T W O

Modeling in the Frequency Domain

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Transfer Functions

Finding each transfer function:

Pot: $\frac{V_i(s)}{\theta_i(s)} = \frac{10}{\pi}$;

Pre-Amp: $\frac{V_p(s)}{V_i(s)} = K$;

Power Amp: $\frac{E_a(s)}{V_p(s)} = \frac{150}{s+150}$

Motor: $J_m = 0.05 + 5\left(\frac{50}{250}\right)^2 = 0.25$

$D_m = 0.01 + 3\left(\frac{50}{250}\right)^2 = 0.13$

$\frac{K_t}{R_a} = \frac{1}{5}$

$\frac{K_t K_b}{R_a} = \frac{1}{5}$

Therefore: $\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s\left(s+\frac{1}{J_m}\left(D_m+\frac{K_t K_b}{R_a}\right)\right)} = \frac{0.8}{s(s+1.32)}$

And: $\frac{\theta_o(s)}{E_a(s)} = \frac{1}{5}\frac{\theta_m(s)}{E_a(s)} = \frac{0.16}{s(s+1.32)}$

Transfer Function of a Nonlinear Electrical Network

Writing the differential equation, $\frac{d(i_0 + \delta i)}{dt} + 2(i_0 + \delta i)^2 - 5 = v(t)$. Linearizing i^2 about i_0,

$(i_0+\delta i)^2 - i_0^2 = 2i\Big|_{i=i_0}\delta i = 2i_0\delta i$. Thus, $(i_0+\delta i)^2 = i_0^2 + 2i_0\delta i$.

Substituting into the differential equation yields, $\frac{d\delta i}{dt} + 2i_0^2 + 4i_0\delta i - 5 = v(t)$. But, the resistor voltage equals the battery voltage at equilibrium when the supply voltage is zero since the voltage across the inductor is zero at dc. Hence, $2i_0^2 = 5$, or $i_0 = 1.58$. Substituting into the linearized differential equation, $\frac{d\delta i}{dt} + 6.32\delta i = v(t)$. Converting to a transfer function, $\frac{\delta i(s)}{V(s)} = \frac{1}{s+6.32}$. Using the linearized i about i_0, and the fact that $v_r(t)$ is 5 volts at equilibrium, the linearized $v_r(t)$ is $v_r(t) = 2i^2 = 2(i_0+\delta i)^2 = 2(i_0^2+2i_0\delta i) = 5+6.32\delta i$. For excursions away from equilibrium, $v_r(t) - 5 = 6.32\delta i = \delta v_r(t)$. Therefore, multiplying the transfer function by 6.32, yields, $\frac{\delta V_r(s)}{V(s)} = \frac{6.32}{s+6.32}$ as the transfer function about $v(t) = 0$.

ANSWERS TO REVIEW QUESTIONS

1. Transfer function

2. Linear time-invariant

3. Laplace

4. $G(s) = C(s)/R(s)$, where $c(t)$ is the output and $r(t)$ is the input.

5. Initial conditions are zero

6. Equations of motion

7. Free body diagram

8. There are direct analogies between the electrical variables and components and the mechanical variables and components.

9. Mechanical advantage for rotating systems

10. Armature inertia, armature damping, load inertia, load damping

11. Multiply the transfer function by the gear ratio relating armature position to load position.

12. (1) Recognize the nonlinear component, (2) Write the nonlinear differential equation, (3) Select the equilibrium solution, (4) Linearize the nonlinear differential equation, (5) Take the Laplace transform of the linearized differential equation, (6) Find the transfer function.

SOLUTIONS TO PROBLEMS

1.

a. $$F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

b. $$F(s) = \int_0^\infty te^{-st}dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_0^\infty = \frac{-(st+1)}{s^2e^{st}}\Bigg|_0^\infty$$

Using L'Hopital's Rule

$$F(s)\Big|_{t\to\infty} = \left.\frac{-s}{s^3 e^{st}}\right|_{t\to\infty} = 0.\ \text{Therefore,}\ F(s) = \frac{1}{s^2}.$$

c. $F(s) = \int_0^\infty \sin \omega t\, e^{-st} dt = \left.\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right|_0^\infty = \frac{\omega}{s^2+\omega^2}$

d. $F(s) = \int_0^\infty \cos \omega t\, e^{-st} dt = \left.\frac{e^{-st}}{s^2+\omega^2}(-s\cos\omega t + \omega\sin\omega t)\right|_0^\infty = \frac{s}{s^2+\omega^2}$

2.

a. Using the frequency shift theorem and the Laplace transform of sin ωt, $F(s) = \frac{\omega}{(s+a)^2+\omega^2}$.

b. Using the frequency shift theorem and the Laplace transform of cos ωt, $F(s) = \frac{(s+a)}{(s+a)^2+\omega^2}$.

c. Using the integration theorem, and successively integrating u(t) three times, $\int dt = t$; $\int t dt = \frac{t^2}{2}$;

$\int \frac{t^2}{2} dt = \frac{t^3}{6}$, the Laplace transform of $t^3u(t)$, $F(s) = \frac{6}{s^4}$.

3.

a. The Laplace transform of the differential equation, assuming zero initial conditions,

is, $(s+7)X(s) = \frac{5s}{s^2+2^2}$. Solving for X(s) and expanding by partial fractions,

$$\frac{5s}{(s+7)(s^2+4)} = -\frac{35}{53}\frac{1}{s+7} + \frac{5}{53}\frac{7s+4}{s^2+4}$$

Or,

$$\frac{5s}{(s+7)(s^2+4)} = -\frac{35}{53}\frac{1}{s+7} + \frac{5}{53}\frac{7s+2\sqrt{4}}{s^2+4}$$

Taking the inverse Laplace transform, $x(t) = -\frac{35}{53}e^{-7t} + (\frac{35}{53}\cos 2t + \frac{10}{53}\sin 2t)$.

b. The Laplace transform of the differential equation, assuming zero initial conditions, is,

$(s^2+6s+8)X(s) = \frac{15}{s^2+9}$.

Solving for X(s)

$$X(s) = \frac{15}{(s^2+9)(s^2+6s+8)}$$

and expanding by partial fractions,

$$X(s) = -\frac{3}{65}\frac{6s+\frac{1}{\sqrt{9}}\sqrt{9}}{s^2+9} - \frac{3}{10}\frac{1}{s+4} + \frac{15}{26}\frac{1}{s+2}$$

Taking the inverse Laplace transform,

$$x(t) = -\frac{18}{65}\cos(3t) - \frac{1}{65}\sin(3t) - \frac{3}{10}e^{-4t} + \frac{15}{26}e^{-2t}$$

c. The Laplace transform of the differential equation is, assuming zero initial conditions, $(s^2+8s+25)x(s) = \frac{10}{s}$. Solving for X(s)

$$X(s) = \frac{10}{s(s^2+8s+25)}$$

and expanding by partial fractions,

$$X(s) = \frac{2}{5}\frac{1}{s} - \frac{2}{5}\frac{1(s+4) + \frac{4}{\sqrt{9}}\sqrt{9}}{s+4^2+9}$$

Taking the inverse Laplace transform,

$$x(t) = \frac{2}{5} - e^{-4t}\left(\frac{8}{15}\sin(3t) + \frac{2}{5}\cos(3t)\right)$$

4.

a. Taking the Laplace transform with initial conditions, $s^2X(s)-2s+3+2sX(s)-4+2X(s) = \frac{2}{s^2+2^2}$.
Solving for X(s),

$$X(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+2s+2)}.$$

Expanding by partial fractions

$$X(s) = -\left(\frac{1}{5}\right)\frac{s+\frac{1}{\sqrt{4}}\sqrt{4}}{s^2+4} + \left(\frac{1}{5}\right)\frac{11(s+1) - \frac{3}{\sqrt{1}}\sqrt{1}}{(s+1)^2+1}$$

Therefore, $x(t) = -0.2\cos 2t - 0.1\sin 2t + e^{-t}(2.2\cos t - 0.6\sin t)$.

b. Taking the Laplace transform with initial conditions, $s^2X(s)-2s-1+2sX(s)-4+X(s) = \frac{5}{s+2} + \frac{1}{s^2}$.
Solving for X(s),

$$X(s) = \frac{2s^4+9s^3+15s^2+s+2}{(s+2)(s+1)^2 s^2}$$

$$X(s) = 5\frac{1}{s+2} - \frac{1}{s+1} + 9\frac{1}{(s+1)^2} - 2\frac{1}{s} + \frac{1}{s^2}$$

Therefore, $x(t) = 5e^{-2t} - e^{-t} + 9te^{-t} - 2 + t$.

c. Taking the Laplace transform with initial conditions, $s^2X(s)-s-2+4X(s) = \frac{2}{s^3}$. Solving for X(s),

$$X(s) = \frac{s^4+2s^3+2}{(s^2+4)s^3}$$

$$X(s) = \frac{1}{8}\frac{9s+8.2}{s^2+4} - \frac{1}{8}\frac{1}{s} + \frac{1}{2}\frac{1}{s^3}$$

Therefore, $x(t) = \frac{9}{8}\cos 2t + \sin 2t - \frac{1}{8} + \frac{1}{4}t^2$.

5.

Program:

```
syms t
f=5*t^2*cos(3*t+45);
pretty(f)
F=laplace(f);
F=simple(F);
pretty(F)
'b'
f=5*t*exp(-2*t)*sin(4*t+60);
pretty(f)
F=laplace(f);
F=simple(F);
pretty(F)
```

Computer response:

```
ans =

a


                            2
                         5 t  cos(3 t + 45)

              3                           2
             s  cos(45) - 27 s cos(45) - 9 s  sin(45) + 27 sin(45)
          10 -----------------------------------------------------
                                       2     3
                                     (s  + 9)

ans =

b


                       5 t exp(-2 t) sin(4 t + 60)

              sin(60)           ((s + 2) sin(60) + 4 cos(60)) (s + 2)
         -5 ------------- + 10 -------------------------------------
                   2                             2      2
            (s + 2)  + 16                ((s + 2)  + 16)
```

6.

Program:

```
syms s
'a'
G=(s^2+3*s+7)*(s+2)/[(s+3)*(s+4)*(s^2+2*s+100)];
pretty(G)
g=ilaplace(G);
pretty(g)
'b'
G=(s^3+4*s^2+6*s+5)/[(s+8)*(s^2+8*s+3)*(s^2+5*s+7)];
pretty(G)
g=ilaplace(G);
pretty(g)
```

Computer response:

```
ans =

a
                                2
```

```
                    (s  + 3 s + 7) (s + 2)
               --------------------------------
                                     2
               (s + 3) (s + 4) (s  + 2 s + 100)

                    11               4681             1/2          1/2
  - 7/103 exp(-3 t) + -- exp(-4 t) - ----- exp(-t) 11    sin(3 11    t)
                    54               61182

         4807                  1/2
       + ---- exp(-t) cos(3 11    t)
         5562
```

```
ans =

b
                         3       2
                        s  + 4 s  + 6 s + 5
             -------------------------------------
                         2                2
             (s + 8) (s  + 8 s + 3) (s  + 5 s + 7)

     299               1367                    1/2
   - --- exp(-8 t) + ---- exp(-4 t) cosh(13    t)
     93                417

         4895              1/2          1/2
       - ---- exp(-4 t) 13    sinh(13    t)
         5421

          232                 1/2             1/2
       - ----- exp(- 5/2 t) 3    sin(1/2 3    t)
         12927

         272                          1/2
       - ---- exp(- 5/2 t) cos(1/2 3    t)
         4309
```

7.

The Laplace transform of the differential equation, assuming zero initial conditions, is,

$(s^3+3s^2+5s+1)Y(s) = (s^3+4s^2+6s+8)X(s)$.

Solving for the transfer function, $\dfrac{Y(s)}{X(s)} = \dfrac{s^3+4s^2+6s+8}{s^3+3s^2+5s+1}$.

8.

a. Cross multiplying, $(s^2+2s+7)X(s) = F(s)$.

Taking the inverse Laplace transform, $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 7x = f(t)$.

b. Cross multiplying after expanding the denominator, $(s^2+15s+56)X(s) = 10F(s)$.

Taking the inverse Laplace transform, $\dfrac{d^2x}{dt^2} + 15\dfrac{dx}{dt} + 56x = 10f(t)$.

c. Cross multiplying, $(s^3+8s^2+9s+15)X(s) = (s+2)F(s)$.

Taking the inverse Laplace transform, $\dfrac{d^3x}{dt^3} + 8\dfrac{d^2x}{dt^2} + 9\dfrac{dx}{dt} + 15x = \dfrac{df(t)}{dt} + 2f(t)$.

9.

The transfer function is $\dfrac{C(s)}{R(s)} = \dfrac{s^5+2s^4+4s^3+s^2+3}{s^6+7s^5+3s^4+2s^3+s^2+3}$.

Cross multiplying, $(s^6+7s^5+3s^4+2s^3+s^2+3)C(s) = (s^5+2s^4+4s^3+s^2+3)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^6c}{dt^6}+7\frac{d^5c}{dt^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+\frac{d^2c}{dt^2}+3c=\frac{d^5r}{dt^5}+2\frac{d^4r}{dt^4}+4\frac{d^3r}{dt^3}+\frac{d^2r}{dt^2}+3r.$$

10.

The transfer function is $\dfrac{C(s)}{R(s)}=\dfrac{s^4+2s^3+5s^2+s+1}{s^5+3s^4+2s^3+4s^2+5s+2}$.

Cross multiplying, $(s^5+3s^4+2s^3+4s^2+5s+2)C(s) = (s^4+2s^3+5s^2+s+1)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^5c}{dc^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+4\frac{d^2c}{dt^2}+5\frac{dc}{dt}+2c=\frac{d^4r}{dt^4}+2\frac{d^3r}{dt^3}+5\frac{d^2r}{dt^2}+\frac{dr}{dt}+r.$$

Substituting $r(t) = t^3$, $\dfrac{d^5c}{dc^5}+3\dfrac{d^4c}{dt^4}+2\dfrac{d^3c}{dt^3}+4\dfrac{d^2c}{dt^2}+5\dfrac{dc}{dt}+2c$

$= 18\delta(t) + (36 + 90t + 9t^2 + 3t^3)\,u(t)$.

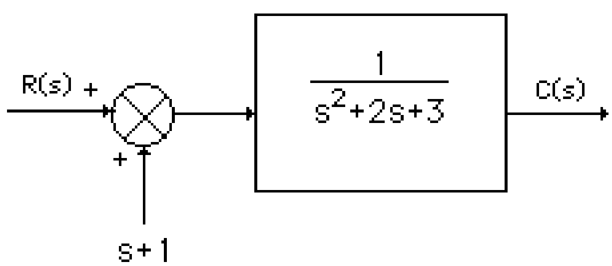
11.

Taking the Laplace transform of the differential equation, $s^2X(s)-s+1+2sX(s)-2+3x(s)=R(s)$.

Collecting terms, $(s^2+2s+3)X(s) = R(s)+s+1$.

Solving for $X(s)$, $X(s)=\dfrac{R(s)}{s^2+2s+3}+\dfrac{s+1}{s^2+2s+3}$.

The block diagram is then,



12.

Program:

```
'Factored'
Gzpk=zpk([-15 -26 -72],[0 -55 roots([1 5 30])' roots([1 27 52])'],5)
'Polynomial'
Gp=tf(Gzpk)
```

Computer response:

```
ans =

Factored

Zero/pole/gain:
           5 (s+15) (s+26) (s+72)
--------------------------------------------
s (s+55) (s+24.91) (s+2.087) (s^2 + 5s + 30)

ans =
```

```
Polynomial

Transfer function:
                 5 s^3 + 565 s^2 + 16710 s + 140400
--------------------------------------------------------------------
s^6 + 87 s^5 + 1977 s^4 + 1.301e004 s^3 + 6.041e004 s^2 + 8.58e004 s
```

13.

Program:

```
'Polynomial'
Gtf=tf([1 25 20 15 42],[1 13 9 37 35 50])
'Factored'
Gzpk=zpk(Gtf)
```

Computer response:

```
ans =

Polynomial

Transfer function:
    s^4 + 25 s^3 + 20 s^2 + 15 s + 42
-----------------------------------------
s^5 + 13 s^4 + 9 s^3 + 37 s^2 + 35 s + 50

ans =

Factored

Zero/pole/gain:
      (s+24.2) (s+1.35) (s^2 - 0.5462s + 1.286)
-------------------------------------------------------
(s+12.5) (s^2 + 1.463s + 1.493) (s^2 - 0.964s + 2.679)
```

14.

Program:

```
numg=[-10 -60];
deng=[0 -40 -30 (roots([1 7 100]))' (roots([1 6 90]))'];
[numg,deng]=zp2tf(numg',deng',1e4);
Gtf=tf(numg,deng)
G=zpk(Gtf)
[r,p,k]=residue(numg,deng)
```

Computer response:

```
Transfer function:
                        10000 s^2 + 700000 s + 6e006
-----------------------------------------------------------------------------
s^7 + 83 s^6 + 2342 s^5 + 33070 s^4 + 3.735e005 s^3 + 2.106e006 s^2

                                                              + 1.08e007 s

Zero/pole/gain:
              10000 (s+60) (s+10)
------------------------------------------------
s (s+40) (s+30) (s^2 + 6s + 90) (s^2 + 7s + 100)

r =

  -0.0073
   0.0313
   2.0431 - 2.0385i
   2.0431 + 2.0385i
  -2.3329 + 2.0690i
  -2.3329 - 2.0690i
   0.5556

p =
```

```
 -40.0000
 -30.0000
  -3.5000 + 9.3675i
  -3.5000 - 9.3675i
  -3.0000 + 9.0000i
  -3.0000 - 9.0000i
        0

k =

     []
```

15.

Program:

```
syms s
'(a)'
Ga=45*[(s^2+37*s+74)*(s^3+28*s^2+32*s+16)]...
/[(s+39)*(s+47)*(s^2+2*s+100)*(s^3+27*s^2+18*s+15)];
'Ga symbolic'
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);
'Ga polynimial'
Ga=tf(numga,denga)
'Ga factored'
Ga=zpk(Ga)
'(b)'
Ga=56*[(s+14)*(s^3+49*s^2+62*s+53)]...
/[(s^2+88*s+33)*(s^2+56*s+77)*(s^3+81*s^2+76*s+65)];
'Ga symbolic'
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);
'Ga polynimial'
Ga=tf(numga,denga)
'Ga factored'
Ga=zpk(Ga)
```

Computer response:

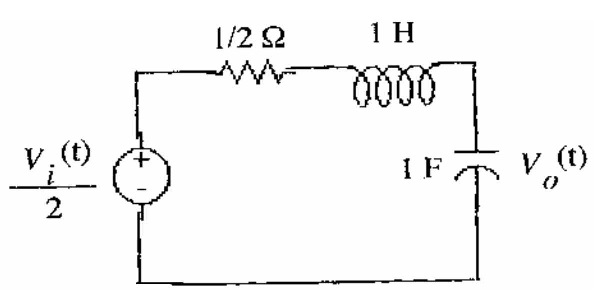
```
ans =

(a)

ans =

Ga symbolic

                      2                 3       2
                    (s  + 37 s + 74) (s  + 28 s  + 32 s + 16)
        45 ----------------------------------------------------------
                               2                 3       2
              (s + 39) (s + 47) (s  + 2 s + 100) (s  + 27 s  + 18 s + 15)

ans =

Ga polynimial

Transfer function:

         45 s^5 + 2925 s^4 + 51390 s^3 + 147240 s^2 + 133200 s + 53280
--------------------------------------------------------------------------------
```

```
s^7 + 115 s^6 + 4499 s^5 + 70700 s^4 + 553692 s^3 + 5.201e006 s^2 + 3.483e006 s

                                                                  + 2.75e006

ans =

Ga factored

Zero/pole/gain:
     45 (s+34.88) (s+26.83) (s+2.122) (s^2 + 1.17s + 0.5964)
-----------------------------------------------------------------
(s+47) (s+39) (s+26.34) (s^2 + 0.6618s + 0.5695) (s^2 + 2s + 100)


ans =

(b)

ans =

Ga symbolic

                                   3        2
                 (s + 14) (s  + 49 s  + 62 s + 53)
         56 ---------------------------------------------------------
              2                2                  3        2
            (s  + 88 s + 33) (s  + 56 s + 77) (s  + 81 s  + 76 s + 65)

ans =

Ga polynimial

Transfer function:

                56 s^4 + 3528 s^3 + 41888 s^2 + 51576 s + 41552
--------------------------------------------------------------------------------
s^7 + 225 s^6 + 16778 s^5 + 427711 s^4 + 1.093e006 s^3 + 1.189e006 s^2

                                                         + 753676 s + 165165

ans =

Ga factored

Zero/pole/gain:
                56 (s+47.72) (s+14) (s^2 + 1.276s + 1.111)
---------------------------------------------------------------------------
(s+87.62) (s+80.06) (s+54.59) (s+1.411) (s+0.3766) (s^2 + 0.9391s + 0.8119)
```

16.

a. Writing the node equations, $\frac{V_o - V_i}{s} + \frac{V_o}{s} + V_o = 0$. Solve for $\frac{V_o}{V_i} = \frac{1}{s+2}$.

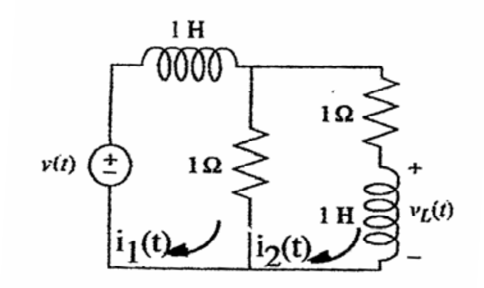
b. Thevenizing,



Using voltage division, $V_o(s) = \frac{V_i(s)}{2}\frac{\frac{1}{s}}{\frac{1}{2}+s+\frac{1}{s}}$. Thus, $\frac{V_o(s)}{V_i(s)} = \frac{1}{2s^2+s+2}$

17.

a.



Writing mesh equations

$$(s+1)I_1(s) - I_2(s) = V_i(s)$$

$$-I_1(s) + (s+2)I_2(s) = 0$$

But, $I_1(s) = (s+2)I_2(s)$. Substituting this in the first equation yields,

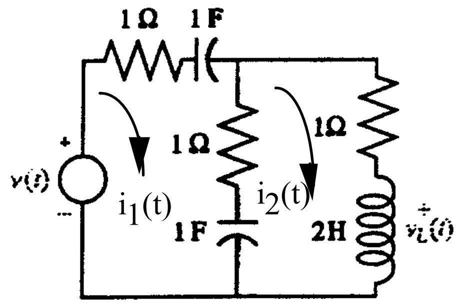
$$(s+1)(s+2)I_2(s) - I_2(s) = V_i(s)$$

or

$$I_2(s)/V_i(s) = 1/(s^2 + 3s + 1)$$

But, $V_L(s) = sI_2(s)$. Therefore, $V_L(s)/V_i(s) = s/(s^2 + 3s + 1)$.

b.



$$(2+\frac{2}{s})I_1(s)-(1+\frac{1}{s})I_2(s)=V(s)$$

$$-(1+\frac{1}{s})I_1(s)+(2+\frac{1}{s}+2s)I_2(s)=0$$

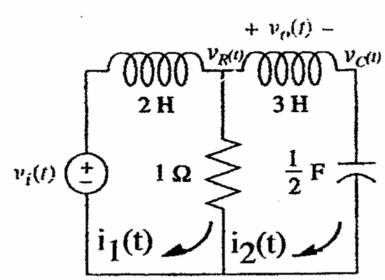
Solving for $I_2(s)$:

$$I_2(s)=\frac{\begin{vmatrix}\frac{2(s+1)}{s} & V(s)\\ -\frac{s+1}{s} & 0\end{vmatrix}}{\begin{vmatrix}\frac{2(s+1)}{s} & -\frac{s+1}{s}\\ -\frac{s+1}{s} & \frac{2s^2+2s+1}{s}\end{vmatrix}}=\frac{V(s)s}{4s^2+3s+1}$$

Therefore, $\frac{V_L(s)}{V(s)}=2s\frac{I_2(s)}{V(s)}=\frac{2s^2}{4s^2+3s+1}$

18.

a.



Writing mesh equations,

$$(2s+1)I_1(s)-I_2(s)=V_i(s)$$

$$-I_1(s)+(3s+1+2/s)I_2(s)=0$$

Solving for $I_2(s)$,

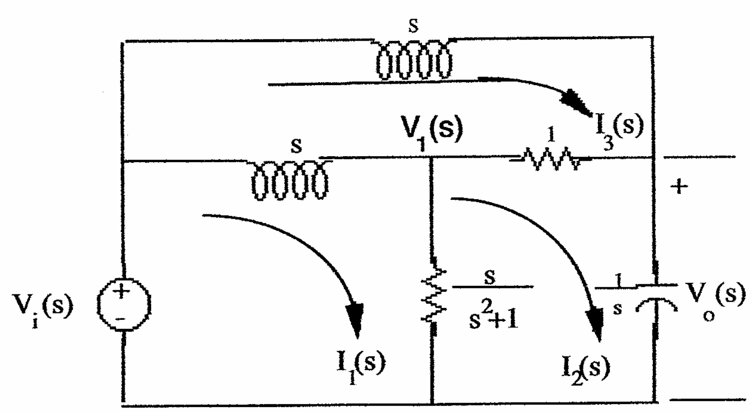
$$I_2(s) = \frac{\begin{vmatrix} 2s+1 & V_i(s) \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2s+1 & -1 \\ -1 & \dfrac{3s^2+s+2}{s} \end{vmatrix}}$$

Solving for $I_2(s)/V_i(s)$,

$$\frac{I_2(s)}{V_i(s)} = \frac{s}{6s^3+5s^2+4s+2}$$

But $V_o(s) = I_2(s)3s$. Therefore , $G(s) = 3s^2/(6s^3 + 5s^2 + 4s + 2)$.

b. Transforming the network yields,



Writing the loop equations,

$$(s+\frac{s}{s^2+1})I_1(s) - \frac{s}{s^2+1}I_2(s) - sI_3(s) = V_i(s)$$
$$-\frac{s}{s^2+1}I_1(s) + (\frac{s}{s^2+1}+1+\frac{1}{s})I_2(s) - I_3(s) = 0$$
$$-sI_1(s) - I_2(s) + (2s+1)I_3(s) = 0$$

Solving for $I_2(s)$,

$$I_2(s) = \frac{s(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}V_i(s)$$

But, $V_o(s) = \dfrac{I_2(s)}{s} = \dfrac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}V_i(s)$. Therefore,

$$\frac{V_o(s)}{V_i(s)} = \frac{s^2+2s+2}{s^4+2s^3+3s^2+3s+2}$$

19.

a. Writing the nodal equations yields,

$$\frac{V_R(s) - V_i(s)}{2s} + \frac{V_R(s)}{1} + \frac{V_R(s) - V_C(s)}{3s} = 0$$

$$-\frac{1}{3s}V_R(s) + \left(\frac{1}{2}s + \frac{1}{3s}\right)V_C(s) = 0$$

Rewriting and simplifying,

$$\frac{6s+5}{6s}V_R(s) - \frac{1}{3s}V_C(s) = \frac{1}{2s}V_i(s)$$

$$-\frac{1}{3s}V_R(s) + \left(\frac{3s^2+2}{6s}\right)V_C(s) = 0$$

Solving for $V_R(s)$ and $V_C(s)$,

$$V_R(s) = \frac{\begin{vmatrix} \frac{1}{2s}V_i(s) & -\frac{1}{3s} \\ 0 & \frac{3s^2+2}{6s} \end{vmatrix}}{\begin{vmatrix} \frac{6s+5}{6s} & -\frac{1}{3s} \\ -\frac{1}{3s} & \frac{3s^2+2}{6s} \end{vmatrix}}; \quad V_C(s) = \frac{\begin{vmatrix} \frac{6s+5}{6s} & \frac{1}{2s}V_i(s) \\ -\frac{1}{3s} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{6s+5}{6s} & -\frac{1}{3s} \\ -\frac{1}{3s} & \frac{3s^2+2}{6s} \end{vmatrix}}$$

Solving for $V_o(s)/V_i(s)$

$$\frac{V_o(s)}{V_i(s)} = \frac{V_R(s) - V_C(s)}{V_i(s)} = \frac{3s^2}{6s^3 + 5s^2 + 4s + 2}$$

b. Writing the nodal equations yields,

$$\frac{(V_1(s) - V_i(s))}{s} + \frac{(s^2+1)}{s}V_1(s) + (V_1(s) - V_o(s)) = 0$$

$$(V_o(s) - V_1(s)) + sV_o(s) + \frac{(V_o(s) - V_i(s))}{s} = 0$$

Rewriting and simplifying,

$$(s + \frac{2}{s} + 1)V_1(s) - V_o(s) = \frac{1}{s}V_i(s)$$

$$V_1(s) + (s + \frac{1}{s} + 1)V_o(s) = \frac{1}{s}V_i(s)$$

Solving for $V_o(s)$

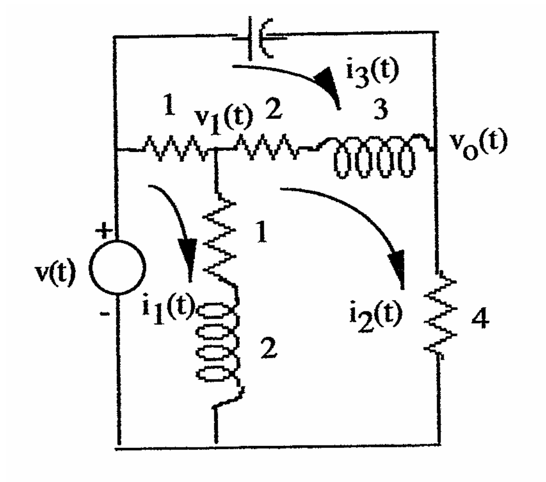
$$V_o(s) = \frac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2} V_i(s).$$

Hence,

$$\frac{V_o(s)}{V_i(s)} = \frac{(s^2+2s+2)}{s^4+2s^3+3s^2+3s+2}$$

20.

a.



Mesh:

$$(2+2s)I_1(s) - (1+2s)I_2(s) - I_3(s) = V(s)$$

$$-(1+2s)I_1(s) + (7+5s)I_2(s) - (2+3s)I_3(s) = 0$$

$$-I_1(s) - (2+3s)I_2(s) + \left(3+3s+\frac{5}{s}\right)I_3(s) = 0$$

Nodal:

$$V_1(s) - V(s) + \frac{V_1(s)}{(1+2s)} + \frac{(V_1(s) - V_o(s))}{2+3s} = 0$$

$$\frac{(V_o(s) - V_1(s))}{2+3s} + \frac{V_o(s)}{4} + \frac{(V_o(s) - V(s))}{\frac{5}{s}} = 0$$

or

$$\frac{6s^2+12s+5}{6s^2+7s+2} V_1(s) - \frac{1}{3s+2} V_o(s) = V(s)$$

$$-\frac{1}{3s+2} V_1(s) + \frac{1}{20}\frac{12s^2+23s+30}{3s+2} V_o(s) = \frac{s}{5} V(s)$$

b.

Program:

```
syms s V                          %Construct symbolic object for frequency
```

```
                              %variable 's' and V.
'Mesh Equations'
A2=[(2+2*s) V -1
-(1+2*s) 0 -(2+3*s)
-1 0 (3+3*s+(5/s))]           %Form Ak = A2.
A=[(2+2*s) -(1+2*s) -1
-(1+2*s) (7+5*s) -(2+3*s)
-1 -(2+3*s) (3+3*s+(5/s))]    %Form A.
I2=det(A2)/det(A);            %Use Cramer's Rule to solve for I2.
G1=I2/V;                      %Form transfer function, G1(s) = I2(s)/V(s).
G=4*G1;                       %Form transfer function, G(s) = V4(s)/V(s).
'G(s) via Mesh Equations'     %Display label.
pretty(G)                     %Pretty print G(s)


'Nodal Equations'
A2=[(6*s^2+12*s+5)/(6*s^2+7*s+2) V
-1/(3*s+2) s*(V/5)]           %Form Ak = A2.
A=[(6*s^2+12*s+5)/(6*s^2+7*s+2) -1/(3*s+2)
-1/(3*s+2) (1/20)*(12*s^2+23*s+30)/(3*s+2)]
                              %Form A.
I2=simple(det(A2))/simple(det(A));
                              %Use Cramer's Rule to solve for I2.
G1=I2/V;                      %Form transfer function, G1(s) = I2(s)/V(s).
'G(s) via Nodal Equations'    %Display label.
pretty(G)                     %Pretty print G(s)
```

Computer response:

```
ans =

Mesh Equations

A2 =

[     2+2*s,         V,        -1]
[    -1-2*s,         0,    -2-3*s]
[        -1,         0, 3+3*s+5/s]

A =

[     2+2*s,    -1-2*s,        -1]
[    -1-2*s,     7+5*s,    -2-3*s]
[        -1,    -2-3*s, 3+3*s+5/s]

ans =

G(s) via Mesh Equations



                                    2          3
                          15 s + 12 s  + 5 + 6 s
                       4 -------------------------
                                    2           3
                         120 s + 78 s  + 65 + 24 s

ans =

Nodal Equations

A2 =

[ (6*s^2+12*s+5)/(2+7*s+6*s^2),                            V]
[                  -1/(2+3*s),                       1/5*s*V]

A =
```

```
[  (6*s^2+12*s+5)/(2+7*s+6*s^2),                          -1/(2+3*s)]
[                      -1/(2+3*s), (3/5*s^2+23/20*s+3/2)/(2+3*s)]

ans =

G(s) via Nodal Equations

                                           2            3
                               15 s + 12 s  + 5 + 6 s
                             4 -------------------------
                                      3        2
                               24 s  + 78 s  + 120 s + 65
```

21.

a.

$$Z_1(s) = 5x10^5 + \frac{10^6}{s}$$

$$Z_2(s) = 10^5 + \frac{10^6}{s}$$

Therefore,

$$-\frac{Z_2(s)}{Z_1(s)} = -\frac{1}{5}\,\frac{s+10}{s+2}$$

b.

$$Z_1(s) = 100000 + \frac{1}{1\times10^{-6}\,s} = 100000\,\frac{s+10}{s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}\,s + \frac{1}{100000}} = 100000\,\frac{s+20}{s+10}$$

Therefore,

$$G(s) = -\frac{Z_2}{Z_1} = -\frac{(s+20)\,s}{(s+10)^2}$$

22.

a.

$$Z_1(s) = 200000 + \frac{1}{1\times10^{-6}\,s}$$

$$Z_2(s) = 100000 + \frac{1}{1\times10^{-6}\,s}$$

Therefore,

$$G(s) = \frac{Z_1+Z_2}{Z_1} = \frac{3}{2}\,\frac{s+\frac{20}{3}}{s+5}.$$

b.

$$Z_1(s) = 2x10^5 + \frac{\frac{5x10^{11}}{s}}{5x10^5 + \frac{10^6}{s}}$$

$$Z_2(s) = 5x10^5 + \frac{\frac{10^{11}}{s}}{10^5 + \frac{10^6}{s}}$$

Therefore,

$$\frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{7}{2}\frac{(s+3.18)(s+11.68)}{(s+7)(s+10)}$$

23.

Writing the equations of motion, where $x_2(t)$ is the displacement of the right member of springr,

$$(s^2+s+1)X_1(s) \;\; -X_2(s) = 0$$

$$-X_1(s) + X_2(s) = F(s)$$

Adding the equations,

$$(s^2+s)X_1(s) \; = F(s)$$

From which, $\frac{X_1(s)}{F(s)} = \frac{1}{s(s+1)}$.

24.

Writing the equations of motion,

$$(s^2 + s + 1)X_1(s) - (s+1)X_2(s) = F(s)$$

$$-(s+1)X_1(s) + (s^2 + s + 1)X_2(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} (s^2+s+1) & F(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -(s+1) \\ -(s+1) & (s^2+s+1) \end{vmatrix}} = \frac{(s+1)F(s)}{s^2(s^2+2s+2)}$$

From which,

$$\frac{X_2(s)}{F(s)} = \frac{(s+1)}{s^2(s^2+2s+2)}.$$

25.

Let $X_1(s)$ be the displacement of the left member of the spring and $X_3(s)$ be the displacement of the mass.

Writing the equations of motion

$$2x_1(s) - 2x_2(s) = F(s)$$
$$-2X_1(s) + (5s+2)X_2(s) - 5sX_3(s) = 0$$
$$-5sX_2(s) + (10s^2 + 7s)X_3(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} 5s^2+10 & F(s) \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} 5s^2+10 & -10 \\ -10 & \frac{1}{5}s+10 \end{vmatrix}} = \frac{10F(s)}{s(s^2+50s+2)}$$

Thus, $\dfrac{X_2(s)}{F(s)} = \dfrac{1}{10}\dfrac{(10s+7)}{s(5s+1)}$

26.

$$\left(s^2+3s+2\right)X_1(s) - (s+1)X_2(s) = 0$$
$$-(s+1)X_1(s) + \left(s^2+2s+1\right)X_2(s) = F(s)$$

Solving for $X_1(s)$; $X_1 = \dfrac{\left|\begin{pmatrix} 0 & -[s+1] \\ F & s^2+2s+1 \end{pmatrix}\right|}{\left|\begin{pmatrix} s^2+3s+2 & -[s+1] \\ -[s+1] & s^2+2s+1 \end{pmatrix}\right|} = \dfrac{F(s)}{s^3+4s^2+4s+1}$. Thus, $\dfrac{X_1}{F(s)} = \dfrac{1}{s^3+4s^2+4s+1}$

27.

Writing the equations of motion,

$$(s^2+s+1)X_1(s) - sX_2(s) = 0$$
$$-sX_1(s) + (s^2+2s+1)X_2(s) - X_3(s) = F(s)$$
$$-X_2(s) + (s^2+s+1)X_3(s) = 0$$

Solving for $X_3(s)$,

$$X_3(s) = \frac{\begin{vmatrix} (s^2+s+1) & -s & 0 \\ -s & (s^2+2s+1) & F(s) \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -s & 0 \\ -s & (s^2+2s+1) & -1 \\ 0 & -1 & (s^2+s+1) \end{vmatrix}} = \frac{F(s)}{s(s^3+3s^2+3s+3)}$$

From which, $\frac{X_3(s)}{F(s)} = \frac{1}{s(s^3+3s^2+3s+3)}$.

28.

a.

$$(s^2+2s+1)X_1(s) - 2sX_2(s) - X_3(s) = F(s)$$
$$-2sX_1(s) + (s^2+4s)X_2(s) - sX_3(s) = 0$$
$$-X_1(s) - sX_2(s) + (s+1)X_3(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} (s^2+2s+1) & F(s) & -1 \\ -2s & 0 & -s \\ -1 & 0 & s+1 \end{vmatrix}}{\Delta} = \frac{-F(s)\begin{vmatrix} -2s & -s \\ -1 & s+1 \end{vmatrix}}{\Delta}$$

or,

$$\frac{X_2(s)}{F(s)} = \frac{2s+3}{s(s^3+6s^2+9s+3)}$$

b.

$$(4s^2+s+4)X_1(s) - (s+1)X_2(s) - 3X_3(s) = 0$$
$$-(s+1)X_1(s) + (2s^2+5s+1)X_2(s) - 4sX_3(s) = F(s)$$
$$-3X_1(s) - 4sX_2(s) + (4s+3)X_3(s) = 0$$

Solving for $X_3(s)$,

$$X_3(s) = \frac{\begin{vmatrix} (4s^2+s+4) & -(s+1) & 0 \\ -(s+1) & (2s^2+5s+1) & F(s) \\ -3 & -4s & 0 \end{vmatrix}}{\Delta} = \frac{-F(s)\begin{vmatrix} (4s^2+s+4) & -(s+1) \\ -3 & -4s \end{vmatrix}}{\Delta}$$

or

$$\frac{X_3(s)}{F(s)} = \frac{16s^3+4s^2+19s+3}{32s^5+48s^4+114s^3+18s^2}$$

29.

Writing the equations of motion,

$$(s^2+2s+2)X_1(s) - X_2(s) - sX_3(s) = 0$$
$$-X_1(s) + (s^2+s+1)X_2(s) - sX_3(s) = F(s)$$
$$-sX_1(s) - sX_2(s) + (s^2+2s+1) = 0$$

30.

a.

Writing the equations of motion,

$$(s^2 + 9s + 8)\theta_1(s) - (2s + 8)\theta_2(s) = 0$$
$$-(2s + 8)\theta_1(s) + (s^2 + 2s + 11)\theta_2(s) = T(s)$$

b.

Defining

$\theta_1(s)$ = rotation of J_1

$\theta_2(s)$ = rotation between K_1 and D_1

$\theta_3(s)$ = rotation of J_3

$\theta_4(s)$ = rotation of right-hand side of K_2

the equations of motion are

$$(J_1 s^2 + K_1)\theta_1(s) - K_1\theta_2(s) = T(s)$$
$$-K_1\theta_1(s) + (D_1 s + K_1)\theta_2(s) - D_1 s\theta_3(s) = 0$$
$$-D_1 s\theta_2(s) + (J_2 s^2 + D_1 s + K_2)\theta_3(s) - K_2\theta_4(s) = 0$$
$$-K_2\theta_3(s) + (D_2 s + (K_2 + K_3))\theta_4(s) = 0$$

31.

Writing the equations of motion,

$$(s^2 + 2s + 1)\theta_1(s) - (s + 1)\theta_2(s) = T(s)$$
$$-(s + 1)\theta_1(s) + (2s + 1)\theta_2(s) = 0$$

Solving for $\theta_2(s)$

$$\theta_2(s) = \frac{\begin{vmatrix} (s^2 + 2s + 1) & T(s) \\ -(s + 1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2 + 2s + 1) & -(s + 1) \\ -(s + 1) & (2s + 1) \end{vmatrix}} = \frac{T(s)}{2s(s + 1)}$$

Hence,

$$\frac{\theta_2(s)}{T(s)} = \frac{1}{2s(s + 1)}$$

32.

Reflecting impedances to θ_3,

$$(J_{eq}s^2+D_{eq}s)\theta_3(s) = T(s)\left(\frac{N_4N_2}{N_3N_1}\right)$$

Thus,

$$\frac{\theta_3(s)}{T(s)} = \frac{\frac{N_4N_2}{N_3N_1}}{J_{eq}s^2+D_{eq}s}$$

where

$$J_{eq} = J_4+J_5+(J_2+J_3)\left(\frac{N_4}{N_3}\right)^2 + J_1\left(\frac{N_4N_2}{N_3N_1}\right)^2, \text{ and}$$

$$D_{eq} = (D_4+D_5)+(D_2+D_3)\left(\frac{N_4}{N_3}\right)^2 + D_1\left(\frac{N_4N_2}{N_3N_1}\right)^2$$

33.

Reflecting all impedances to $\theta_2(s)$,

$$\left\{\left[J_2+J_1\left(\frac{N_2}{N_1}\right)^2+J_3\left(\frac{N_3}{N_4}\right)^2\right]s^2+\left[f_2+f_1\left(\frac{N_2}{N_1}\right)^2+f_3\left(\frac{N_3}{N_4}\right)^2\right]s+\left[K\left(\frac{N_3}{N_4}\right)^2\right]\right\}\theta_2(s) = T(s)\frac{N_2}{N_1}$$

Substituting values,

$$\left\{\left[1+2(3)^2+16\left(\frac{1}{4}\right)^2\right]s^2+\left[2+1(3)^2+32\left(\frac{1}{4}\right)^2\right]s+64\left(\frac{1}{4}\right)^2\right\}\theta_2(s) = T(s)(3)$$

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{3}{20s^2+13s+4}$$

34.

Reflecting impedances to θ_2,

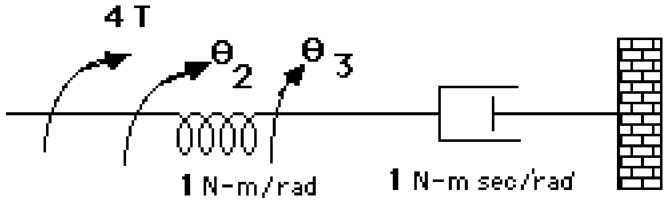
$$\left[200+3\left(\frac{50}{5}\right)^2+200\left(\frac{5}{25}x\frac{50}{5}\right)^2\right]s^2+\left[1000\left(\frac{5}{25}x\frac{50}{5}\right)^2\right]s+\left[250+3\left(\frac{50}{5}\right)^2\right]=\left(\frac{50}{5}\right)T(s)$$

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{10}{1300s^2+4000s+550}$$

35.

Reflecting impedances and applied torque to respective sides of the spring yields the following equivalent circuit:



Writing the equations of motion,

$$\theta_2(s) - \theta_3(s) = 4T(s)$$

$$-\theta_2(s) + (s+1)\theta_3(s) = 0$$

Solving for $\theta_3(s)$,

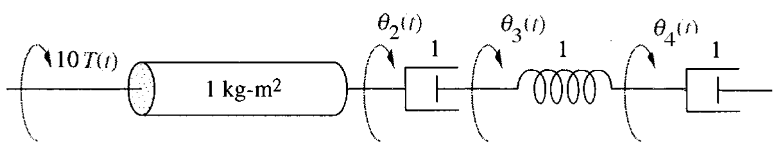
$$\theta_3(s) = \frac{\begin{vmatrix} 1 & 4T(s) \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -1 & (s+1) \end{vmatrix}} = \frac{4T(s)}{s}$$

Hence, $\frac{\theta_3(s)}{T(s)} = \frac{4}{s}$. But, $\theta_4(s) = \frac{1}{5}\,\theta_3(s)$. Thus, $\frac{\theta_4(s)}{T(s)} = \frac{4/5}{s}$.

36.

Reflecting impedances and applied torque to respective sides of the viscous damper yields the following

equivalent circuit:



Writing the equations of motion,

$$(s^2+s)\theta_2(s) - s\theta_3(s) = 10T(s)$$
$$-s\theta_2(s) + (s+1)\theta_3(s) - \theta_4(s) = 0$$
$$-\theta_3(s) + (s+1)\theta_4(s) = 0$$

Solving for $\theta_4(s)$,

$$\theta_4(s) = \frac{\begin{vmatrix} s(s+1) & -s & 10T(s) \\ -s & (s+1) & 0 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} s(s+1) & -s & 0 \\ -s & (s+1) & -1 \\ 0 & -1 & (s+1) \end{vmatrix}} = \frac{s10T(s)}{\begin{vmatrix} s(s+1) & -s & 0 \\ -s & (s+1) & -1 \\ 0 & -1 & (s+1) \end{vmatrix}}$$

Thus,

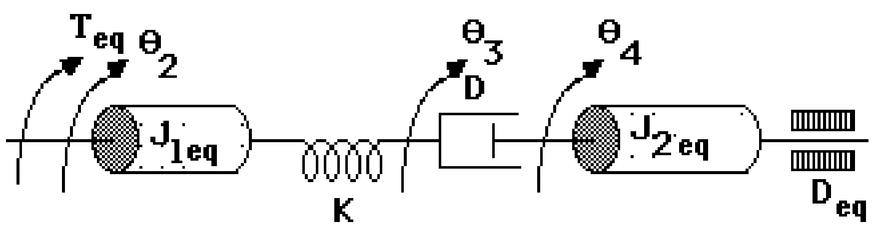
$$\frac{\theta_4(s)}{T(s)} = \frac{10}{s(s+1)^2}$$

But, $\theta_L(s) = 5\theta_4(s)$. Hence,

$$\frac{\theta_4(s)}{T(s)} = \frac{50}{s(s+1)^2}$$

37.

Reflect all impedances on the right to the viscous damper and reflect all impedances and torques on the left to the spring and obtain the following equivalent circuit:



Writing the equations of motion,

$$(J_{1eq}s^2+K)\theta_2(s) - K\theta_3(s) = T_{eq}(s)$$

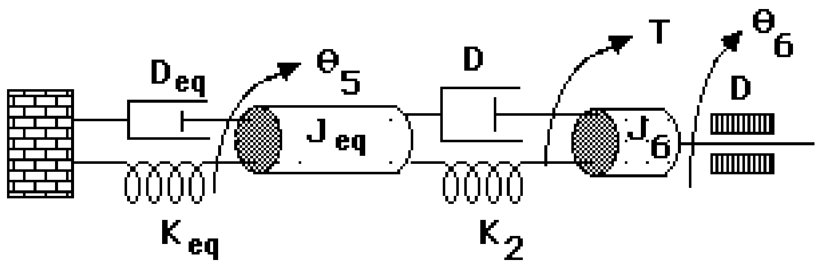
$$-K\theta_2(s)+(Ds+K)\theta_3(s) - Ds\theta_4(s) = 0$$

$$-Ds\theta_3(s) + [J_{2eq}s^2 + (D+D_{eq})s]\theta_4(s) = 0$$

where: $J_{1eq} = J_2+(J_a+J_1)\left(\frac{N_2}{N_1}\right)^2$; $J_{2eq} = J_3+(J_L+J_4)\left(\frac{N_3}{N_4}\right)^2$; $D_{eq} = D_L\left(\frac{N_3}{N_4}\right)^2$; $\theta_2(s) = \theta_1(s)\frac{N_1}{N_2}$.

38.

Reflect impedances to the left of J_5 to J_5 and obtain the following equivalent circuit:



Writing the equations of motion,

$$[J_{eq}s^2+(D_{eq}+D)s+(K_2+K_{eq})]\theta_5(s) \qquad -[Ds+K_2]\theta_6(s) = 0$$

$$-[K_2+Ds]\theta_5(s) + [J_6s^2+2Ds+K_2]\theta_6(s) = T(s)$$

From the first equation, $\dfrac{\theta_6(s)}{\theta_5(s)} = \dfrac{J_{eq}s^2+(D_{eq}+D)s+ (K_2+K_{eq})}{Ds+K_2}$. But, $\dfrac{\theta_5(s)}{\theta_1(s)} = \dfrac{N_1N_3}{N_2N_4}$. Therefore,

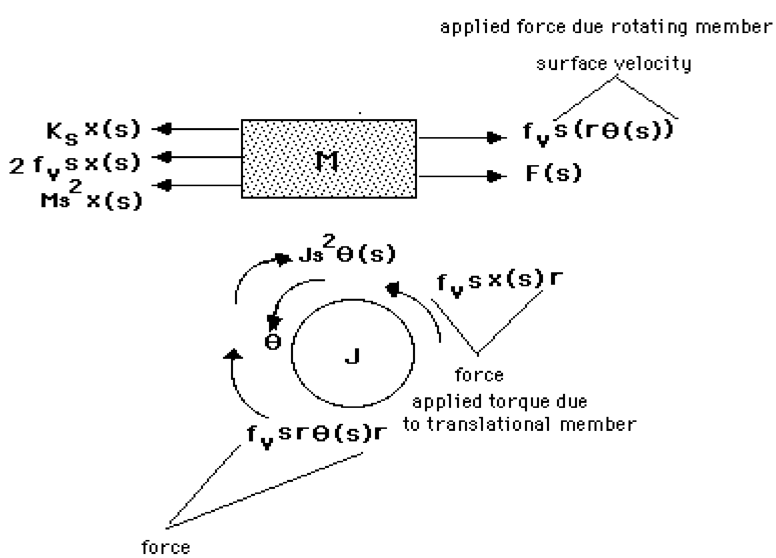
$$\frac{\theta_6(s)}{\theta_1(s)} = \frac{N_1N_3}{N_2N_4}\left(\frac{J_{eq}s^2+(D_{eq}+D)s+ (K_2+K_{eq})}{Ds+K_2}\right),$$

where $J_{eq} = \left[J_1\left(\frac{N_4N_2}{N_3N_1}\right)^2 + (J_2+J_3)\left(\frac{N_4}{N_3}\right)^2 + (J_4+J_5)\right]$, $K_{eq} = K_1\left(\frac{N_4}{N_3}\right)^2$, and

$D_{eq} = D\left[\left(\frac{N_4N_2}{N_3N_1}\right)^2 + \left(\frac{N_4}{N_3}\right)^2 + 1\right]$.

39.

Draw the freebody diagrams,



Write the equations of motion from the translational and rotational freebody diagrams,

$$(Ms^2+2f_v s+K_2)X(s) \qquad -f_v rs\theta(s) = F(s)$$

$$-f_v rsX(s) +(Js^2+f_v r^2 s)\theta(s) = 0$$

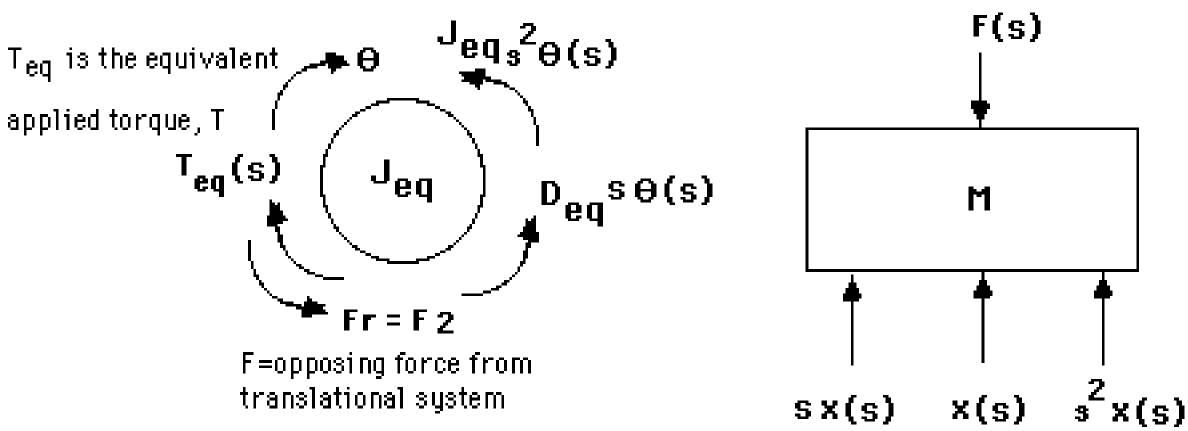
Solve for $\theta(s)$,

$$\theta(s) = \frac{\begin{vmatrix} Ms^2+2f_v s+K_2 & F(s) \\ -f_v rs & 0 \end{vmatrix}}{\begin{vmatrix} Ms^2+2f_v s+K_2 & -f_v rs \\ -f_v rs & Js^2+f_v r^2 s \end{vmatrix}} = \frac{f_v rF(s)}{JMs^3+(2Jf_v+Mf_v r^2)s^2+(JK_2+f_v^2 r^2)s+K_2 f_v r^2}$$

From which, $\dfrac{\theta(s)}{F(s)} = \dfrac{f_v r}{JMs^3+(2Jf_v+Mf_v r^2)s^2+(JK_2+f_v^2 r^2)s+K_2 f_v r^2}$.

40.

Draw a freebody diagram of the translational system and the rotating member connected to the translational system.



From the freebody diagram of the mass, $F(s) = (s^2+s+1)X(s)$. Summing torques on the rotating member,

$(J_{eq}s^2 +D_{eq}s)\theta(s) + F(s)2 = T_{eq}(s)$. Substituting F(s) above, $(J_{eq}s^2 +D_{eq}s)\theta(s) + (2s^2+2s+2)X(s) = T_{eq}(s)$. However, $\theta(s) = \dfrac{X(s)}{2}$. Substituting and simplifying,

$$T_{eq} = \left[\left(\frac{J_{eq}}{2} +2\right)s^2 +\left(\frac{D_{eq}}{2} +2\right)s+2\right]X(s)$$

But, $J_{eq} = 1+1(4)^2 = 17$, $D_{eq} = 1(2)^2 + 1 = 5$, and $T_{eq}(s) = 4T(s)$. Therefore, $\left[\frac{21}{2}s^2+\frac{9}{2}s+2\right]X(s) = 4T(s)$. Finally, $\frac{X(s)}{T(s)} = \frac{\frac{8}{21}}{s^2+\frac{9}{21}s+\frac{4}{21}}$.

41.

Writing the equations of motion,

$$\begin{aligned}(J_1s^2+K_1)\theta_1(s) \quad - K_1\theta_2(s) \quad &= T(s)\\ -K_1\theta_1(s) + (J_2s^2+D_3s+K_1)\theta_2(s) + F(s)r \quad -D_3s\theta_3(s) &= 0\\ -D_3s\theta_2(s) + (J_2s^2+D_3s)\theta_3(s) &= 0\end{aligned}$$

where F(s) is the opposing force on J_2 due to the translational member and r is the radius of J_2. But, for the translational member,

$$F(s) = (Ms^2+f_vs+K_2)X(s) = (Ms^2+f_vs+K_2)r\theta(s)$$

Substituting F(s) back into the second equation of motion,

$$\begin{aligned}(J_1s^2+K_1)\theta_1(s) \quad - K_1\theta_2(s) \quad &= T(s)\\ -K_1\theta_1(s) + [(J_2 + Mr^2)s^2+(D_3 + f_vr^2)s+(K_1 + K_2r^2)]\theta_2(s) \quad -D_3s\theta_3(s) &= 0\\ -D_3s\theta_2(s) + (J_2s^2+D_3s)\theta_3(s) &= 0\end{aligned}$$

Notice that the translational components were reflected as equivalent rotational components by the square of the radius. Solving for $\theta_2(s)$, $\theta_2(s) = \frac{K_1(J_3s^2 + D_3s)T(s)}{\Delta}$, where Δ is the determinant formed from the coefficients of the three equations of motion. Hence,

$$\frac{\theta_2(s)}{T(s)} = \frac{K_1(J_3s^2 + D_3s)}{\Delta}$$

Since

$$X(s) = r\theta_2(s), \quad \frac{X(s)}{T(s)} = \frac{rK_1(J_3s^2 + D_3s)}{\Delta}$$

42.

$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{100}{50} = 2\ ;\ K_b = \frac{E_a}{\omega_{no-load}} = \frac{50}{150} = \frac{1}{3}$$

Also,

$$J_m = 2+18\left(\frac{1}{3}\right)^2 = 4;\ D_m = 2+36\left(\frac{1}{3}\right)^2 = 6.$$

Thus,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{2/4}{s(s+\frac{1}{4}(6+\frac{2}{3}))} = \frac{1/2}{s(s+\frac{5}{3})}$$

Since $\theta_L(s) = \frac{1}{3}\,\theta_m(s)$,

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{1}{6}}{s(s+\frac{5}{3})}.$$

43.

The parameters are:

$$\frac{K_t}{R_a} = \frac{T_s}{E_a} = \frac{5}{5} = 1;\ K_b = \frac{E_a}{\omega} = \frac{5}{\frac{600}{\pi}2\pi\frac{1}{60}} = \frac{1}{4};\ J_m = 16\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{2}\right)^2 + 1 = 3;\ D_m = 32\left(\frac{1}{4}\right)^2 = 2$$

Thus,

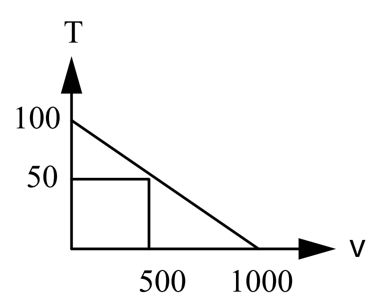
$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{1}{3}}{s(s+\frac{1}{3}(2+(1)(\frac{1}{4})))} = \frac{\frac{1}{3}}{s(s+0.75)}$$

Since $\theta_2(s) = \frac{1}{4}\,\theta_m(s)$,

$$\frac{\theta_2(s)}{E_a(s)} = \frac{\frac{1}{12}}{s(s+0.75)}.$$

44.

The following torque-speed curve can be drawn from the data given:



Therefore, $\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{100}{10} = 10$; $K_b = \frac{E_a}{\omega_{no-load}} = \frac{10}{1000} = \frac{1}{100}$. Also, $J_m = 5+100\left(\frac{1}{5}\right)^2 = 9$;

$D_m = 1$. Thus,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{10}{9}}{s(s+\frac{1}{9}(1+0.1))} = \frac{\frac{10}{9}}{s(s+0.122)}$$. Since $\theta_L(s) = \frac{1}{5}\,\theta_m(s)$, $$\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{10}{45}}{s(s+0.122)} = \frac{0.222}{s(s+0.122)}.$$

45.

From Eqs. (2.45) and (2.46),

$$R_a I_a(s) + K_b s\theta(s) = E_a(s) \qquad (1)$$

Also,

$T_m(s) = K_t I_a(s) = (J_m s^2 + D_m s)\theta(s)$. Solving for $\theta(s)$ and substituting into Eq. (1), and simplifying yields

$$\frac{I_a(s)}{E_a(s)} = \frac{1}{R_a}\frac{(s+\frac{D_m}{J_m})}{s+\frac{R_a D_m + K_b K_t}{R_a J_m}} \qquad (2)$$

Using $T_m(s) = K_t I_a(s)$ in Eq. (2),

$$\frac{T_m(s)}{E_a(s)} = \frac{K_t}{R_a}\frac{(s+\frac{D_m}{J_m})}{s+\frac{R_a D_m + K_b K_t}{R_a J_m}}$$

46.

For the rotating load, assuming all inertia and damping has been reflected to the load, $(J_{eqL}s^2+D_{eqL}s)\theta_L(s) + F(s)r = T_{eq}(s)$, where F(s) is the force from the translational system, r=2 is the radius of the rotational member, J_{eqL} is the equivalent inertia at the load of the rotational load and the armature, and D_{eqL} is the equivalent damping at the load of the rotational load and the armature. Since $J_{eqL} = 1(2)^2 + 1 = 5$, and $D_{eqL} = 1(2)^2 + 1 = 5$, the equation of motion becomes, $(5s^2+5s)\theta_L(s) + F(s)r = T_{eq}(s)$. For the translational system, $(s^2+s)X(s) = F(s)$. Since $X(s) = 2\theta_L(s)$, $F(s) = (s^2+s)2\theta_L(s)$. Substituting F(s) into the rotational equation, $(9s^2+9s)\,\theta_L(s) = T_{eq}(s)$. Thus, the equivalent inertia at the load is 9, and the equivalent damping at the load is 9. Reflecting these back to the armature, yields an equivalent inertia of $\frac{9}{4}$ and an equivalent damping of $\frac{9}{4}$. Finally, $\frac{K_t}{R_a} = 1$;

$K_b = 1$. Hence, $\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{4}{9}}{s(s+\frac{4}{9}(\frac{9}{4}+1))} = \frac{\frac{4}{9}}{s(s+\frac{13}{9})}$. Since $\theta_L(s) = \frac{1}{2}\ \theta_m(s)$, $\frac{\theta_L(s)}{E_a(s)} = \frac{\frac{2}{9}}{s(s+\frac{13}{9})}$. But

$X(s) = r\theta_L(s) = 2\theta_L(s)$. therefore, $\frac{X(s)}{E_a(s)} = \frac{\frac{4}{9}}{s(s+\frac{13}{9})}$.

47.

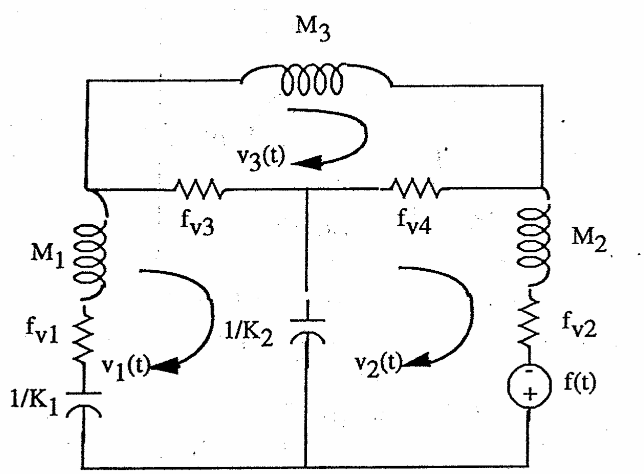
The equations of motion in terms of velocity are:

$$[M_1 s + (f_{v1} + f_{v3}) + \frac{K_1}{s} + \frac{K_2}{s}]V_1(s) - \frac{K_2}{s}V_2(s) - f_{v3}V_3(s) = 0$$

$$-\frac{K_2}{s}V_1(s) + [M_2 s + (f_{v2} + f_{v4}) + \frac{K_2}{s}]V_2(s) - f_{v4}V_3(s) = F(s)$$

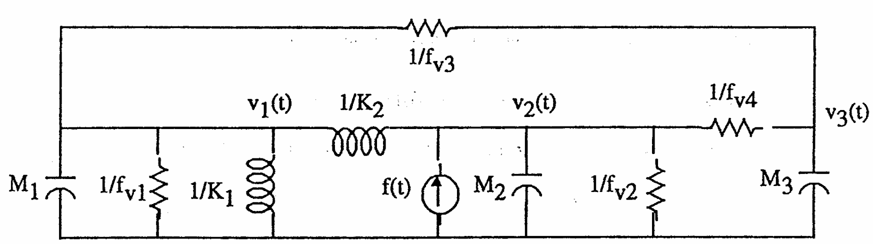
$$-f_{v3}V_1(s) - f_{v4}V_2(s) + [M_3 s + f_{v3} + f_{v4}]V_3(S) = 0$$

For the series analogy, treating the equations of motion as mesh equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

48.

Writing the equations of motion in terms of angular velocity, $\Omega(s)$ yields

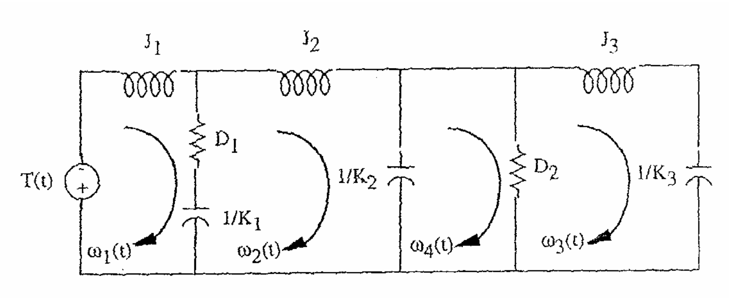
$$(J_1 s + D_1 + \frac{K_1}{s})\Omega_1(s) - (D_1 + \frac{K_1}{s})\Omega_2(s) = T(s)$$

$$-(D_1 + \frac{K_1}{s})\Omega_1(s) + (J_2 s + D_1 + \frac{(K_1 + K_2)}{s})\Omega_2(s) = 0$$

$$-\frac{K_2}{s}\Omega_2(s) - D_2\Omega_3(s) + (D_2 + \frac{K_2}{s})\Omega_4(s) = 0$$

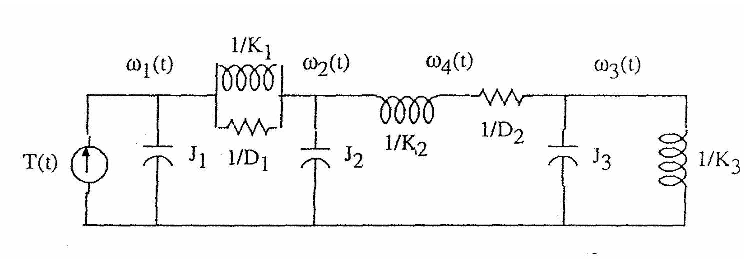
$$(J_3 s + D_2 + \frac{K_3}{s})\Omega_3(s) - D_2\Omega_4(s) = 0$$

For the series analogy, treating the equations of motion as mesh equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields



In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

49.

An input r_1 yields $c_1 = 5r_1+7$. An input r_2 yields $c_2 = 5r_2 +7$. An input $r_1 +r_2$ yields, $5(r_1+r_2)+7 = 5r_1+7+5r_2 = c_1+c_2-7$. Therefore, not additive. What about homogeneity? An input of Kr_1 yields $c = 5Kr_1+7 \neq Kc_1$. Therefore, not homogeneous. The system is not linear.

50.

a. Let $x = \delta x+0$. Therefore,

$\ddot{\delta x}+3\dot{\delta x}+2\delta x = \sin(0+\delta x)$

But, $\sin(0+\delta x) = \sin 0 + \left.\frac{d\sin x}{dx}\right|_{x=0} \delta x = 0+\left.\cos x\right|_{x=0} \delta x = \delta x$

Therefore, $\ddot{\delta x}+3\dot{\delta x}+2\delta x = \delta x$. Collecting terms, $\ddot{\delta x}+3\dot{\delta x}+\delta x = 0$.

b. Let $x = \delta x+\pi$. Therefore,

$\ddot{\delta x}+3\dot{\delta x}+2\delta x = \sin(\pi+\delta x)$

But, $\sin(\pi+\delta x) = \sin\pi + \left.\frac{d\sin x}{dx}\right|_{x=\pi} \delta x = 0+\left.\cos x\right|_{x=\pi} \delta x = -\delta x$

Therefore, $\ddot{\delta x}+3\dot{\delta x}+2\delta x = -\delta x$. Collecting terms, $\ddot{\delta x}+3\dot{\delta x}+3\delta x = 0$.

51.

If $x = 0 + \delta x$,

$\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 30\delta x = e^{-(\delta x)}$

But $e^{-(\delta x)} = e^{-0} + \left.\frac{de^{-x}}{dx}\right|_{x=0} \delta x = 1 - \left.e^{-x}\right|_{x=0} \delta x = 1 - \delta x$

Therefore, $\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 30\delta x = 1 - \delta x$, or, $\dddot{\delta x} + 10\ddot{\delta x} + 31\dot{\delta x} + 31\delta x = 1$.

52.

The given curve can be described as follows:

$f(x) = -4$; $-\infty<x<-2$;

$f(x) = 2x$; $-2<x<2$;

$f(x) = 4$; $2<x<+\infty$

Thus,

a. $\ddot{x} + 15\dot{x} + 50x = -4$

b. $\ddot{x} + 15\dot{x} + 50x = 2x$, or $\ddot{x} + 15\dot{x} + 48x = 0$

c. $\ddot{x} + 15\dot{x} + 50x = 4$

53.

The relationship between the nonlinear spring's displacement, $x_s(t)$ and its force, $f_s(t)$ is

$x_s(t) = 1 - e^{-f_s(t)}$

Solving for the force,

$$f_s(t) = -\ln(1 - x_s(t)) \tag{1}$$

Writing the differential equation for the system by summing forces,

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} - \ln(1 - x(t)) = f(t) \tag{2}$$

Letting $x(t) = x_0 + \delta x$ and $f(t) = 1 + \delta f$, linearize $\ln(1 - x(t))$.

$$\ln(1-x) - \ln(1-x_0) = \left.\frac{d\ln(1-x)}{dx}\right|_{x=x_0} \delta x$$

Solving for $\ln(1 - x)$,

$$\ln(1-x) = \ln(1-x_0) - \left.\frac{1}{1-x}\right|_{x=x_0} \delta x = \ln(1-x_0) - \frac{1}{1-x_0}\delta x \qquad (3)$$

When f = 1, δx = 0. Thus from Eq. (1), $1 = -\ln(1 - x_0)$. Solving for x_0,

$1 - x_0 = e^{-1}$, or $x_0 = 0.6321$.

Substituting $x_0 = 0.6321$ into Eq. (3),

$$\ln(1-x) = \ln(1 - 0.6321) - \frac{1}{1-0.6321}\delta x = -1 - 2.718\delta x$$

Placing this value into Eq. (2) along with $x(t) = x_0 + \delta x$ and $f(t) = 1 + \delta f$, yields the linearized differential equation,

$$\frac{d^2\delta x}{dt^2} + \frac{d\delta x}{dt} + 1 + 2.718\delta x = 1 + \delta f$$

or

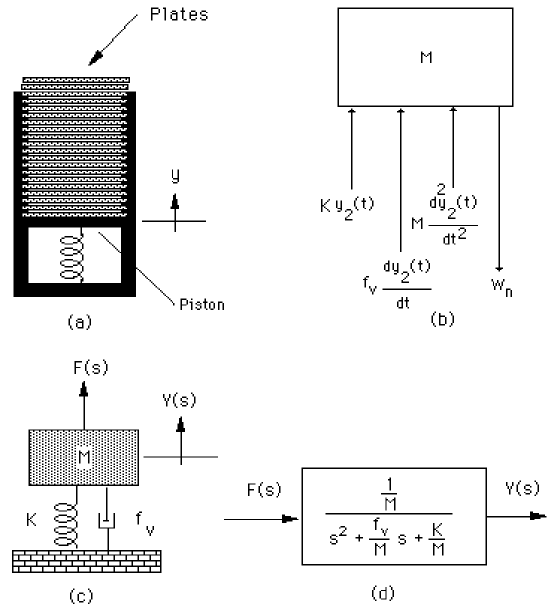
$$\frac{d^2\delta x}{dt^2} + \frac{d\delta x}{dt} + 2.718\delta x = \delta f$$

Taking the Laplace transform and rearranging yields the transfer function,

$$\frac{\delta x(s)}{\delta f(s)} = \frac{1}{s^2 + s + 2.718}$$

54.

First assume there are n plates without the top plate positioned at a displacement of $y_2(t)$ where $y_2(t) = 0$ is the position of the unstretched spring. Assume the system consists of mass M, where M is the mass of the dispensing system and the n plates, viscous friction, f_v, where the viscous friction originates where the piston meets the sides of the cylinder, and of course the spring with spring constant, K. Now, draw the freebody diagram shown in Figure (b) where W_n is the total weight of the n dishes and the piston. If we now consider the current position, $y_2(0)$,



Restaurant Plate Dispenser

the equilibrium point and define a new displacement, $y_1(t)$, which is measured from equilibrium, we can write the force in the spring as $Ky_2(t) = Ky_2(0) + Ky_1(t)$. Changing variables from $y_2(t)$ to $y_1(t)$, we sum forces and get,

$$M\frac{d^2y_1}{dt^2} + f_v\frac{dy_1}{dt} + Ky_1 + Ky_2(0) + W_n = 0 \qquad (1)$$

where $\frac{d^2y_2}{dt^2} = \frac{d^2y_1}{dt^2}$ and $\frac{dy_2}{dt} = \frac{dy_1}{dt}$. But, $Ky_2(0) = -W_n$, since it is the component of the spring force that balances the weight at equilibrium when $y_1 = 0$. Thus, the differential equation becomes,

$$M\frac{d^2y_1}{dt^2} + f_v\frac{dy_1}{dt} + Ky_1 = 0 \qquad (2)$$

When the top plate is added, the spring is further compressed by an amount, $\frac{W_D}{K}$, where W_D is the weight of the single dish, and K is the spring constant. We can think of this displacement as an initial condition. Thus, $y_1(0-) = -\frac{W_D}{K}$ and $\frac{dy_1}{dt}(0-) = 0$, and $y_1(t) = 0$ is the equilibrium position of the spring with n plates rather than the unstretched position. Taking the Laplace transform of equation (2), using the initial conditions,

$$M(s^2Y_1(s) + s\frac{W_D}{K}) + f_v(sY_1(s) + \frac{W_D}{K}) + KY_1(s) = 0 \qquad (3)$$

or

$$(Ms^2 + f_vs + K)Y_1(s) = -\frac{W_D}{K}(Ms + f_v) \qquad (4)$$

Now define a new position reference, Y(s), which is zero when the spring is compressed with the initial condition,

$$Y(s) = Y_1(s) + \frac{W_D}{Ks} \qquad (5)$$

or

$$Y_1(s) = Y(s) - \frac{W_D}{Ks} \qquad (6)$$

Substituting $Y_1(s)$ in Equation (4), we obtain,

$$(Ms^2 + f_vs + K)Y(s) = \frac{W_D}{s} = F(s) \qquad (7)$$

a differential equation that has an input and zero initial conditions. The schematic is shown in Figure (c). Forming the transfer function, $\frac{Y(s)}{F(s)}$, we show the final result in Figure (d), where for the removal of the top plate, F(s) is always a step, $F(s) = \frac{W_D}{s}$.

55.

Writing the equations of motion,

$$(17.2s^2 + 160s + 7000)Y_f(s) \quad - (130s + 7000)Y_h(s) \quad - 0Y_{cat}(s) = F_{up}(s)$$

$$-(130s+7000)Y_f(s) \quad + (9.1s^s + 130s + 89300)Y_h(s) \quad - 82300Y_{cat}(s) = 0$$

$$-0Y_f(s) \quad - 82300Y_h(s) \quad + 1.6173 \times 10^6\, Y_{cat}(s) = 0$$

These equations are in the form **AY=F**, where det(**A**) = $2.5314 \times 10^8\ (s^2 + 15.47 s + 9283)(s^2 + 8.119 s + 376.3)$

Using Cramer's rule:

$$\frac{Y_{cat}(s)}{F_{up}(s)} = \frac{0.04227(s + 53.85)}{(s^2 + 15.47s + 9283)(s^2 + 8.119s + 376.3)}$$

$$\frac{Y_h(s)}{F_{up}(s)} = \frac{0.8306(s + 53.85)}{(s^2 + 15.47s + 9283)(s^2 + 8.119s + 376.3)}$$

$$\frac{Y_h(s) - Y_{cat}(s)}{F_{up}(s)} = \frac{0.7883(s + 53.85)}{(s^2 + 15.47s + 9283)(s^2 + 8.119s + 376.3)}$$