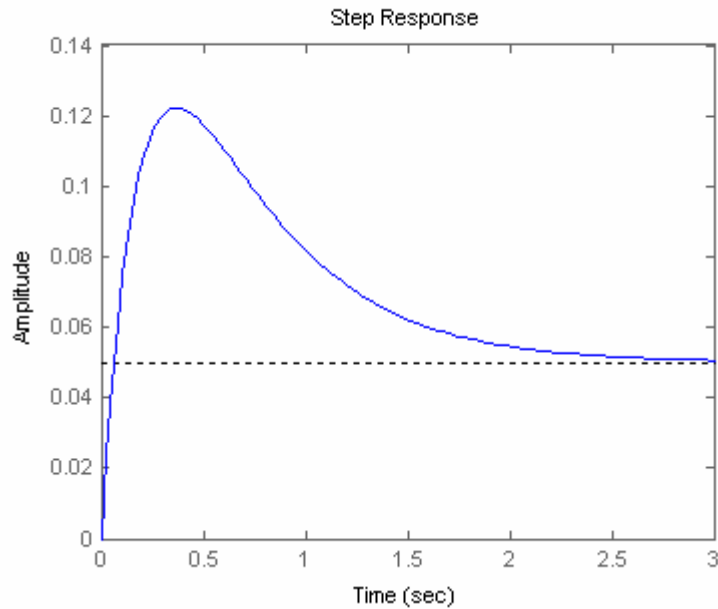
```
a =
      x1      x2
x1      0      1
x2     -10     -7
```

```
b =
      u1
x1      0
x2      1
```

```
c =
      x1      x2
y1    0.5      1
```

```
d =
      u1
y1      0
```

Continuous-time model.



50.

a. $\omega_n = \sqrt{10} = 3.16$; $2\zeta\omega_n = 4$. Therefore $\zeta = 0.632$. $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} * 100 = 7.69\%$.

$T_s = \frac{4}{\zeta\omega_n} = 2$ seconds. $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.28$ seconds. From Figure 4.16, $T_r\omega_n = 1.93$.

Thus, $T_r = 0.611$ second. To justify second-order assumption, we see that the dominant poles are at –

$2 \pm j2.449$. The third pole is at -10, or 5 times further. The second-order approximation is valid.

b. $G_c(s) = \frac{K}{(s+10)(s^2+4s+10)} = \frac{K}{s^3+14s^2+50s+100}$. Representing the system in phase-variable form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -50 & -14 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}; \mathbf{C} = [1 \quad 0 \quad 0]$$

c.

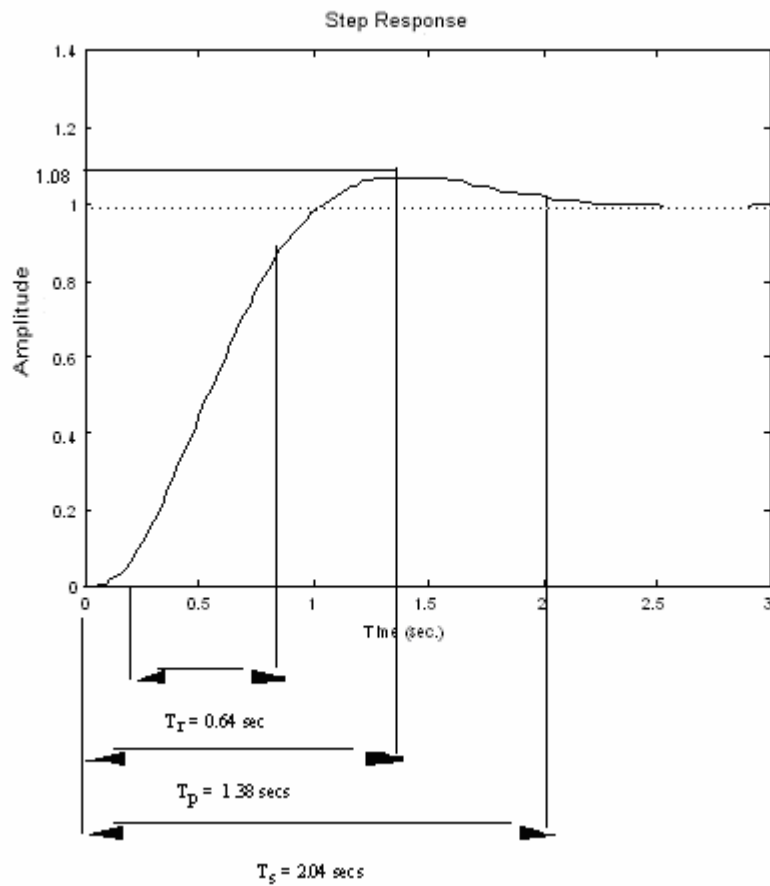
Program:

```
numg=100;
deng=conv([1 10],[1 4 10]);
G=tf(numg,deng)
step(G)
```

Computer response:

Transfer function:

```
100
-----
s^3 + 14 s^2 + 50 s + 100
```



$$\%OS = \frac{(1.08-1)}{1} * 100 = 8\%$$

51.

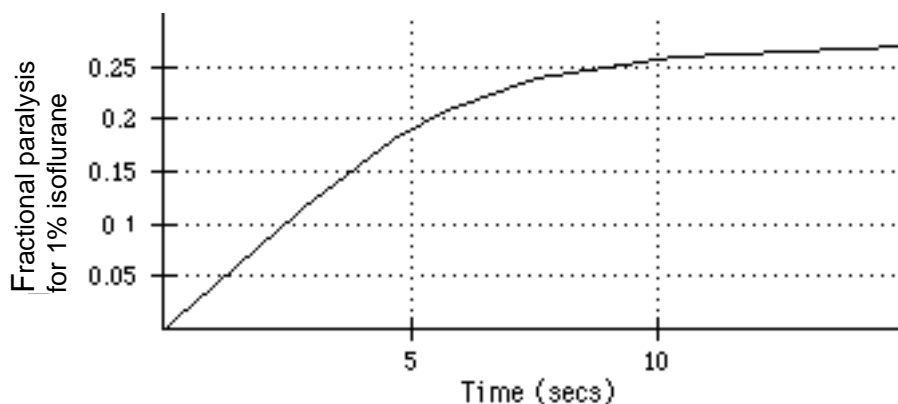
a. $\omega_n = \sqrt{0.28} = 0.529$; $2\zeta\omega_n = 1.15$. Therefore $\zeta = 1.087$.

b. $P(s) = U(s) \frac{7.63 \times 10^{-2}}{s^2 + 1.15s + 0.28}$, where $U(s) = \frac{2}{s}$. Expanding by partial fractions, $P(s) = \frac{0.545}{s} +$

natural response terms. Thus percent paralysis = 54.5%.

c. $P(s) = \frac{7.63 \times 10^{-2}}{s(s^2 + 1.15s + 0.28)} = \frac{0.2725}{s} - \frac{0.48444}{s+0.35} + \frac{0.21194}{s+0.8}$.

Hence, $p(t) = 0.2725 - 0.48444e^{-0.35t} + 0.21194e^{-0.8t}$. Plotting,



d. $P(s) = \frac{K}{s} * \frac{7.63 \times 10^{-2}}{s^2 + 1.15s + 0.28} = \frac{1}{s} + \text{natural response terms.}$ Therefore, $\frac{7.63 \times 10^{-2} K}{0.28} = 1$. Solving

for K, $K = 3.67\%$.

52.

a. Writing the differential equation,

$$\frac{dc(t)}{dt} = -k_{10}c(t) + \frac{i(t)}{V_d}$$

Taking the Laplace transform and rearranging,

$$(s + k_{10})C(s) = \frac{I(s)}{V_d}$$

from which the transfer function is found to be

$$\frac{C(s)}{I(s)} = \frac{\frac{1}{V_d}}{s + k_{10}}$$

For a step input, $I(s) = \frac{I_0}{s}$. Thus the response is

$$C(s) = \frac{\frac{I_0}{V_d}}{s(s + k_{10})} = \frac{I_0}{k_{10}V_d} \left(\frac{1}{s} - \frac{1}{s + k_{10}} \right)$$

Taking the inverse Laplace transform,

$$c(t) = \frac{I_0}{k_{10}V_d} (1 - e^{-k_{10}t})$$

where the steady-state value, C_D , is

$$C_D = \frac{I_0}{k_{10}V_d}$$

Solving for $I_R = I_0$,

$$I_R = C_D k_{10} V_d$$

b. $T_r = \frac{2.2}{k_{10}} ; T_s = \frac{4}{k_{10}}$

c. $I_R = C_D k_{10} V_d = 12 \frac{\mu\text{g}}{\text{ml}} \times 0.07 \text{ hr}^{-1} \times 0.6 \text{ liters} = 0.504 \frac{\text{mg}}{\text{h}}$

d. Using the equations of part b, where $k_{10} = 0.07$, $T_r = 31.43 \text{ hrs}$, and $T_s = 57.14 \text{ hrs}$.

SOLUTIONS TO DESIGN PROBLEMS

53.

Writing the equation of motion, $(f_v s + 2)X(s) = F(s)$. Thus, the transfer function is

$$\frac{X(s)}{F(s)} = \frac{1/f_v}{s + \frac{2}{f_v}}. \text{ Hence, } T_s = \frac{4}{a} = \frac{4}{\frac{2}{f_v}} = 2f_v, \text{ or } f_v = \frac{T_s}{2}.$$

54.

The transfer function is, $F(s) = \frac{1/M}{s^2 + \frac{1}{M}s + \frac{K}{M}}$. Now, $T_s = 2 = \frac{4}{|\text{Re}|} = \frac{4}{\frac{1}{2M}}$. Thus,

$M = \frac{1}{4}$. Substituting the value of M in the denominator of the transfer function yields,

$s^2 + 4s + 4K$. Identify the roots $s_{1,2} = -2 \pm j2\sqrt{K-1}$. Using the imaginary part and substituting into the peak time equation yields $T_p = 1 = \frac{\pi}{|\text{Im}|} = \frac{\pi}{2\sqrt{K-1}}$, from which

$$K = 3.467.$$

55.

Writing the equation of motion, $(Ms^2 + f_v s + 1)X(s) = F(s)$. Thus, the transfer function is

$$\frac{X(s)}{F(s)} = \frac{1/M}{s^2 + \frac{f_v}{M}s + \frac{1}{M}}. \text{ Since } T_s = 10 = \frac{4}{\zeta\omega_n}, \zeta\omega_n = 0.4. \text{ But, } \frac{f_v}{M} = 2\zeta\omega_n = 0.8. \text{ Also,}$$

from Eq. (4.39) 30% overshoot implies $\zeta = 0.358$. Hence, $\omega_n = 1.117$. Now, $1/M = \omega_n^2 = 1.248$.

Therefore, $M = 0.801$. Since $\frac{f_v}{M} = 2\zeta\omega_n = 0.8$, $f_v = 0.641$.

56.

Writing the equation of motion: $(Js^2 + s + K)\theta(s) = T(s)$. Therefore the transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{\frac{1}{J}}{s^2 + \frac{1}{J}s + \frac{K}{J}}.$$

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.358.$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\frac{1}{2J}} = 8J = 4.$$

Therefore $J = \frac{1}{2}$. Also, $T_s = 4 = \frac{4}{\zeta\omega_n} = \frac{4}{(0.358)\omega_n}$. Hence, $\omega_n = 2.793$. Now, $\frac{K}{J} = \omega_n^2 = 7.803$.

Finally, $K = 3.901$.

57.

Writing the equation of motion

$$[s^2 + D(5)^2 s + \frac{1}{4}(10)^2] \theta(s) = T(s)$$

The transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{1}{s^2 + 25Ds + 25}$$

Also,

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.358$$

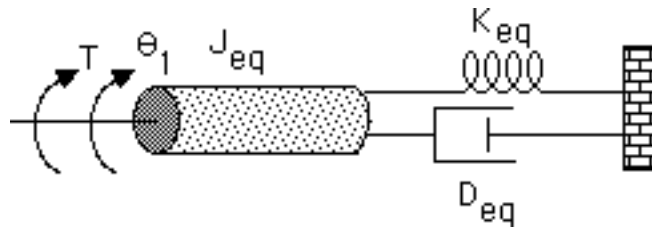
and

$$2\zeta\omega_n = 2(0.358)(5) = 25D$$

Therefore $D = 0.14$.

58.

The equivalent circuit is:



where $J_{eq} = 1 + \left(\frac{N_1}{N_2}\right)^2$; $D_{eq} = \left(\frac{N_1}{N_2}\right)^2$; $K_{eq} = \left(\frac{N_1}{N_2}\right)^2$. Thus,

$$\frac{\theta_1(s)}{T(s)} = \frac{1}{J_{eq}s^2 + D_{eq}s + K_{eq}}. \text{ Letting } \frac{N_1}{N_2} = n \text{ and substituting the above values into the transfer}$$

function,

$$\frac{\theta_1(s)}{T(s)} = \frac{\frac{1}{1+n^2}}{s^2 + \frac{n^2}{1+n^2}s + \frac{n^2}{1+n^2}}. \text{ Therefore, } \zeta\omega_n = \frac{n^2}{2(1+n^2)}. \text{ Finally, } T_s = \frac{4}{\zeta\omega_n} = \frac{8(1+n^2)}{n^2} = 16. \text{ Thus}$$

$n = 1$.

59.

Let the rotation of the shaft with gear N_2 be $\theta_L(s)$. Assuming that all rotating load has been reflected to the N_2 shaft, $(J_{eqL}s^2 + D_{eqL}s + K)\theta_L(s) + F(s)r = T_{eq}(s)$, where $F(s)$ is the force from the translational system, $r = 2$ is the radius of the rotational member, J_{eqL} is the equivalent inertia at the N_2 shaft, and D_{eqL} is the equivalent damping at the N_2 shaft. Since $J_{eqL} = 1(2)^2 + 1 = 5$ and $D_{eqL} =$

$1(2)^2 = 4$, the equation of motion becomes, $(5s^2 + 4s + K)\theta_L(s) + 2F(s) = T_{eq}(s)$. For the translational system $(Ms^2 + s)X(s) = F(s)$. Substituting $F(s)$ into the rotational equation of motion, $(5s^2 + 4s + K)\theta_L(s) + (Ms^2 + s)2X(s) = T_{eq}(s)$.

But, $\theta_L(s) = \frac{X(s)}{r} = \frac{X(s)}{2}$ and $T_{eq}(s) = 2T(s)$. Substituting these quantities in the equation

above yields $((5 + 4M)s^2 + 8s + K)\frac{X(s)}{4} = T(s)$. Thus, the transfer function is

$$\frac{X(s)}{T(s)} = \frac{4/(5 + 4M)}{s^2 + \frac{8}{(5 + 4M)}s + \frac{K}{(5 + 4M)}}. \text{ Now, } T_s = 10 = \frac{4}{\text{Re}} = \frac{4}{\frac{8}{2(5 + 4M)}} = (5 + 4M).$$

Hence, $M = 5/4$. For 10% overshoot, $\zeta = 0.5912$ from Eq. (4.39). Hence,

$$2\zeta\omega_n = \frac{8}{(5 + 4M)} = 0.8. \text{ Solving for } \omega_n \text{ yields } \omega_n = 0.6766. \text{ But,}$$

$$\omega_n = \sqrt{\frac{K}{(5 + 4M)}} = \sqrt{\frac{K}{10}} = 0.6766. \text{ Thus, } K = 4.578.$$

60.

The transfer function for the capacitor voltage is $\frac{V_C(s)}{V(s)} = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{10^6}{s^2 + Rs + 10^6}$.

$$\text{For 20\% overshoot, } \zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.456. \text{ Therefore, } 2\zeta\omega_n = R = 2(0.456)(10^3) =$$

912Ω.

61.

Solving for the capacitor voltage using voltage division, $V_C(s) = V_i(s) \frac{1/(CS)}{R + Ls + \frac{1}{CS}}$. Thus, the

transfer function is $\frac{V_C(s)}{V_i(s)} = \frac{1/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$. Since $T_s = \frac{4}{|\text{Re}|} = 10^{-3}$, $|\text{Re}| = \frac{R}{2L} = 4000$. Thus

$R = 8 \text{ K}\Omega$. Also, since 20% overshoot implies a damping ratio of 0.46 and $2\zeta\omega_n = 8000$, $\omega_n = 8695.65 = \frac{1}{\sqrt{LC}}$. Hence, $C = 0.013 \mu\text{F}$.

62.

Using voltage division the transfer function is,

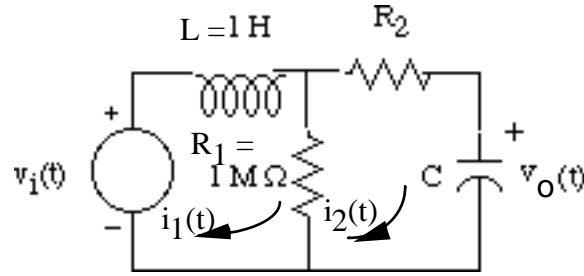
$$\frac{V_C(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Also, $T_s = 2 \times 10^{-3} = \frac{4}{\text{Re}} = \frac{4}{\frac{R}{2L}} = \frac{8L}{R}$. Thus, $\frac{R}{L} = 4000$. Using Eq. (4.39) with 15% overshoot,

$\zeta = 0.5169$. But, $2\zeta\omega_n = R/L$. Thus, $\omega_n = 3869 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{L(10^{-5})}}$. Therefore, $L = 6.7 \text{ mH}$ and $R = 26.72 \Omega$.

63.

For the circuit shown below



write the loop equations as

$$\begin{aligned} (R_1 + Ls)I_1(s) - R_1 I_2(s) &= V_i(s) \\ -R_1 I_1(s) + \left(R_1 + R_2 + \frac{1}{Cs}\right)I_2(s) &= 0 \end{aligned}$$

Solving for $I_2(s)$

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V_i(s) \\ -R_1 & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -R_1 \\ -R_1 & R_1 + R_2 + \frac{1}{Cs} \end{vmatrix}}$$

But, $V_o(s) = \frac{1}{Cs} I_2(s)$. Thus,

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1}{(R_2 + R_1)Cs^2 + (CR_2R_1 + L)s + R_1}$$

Substituting component values,

$$\frac{V_o(s)}{V_i(s)} = 1000000 \frac{1}{s^2 + \frac{(1000000CR_2 + 1)}{(R_2 + 1000000)C}s + 1000000 \frac{1}{(R_2 + 1000000)C}}$$

For 15% overshoot, $\zeta = 0.517$. For $T_s = 0.001$, $\zeta\omega_n = \frac{4}{0.001} = 4000$. Hence, $\omega_n = 7736.9$. Thus,

$$1000000 \frac{1}{(R_2 + 1000000)C} = 7736.9^2$$

or,

$$C = 0.016706 \frac{1}{R_2 + 1000000} \quad (1)$$

Also,

$$\frac{1000000 C R_2 + 1}{(R_2 + 1000000) C} = 8000 \quad (2)$$

Solving (1) and (2) simultaneously, $R_2 = 8003.7 \Omega$, and $C = 1.6573 \times 10^{-2} \mu\text{F}$.

64.

$$\begin{aligned} s\mathbf{I} - \mathbf{A} &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} (3.45 - 14000K_c) & -0.255 \times 10^{-9} \\ 0.499 \times 10^{11} & -3.68 \end{bmatrix} \\ &= \begin{bmatrix} s - (3.45 - 14000K_c) & 0.255 \times 10^{-9} \\ -0.499 \times 10^{11} & s + 3.68 \end{bmatrix} \\ |s\mathbf{I} - \mathbf{A}| &= s^2 + (0.23 + 0.14 \times 10^5 K_c)s + (51520K_c + 0.0285) \\ (2\zeta\omega_n)^2 &= [2 * 0.9]^2 * (51520K_c + 0.0285) = (0.23 + 0.14 \times 10^5 K_c)^2 \end{aligned}$$

or

$$K_c^2 - 8.187 \times 10^{-4} K_c - 2.0122 \times 10^{-10} = 0$$

Solving for K_c ,

$$K_c = 8.189 \times 10^{-4}$$

65.

a. The transfer function from Chapter 2 is,

$$\frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s + 53.85)}{(s^2 + 15.47s + 9283)(s^2 + 8.119s + 376.3)}$$

The dominant poles come from $s^2 + 8.119s + 376.3$. Using this polynomial,

$2\zeta\omega_n = 8.119$, and $\omega_n^2 = 376.3$. Thus, $\omega_n = 19.4$ and $\zeta = 0.209$. Using Eq. (4.38), %OS =

51.05%. Also, $T_s = \frac{4}{\zeta\omega_n} = 0.985$ s, and $T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 0.166$ s. To find rise time, use

Figure 4.16. Thus, $\omega_n T_r = 1.2136$ or $T_r = 0.0626$ s.

b. The other poles have a real part of $15.47/2 = 7.735$. Dominant poles have a real part of $8.119/2 = 4.06$. Thus, $7.735/4.06 = 1.91$. This is not at least 5 times.

c.

Program:

```
syms s
numg=0.7883*(s+53.85);
deng=(s^2+15.47*s+9283)*(s^2+8.119*s+376.3);
'G(s) transfer function'
G=vpa(numg/deng,3);
pretty(G)
numg=sym2poly(numg);
deng=sym2poly(deng);
G=tf(numg,deng)
```

```
step(G)
```

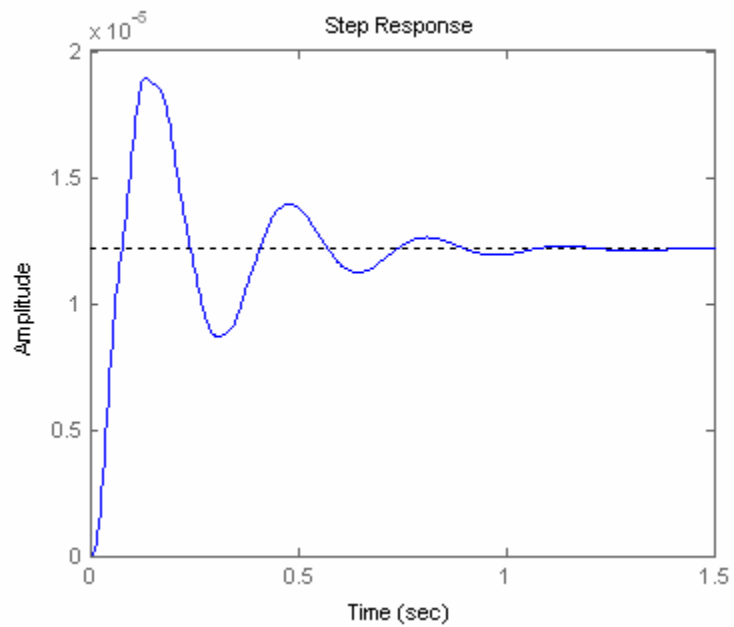
Computer response:

```
ans =  
G(s) transfer function
```

$$\frac{.788 s + 42.4}{(s^2 + 15.5 s + 9280.) (s^2 + 8.12 s + 376.)}$$

```
Transfer function:
```

$$\frac{0.7883 s + 42.45}{s^4 + 23.59 s^3 + 9785 s^2 + 8.119e004 s + 3.493e006}$$



The time response shows 58 percent overshoot, $T_s = 0.86$ s, $T_p = 0.13$ s, $T_r = 0.05$ s.