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Adaptive Control Systems

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GANG FENG and ROGELIO LOZANO

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Preface

Adaptive control has been extensively investigated and developed in both theory and application during the past few decades, and it is still a very active research field. In the earlier stage, most studies in adaptive control concentrated on linear systems. A remarkable development of the adaptive control theory is the resolution of the so-called ideal problem, that is, the proof that several adaptive control systems are globally stable under certain ideal conditions. Then the robustness issues of adaptive control with respect to nonideal conditions such as external disturbances and unmodelled dynamics were addressed which resulted in many different robust adaptive control algorithms. These robust algorithms include dead zone, normalization, ε-modification, e_1-modification among many others. At the same time, extensive study has been carried out for reducing a priori knowledge of the systems and improving the transient performance of adaptive control systems. Most recently, adaptive control of nonlinear systems has received great attention and a number of significant results have been obtained.

In this book, we have compiled some of the most recent developments of adaptive control for both linear and nonlinear systems from leading world researchers in the field. These include various robust techniques, performance enhancement techniques, techniques with less a priori knowledge, adaptive switching techniques, nonlinear adaptive control techniques and intelligent adaptive control techniques. Each technique described has been developed to provide a practical solution to a real-life problem. This volume will therefore not only advance the field of adaptive control as an area of study, but will also show how the potential of this technology can be realized and offer significant benefits.

The first contribution in this book is 'Adaptive internal model control' by A. Datta and L. Xing. It develops a systematic theory for the design and analysis of adaptive internal model control schemes. The ubiquitous certainty equivalence principle of adaptive control is used to combine a robust adaptive law with robust internal model controllers to obtain adaptive internal model control schemes which can be proven to be robustly stable. Specific controller structures considered include those of the model reference, partial pole placement, and H_2 and H_∞ optimal control types. The results here not only provide a theoretical basis for analytically justifying some of the reported industrial successes of existing adaptive internal model control schemes but also open up the possibility of synthesizing new ones by simply combining a robust adaptive law with a robust internal model controller structure.

The next contribution is 'An algorithm for robust direct adaptive control with less prior knowledge' by G. Feng, Y. A. Jiang and R. Zmood. It discusses several approaches to minimizing a priori knowledge required on the unknown plants for robust adaptive control. It takes a discrete time robust direct adaptive control algorithm with a dead zone as an example. It shows that for a class of unmodelled dynamics and bounded disturbances, no knowledge of the parameters of the upper bounding function on the unmodelled dynamics and disturbances is required a priori. Furthermore it shows that a correction procedure can be employed in the least squares estimation algorithm so that no knowledge of the lower bound on the leading coefficient of the plant numerator polynomial is required to achieve the singularity free adaptive control law. The global stability and convergence results of the algorithm are established.

The next contribution is 'Adaptive variable structure control' by C. J. Chiang and Lichen Fu. A unified algorithm is presented to develop the variable structure MRAC for an SISO system with unmodelled dynamics and output measurement noises. The proposed algorithm solves the robustness and performance problem of the traditional MRAC with arbitrary relative degree. It is shown that without any persistent excitation the output tracking error can be driven to zero for relative degree-one plants and driven to a small residual set asymptotically for plants with any higher relative degree. Furthermore, under suitable choice of initial conditions on control parameters, the tracking performance can be improved, which is hardly achievable by the traditional MRAC schemes, especially for plants with uncertainties.

The next contribution is 'Indirect adaptive periodic control' by D. Dimogianopoulos, R. Lozano and A. Ailon. This new, indirect adaptive control method is based on a lifted representation of the plant which can be stabilized using a simple performant periodic control scheme. The controller parameters computation involves the inverse of the controllability/observability matrix. Potential singularities of this matrix are avoided by means of an appropriate estimates modification. This estimates transformation is linked to the covariance matrix properties and hence it preserves the convergence properties of the original estimates. This modification involves the singular value decomposition of the controllability/observability matrix's estimate. As compared to previous studies in the subject the controller proposed here does not require the frequent introduction of periodic n-length sequences of zero inputs. Therefore the new controller is such that the system is almost always operating in closed loop which should lead to better performance characteristics.

The next contribution is ‘Adaptive stabilization of uncertain discrete-time systems via switching control: the method of localization’ by P. V. Zhivoglyadov, R. Middleton and M. Fu. It presents a new systematic switching control approach to adaptive stabilization of uncertain discrete-time systems. The approach is based on a method of localization which is conceptually different from supervisory adaptive control schemes and other existing switching control schemes. The proposed approach allows for slow parameter drifting, infrequent large parameter jumps and unknown bound on exogenous disturbances. The unique feature of the localization-based switching adaptive control proposed here is its rapid model falsification capability. In the LTI case this is manifested in the ability of the switching controller to quickly converge to a suitable stabilizing controller. It is believed that the approach is applicable to a wide class of linear time invariant and time-varying systems with good transient performance.

The next contribution is ‘Adaptive nonlinear control: passivation and small gain techniques’ by Z. P. Jiang and D. Hill. It proposes methods to systematically design stabilizing adaptive controllers for new classes of nonlinear systems by using passivation and small gain techniques. It is shown that for a class of linearly parametrized nonlinear systems with only unknown parameters, the concept of adaptive passivation can be used to unify and extend most of the known adaptive nonlinear control algorithms based on Lyapunov methods. A novel recursive robust adaptive control method by means of backstepping and small gain techniques is also developed to generate a new class of adaptive nonlinear controllers with robustness to nonlinear unmodelled dynamics.

The next contribution is ‘Active identification for control of discrete-time uncertain nonlinear systems’ by J. Zhao and I. Kanellakopoulos. A novel approach is proposed to remove the restrictive growth conditions of the nonlinearities and to yield global stability and tracking for systems that can be transformed into an output-feedback canonical form. The main novelties of the design are (i) the temporal and algorithmic separation of the parameter estimation task from the control task and (ii) the development of an active identification procedure, which uses the control input to actively drive the system state to points in the state space that allow the orthogonalized projection estimator to acquire all the necessary information about the unknown parameters. It is proved that the proposed algorithm guarantees complete identification in a finite time interval and global stability and tracking.

The next contribution is 'Optimal adaptive tracking for nonlinear systems' by M. Krštić and Z. H. Li. In this chapter an 'inverse optimal' adaptive tracking problem for nonlinear systems with unknown parameters is defined and solved. The basis of the proposed method is an adaptive tracking control Lyapunov function (atclf) whose existence guarantees the solvability of the inverse optimal problem. The controllers designed are not of certainty equivalence type. Even in the linear case they would not be a result of solving a Riccati equation for a given value of the parameter estimate. Inverse optimality is combined with backstepping to design a new class of adaptive controllers for strict-feedback systems. These controllers solve a problem left open in the previous adaptive backstepping designs – getting transient performance bounds that include an estimate of control effort.

The next contribution is 'Stable adaptive systems in the presence of nonlinear parameterization' by A. M. Annaswamy and A. P. Loh. This chapter addresses the problem of adaptive control when the unknown parameters occur nonlinearly in a dynamic system. The traditional approach used in linearly parameterized systems employs a gradient-search principle in estimating the unknown parameters. Such an approach is not sufficient for nonlinearly parametrized systems. Instead, a new algorithm based on a min–max optimization scheme is developed to address nonlinearly parametrized adaptive systems. It is shown that this algorithm results in globally stable closed loop systems when the states of the plant are accessible for measurement.

The next contribution is 'Adaptive inverse for actuator compensation' by G. Tao. A general adaptive inverse approach is developed for control of plants with actuator imperfections caused by nonsmooth nonlinearities such as deadzone, backlash, hysteresis and other piecewise-linear characteristics. An adaptive inverse is employed for cancelling the effect of an unknown actuator nonlinearity, and a linear feedback control law is used for controlling the dynamics of a known linear or smooth nonlinear part following the actuator nonlinearity. State feedback and output feedback control designs are presented which all lead to linearly parametrized error models suitable for the development of adaptive laws to update the inverse parameters. This approach suggests that control systems with commonly used linear or nonlinear feedback controllers such as those with an LQ, model reference, PID, pole placement or other dynamic compensation design can be combined with an adaptive inverse for improving system tracking performance despite the presence of actuator imperfections.

The next contribution is 'Stable multi-input multi-output adaptive fuzzy/neural control' by R. Ordóñez and K. Passino. In this chapter, stable direct and indirect adaptive controllers are presented which use Takagi–Sugeno fuzzy systems, conventional fuzzy systems, or a class of neural networks to provide asymptotic tracking of a reference signal vector for a class of continuous time multi-input multi-output (MIMO) square nonlinear plants with poorly understood dynamics. The direct adaptive scheme allows for the inclusion of a priori knowledge about the control input in terms of exact mathematical equations or linguistics, while the indirect adaptive controller permits the explicit use of equations to represent portions of the plant dynamics. It is shown that with or without such knowledge the adaptive schemes can 'learn' how to control the plant, provide for bounded internal signals, and achieve asymptotically stable tracking of the reference inputs. No initialization condition needs to be imposed on the controllers, and convergence of the tracking error to zero is guaranteed.

The final contribution is 'Adaptive robust control scheme with an application to PM synchronous motors' by J. X. Xu, Q. W. Jia and T. H. Lee. A new, adaptive, robust control scheme for a class of nonlinear uncertain dynamical systems is presented. To reduce the robust control gain and widen the application scope of adaptive techniques, the system uncertainties are classified into two different categories: the structured and nonstructured uncertainties with partially known bounding functions. The structured uncertainty is estimated with adaptation and compensated. Meanwhile, the adaptive robust method is applied to deal with the non-structured uncertainty by estimating unknown parameters in the upper bounding function. It is shown that the new control scheme guarantees the uniform boundedness of the system and assures the tracking error entering an arbitrarily designated zone in a finite time. The effectiveness of the proposed method is demonstrated by the application to PM synchronous motors.

1

Adaptive internal model control

A. Datta and L. Xing

Abstract

This chapter develops a systematic theory for the design and analysis of adaptive internal model control schemes. The principal motivation stems from the fact that despite the reported industrial successes of adaptive internal model control schemes, there currently does not exist a design methodology capable of providing theoretical guarantess of stability and robustness. The ubiquitous certainty equivalence principle of adaptive control is used to combine a robust adaptive law with robust internal model controllers to obtain adaptive internal model control schemes which can be proven to be robustly stable. Specific controller structures considered include those of the model reference, 'partial' pole placement, and H_2 and H_∞ optimal control types. The results here not only provide a theoretical basis for analytically justifying some of the reported industrial successes of existing adaptive internal model control schemes but also open up the possibility of synthesizing new ones by simply combining a robust adaptive law with a robust internal model controller structure.

1.1 Introduction

Internal model control (IMC) schemes, where the controller implementation includes an *explicit* model of the plant, continue to enjoy widespread popularity in industrial process control applications [1]. Such schemes can guarantee internal stability for only *open loop stable* plants; since most plants encountered in process control are anyway open loop stable, this really does not impose any significant restriction.

As already mentioned, the main feature of IMC is that its implementation requires an explicit model of the plant to be used as part of the controller. When the plant itself happens to be unknown, or the plant parameters vary slowly with time due to ageing, no such model is directly available a priori and one has to resort to identification techniques to come up with an appropriate plant model on-line. Several empirical studies, e.g. [2], [3] have demonstrated the feasibility of such an approach. However, what is, by and large, lacking in the process control literature is the availability of results with solid theoretical guarantees of stability and performance.

Motivated by this fact, in [4], [5], we presented designs of adaptive IMC schemes with provable guarantees of stability and robustness. The scheme in [4] involved on-line adaptation of only the internal model while in [5], in addition to adapting the internal model on-line, the IMC parameter was chosen in a certainty equivalence fashion to pointwise optimize an H_2 performance index. In this chapter, it is shown that the approach of [5] can be adapted to design and analyse a class of adaptive H_∞ optimal control schemes that are likely to arise in process control applications. This class specifically consists of those H_∞ norm minimization problems that involve only *one interpolation constraint*. Additionally, we reinterpret the scheme of [4] as an adaptive 'partial' pole-placement control scheme and consider the design and analysis of a model reference adaptive control scheme based on the IMC structure. In other words, this chapter considers the design and analysis of popular adaptive control schemes from the literature within the context of the IMC configuration. A single, unified, analysis procedure, applicable to each of the schemes considered, is also presented.

The chapter is organized as follows. In Section 1.2, we present several nonadaptive control schemes utilizing the IMC configuration. Their adaptive certainty equivalence versions are presented in Section 1.3. A unified stability and robustness analysis encompassing all of the schemes of Section 1.3 is presented in Section 1.4. In Section 1.5, we present simulation examples to demonstrate the efficacy of our adaptive IMC designs. Section 1.6 concludes the chapter by summarizing the main results and outlining their expected significance.

1.2 Internal model control (IMC) schemes: known parameters

In this section, we present several nonadaptive control schemes utilizing the IMC structure. To this end, we consider the IMC configuration for a stable plant $P(s)$ as shown in Figure 1.1. The IMC controller consists of a stable 'IMC parameter' $Q(s)$ and a model of the plant which is usually referred to as the 'internal model'. It can be shown [1, 4] that if the plant $P(s)$ is stable and

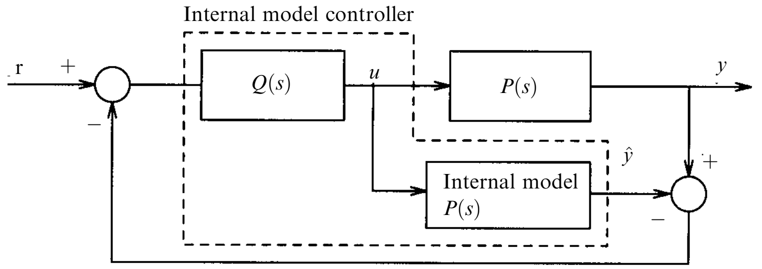

Figure 1.1 *The IMC configuration*

the internal model is an exact replica of the plant, then the stability of the IMC parameter is equivalent to the internal stability of the configuration in Figure 1.1. Indeed, the IMC parameter is really the Youla parameter [6] that appears in a special case of the YJBK parametrization of all stabilizing controllers [4]. Because of this, internal stability is assured as long as $Q(s)$ is chosen to be any stable rational transfer function. We now show that different choices of stable $Q(s)$ lead to some familiar control schemes.

1.2.1 Partial pole placement control

From Figure 1.1, it is clear that if the internal model is an exact replica of the plant, then there is no feedback signal in the loop. Consequently the poles of the closed loop system are made up of the open loop poles of the plant and the poles of the IMC parameter $Q(s)$. Thus, in this case, a 'complete' pole placement as in traditional pole placement control schemes is not possible. Instead, one can only choose the poles of the IMC parameter $Q(s)$ to be in some desired locations in the left half plane while leaving the remaining poles at the plant open loop pole locations. Such a control scheme, where $Q(s)$ is chosen to inject an additional set of poles at some desired locations in the complex plane, is referred to as 'partial' pole placement.

1.2.2 Model reference control

The objective in model reference control is to design a differentiator-free controller so that the output y of the controlled plant $P(s)$ asymptotically tracks the output of a stable reference model $W_m(s)$ for *all* piecewise continuous reference input signals $r(t)$. In order to meet the control objective, we make the following assumptions which are by now standard in the model reference control literature:

(M1) The plant $P(s)$ is minimum phase; and

(M2) The relative degree of the reference model transfer function $W_m(s)$ is greater than or equal to that of the plant transfer function $P(s)$.

Assumption (M1) above is necessary for ensuring internal stability since satisfaction of the model reference control objective requires cancellation of the plant zeros. Assumption (M2), on the other hand, permits the design of a differentiator-free controller to meet the control objective. If assumptions (M1) and (M2) are satisfied, it is easy to verify from Figure 1.1 that the choice

$$Q(s) = W_m(s)P^{-1}(s) \tag{1.1}$$

for the IMC parameter guarantees the satisfaction of the model reference control objective in the ideal case, i.e. in the absence of plant modelling errors.

1.2.3 H_2 optimal control

In H_2 optimal control, one chooses $Q(s)$ to minimize the L_2 norm of the tracking error $r - y$ provided $r - y \in L_2$. From Figure 1.1, we obtain

$$y = P(s)Q(s)[r]$$
$$\Rightarrow r - y = [1 - P(s)Q(s)][r]$$
$$\Rightarrow \int_0^{\infty} (r(\tau) - y(\tau))^2 d\tau = (\|[1 - P(s)Q(s)]R(s)\|_2)^2 \text{ (using Parseval's Theorem)}$$

where $R(s)$ is the Laplace transform of $r(t)$ and $\|\cdot(s)\|_2$ denotes the standard H_2 norm. Thus the mathematical problem of interest here is to choose $Q(s)$ to minimize $\|[1 - P(s)Q(s)]R(s)\|_2$. The following theorem gives the analytical expression for the minimizing $Q(s)$. The detailed derivation can be found in [1].

Theorem 2.1 Let $P(s)$ be the stable plant to be controlled and let $R(s)$ be the Laplace Transform of the external input signal $r(t)$[1]. Suppose that $R(s)$ has no poles in the open right half plane[2] and that there exists at least one choice, say $Q_0(s)$, of the stable IMC parameter $Q(s)$ such that $[1 - P(s)Q_0(s)]R(s)$ is stable[3]. Let $z_{p_1}, z_{p_2}, \ldots, z_{p_l}$ be the open right half plane zeros of $P(s)$ and define the Blaschke product[4]

$$B_P(s) = \frac{(-s + z_{p_1})(-s + z_{p_2}) \ldots (-s + z_{p_l})}{(s + \bar{z}_{p_1})(s + \bar{z}_{p_2}) \ldots (s + \bar{z}_{p_l})}$$

so that $P(s)$ can be rewritten as

$$P(s) = B_P(s)P_M(s)$$

[1] For the sake of simplicity, both $P(s)$ and $R(s)$ are assumed to be rational transfer functions. The theorem statement can be appropriately modified for the case where $P(s)$ and/or $R(s)$ contain all-pass time delay factors [1]

[2] This assumption is reasonable since otherwise the external input would be unbounded.

[3] The final construction of the H_2 optimal controller serves as proof for the existence of a $Q_0(s)$ with such properties.

[4] Here $(\bar{\cdot})$ denotes complex conjugation.

where $P_M(s)$ is minimum phase. Similarly, let $z_{r_1}, z_{r_2}, \ldots, z_{r_k}$ be the open right half plane zeros of $R(s)$ and define the Blashcke product

$$B_R(s) = \frac{(-s + z_{r_1})(-s + z_{r_2}) \ldots (-s + z_{r_k})}{(s + \bar{z_{r_1}})(s + \bar{z_{r_2}}) \ldots (s + \bar{z_{r_k}})}$$

so that $R(s)$ can be rewritten as

$$R(s) = B_R(s) R_M(s)$$

where $R_M(s)$ is minimum phase. Then the $Q(s)$ which minimizes $\|[1 - P(s)Q(s)]R(s)\|_2$ is given by

$$Q(s) = P_M^{-1}(s) R_M^{-1}(s) \left[B_P^{-1}(s) R_M(s) \right]_* \tag{1.2}$$

where $[\cdot]_*$ denotes that after a partial fraction expansion, the terms corresponding to the poles of $B_P^{-1}(s)$ are removed.

Remark 2.1 The optimal $Q(s)$ defined in (1.2) is usually improper. So it is customary to make $Q(s)$ proper by introducing sufficient high frequency attenuation via what is called the 'IMC Filter' $F(s)$ [1]. Instead of the optimal $Q(s)$ in (1.2), the $Q(s)$ to be implemented is given by

$$Q(s) = P_M^{-1}(s) R_M^{-1}(s) \left[B_P^{-1}(s) R_M(s) \right]_* F(s) \tag{1.3}$$

where $F(s)$ is the stable IMC filter. The design of the IMC filter for H_2 optimal control depends on the choice of the input $R(s)$. Although this design is carried out in a somewhat ad hoc fashion, care is taken to ensure that the original asymptotic tracking properties of the controller are preserved. This is because otherwise $[1 - P(s)Q(s)]R(s)$ may no longer be a function in H_2. As a specific example, suppose that the system is of Type 1.[5] Then, a possible choice for the IMC filter to ensure retention of asymptotic tracking properties is $F(s) = \frac{1}{(\tau s + 1)^{n^*}}$, $\tau > 0$ where n^* is chosen to be a large enough positive integer to make $Q(s)$ proper. As shown in [1], the parameter τ represents a trade-off between tracking performance and robustness to modelling errors.

1.2.4 H_∞ optimal control

The sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ for the IMC configuration in Figure 1.1 are given by $S(s) = 1 - P(s)Q(s)$ and $T(s) = P(s)Q(s)$ respectively [1]. Since the plant $P(s)$ is open loop stable, it follows that the H_∞ norm of the complementary sensitivity function $T(s)$ can be made *arbitrarily small* by simply choosing $Q(s) = \frac{1}{k}$ and letting k tend to

[5] Other system types can also be handled as in [1].

infinity. Thus minimizing the H_∞ norm of $T(s)$ does not make much sense since the infimum value of zero is unattainable.

On the other hand, if we consider the weighted sensitivity minimization problem where we seek to minimize $\|W(s)S(s)\|_\infty$ for some stable, minimum phase, rational weighting transfer function $W(s)$, then we have an interesting H_∞ minimization problem, i.e. choose a stable $Q(s)$ to minimize $\|W(s)[1 - P(s)Q(s)]\|_\infty$.The solution to this problem depends on the number of open right half plane zeros of the plant $P(s)$ and involves the use of Nevanlinna–Pick interpolation when the plant $P(s)$ has more than one right half plane zero [7]. However, when the plant has only one right half plane zero b_1 and none on the imaginary axis, there is only one interpolation constraint and the closed form solution is given by [7]

$$Q(s) = [1 - \frac{W(b_1)}{W(s)}]P^{-1}(s) \tag{1.4}$$

Fortunately, this case covers a large number of process control applications where plants are typically modelled as minimum phase first or second order transfer functions with time delays. Since approximating a delay using a first order Padé approximation introduces one right half plane zero, the resulting rational approximation will satisfy the one right half plane zero assumption.

Remark 2.2 As in the case of H_2 optimal control, the optimal $Q(s)$ defined by (1.4) is usually improper. This situation can be handled as in Remark 2.1 so that the $Q(s)$ to be implemented becomes

$$Q(s) = \left[1 - \frac{W(b_1)}{W(s)}\right]P^{-1}(s)F(s) \tag{1.5}$$

where $F(s)$ is a stable IMC filter. In this case, however, there is more freedom in the choice of $F(s)$ since the H_∞ optimal controller (1.4) does not necessarily guarantee any asymptotic tracking properties to start with.

1.2.5 Robustness to uncertainties (small gain theorem)

In the next section, we will be combining the above schemes with a robust adaptive law to obtain adaptive IMC schemes. If the above IMC schemes are unable to tolerate uncertainty in the case where all the plant parameters are known, then there is little or no hope that certainty equivalence designs based on them will do any better when additionally the plant parameters are unknown and have to be estimated using an adaptive law. Accordingly, we now establish the robustness of the nonadaptive IMC schemes to the presence of plant modelling errors. Without any loss of generality let us suppose that the uncertainty is of the multiplicative type, i.e.

$$P(s) = P_0(s)(1 + \mu\Delta_m(s)) \tag{1.6}$$

where $P_0(s)$ is the modelled part of the plant and $\mu\Delta_m(s)$ is a stable multiplicative uncertainty such that $P_0(s)\Delta_m(s)$is strictly proper. Then we can state the following robustness result which follows immediately from the small gain theorem [8]. A detailed proof can also be found in [1].

Theorem 2.2 Suppose $P_0(s)$ and $Q(s)$ are stable transfer functions so that the IMC configuration in Figure 1.1 is stable for $P(s) = P_0(s)$. Then the IMC configuration with the actual plant given by (1.6) is still stable provided $\mu \in [0, \mu^*)$ where $\mu^* = \dfrac{1}{\|P_0(s)Q(s)\Delta_m(s)\|_\infty}$.

1.3 Adaptive internal model control schemes

In order to implement the IMC-based controllers of the last section, the plant must be known a priori so that the 'internal model' can be designed and the IMC parameter $Q(s)$ calculated. When the plant itself is unknown, the IMC-based controllers cannot be implemented. In this case, the natural approach to follow is to retain the same controller structure as in Figure 1.1, with the internal model being adapted on-line based on some kind of parameter estimation mechanism, and the IMC parameter $Q(s)$ being updated pointwise using one of the above control laws. This is the standard certainty equivalence approach of adaptive control and results in what are called adaptive internal model control schemes. Although such adaptive IMC schemes have been empirically studied inthe literature, e.g. [2, 3], our objective here is to develop adaptive IMC schemes with provable guarantees of stability and robustness.

To this end, we assume that the stable plant to be controlled is described by

$$P(s) = \frac{Z_0(s)}{R_0(s)}[1 + \mu\Delta_m(s)], \quad \mu > 0 \tag{1.7}$$

where $R_0(s)$ is a monic Hurwitz polynomial of degree n; $Z_0(s)$ is a polynomial of degree l with $l < n$; $\dfrac{Z_0(s)}{R_0(s)}$ represents the modelled part of the plant; and $\mu\Delta_m(s)$ is a stable multiplicative uncertainty such that $\dfrac{Z_0(s)}{R_0(s)}\Delta_m(s)$ is strictly proper. We next present the design of the robust adaptive law which is carried out using a standard approach from the robust adaptive control literature [9].

1.3.1 Design of the robust adaptive law

We start with the plant equation

$$y = \frac{Z_0(s)}{R_0(s)}[1 + \mu\Delta_m(s)][u], \quad \mu > 0 \tag{1.8}$$

where u, y are the plant input and output signals. This equation can be rewritten as

$$R_0(s)[y] = Z_0(s)[u] + \mu\Delta_m(s)Z_0(s)[u]$$

Filtering both sides by $\frac{1}{\Lambda(s)}$, where $\Lambda(s)$ is an arbitrary, monic, Hurwitz polynomial of degree n, we obtain

$$y = \frac{\Lambda(s) - R_0(s)}{\Lambda(s)}[y] + \frac{Z_0(s)}{\Lambda(s)}[u] + \frac{\mu\Delta_m(s)Z_0(s)}{\Lambda(s)}[u] \tag{1.9}$$

The above equation can be rewritten as

$$y = \theta^{*^T}\phi + \mu\eta \tag{1.10}$$

where $\theta^* = [\theta_1^{*^T},\ \theta_2^{*^T}]^T$; θ_1^*, θ_2^* are vectors containing the coefficients of $[\Lambda(s) - R_0(s)]$ and $Z_0(s)$ respectively; $\phi = [\phi_1^T, \phi_2^T]^T$; $\phi_1 = \frac{a_{n-1}(s)}{\Lambda(s)}[y]$, $\phi_2 = \frac{a_l(s)}{\Lambda(s)}[u]$;

$$a_{n-1}(s) = \left[s^{n-1}, s^{n-2}, \ldots, 1\right]^T$$

$$a_l(s) = \left[s^l, s^{l-1}, \ldots, 1\right]^T$$

and

$$\eta \triangleq \frac{\Delta_m(s)Z_0(s)}{\Lambda(s)}[u] \tag{1.11}$$

Equation (1.10) is exactly in the form of the linear parametric model with modelling error for which a large class of robust adaptive laws can be developed. In particular, using the gradient method with normalization and parameter projection, we obtain the following robust adaptive law [9]

$$\dot{\theta} = Pr[\gamma\varepsilon\phi],\ \theta(0) \in \mathcal{C}_\theta \tag{1.12}$$

$$\varepsilon = \frac{y - \hat{y}}{m^2} \tag{1.13}$$

$$\hat{y} = \theta^T\phi \tag{1.14}$$

$$m^2 = 1 + n_s^2, \quad n_s^2 = m_s \tag{1.15}$$

$$\dot{m}_s = -\delta_0 m_s + u^2 + y^2, \quad m_s(0) = 0 \tag{1.16}$$

where $\gamma > 0$ is an adaptive gain; $\mathcal{C}_\theta$ is a known compact convex set containing θ^*; $Pr[\cdot]$ is the standard projection operator which guarantees that the parameter estimate $\theta(t)$ does not exit the set $\mathcal{C}_\theta$ and $\delta_0 > 0$ is a constant chosen so that $\Delta_m(s)$, $\frac{1}{\Lambda(s)}$ are analytic in $\mathcal{R}e[s] \geq -\frac{\delta_0}{2}$. This choice of δ_o, of course, necessitates some a priori knowledge about the stability margin of the unmodelled dynamics, an assumption which has by now become fairly standard in the robust adaptive control literature [9]. The robust adaptive IMC schemes are obtained by replacing the internal model in Figure 1.1 by that obtained from equation (1.14), and the IMC parameters $Q(s)$ by time-varying operators which implement the certainty equivalence versions of the controller structures considered in the last section. The design of these certainty equivalence controllers is discussed next.

1.3.2 Certainty equivalence control laws

We first outline the steps involved in designing a general certainty equivalence adaptive IMC scheme. Thereafter, additional simplifications or complexities that result from the use of a particular control law will be discussed.

- *Step 1:* First use the parameter estimate $\theta(t)$ obtained from the robust adaptive law (1.12)–(1.16) to generate estimates of the numerator and denominator polynomials for the modelled part of the plant[6]

$$\hat{Z}_0(s,t) = \theta_2^T(t)a_l(s)$$
$$\hat{R}_0(s,t) = \Lambda(s) - \theta_1^T(t)a_{n-1}(s)$$

- *Step 2:* Using the frozen time plant $\hat{P}(s,t) = \dfrac{\hat{Z}_0(s,t)}{\hat{R}_0(s,t)}$, calculate the appropriate $\hat{Q}(s,t)$ using the results developed in Section 1.2.
- *Step 3:* Express $\hat{Q}(s,t)$ as $\hat{Q}(s,t) = \dfrac{\hat{Q}_n(s,t)}{\hat{Q}_d(s,t)}$ where $\hat{Q}_n(s,t)$ and $\hat{Q}_d(s,t)$ are time-varying polynomials with $\hat{Q}_d(s,t)$ *being monic*.
- *Step 4:* Choose $\Lambda_1(s)$ to be an arbitrary monic Hurwitz polynomial of degree equal to that of $\hat{Q}_d(s,t)$, and let this degree be denoted by n_d.
- *Step 5:* The certainty equivalence control law is given by

$$u = q_d^T(t)\frac{a_{n_d-1}(s)}{\Lambda_1(s)}[u] + q_n^T(t)\frac{a_{n_d}(s)}{\Lambda_1(s)}[r - \varepsilon m^2] \tag{1.17}$$

  where $q_d(t)$ is the vector of coefficients of $\Lambda_1(s) - \hat{Q}_d(s,t)$; $q_n(t)$ is the vector of coefficients of $\hat{Q}_n(s,t)$; $a_{n_d}(s) = [s^{n_d}, s^{n_d-1}, \ldots, 1]^T$ and $a_{n_d-1}(s) = [s^{n_d-1}, s^{n_d-2}, \ldots, 1]^T$.

The robust adaptive IMC scheme resulting from combining the control law (1.17) with the robust adaptive law (1.12)–(1.16) is schematically depicted in

[6] In the rest of this chapter, the 'hats' denote the time varying polynomials/frozen time 'transfer functions' that result from replacing the time-invariant coefficients of a 'hat-free' polynomial/transfer function by their corresponding time-varying values obtained from adaptation and/or certainty equivalence control.

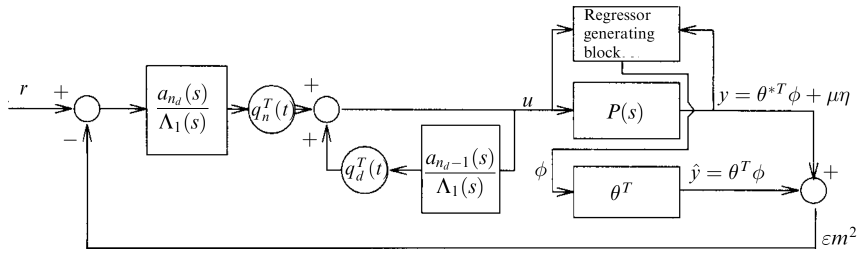

Figure 1.2 *Robust adaptive IMC scheme*

Figure 1.2. We now proceed to discuss the simplifications or additional complexities that result from the use of each of the controller structures presented in Section 1.2.

1.3.2.1 Partial adaptive pole placement

In this case, the design of the IMC parameter does not depend on the estimated plant. Indeed, $Q(s)$ is a fixed stable transfer function and not a time-varying operator so that we essentially recover the scheme presented in [4]. Consequently, this scheme admits a simpler stability analysis as in [4] although the general analysis procedure to be presented in the next section is also applicable.

1.3.2.2 Model reference adaptive control

In this case from (1.1), we see that the $\hat{Q}(s,t)$ in Step 2 of the certainty equivalence design becomes

$$\hat{Q}(s,t) = W_m(s)\left[\hat{P}(s,t)\right]^{-1} \tag{1.18}$$

Our stability analysis to be presented in the next section is based on results in the area of slowly time-varying systems. In order for these results to be applicable, it is required that the operator $\hat{Q}(s,t)$ be pointwise stable and also that the degree of $\hat{Q}_d(s,t)$ in Step 3 of the certainty equivalence design not change with time. These two requirements can be satisfied as follows:

- The pointwise stability of $\hat{Q}(s,t)$ can be guaranteed by ensuring that the frozen time estimated plant is minimum phase, i.e. $\hat{Z}_0(s,t)$ is Hurwitz stable for every fixed t. To guarantee such a property for $\hat{Z}_0(s,t)$, the projection set $\mathcal{C}_\theta$ in (1.12)–(1.16) is chosen so that $\forall\, \theta \in \mathcal{C}_\theta$, the corresponding $Z_0(s) = \theta_2^T a_l(s)$ is Hurwitz stable. By restricting $\mathcal{C}_\theta$ to be a subset of a Cartesian product of closed intervals, results from Kharitonov Theory [10] can be used to ensure that $\mathcal{C}_\theta$ satisfies such a requirement. Also, when the projection set $\mathcal{C}_\theta$ cannot be specified as a single convex set, results from *hysteresis switching* using a finite number of convex sets [11] can be used.

- The degree of $\hat{Q}_d(s,t)$ can also be rendered time invariant by ensuring that the leading coefficient of $\hat{Z}_0(s,t)$ is not allowed to pass through zero. This feature can be built into the adaptive law by assuming some knowledge about the sign and a lower bound on the absolute value of the leading coefficient of $Z_0(s)$. Projection techniques, appropriately utilizing this knowledge, are by now standard in the adaptive control literature [12].

We will therefore assume that for IMC-based model reference adaptive control, the set $\mathcal{C}_\theta$ has been suitably chosen to guarantee that the estimate $\theta(t)$ obtained from (1.12)–(1.16) actually satisfies both of the properties mentioned above.

1.3.2.3 Adaptive H_2 optimal control

In this case, $\hat{Q}(s,t)$ is obtained by substituting $\hat{P}_M^{-1}(s,t)$, $\hat{B}_P^{-1}(s,t)$ into the right-hand side of (1.3) where $\hat{P}_M(s,t)$ is the minimum phase portion of $\hat{P}(s,t)$ and $\hat{B}_P(s,t)$ is the Blaschke product containing the open right-half plane zeros of $\hat{Z}_0(s,t)$. Thus $\hat{Q}(s,t)$ is given by

$$\hat{Q}(s,t) = \hat{P}_M^{-1}(s,t)R_M^{-1}(s)[\hat{B}_P^{-1}(s,t)R_M(s)]_* F(s) \tag{1.19}$$

where $[\cdot]_*$ denotes that after a partial fraction expansion, the terms corresponding to the poles of $\hat{B}_P^{-1}(s,t)$ are removed, and $F(s)$ is an IMC filter used to force $\hat{Q}(s,t)$ to be proper. As will be seen in the next section, specifically Lemma 4.1, the degree of $\hat{Q}_d(s,t)$ in Step 3 of the certainty equivalence design can be kept constant *using a single fixed* $F(s)$ provided the leading coefficient of $\hat{Z}_0(s,t)$ is not allowed to pass through zero. Additionally $\hat{Z}_0(s,t)$ should not have any zeros on the imaginary axis. A parameter projection modification, as in the case of model reference adaptive control, can be incorporated into the adaptive law (1.12)–(1.16) to guarantee both of these properties.

1.3.2.4 Adaptive H_∞ optimal control

In this case, $\hat{Q}(s,t)$ is obtained by substituting $\hat{P}(s,t)$ into the right-hand side of (1.5), i.e.

$$\hat{Q}(s,t) = \left[1 - \frac{W(\hat{b}_1)}{W(s)}\right]\hat{P}^{-1}(s,t)F(s) \tag{1.20}$$

where $\hat{b}_1$ is the open right half plane zero of $\hat{Z}_0(s,t)$ and $F(s)$ is the IMC filter. Since (1.20) assumes the presence of only one open right half plane zero, the estimated polynomial $\hat{Z}_0(s,t)$ must have only one open right half plane zero and none on the imaginary axis. Additionally the leading coefficient of $\hat{Z}_0(s,t)$ should not be allowed to pass through zero so that the degree of $\hat{Q}_d(s,t)$ in Step

3 of the certainty equivalence design can be kept fixed using a single fixed $F(s)$. Once again, both of these properties can be guaranteed by the adaptive law by appropriately choosing the set $\mathcal{C}_\theta$.

Remark 3.1 The actual construction of the sets $\mathcal{C}_\theta$ for adaptive model reference, adaptive H_2 and adaptive H_∞ optimal control may not be straightforward especially for higher order plants. However, this is a well-known problem that arises in any certainty equivalence control scheme based on the estimated plant and is really not a drawback associated with the IMC design methodology. Although from time to time a lot of possible solutions to this problem have been proposed in the adaptive literature, it would be fair to say that, by and large, no satisfactory solution is currently available.

1.4 Stability and robustness analysis

Before embarking on the stability and robustness analysis for the adaptive IMC schemes just proposed, we first introduce some definitions [9, 4] and state and prove two lemmas which play a pivotal role in the subsequent analysis.

Definition 4.1 For any signal $x : [0,\ \infty) \to R^n$, x_t denotes the truncation of x to the interval $[0, t]$ and is defined as

$$x_t(\tau) = \begin{cases} x(\tau) & \text{if } \ \tau \leq t \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

Definition 4.2 For any signal $x : [0,\ \infty) \to R^n$, and for any $\delta \geq 0$, $t \geq 0$, $\|x_t\|_2^\delta$ is defined as

$$\|x_t\|_2^\delta \triangleq \left(\int_0^t e^{-\delta(t-\tau)} [x^T(\tau) x(\tau)]\, d\tau \right)^{\frac{1}{2}} \tag{1.22}$$

The $\|(\cdot)_t\|_2^\delta$ represents the exponentially weighted L_2 norm of the signal truncated to $[0, t]$. When $\delta = 0$ and $t = \infty$, $\|(.)_t\|_2^\delta$ becomes the usual L_2 norm and will be denoted by $\|.\|_2$. It can be shown that $\|.\|_2^\delta$ satisfies the usual properties of the vector norm.

Definition 4.3 Consider the signals $x : [0, \infty) \to R^n$, $y : [0, \infty) \to R^+$ and the set

$$\mathcal{S}(y) = \left\{ x : [0, \infty) \to R^n | \int_t^{t+T} x^T(\tau) x(\tau)\, d\tau \leq \int_t^{t+T} y(\tau)\, d\tau + c \right\}$$

for some $c \geq 0$ and $\forall\ t,\ T \geq 0$. We say that x is y-small in the mean if $x \in S(y)$.

Lemma 4.1 In each of the adaptive IMC schemes presented in the last section, the degree of $\hat{Q}_d(s, t)$ in Step 3 of the certainty equivalence design can be made time invariant. Furthermore, for the adaptive H_2 and H_∞ designs, this can be done using a single fixed $F(s)$.

Proof The proof of this lemma is relatively straightforward except in the case of adaptive H_2 optimal control. Accordingly, we first discuss the simpler cases before giving a detailed treatment of the more involved one.

For adaptive partial pole placement, the time invariance of the degree of $\hat{Q}_d(s,t)$ follows trivially from the fact that the IMC parameter in this case is time invariant. For model reference adaptive control, the fact that the leading coefficient of $\hat{Z}_0(s,t)$ is not allowed to pass through zero guarantees that the degree of $\hat{Q}_d(s,t)$ is time invariant. Finally, for adaptive H_∞ optimal control, the result follows from the fact that the leading coefficient of $\hat{Z}_0(s,t)$ is not allowed to pass through zero.

We now present the detailed proof for the case of adaptive H_2 optimal control. Let n_r, m_r be the degrees of the denominator and numerator polynomials respectively of $R(s)$. Then, in the expression for $\hat{Q}(s,t)$ in (1.19), it is clear that $\hat{P}_M^{-1}(s,t) = \dfrac{n\text{th order polynomial}}{l\text{th order polynomial}}$ while $R_M^{-1}(s) = \dfrac{n_r\text{th order polynomial}}{m_r\text{th order polynomial}}$. Also $\left[\hat{B}_P^{-1}(s,t)R_M(s)\right]_* = \dfrac{(\bar{n}-1)\text{th order polynomial}}{\bar{n}\text{th order polynomial}}$ where $\bar{n} \leq n_r$, strict inequality being attained when some of the poles of $R_M(s)$ coincide with some of the stable zeros of $\hat{B}_P^{-1}(s,t)$. Moreover, in any case, the $\bar{n}$th order denominator polynomial of $[\hat{B}_P^{-1}(s,t)R_M(s)]_*$ is a factor of the n_rth order numerator polynomial of $R_M^{-1}(s)$. Thus for the $\hat{Q}(s,t)$ given in (1.19), if we disregard $F(s)$, then the degree of the numerator polynomial is $n + n_r - 1$ while that of the denominator polynomial is $l + m_r \leqslant n + n_r - 1$. Hence, the degree of $\hat{Q}_d(s,t)$ in Step 3 of the certainty equivalence design can be kept fixed at $(n + n_r - 1)$, and this can be achieved with *a single fixed* $F(s)$ of relative degree $n - l + n_r - m_r - 1$, provided that the leading coefficient of $\hat{Z}_0(s,t)$ is appropriately constrained.

Remark 4.1 Lemma 4.1 tells us that the degree of each of the certainty equivalence controllers presented in the last section can be made time invariant. This is important because, as we will see, it makes it possible to carry out the analysis using standard state-space results on slowly time-varying systems.

Lemma 4.2 At any fixed time t, the coefficients of $\hat{Q}_d(s,t)$, $\hat{Q}_n(s,t)$, and hence the vectors $q_d(t)$, $q_n(t)$, are continuous functions of the estimate $\theta(t)$.

Proof Once again, the proof of this lemma is relatively straightforward except in the case of adaptive H_2 optimal control. Accordingly, we first discuss the simpler cases before giving a detailed treatment of the more involved one.

For the case of adaptive partial pole placement control, the continuity follows trivially from the fact that the IMC parameter is independent of $\theta(t)$. For model reference adaptive control, the continuity is immediate from (1.18) and the fact that the leading coefficient of $\hat{Z}_0(s,t)$ is not allowed to pass through zero. Finally for adaptive H_∞ optimal control, we note that the right half plane zero $\hat{b}_1$ of $\hat{Z}_0(s,t)$ is a continuous function of $\theta(t)$. This is a consequence of the fact that the degree of $\hat{Z}_0(s,t)$ cannot drop since its leading coefficient is not allowed to pass through zero. The desired continuity now follows from (1.20).

We now present the detailed proof for the H_2 optimal control case. Since the leading coefficient of $\hat{Z}_0(s,t)$ has been constrained so as not to pass through zero then, for any fixed t, the roots of $\hat{Z}_0(s,t)$ are continuous functions of $\theta(t)$. Hence, it follows that the coefficients of the numerator and denominator polynomials of $[\hat{P}_M(s,t)]^{-1} = [\hat{B}_P(s,t)][\hat{P}(s,t)]^{-1}$ are continuous functions of $\theta(t)$. Moreover, $[[\hat{B}_P(s,t)]^{-1}R_M(s)]_*$ is the sum of the residues of $[\hat{B}_P(s,t)]^{-1}R_M(s)$ at the poles of $R_M(s)$, which clearly depends continuously on $\theta(t)$ (through the factor $[\hat{B}_P(s,t)]^{-1}$). Since $F(s)$ is fixed and independent of θ, it follows from (1.19) that the coefficients of $\hat{Q}_d(s,t)$, $\hat{Q}_n(s,t)$ depend continuously on $\theta(t)$.

Remark 4.2 Lemma 4.2 is important because it allows one to translate slow variation of the estimated parameter vector $\theta(t)$ to slow variation of the controller parameters. Since the stability and robustness proofs of most adaptive schemes rely on results from the stability of slowly time-varying systems, establishing continuity of the controller parameters as a function of the estimated plant parameters (which are known to vary slowly) is a crucial ingredient of the analysis.

The following theorem describes the stability and robustness properties of the adaptive IMC schemes presented in this chapter.

Theorem 4.1 Consider the plant (1.8) subject to the robust adaptive IMC control law (1.12)–(1.16), (1.17), where (1.17) corresponds to any one of the adaptive IMC schemes considered in the last section and $r(t)$ is a bounded external signal. Then, $\exists\, \mu^* > 0$ such that $\forall\, \mu \in [0, \mu^*)$, all the signals in the closed loop system are uniformly bounded and the error $y - \hat{y} \in S\left(c\frac{\mu^2\eta^2}{m^2}\right)$ for some $c > 0$[7]

[7] In the rest of this chapter, 'c' is the generic symbol for a positive constant. The exact value of such a constant can be determined (for a quantitative robustness result) as in [13, 9]. However, for the qualitative presentation here, the exact values of these constants are not important.

Proof The proof is obtained by combining the properties of the robust adaptive law (1.12)–(1.16) with the properties of the IMC-based controller structure. We first analyse the properties of the adaptive law.

From (1.10), (1.13) and (1.14), we obtain

$$\varepsilon = \frac{-\tilde{\theta}^T \phi + \mu\eta}{m^2}, \quad \tilde{\theta} \triangleq \theta - \theta^* \tag{1.23}$$

Consider the positive definite function

$$V(\tilde{\theta}) = \frac{\tilde{\theta}^T \tilde{\theta}}{2\gamma}$$

Then, along the solution of (1.12), it can be shown that [9]

$$\begin{aligned} \dot{V} &\le \tilde{\theta}^T \varepsilon\phi \\ &= \varepsilon\,[-\varepsilon m^2 + \mu\eta] \quad \text{(using (1.23)} \\ &\le -\frac{1}{2}\varepsilon^2 m^2 + \frac{1}{2}\frac{\mu^2\eta^2}{m^2} \quad \text{(completing the squares)} \end{aligned} \tag{1.24}$$

From (1.11), (1.15), (1.16), using Lemma 2.1 (Equation (7)) in [4], it follows that $\frac{\eta}{m} \in L_\infty$. Now, the parameter projection guarantees that $V, \tilde{\theta}, \theta \in L_\infty$. Hence integrating both sides of (1.24) from t to $t + T$, we obtain

$$\varepsilon m \in S\left(\frac{\mu^2\eta^2}{m^2}\right)$$

Also from (1.12)

$$|\dot{\theta}| \le \gamma|\varepsilon m|\frac{|\phi|}{m} \tag{1.25}$$

From the definition of ϕ, it follows using Lemma 2.1 (equation (7)) in [4] that $\frac{\phi}{m} \in L_\infty$, which in turn implies that $\dot{\theta} \in S\left(c\,\frac{\mu^2\eta^2}{m^2}\right)$. This completes the analysis of the properties of the robust adaptive law. To complete the stability proof, we now turn to the properties of the IMC-based controller structure.

The certainty equivalence control law (1.17) can be rewritten as

$$\frac{s^{n_d}}{\Lambda_1(s)}[u] + \beta_1(t)\frac{s^{n_d-1}}{\Lambda_1(s)}[u] + \ldots + \beta_{n_d}(t)\frac{1}{\Lambda_1(s)}[u] = q_n^T(t)\frac{a_{n_d}(s)}{\Lambda_1(s)}[r - \varepsilon m^2]$$

where $\beta_1(t)$, $\beta_2(t)$,..., $\beta_{n_d}(t)$ are the time-varying coefficients of $\hat{Q}_d(s,t)$. Defining $x_1 = \frac{1}{\Lambda_1(s)}[u], x_2 = \frac{s}{\Lambda_1(s)}[u], \ldots, x_{n_d} = \frac{s^{n_d-1}}{\Lambda_1(s)}[u], X \triangleq [x_1, x_2, \ldots, x_{n_d}]^T$, the above equation can be rewritten as

$$\dot{X} = A(t)X + Bq_n^T(t)\frac{a_{n_d}(s)}{\Lambda_1(s)}[r - \varepsilon m^2] \tag{1.26}$$

where

$$A(t) \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\beta_{n_d}(t) & -\beta_{n_d-1}(t) & \cdot & \cdot & \cdot & -\beta_1(t) \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

Since the time-varying polynomial $\hat{Q}_d(s, t)$ is pointwise Hurwitz, it follows that for any *fixed* t, the eigenvalues of $A(t)$ are in the open left half plane. Moreover, since the coefficients of $\hat{Q}_d(s, t)$ are continuous functions of $\theta(t)$ (Lemma 4.2) and $\theta(t) \in \mathcal{C}_\theta$, a compact set, it follows that $\exists\ \sigma_s > 0$ such that

$$Re\{\lambda_i(A(t))\} \leq -\sigma_s\ \forall\ t \geq 0 \quad \text{and} \quad i = 1, 2, \ldots, n_d$$

The continuity of the elements of $A(t)$ with respect to $\theta(t)$ and the fact that $\dot{\theta} \in \mathcal{S}\left(c\frac{\mu^2\eta^2}{m^2}\right)$ together imply that $\dot{A}(t) \in \mathcal{S}\left(c\frac{\mu^2\eta^2}{m^2}\right)$. Hence, using the fact that $\frac{\eta}{m} \in L_\infty$, it follows from Lemma 3.1 in [9] that $\exists\ \mu_1^* > 0$ such that $\forall\ \mu \in [0, \mu_1^*)$, the equilibrium state $x_e = 0$ of $\dot{x} = A(t)x$ is exponentially stable, i.e. there exist $c_0, p_0 > 0$ such that the state transition matrix $\Phi(t, \tau)$ corresponding to the homogeneous part of (1.26) satisfies

$$\|\Phi(t, \tau)\| \leq c_0 e^{-p_0(t-\tau)}\ \forall\ t \geq \tau \tag{1.27}$$

From the identity $u = \frac{\Lambda_1(s)}{\Lambda_1(s)}[u]$, it is easy to see that the control input u can be rewritten as

$$u = v^T(t)X + q_n^T(t)\frac{a_{n_d}(s)}{\Lambda_1(s)}[r - \varepsilon m^2] \tag{1.28}$$

where

$$v(t) = [\lambda_{n_d} - \beta_{n_d}(t), \lambda_{n_{d-1}} - \beta_{n_{d-1}}(t), \cdots, \lambda_1 - \beta_1(t)]^T$$

and

$$\Lambda_1(s) = s^{n_d} + \lambda_1 s^{n_{d-1}} + \ldots + \lambda_{n_d}$$

Also, using (1.28) in the plant equation (1.8), we obtain

$$y = \frac{Z_0(s)}{R_0(s)}[1+\mu\Delta_m(s)]\left[v^T(t)X + q_n^T(t)\frac{a_{n_d}(s)}{\Lambda_1(s)}[r-\varepsilon m^2]\right] \tag{1.29}$$

Now let $\delta \in (0, \min[\delta_0, p_0])$ be chosen such that $R_0(s)$, $\Lambda_1(s)$ are analytic in $\mathcal{R}e[s] \geq -\frac{\delta}{2}$, and define the fictitious normalizing signal $m_f(t)$ by

$$m_f(t) = 1.0 + \|u_t\|_2^\delta + \|y_t\|_2^\delta \tag{1.30}$$

As in [9], we take truncated exponentially weighted norms on both sides of (1.28), (1.29) and make use of Lemma 3.3 in [9] and Lemma 2.1 (equation (6)) in [4], while observing that $v(t)$, $q_n(t)$, $r(t) \in L_\infty$, to obtain

$$\|u_t\|_2^\delta \leq c + c\|(\varepsilon m^2)_t\|_2^\delta \tag{1.31}$$

$$\|y_t\|_2^\delta \leq c + c\|(\varepsilon m^2)_t\|_2^\delta \tag{1.32}$$

which together with (1.30) imply that

$$m_f(t) \leq c + c\|(\varepsilon m^2)_t\|_2^\delta \tag{1.33}$$

Now squaring both sides of (1.33) we obtain

$$m_f^2(t) \leq c + c\int_0^t e^{-\delta(t-\tau)}\varepsilon^2 m^2 m_f^2(\tau)d\tau \quad (\text{since } m(t) \leq m_f(t)$$

$$\Rightarrow m_f^2(t) \leq c + c\int_0^t e^{-\delta(t-s)}\varepsilon^2(s)m^2(s)\left(e^{c\int_s^t \varepsilon^2 m^2 d\tau}\right)ds$$

(using the Bellman-Gronwall lemma [8])

Since $\varepsilon m \in S\left(\frac{\mu^2\eta^2}{m^2}\right)$ and $\frac{\eta}{m}$ is bounded, it follows using Lemma 2.2 in [4] that $\exists\, \mu^* \in (0,\ \mu_1^*)$ such that $\forall\, \mu \in [0, \mu^*)$, $m_f \in L_\infty$, which in turn implies that $m \in L_\infty$. Since $\frac{\phi}{m}, \frac{\eta}{m}$ are bounded, it follows that ϕ, $\eta \in L_\infty$. Thus $\varepsilon m^2 = -\tilde{\theta}^T\phi + \mu\eta$ is also bounded so that from (1.26), we obtain $X \in L_\infty$. From (1.28), (1.29), we can now conclude that u, $y \in L_\infty$. This establishes the boundedness of all the closed loop signals in the adaptive IMC scheme. Since $y - \hat{y} = \varepsilon m^2$ and $\varepsilon m \in S\left(\frac{\mu^2\eta^2}{m^2}\right)$, $m \in L_\infty$, it follows that $y - \hat{y} \in S\left(c\frac{\mu^2\eta^2}{m^2}\right)$ as claimed and, therefore, the proof is complete.

Remark 4.3 The robust adaptive IMC schemes of this chapter recover the performance properties of the ideal case if the modelling error disappears, i.e. we can show that if $\mu = 0$ then $y - \hat{y} \to 0$ as $t \to \infty$. This can be established using standard arguments from the robust adaptive control literature, and is a consequence of the use of parameter projection as the robustifying modification in the adaptive law [9]. An alternative robustifying modification which can guarantee a similar property is the switching-σ modification [14].

1.5 Simulation examples

In this section, we present some simulation examples to demonstrate the efficacy of the adaptive IMC schemes proposed.

We first consider the plant (1.7) with $Z_0(s) = s + 2$, $R_0(s) = s^2 + s + 1$, $\Delta_m = \frac{s+1}{s+3}$ and $\mu = 0.01$. Choosing $\delta_0 = 0.1$, $\gamma = 1$, $\Lambda(s) = s^2 + 2s + 2$, $\mathcal{C}_\theta = [-5.0,\ 5.0] \times [-4.0,\ 4.0] \times [0.1,\ 6.0] \times [-6.0,\ 6.0]$, $Q(s) = \frac{1}{s+4}$ and implementing the adaptive partial pole placement control scheme (1.12)–(1.16), (1.17), with $\theta(0) = [-1.0,\ 2.0,\ 3.0,\ 1.0]^T$ and all other initial conditions set to zero, we obtained the plots in Figure 1.3 for $r(t) = 1.0$ and $r(t) = \sin(0.2t)$. From these plots, it is clear that $y(t)$ tracks $\frac{s+2}{(s^2+s+1)(s+4)}[r]$ quite well.

Let us now consider the design of an adaptive model reference control scheme for the same plant where the reference model is given by $W_m(s) = \frac{1}{s^2+2s+1}$. The adaptive law (1.12)–(1.16) must now guarantee that the estimated plant is pointwise minimum phase, to ensure which, we now choose the set $\mathcal{C}_\theta$ as $\mathcal{C}_\theta = [-5.0,\ 5.0] \times [-4.0,\ 4.0] \times [0.1,\ 6.0] \times [0.1,\ 6.0]$. All the other design parameters are exactly the same as before except that now (1.17) implements the IMC control law (1.18) and $\Lambda_1(s) = s^3 + 2s^2 + 2s + 2$. The resulting plots are shown in Figure 1.4 for $r(t) = 1.0$ and $r(t) = \sin(0.2t)$. From these plots, it is clear that the adaptive IMC scheme does achieve model following.

The modelled part of the plant we have considered so far is minimum phase which would not lead to an interesting H_2 or H_∞ optimal control problem. Thus, for H_2 and H_∞ optimal control, we consider the plant (1.7) with $Z_0(s) = -s + 1$, $R_0(s) = s^2 + 3s + 2$, $\Delta_m = \frac{s+1}{s+3}$ and $\mu = 0.01$. Choosing $\delta_0 = 0.1$, $\gamma = 1$, $\Lambda(s) = s^2 + 2s + 2$, $\Lambda_1(s) = s^2 + 2s + 2$, $\mathcal{C}_\theta = [-5.0,\ 5.0] \times [-4.0,\ 4.0] \times [-6.0,\ -0.1] \times [-6.0,\ 6.0]$, $F(s) = \frac{1}{(s+1)^2}$ and implementing the adaptive H_2 optimal control scheme (1.12)–(1.16), (1.17), with $\theta(0) = [2.0,\ 2.0,\ -2.0,\ 2.0]^T$ and all other initial conditions set to zero, we obtained the plot shown in Figure 1.5. From Figure 1.5, it is clear that $y(t)$

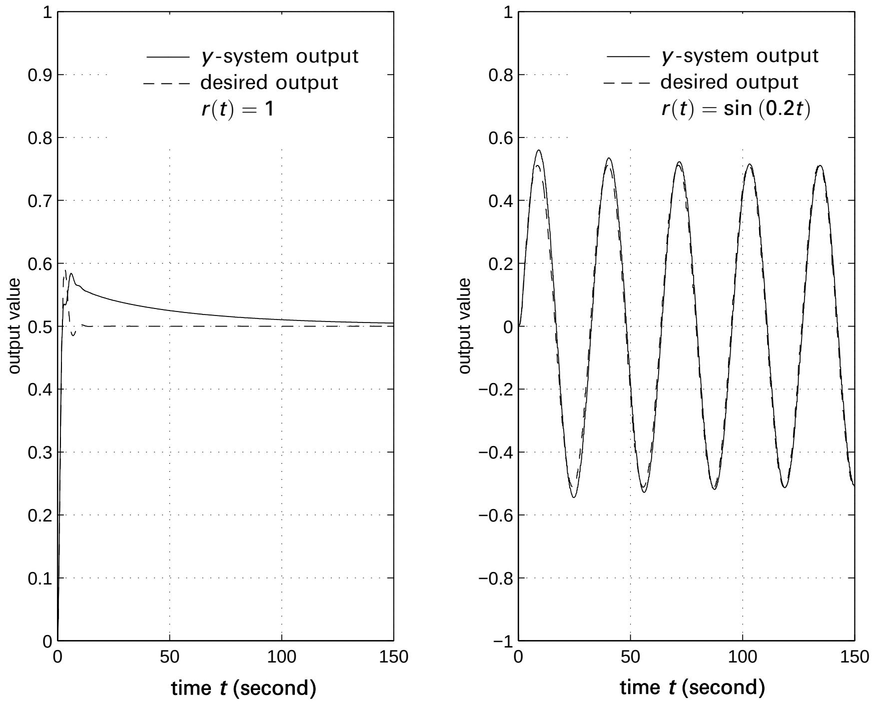

Figure 1.3 *PPAC IMC simulation*

asymptotically tracks $r(t)$ quite well. Note that the projection set $\mathcal{C}_\theta$ here has been chosen to ensure that the degree of $\hat{Z}_0(s,t)$ does not drop.

Finally, we simulated an H_∞ optimal controller for the same plant used for the H_2 design.The weighting $W(s)$ was chosen as $W(s) = \dfrac{0.01}{s+0.01}$ and the set $\mathcal{C}_\theta$ was taken as $\mathcal{C}_\theta = [-5.0,\ 5.0] \times [-4.0,\ 4.0] \times [-6.0,\ -0.1] \times [0.1,\ 6.0]$. This choice of $\mathcal{C}_\theta$ ensures that the estimated plant has one and only one right half plane zero. Keeping all the other design parameters the same as in the H_2 optimal control case and choosing $r(t) = 1.0$ and $r(t) = 0.8\sin(0.2t)$, we obtained the plots shown in Figure 1.6. From these plots, we see that the adaptive H_∞-optimal controller does produce reasonably good tracking.

1.6 Concluding remarks

In this chapter, we have presented a general systematic theory for the design and analysis of robust adaptive internal model control schemes. The certainty

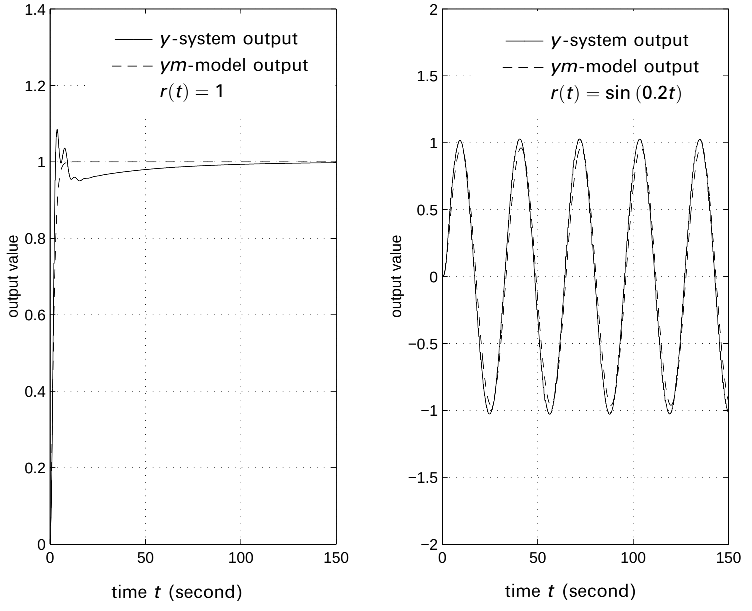

Figure 1.4 *MRAC IMC simulation*

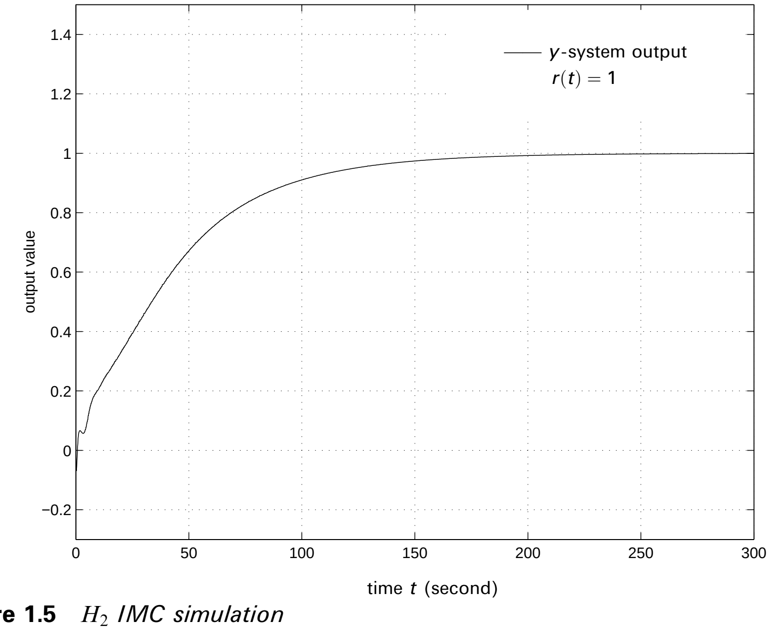

Figure 1.5 *H_2 IMC simulation*

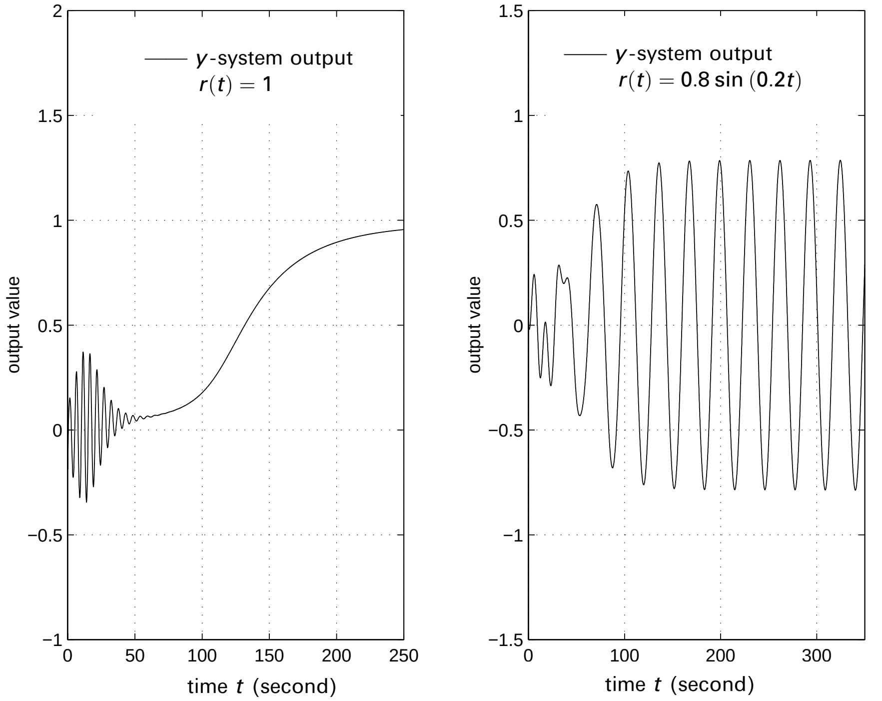

Figure 1.6 H_∞ *IMC simulation*

equivalence approach of adaptive control was used to combine a robust adaptive law with robust internal model controller structures to obtain adaptive internal model control schemes with provable guarantees of robustness. Some specific adaptive IMC schemes that were considered here include those of the partial pole placement, model reference, H_2 optimal and H_∞ optimal control types. A single analysis procedure encompassing all of these schemes was presented.

We do believe that the results of this chapter complete our earlier work on adaptive IMC [4, 5] in the sense that a proper bridge has now been established between adaptive control theory and some of its industrial applications. It is our hope that both adaptive control theorists as well as industrial practitioners will derive some benefit by traversing this bridge.

References

[1] Morari, M. and Zafiriou, E. (1989). *Robust Process Control*. Prentice-Hall, Englewood Cliffs, NJ.

[2] Takamatsu, T., Shioya, S. and Okada, Y. (1985). 'Adaptive Internal Model Control and its Application to a Batch Polymerization Reactor', *IFAC Symposium on Adaptive Control of Chemical Processes*, Frankfurt am Main.

[3] Soper, R. A., Mellichamp, D. A. and Seborg, D. E. (1993). 'An Adaptive Nonlinear Control Strategy for Photolithography', *Proc. Amer. Control Conf.*

[4] Datta, A. and Ochoa, J. (1996). 'Adaptive Internal Model Control: Design and Stability Analysis', *Automatica*, Vol. 32, No. 2, 261–266, Feb.

[5] Datta, A. and Ochoa, J. (1998). 'Adaptive Internal Model Control: H_2 Optimization for Stable Plants', *Automatica*. Vol. 34, No. 1, 75–82, Jan.

[6] Youla, D. C., Jabr, H. A. and Bongiorno, J. J. (1976). 'Modern Wiener-Hopf Design of Optimal Controllers – Part II: The Multivariable Case', *IEEE Trans. Automat. Contr.*, Vol. AC-21, 319–338.

[7] Zames, G. and Francis, B. A. (1983). 'Feedback, Minimax Sensitivity, and Optimal Robustness', *IEEE Trans. Automat. Contr.*, Vol. AC-28, 585–601.

[8] Desoer, C. A. and Vidyasagar, M. (1975). *Feedback Systems: Input-Output Properties*, Academic Press, New York.

[9] Ioannou, P. A. and Datta, A. (1991). 'Robust Adaptive Control: A Unified Approach', *Proc. of the IEEE*, Vol. 79, No. 12, 1736–1768, Dec.

[10] Kharitonov, V. L. (1978). 'Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations', *Differentsial' nye Uravneniya*, Vol. 14, 2086–2088.

[11] Middleton, R. H., Goodwin, G. C., Hill, D. J. and Mayne, D. Q. (1988) 'Design Issues in Adaptive Control', *IEEE Trans. on Automat. Contr.*, Vol. AC-33, 50–58.

[12] Ioannou, P. A. and Sun, J. (1996). *Robust Adaptive Control*, Prentice Hall, Englewood Cliffs, NJ.

[13] Tsakalis, K. S. (1992). 'Robustness of Model Reference Adaptive Controllers: An Input-Output Approach', *IEEE Trans. on Automat. Contr.*, Vol. AC-37, 556–565.

[14] Ioannou, P. A. and Tsakalis, K. S. (1986). 'A Robust Direct Adaptive Controller,' *IEEE Trans. on Automat. Contr.*, Vol. AC-31, 1033–1043, Nov.

2

An algorithm for robust adaptive control with less prior knowledge

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Abstract

A new robust discrete-time singularity free direct adaptive control scheme is proposed with respect to a class of modelling uncertainties in this chapter. Two key features of this scheme are that a relative dead zone is used but no knowledge of the parameters of the upper bounding function on the class of modelling uncertainties is required, and no knowledge of the lower bound on the leading coefficient of the parameter vector is required to ensure the control law singularity free. Global stability and convergence results of the scheme are provided.

2.1 Introduction

Since it was shown (e.g. [1], [2]) that unmodelled dynamics or even a small bounded disturbance could cause most of the adaptive control algorithms to go unstable, much effort has been devoted to developing robust adaptive control algorithms to account for the system uncertainties. As a consequence, a number of adaptive control algorithms have been developed, for example, see [3] and references therein. Among those algorithms are simple projection (e.g. [4], [5]), normalization (e.g. [6], [7]), dead zone (e.g. [8–12]), adaptive law modification (e.g. [13], [14]), σ-modification (e.g. [15], [16]), as well as persistent excitation (e.g. [17], [18]).

In the case of the dead zone based methods, a fixed dead zone can be used [6–8] in the presence of only bounded disturbance, which turns off the algorithm when the identification error is smaller than a certain threshold. In order to choose an appropriate size of the dead zone, an upper bound on the disturbance must be known.

When unmodelled dynamics are present, a relative dead zone modification should be employed [11], [12]. Here the knowledge of the parameters of bounding function on the unmodelled dynamics and bounded disturbances is required.

However, such knowledge, especially knowledge of the nonconservative upper bound or the parameters of the upper bounding function, can be hardly obtained in practice. Therefore, the robust adaptive control algorithm which does not rely on such knowledge is in demand but remains absent in the literature. One may argue that the robustness of the adaptive control algorithms can be achieved with only simple projection techniques in parameter estimation [4], [5]. However, it should be noted that using the robust adaptive control algorithms such as the dead zone, the robustness of the resulting adaptive control systems will be improved in the sense that the tolerable unmodelled dynamics can be enlarged [19]. Therefore, discussion of the robust adaptive control approaches such as those based on the dead zone technique is still of interest and the topic of this chapter.

Another potential problem associated with adaptive control is its control law singularity. The estimated plant model could be in such a form that the pole-zero cancellations occur or the leading coefficient of the estimated parameter vector is zero. In such cases, the control law becomes singular and thus cannot be implemented. In order to secure the adaptive control law singularity free, various approaches have been developed. These approaches can be classified into two categories. One relies on persistent excitation. The other depends on modifications of the parameter estimation schemes.

In the latter case, the most popular method is to hypothesize the existence of a known convex region in which no pole-zero cancellations occur and then to develop a convergent adaptive control scheme by constraining the parameter estimates inside this region (e.g. [11], [20–22]) for pole placement design; or to hypothesize the existence of a known lower bound on the leading coefficient of the parameter vector and then to use an ad hoc projection procedure to secure the estimated leading coefficient bounded away from zero and thus achieve the convergence and stability of the direct adaptive control system. However, such methods suffer the problem of requirement for significant a priori knowledge about the plant.

Recently, another approach has been developed which also modifies the parameter estimation algorithm. This approach is to re-express the plant model in a special input–output representation and then use a correction procedure in the estimation algorithm to secure the controllability and observability of the estimated model of the system [23–24]. They also addressed the robustness problem of such algorithms with respect to bounded disturbance [25] using the dead zone technique. They did not address the robustness problem with respect to unmodelled dynamics. Moreover, those algorithms also suffer the same problem as the usual dead zone based robust adaptive control algorithms. That is, they still require the knowledge of the upper bound on the disturbance or the parameters of the upper bounding function on the unmodelled dynamics and disturbances.

In this chapter, a new robust direct adaptive control algorithm will be proposed which does use dead zone but does not require the knowledge of the parameters of the upper bounding functions on the unmodelled dynamics and the disturbance. It has also been shown that our algorithm can be combined with the parameter estimate correction procedure, which was originated in [24] to ensure the control law singularity free, so that the least a priori information is required on the plant.

The chapter is organized as follows. The problem is formulated in Section 2.2. Ordinary discrete time direct adaptive control algorithm with dead zone is reviewed in Section 2.3. Our main results, a new robust direct adaptive control algorithm and its improved version with control law singularity free are presented in Section 2.4 and Section 2.5 respectively. Section 2.6 presents one simulation examples to illustrate the proposed adaptive control algorithms, which is followed by some concluding remarks in Section 2.7.

2.2 Problem formulation

Consider a discrete time single input single output plant

$$y(t) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}u(t) + v(t) \tag{2.1}$$

where $y(t)$ and $u(t)$ are plant output and input respectively, $v(t)$ represents the class of unmodelled dynamics and bounded disturbances and d is the time delay. $A(z^{-1})$ and $B(z^{-1})$ are polynomials in z^{-1}, written as

$$A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n}$$
$$B(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m}$$

Specify a reference model as

$$E(z^{-1})y*(t) = z^{-d}R(z^{-1})r(t) \tag{2.2}$$

where $E(z^{-1})$ is a strictly stable monic polynomial written as

$$E(z^{-1}) = 1 + e_1 z^{-1} + \ldots + e_{\bar{n}} z^{-\bar{n}}$$

Then, there exist unique polynomials $F(z^{-1})$ and $G(z^{-1})$ written as

$$F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_{d-1} z^{-d+1}$$

$$G(z^{-1}) = g_0 + g_1 z^{-1} + \ldots + g_{n-1} z^{-n+1}$$

such that

$$E(z^{-1}) = F(z^{-1})A(z^{-1}) + z^{-d}G(z^{-1}) \tag{2.3}$$

Using equation (2.3), it can be shown that the plant equation (2.1) can be rewritten as

$$\begin{aligned} \bar{y}(t+d) &= \alpha(z^{-1})y(t) + \beta(z^{-1})u(t) \\ &= \theta^T \phi(t) + \eta(t+d) \end{aligned} \tag{2.4}$$

where

$$\bar{y}(t+d) = E(z^{-1})y(t+d)$$

$$\eta(t+d) = F(z^{-1})A(z^{-1})v(t+d)$$

$$\begin{aligned} \phi(t)^T &= [u(t), u(t-1), \ldots, u(t-m-d+1), y(t), y(t-1), \ldots, y(t-n+1)] \\ &:= [u(t), \phi'(t)^T] \\ \theta^T &= [\theta^1, \ldots \theta^{n+m+d}] := [\theta_1, \theta'^T] \\ \alpha(z^{-1}) &= G(z^{-1}) \\ \beta(z^{-1}) &= F(z^{-1})B(z^{-1}) \end{aligned}$$

We make the following standard assumptions [26], [18].

(A1) The time delay d and the plant order n are known.
(A2) The plant is minimum phase.

For the modelling uncertainties, we assume only:

(A3) There exists a function [11] $\gamma(t)$ such that

$$|\eta(t)|^2 \le \gamma(t)$$

where $\gamma(t)$ satisfies

$$\gamma(t) \le \varepsilon_1 \sup_{0 \le \tau \le t} ||x(\tau)||^2 + \varepsilon_2$$

for some *unknown* constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and x(t) is defined as

$$x(t) = [y(t-1), \ldots, y(t-n), u(t-1), \ldots, u(t-m-d)]^T$$

For the usual direct adaptive control, in order to facilitate the implementation of projection procedure to secure the control law singularity free, the following assumption is required.

(A4) There is a known constant θ_m^1 satisfying

$$|\theta_m^1| \le |\theta^1| \quad \text{and} \quad \theta_m^1\theta^1 > 0$$

For the usual relative dead zone based direct adaptive control algorithm, another assumption is needed as follows:

(A5) The constants $\varepsilon_1$and ε_2 in (A3) are known a priori.

Remark 2.1 It should be noted that the assumptions (A4) and (A5) will not be required in our new adaptive control algorithm to be developed in the next few sections. It is believed that the elimination of assumptions (A4) and (A5) will improve the applicability of the adaptive control systems.

2.3 Ordinary direct adaptive control with dead zone

Let $\hat{\theta}(t)$ denote the estimate of the unknown parameter θ for the plant model (2.4). Defining the estimation error as

$$e(t) = \bar{y}(t) - \phi^T(t-d)\hat{\theta}(t-1) \tag{2.5}$$

and a dead zone function as

$$f(g,e) = \begin{cases} e-g & \text{if } e > g \\ 0 & \text{if } |e| \le g \quad 0 < g < \infty \\ e+g & \text{if } e < -g \end{cases} \tag{2.6}$$

then the following least squares algorithm with a relative dead zone can be used for parameter estimation

$$\begin{aligned} \hat{\theta}(t) &= \text{proj}\{\hat{\theta}(t-1) + a(t)\frac{P(t-1)\phi(t-d)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}e(t)\} \\ P(t) &= P(t-1) - a(t)\frac{P(t-1)\phi(t-d)\phi(t-d)^T P(t-1)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\ P(-1) &= k_0 I, \quad k_0 > 0 \end{aligned} \tag{2.7}$$

where the term $a(t)$ is a dead zone, which is defined as follows:

$$a(t) = \begin{cases} 0 & \text{if } |e(t)|^2 \le \xi\gamma(t) \\ \alpha f(\xi^{1/2}\gamma(t)^{1/2}, e(t))/e(t) & \text{otherwise} \end{cases} \tag{2.8}$$

with $0 < \alpha < 1$, $\xi = \dfrac{\xi_0}{1-\alpha}$, $\xi_0 > 1$, and proj is the projection operator [26] such that

$$\hat{\theta}^1(t) = \begin{cases} \hat{\theta}^1(t) & \text{if } \hat{\theta}^1(t)\,\text{sgn}\,(\theta^1_m) \geq |\theta^1_m| \\ \theta^1_m & \text{otherwise} \end{cases} \tag{2.9}$$

It has been shown that the above parameter estimation algorithm has the following properties:

(i) $\hat{\theta}(t)$ is bounded

(ii) $|\hat{\theta}^1(t) \geq |\theta^1_m|$ and $\dfrac{1}{\hat{\theta}^1(t)} = \left|\dfrac{\theta^1}{\hat{\theta}^1(t)}\right| \dfrac{1}{\theta^1}$

(iii) $\dfrac{f(\xi^{1/2}\gamma(t)^{1/2}, e(t))^2}{1 + \phi(t-d)^T P(t-1)\phi(t-d)} \in l_2$

(iv) $\dfrac{f(\xi^{1/2}\gamma(t)^{1/2}, \varepsilon(t))^2}{1 + \phi(t-d)^T P(t-1)\phi(t-d)} \in l_2$

where

$$\varepsilon(t) = \bar{y}(t) - \hat{\theta}(t-d)^T \phi(t-d) \tag{2.10}$$

The direct adaptive control law can be written as

$$u(t) = \frac{r_f(t) - \hat{\theta}'(t)^T \phi'(t)}{\hat{\theta}^1(t)} \tag{2.11}$$

where

$$r_f(t) = R(z^{-1})r(t) \tag{2.12}$$

with $r(t)$ a reference input and $R(z^{-1})$ is specified by the reference model equation (2.2).

Then with the parameter estimation properties (i)–(iv), the global stability and convergence results of the adaptive control systems can be established as in [26], [11], which are summarized in the following theorem.

Theorem 3.1 The direct adaptive control system satisfying assumptions (A1)–(A5) with adaptive controller described in equations (2.7)–(2.9) and (2.11) is globally stable in the sense that all the signals in the loop remain bounded.

However, as discussed in the first section, the requirement for knowledge of the parameters of the upper bounding function on the unmodelled dynamics and bounded disturbances is very restrictive. In the next section, we are attempting to propose a new approach to get rid of such a requirement.

2.4 New robust direct adaptive control

Here we develop a new robust adaptive control algorithm which does not need such knowledge. That is, we drop the assumption (A5).

The key idea is to use an adaptation law to update those parameters. The new parameter estimation algorithm is the same as equations (2.7–2.9) but with a different dead zone $a(t)$, where

$$a(t) = \begin{cases} 0 & \text{if } |e(t)|^2 \le \xi(\hat{\gamma}(t) + Q(t)) \\ \alpha f(\xi^{1/2}(\hat{\gamma}(t) + Q(t))^{1/2}, e(t))/e(t) & \text{otherwise} \end{cases} \tag{2.13}$$

and

$$Q(t) = \frac{\alpha\beta}{4(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d)} \begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix}^T \begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix} \tag{2.14}$$

And $\hat{\gamma}(t)$is calculated by

$$\hat{\gamma}(t) = \hat{C}(t)^T \begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix} \tag{2.15}$$

$$\hat{C}(t) = \hat{C}(t-1) + \frac{a(t)\beta}{2(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))} \begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix}$$

$$\beta > 0 \tag{2.16}$$

where

$$\hat{C}(t)^T = [\hat{\varepsilon}_1 \quad \hat{\varepsilon}_2]$$

with zero initial condition. It should be noted that $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ will be always positive and non decreasing.

As shown in [23], the projection operation does not alter the convergence properties of the original parameter estimation algorithms. Therefore, in the following analysis, the projection operation will be neglected.

The properties of the above modified least squares parameter estimator with a relative dead zone are summarized in the following lemma.

Lemma 4.1 The least squares algorithm with equations (2.7), (2.9), (2.13)–(2.16) applied to any system has the following properties irrespective of the control law:

(i) $\hat{\theta}(t)$ is bounded

(ii) $\hat{C}(t)$ is bounded and non-decreasing, and thus $\hat{\varepsilon}_1$ converges to a constant, say $\bar{\varepsilon}_1$

(iii) $|\hat{\theta}^1(t) \ge |\theta^1_m|$ and $\dfrac{1}{\hat{\theta}^1(t)} = \left|\dfrac{\theta^1}{\hat{\theta}^1(t)}\right| \dfrac{1}{\theta^1}$

(iv) $||\hat{\theta}(t) - \hat{\theta}(t-1)|| \in l_2$

(v) $\tilde{f}(t)^2 := \dfrac{f(t)^2}{1+\phi(t-d)^T P(t-1)\phi(t-d)} := \dfrac{f(\xi^{1/2}(\hat{\gamma}(t)+Q(t))^{1/2}, e(t))^2}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \in l_2$

Proof Define a Lyapunov function candidate

$$V(t+1) = \tfrac{1}{2}(\tilde{\theta}(t)^T P(t-1)^{-1}\tilde{\theta}(t) + \tilde{C}(t+1)^T \beta^{-1}\tilde{C}(t+1)) \tag{2.17}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$, $\tilde{C}(t+1) = \hat{C}(t+1) - [\varepsilon_1 \quad \varepsilon_2]$. Then, its difference becomes

$$\begin{aligned}
V(t+1) - V(t) &= \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\
&\quad \times \left[\frac{1+\phi(t-d)^T P(t-1)\phi(t-d)}{1+(1-a(t))\phi(t-d)^T P(t-1)\phi(t-d)}\eta(t)^2 - e(t)^2\right] \\
&\quad + \frac{a(t)\tilde{C}(t)^T}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))}\begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} \\
&\quad + \frac{a(t)^2\beta}{4(1-\alpha)^2(1+\phi(t-d)^T P(t-1)\phi(t-d))^2} \\
&\quad \times \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix}^T \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\
&\quad \times \left[\frac{1+\phi(t-d)^T P(t-1)\phi(t-d)}{1+(1-a(t))\phi(t-d)^T P(t-1)\phi(t-d)}\eta(t)^2 - e(t)^2\right] \\
&\quad + \frac{a(t)}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))}(\tilde{C}(t)^T \\
&\quad \times \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} + Q(t)) \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[\frac{1}{(1-\alpha)}\eta(t)^2 - e(t)^2\right] \\
&\quad + \frac{a(t)}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))}(\hat{\gamma}(t) - \gamma(t) + Q(t))
\end{aligned}$$

$$\leq \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}$$
$$\times \left[\frac{1}{(1-\alpha)}\eta(t)^2 - e(t)^2 + \frac{1}{(1-\alpha)}(\hat{\gamma}(t) - \gamma(t) + Q(t))\right]$$
$$\leq \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-e(t)^2 + \frac{1}{(1-\alpha)}(\hat{\gamma}(t) + Q(t))\right]$$
$$\leq \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-e(t)^2 + \frac{1}{(1-\alpha)}\frac{1}{\xi}e(t)^2\right]$$
$$\leq \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-e(t)^2 + \frac{1}{\xi_0}e(t)^2\right]$$
$$\leq -\frac{\xi_0 - 1}{\xi_0}\frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}e(t)^2$$
$$\leq -\frac{\xi_0 - 1}{\xi_0}\frac{f(t)^2}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \tag{2.18}$$

where the fact that $a(t)e(t)^2 = f(t)e(t) \geq f(t)^2$ has been used. Therefore, following the same arguments in [20], [21], [23], the results in Lemma 4.1 are thus proved.

If using the same adaptive control law as in equation (2.11), then with the parameter estimation properties (i)–(v) in Lemma 4.1, the global stability and convergence results of the new adaptive control system can be established as in [26], [11] as long as the estimated $\bar{\varepsilon}_1$ is small enough, which are summarized in the following theorem.

Theorem 4.1 The direct adaptive control system satisfying assumptions (A1)–(A4) with the adaptive controller described in equations (2.7), (2.9), (2.13)–(2.16) and (2.11) is globally stable in the sense that all the signals in the loop remain bounded.

In this approach, we have eliminated the requirement for the knowledge of the parameters of the upper bounding function on the modelling uncertainties. But the requirement for the knowledge of the lower bound on the leading coefficient of the parameter vector, i.e. assumption (A4) is still there. In the next section, the technique of the parameter correction procedure will be combined with the algorithm developed in this section to ensure the least prior knowledge on the plant. That is, only assumptions (A1)–(A3) are needed.

2.5 Robust adaptive control with least prior knowledge

The following modified least squares algorithm will be used for robust parameter estimation:

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + a(t)\frac{P(t-1)\phi(t-d)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\bar{e}(t) \\
P(t) &= P(t-1) - a(t)\frac{P(t-1)\phi(t-d)\phi(t-d)^T P(t-1)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \qquad (2.19) \\
P(-1) &= k_0 I, \quad k_0 > 0
\end{aligned}$$

and the parameter estimate is then corrected [24] as

$$\bar{\theta}(t) = \hat{\theta}(t) + P(t)\beta(t) \qquad (2.20)$$

where

$$\bar{e}(t) = \bar{y}(t) - \bar{\theta}(t-1)^T\phi(t-d) \qquad (2.21)$$

the vector $\beta(t)$ is described in Figure 2.1

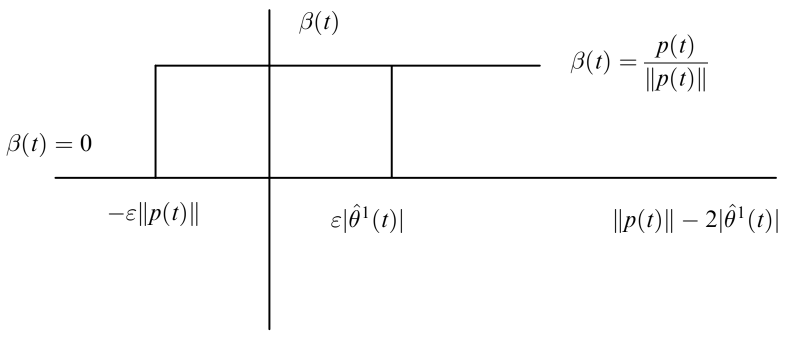

Figure 2.1 *Parameter correction vector*

where $p(t)$ is the first column of the covariance matrix $P(t)$, and the term $a(t)$ is now defined as follows:

$$a(t) = \begin{cases} 0 & \text{if } \bar{e}(t)|^2 \le \xi(\hat{\gamma}(t) + Q(t)) \\ \alpha f(\xi^{1/2}(\hat{\gamma}(t)+Q(t))^{1/2}, \bar{e}(t))/\bar{e}(t) & \text{otherwise} \end{cases}.$$

with $0 < \alpha < 1$, $\xi = \dfrac{2\xi_0}{1-\alpha}$, $\xi_0 > 1$, and

$$\begin{aligned}
Q(t) &= [\beta(t-1)^T P(t-1)\phi(t-d)]^2 \\
&\quad + \frac{\alpha\lambda}{2(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d)}\begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix}^T \begin{bmatrix} \sup\limits_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix}
\end{aligned}$$

And $\hat{\gamma}(t)$is calculated by

$$\hat{\gamma}(t) = \hat{C}(t)^T \begin{bmatrix} \sup_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix} \tag{2.22}$$

$$\hat{C}(t) = \hat{C}(t-1) + \frac{a(t)\lambda}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d)} \begin{bmatrix} \sup_{0\le\tau\le t} ||x(\tau)||^2 \\ 1 \end{bmatrix}, \quad \lambda > 0 \tag{2.23}$$

where

$$\hat{C}(t)^T = [\hat{\varepsilon}_1 \quad \hat{\varepsilon}_2]$$

with zero initial condition. It should be noted that $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$will be always positive and non-decreasing.

Remark 5.1 The prediction error $\bar{e}(t)$ is used in the modified least squares algorithm to ensure that the estimator property (iii) in the following lemma can be established.

The properties of the above modified least squares parameter estimator are summarized in the following lemma.

Lemma 5.1 If the plant satisfies the assumptions (A1)–(A3), the least squares algorithm (2.19)–(2.23) has the following properties:

(i) $\hat{\theta}(t)$ is bounded, and $||\hat{\theta}(t) - \hat{\theta}(t-1)|| \in l_2$.
(ii) $\hat{C}(t)$ is bounded and non-decreasing, thus converges
(iii) $\dfrac{f(\xi^{1/2}(\hat{\gamma}(t) + Q(t))^{1/2}, \bar{e}(t))^2}{1 + \phi(t-d)^T P(t-1)\phi(t-d)} \in l_2$
(iv) $||p(t)|| + |\hat{\theta}^1(t)| > b_{\min}$ where

$$b_{\min} = \frac{|\theta^1|}{\max(1, ||\beta^*||)}$$

with β^* defined such that

$$\theta^* = \hat{\theta}(t) = P(t)\beta^*$$

(v) $|\bar{\theta}^1(t)| > \dfrac{1-\varepsilon}{3+\varepsilon} b_{\min}$
(vi) $\bar{\theta}(t)$ is bounded, and $||\bar{\theta}(t) - \bar{\theta}(t-1)|| \in l_2$

Proof Define a Lyapunov function candidate

$$V(t+1) = \tfrac{1}{2}(\tilde{\theta}(t)^T P(t)^{-1}\tilde{\theta}(t) + \tilde{C}(t+1)^T \lambda^{-1}\tilde{C}(t+1)) \tag{2.24}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$, $\tilde{C}(t+1) = \hat{C}(t+1) - [\varepsilon_1 \quad \varepsilon_2]^T$. Noting that

$$\begin{aligned}\bar{e}(t) &= \bar{y}(t) - \bar{\theta}(t-1)^T\phi(t-d) \\ &= \bar{y}(t) - \hat{\theta}(t-1)\phi(t-d) - \beta(t-1)^T P(t-1)\phi(t-d) \\ &= e(t) - \beta(t-1)^T P(t-1)\phi(t-d)\end{aligned} \tag{2.25}$$

Then, the difference of the Lyapunov function candidate becomes

$$\begin{aligned}V(t+1) - V(t) = &\frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\ &\times \Bigg[\frac{1+\phi(t-d)^T P(t-1)\phi(t-d)}{1+(1-a(t))\phi(t-d)^T P(t-1)\phi(t-d)} \\ &\times (\eta(t) - \beta(t-1)^T P(t-1)\phi(t-d))^2 - \bar{e}(t)^2\Bigg] \\ &+ \frac{a(t)\tilde{C}(t)^T}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))}\begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} \\ &+ \frac{a(t)^2\lambda}{(1-\alpha)^2(1+\phi(t-d)^T P(t-1)\phi(t-d))^2} \\ &\times \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix}^T \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} \\ \le &\frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\ &\times \Big[\frac{1}{1-\alpha}(\eta(t) - \beta(t-1)^T P(t-1)\phi(t-d))^2 - \bar{e}(t)^2\Big] \\ &+ \frac{2a(t)(\hat{\gamma}(t) - \gamma(t))}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))} \\ &+ \frac{a(t)^2\lambda}{(1-\alpha)^2(1+\phi(t-d)^T P(t-1)\phi(t-d))^2} \\ &\times \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix}^T \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\
&\quad \times [\frac{2}{1-\alpha}\eta(t)^2 - \frac{2}{1-\alpha}(\beta(t-1)^T P(t-1)\phi(t-d))^2 - \bar{e}(t)^2\Big] \\
&\quad + \frac{2a(t)(\hat{\gamma}(t)-\gamma(t))}{(1-\alpha)(1+\phi(t-d)^T P(t-1)\phi(t-d))} \\
&\quad + \frac{a(t)^2\lambda}{(1-\alpha)^2(1+\phi(t-d)^T P(t-1)\phi(t-d))^2}\begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix}^T \begin{bmatrix}\sup\limits_{0\le\tau\le t}||x(\tau)||^2 \\ 1\end{bmatrix} \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\
&\quad \times \left[\frac{2}{1-\alpha}\eta(t)^2 - \bar{e}(t)^2 + \frac{2}{1-\alpha}(\hat{\gamma}(t)-\gamma(t)+Q(t))\right] \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-\bar{e}(t)^2 + \frac{2}{1-\alpha}(\hat{\gamma}(t)+Q(t))\right] \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-\bar{e}(t)^2 + \frac{2}{1-\alpha}\frac{1}{\xi}\bar{e}(t)^2\right] \\
&\le \frac{a(t)}{1+\phi(t-d)^T P(t-1)\phi(t-d)}\left[-\bar{e}(t)^2 + \frac{1}{\xi_0}\bar{e}(t)^2\right] \\
&\le -\frac{\xi_0-1}{\xi_0}\frac{a(t)\bar{e}(t)^2}{1+\phi(t-d)^T P(t-1)\phi(t-d)} \\
&\le -\frac{\xi_0-1}{\xi_0}\frac{f(\xi^{1/2}(\hat{\gamma}(t)+Q(t))^{1/2},\bar{e}(t))^2}{1+\phi(t-d)^T P(t-1)\phi(t-d)}
\end{aligned}
\tag{2.26}
$$

where the fact that $a(t)e(t)^2 = f(t)e(t) \ge f(t)^2$ has been used. Therefore, following the same arguments in [26], [11], [21], the results (i)–(iii) in Lemma 5.1 are thus proved. The properties (iv)–(vi) in the lemma can also be obtained directly from the results in [23].

If using the same adaptive control law as in equation (2.11), then with the parameter estimation properties (i)–(vi), the global stability and convergence results of the new adaptive control system can be established as in [26], [11] as long as the estimated $\bar{\varepsilon}_1$ is small enough, which are summarized in the following theorem.

Theorem 5.1 The direct adaptive control system satisfying assumptions (A1)–(A3) with the adaptive controller described in equations (2.19)–(2.23) and (2.11) is globally stable in the sense that all the signals in the loop remain bounded.

2.6 Simulation example

In this section, one numerical example is presented to demonstrate the performance of the proposed algorithm. A fourth order plant is given by the transfer function as

$$G(s) = G_n(s)\, G_u(s)$$

with

$$G_n(s) = \frac{5(s+2)}{s(s+1)}$$

as a nominal part, and

$$G_u(s) = \frac{229}{s^2 + 30s + 229}$$

as the unmodelled dynamics.

With the sampling period $T = 0.1$ second, we have the following corresponding discrete-time model

$$G(q^{-1}) = \frac{0.09784q^{-1} + 0.1206q^{-2} - 0.1414q^{-3} - 0.01878q^{-4}}{1 - 2.3422q^{-1} + 1.0788q^{-2} - 0.4906q^{-3} + 0.04505q^{-4}}$$

The reference model is chosen as

$$G_m(s) = \frac{1}{0.2s + 1}$$

whose corresponding discrete-time model is

$$G_m(q^{-1}) = \frac{0.3935q^{-1}}{1 - 0.6065q^{-1}}$$

We have chosen $k_0 = 1$, and $\hat{\theta}(0) = [0.6 \quad 0 \quad 0 \quad 0]^T$. If no dead zone is used, the simulation results are divergent. If using the algorithm developed in this chapter with $\lambda = 10^{-5}$, the simulation results are shown as in Figure 2.2, where (a) represents the system output $y(t)$ and reference model output $y^*(t)$, (b) is the control signal $u(t)$, (c) is the estimated parameter $\hat{\theta}^1$, and (d) denotes the estimated bounding parameters $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$.

In order to demonstrate the effect of the update rate parameter λ, the following simulation with $\lambda = 1.4 \times 10^{-5}$ was also conducted. The result is shown in Figure 2.3.

The steady state values of the several important parameters and the tracking error in both cases are summarized in Table 2.1.

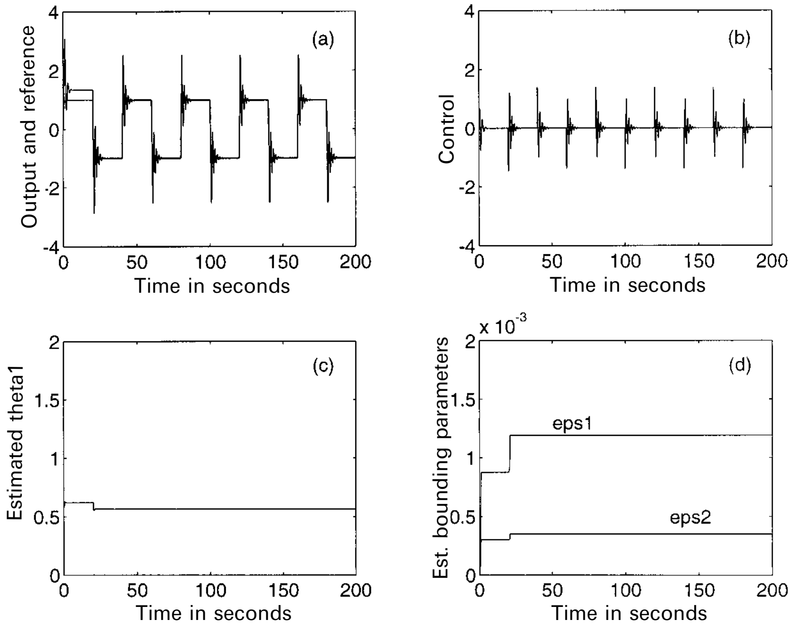

Figure 2.2 *Robust adaptive control with* $\lambda = 10^{-5}$

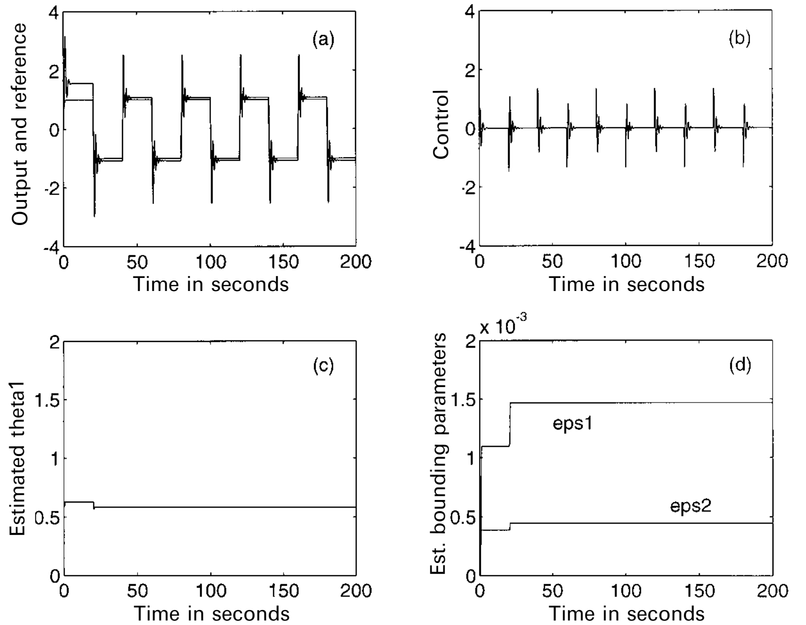

Figure 2.3 *Robust adaptive control with* $\lambda = 1.4 \times 10^{-5}$

Table 2.1 *Steady state values*

	$\lambda = 10^{-5}$	$\lambda = 1.4 \times 10^{-5}$
$\hat{\varepsilon}_1$	1.1898×10^{-3}	1.4682×10^{-3}
$\hat{\varepsilon}_2$	0.3509×10^{-3}	0.4467×10^{-3}
$\hat{\theta}^1$	0.5649	0.5833
$\lvert y - y^* \rvert$	0.0179	0.07587

It can be observed from the above simulation results that the algorithm developed in this chapter can guarantee the stability of the adaptive system in the presence of the modelling uncertainties, and the smaller tracking error could be achieved with smaller update rate parameter λ.

Most importantly, the knowledge of the parameters ε_1 and ε_2 of the upper bounding function and the knowledge of the leading coefficient of the parameter vector θ^1 are not required a priori.

2.7 Conclusions

In this chapter, a new robust discrete-time direct adaptive control algorithm is proposed with respect to a class of unmodelled dynamics and bounded disturbances. Dead zone is indeed used but no knowledge of the parameters of the upper bounding function on the unmodelled dynamics and disturbances is required a priori. Another feature of the algorithm is that a correction procedure is employed in the least squares estimation algorithm so that no knowledge of the lower bound on the leading coefficient of the plant numerator polynomial is required to achieve the singularity free adaptive control law. The global stability and convergence results of the algorithm are established.

References

[1] Rohrs, C., Valavani, L., Athans, M. and Stein, G. (1985). 'Robustness of Adaptive Control Algorithms in the Presence of Unmodelled Dynamics', *IEEE Trans. Automat. Contr.*, Vol. AC-30, 881–889.

[2] Egardt, B. (1979). *Stability of Adaptive Controllers*, Lecture Notes in Control and Information Sciences, New York, Springer Verlag.

[3] Ortega, R. and Tang, Y. (1989). 'Robustness of Adaptive Controllers – A Survey', *Automatica*, Vol. 25, 651–677.

[4] Ydstie, B. E. (1989). 'Stability of Discrete MRAC-revisited', *Systems and Control Letters*, Vol. 13, 429–438.

[5] Naik, S. M., Kumar, P. R., Ydstie, B. E. (1992). 'Robust Continuous-time Adaptive Control by Parameter Projection', *IEEE Trans. Automat. Contr.*, Vol. AC-37, No. 2, 182–197.

[6] Praly, L. (1983). 'Robustness of Model Reference Adaptive Control', *Proc. 3rd Yale Workshop on Application of Adaptive System Theory*, New Haven, Connecticut.

[7] Praly, L. (1987). 'Unmodelled Dynamics and Robustness of Adaptive Controllers', presented at the *Workshop on Linear Robust and Adaptive Control*, Oaxaca, Mexico.

[8] Petersen, B. B. and Narendra, K. S. (1982). 'Bounded Error Adaptive Control', *IEEE Trans. Automat. Contr.*, Vol. AC-27, 1161–1168.

[9] Samson, C. (1983). 'Stability Analysis of Adaptively Controlled System Subject to Bounded Disturbances', *Automatica*, Vol. 19, 81–86.

[10] Egardt, B. (1980). 'Global Stability of Adaptive Control Systems with Disturbances', *Proc. JACC*, San Francisco, CA.

[11] Middleton, R. H., Goodwin, G. C., Hill, D. J. and Mayne, D. Q. (1988). 'Design Issues in Adaptive Control', *IEEE Trans. Automat. Contr.*, Vol. AC-33, 50–58.

[12] Kreisselmeier, G. and Anderson, B. D. O. (1986). 'Robust Model Reference Adaptive Control', *IEEE Trans. Automat. Contr.*, Vol. AC-31, 127–133.

[13] Kreisselmeier, G. and Narendra, K. S. (1982). 'Stable Model Reference Adaptive Control in the Presence of Bounded Disturbances', *IEEE Trans. Automat. Contr.*, Vol. AC-27, 1169–1175.

[14] Iounnou, P. A. (1984). 'Robust Adaptive Control', *Proc. Amer. Contr. Conf.*, San Diego, CA.

[15] Ioannou, P. and Kokotovic, P. V. (1984). 'Robust Redesign of Adaptive Control', *IEEE Trans. Automat. Contr.*, Vol. AC-29, 202–211.

[16] Iounnou, P. A. (1986). 'Robust Adaptive Controller with Zero Residual Tracking Error', *IEEE Trans. Automat. Contr.*, Vol. AC-31, 773–776.

[17] Anderson, B. D. O. (1981). 'Exponential Convergence and Persistent Excitation', *Proc. 20th IEEE Conf. Decision Contr.*, San Diego, CA.

[18] Narendra, K. S. and Annaswamy, A. M. (1989). *Stable Adaptive Systems*, Prentice-Hall, NJ.

[19] Feng, G. and Palaniswami, M. (1994). 'Robust Direct Adaptive Controllers with a New Normalization Technique', *IEEE Trans. Automat. Contr.*, Vol. 39, 2330–2334.

[20] Goodwin, G. C. and Sin, K. S. (1981) 'Adaptive Control of Nonminimum Phase Systems', *IEEE Trans. Automat. Contr.*, Vol. AC-26, 478–483.

[21] Feng, G. and Palaniswami, M. (1992). 'A Stable Implementation of the Internal Model Principle', *IEEE Trans. Automat. Contr.*, Vol. AC-37, 1220–1225.

[22] Feng, G., Palaniswami, M. and Zhu, Y. (1992). 'Stability of Rate Constrained Robust Pole Placement Adaptive Control Systems', *Systems and Control Letters*, Vol. 18, 99–107.

[23] Lazono-Leal, R. and Goodwin, G. C. (1985). 'A Globally Convergent Adaptive Pole Placement Algorithm without a Persistency of Excitation Requirement', *IEEE Trans. Automat. Contr.*, Vol. AC-30, 795–799.

[24] Lazono-Leal, R., Dion, J. and Dugard, L. (1993). 'Singularity Free Adaptive Pole Placement Using Periodic Controllers', *IEEE Trans. Automat. Contr.*, Vol. AC-38, 104–108.

[25] Lazono-Leal, R. and Collado, J. (1989). 'Adaptive Control for Systems with Bounded Disturbances', *IEEE Trans. Automat. Contr.*, Vol. AC-34, 225–228.

[26] Goodwin, G. C. and Sin, K. S. (1984). *Adaptive Filtering, Prediction and Control*, Prentice-Hall, NJ.

[27] Lazono-Leal, R. (1989). 'Robust Adaptive Regulation without Persistent Excitation', *IEEE Trans. Automat. Contr.*, Vol. AC-34, 1260–1267.

3

Adaptive variable structure control

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3.1 Introduction

In the past two decades, model reference adaptive control (MRAC) using only input/output measurements has evolved as one of the most soundly developed adaptive control techniques. Not only has the stability property been rigorously established [17], [19] but also the robustness issue due to unmodelled dynamics and input/output disturbance has been successfully solved [15], [18]. However, several limitations on MRAC remain to be relaxed, especially the problem of unpredictable transient response and tracking performance which has recently become one of the challenging research topics in the field of MRAC. A considerable amount of effort has been made to improve these schemes to obtain better control effects [6], [9], [11], [22]. One effort out of several is to try to incorporate the variable structure design (VSD) [9], [11] concept into the traditional model reference adaptive controller structure. Notably, Hsu and Costa [11] have first successfully proposed a plausible scheme in this line, which was then followed by a series of more general results [12], [13], [14]. Aside from those, Fu [9], [10] has taken up a different approach in placing the variable structure design in the overall resulting adaptive controller. An offspring of the work [9] and part of the work [12] include various versions of results respectively applied to SISO [20], [23], MIMO [2], [5], time-varying [4], decentralized [24] and affine nonlinear [3] systems.

It is well known that a main difficulty for the design of the variable structure MRAC system is the so-called general case when relative degree of the plant is greater than one. In this chapter, we present a new algorithm to solve the variable structure model reference adaptive control for a single input single output system with unmodelled dynamics and output disturbances. The design concept will be first introduced for relative degree-one plants and then be extended to the general case. Compared with the previous works, which used adaptive variable structure design or traditional robust adaptive approaches for the MRAC problem, this algorithm has the following special features:

(1) This control algorithm successfully applies the variable structure adaptive controller for the general case under robustness consideration.
(2) The control strategy using the concept of 'average control' rather than that of 'equivalent control' is thoroughly analysed.
(3) A systematic design approach is proposed and a new adaptation mechanism is developed so that the prior upper bounds on some appropriately defined but unavailable system parameters are not needed. It is shown that without any persistent excitation the global stability and robustness with asymptotic tracking performance can be guaranteed. The output tracking error can be driven to zero for relative degree-one plants and to a small residual set (whose size depends on the level of magnitude of some design parameter) for plants with any higher relative degree. Both results are achieved even when the unmodelled dynamic and output disturbance are present.
(4) If the aforementioned bounds on the system parameters are available by some means before controller design, then with a suitable choice of initial control parameters, the output tracking error can even be driven to zero in finite time for relative degree-one plants and to a small residual set exponentially for plants with any higher relative degree. It is noted that these bounds are usually assumed to be known before the construction of the variable structure controller or the robust adaptation law.

In order to make a comparison between the proposed adaptive variable structure scheme and the traditional approaches, some computer simulations are made to illustrate the differences of the tracking performance. The simulations will clearly demonstrate the excellent transient responses as well as tracking performance, which are almost never possible to achieve when traditional MRAC schemes are employed [19].

The theoretical framework in this chapter is developed based on Filippov's solution concept for a differential equation with discontinuous right-hand side [8]. In the subsequent discussions, the following notations will be used:

(1) $P(s)[u](t)$: denotes the filtered version of $u(t)$ with any proper or strictly proper transfer function $P(s)$.
(2) $|\cdot|$: denotes the absolute value of any scalar or the Euclidean norm of any vector or matrix.
(3) $\|(\cdot)_t\|_\infty = \sup_{\tau \le t} |(\cdot)(\tau)|$: denotes the truncated L_∞ norm of the argument function or vector.
(4) $\|P(s)\|_\infty$: denotes the H_∞ norm of the transfer function $P(s)$.

The chapter is organized as follows. In Section 3.2, we give the plant description, control objective and then derive the MRAC based error model. In Section 3.3, the adaptive variable structure controller for relative degree-one plants is proposed with stability and performance analysis. The extension to plants with relative degree greater than one is presented in Section 3.4. Section 3.5 gives simulation results to demonstrate the effectiveness of the adaptive variable structure controller. Finally, a conclusion is made in Section 3.6.

3.2 Problem formulation

3.2.1 Plant description and control objective

In this chapter, we consider the following SISO linear time-invariant plant described by the equation:

$$y_p(t) = P(s)\big(1 + \mu P_u(s)\big)[u_p](t) + d_o(t) \tag{3.1}$$

where $u_p(t)$ and $y_p(t)$ are plant input and plant output respectively, $\mu P_u(s)$ is the multiplicative unmodelled dynamics with some $\mu \in R^+$, and d_o is the output disturbance. Here, $P(s)$ represents the strictly proper rational transfer function of the nominal plant which is described by

$$P(s) = k_p \frac{n_p(s)}{d_p(s)} \tag{3.2}$$

where $n_p(s)$ and $d_p(s)$ are some monic coprime polynomials and k_p is the high frequency gain. Now suppose that the plant (3.1) is not precisely known but some prior knowledge about the transfer function may be available. The control objective is to design an adaptive variable structure control scheme such that the output $y_p(t)$ of the plant will track the output $y_m(t)$ of a linear time-invariant reference model described by

$$y_m(t) = M(s)[r_m](t) = k_m \frac{n_m(s)}{d_m(s)}[r_m](t) \tag{3.3}$$

where $M(s)$ is a stable transfer function and $r_m(t)$ is a uniformly bounded reference input. In order to achieve such an objective, we need some assumptions on the modelled part of the plant and the reference model as well as the unmodelled part of the plant. These assumptions are made in the following.

For the modelled part of the plant and reference model:

(A1) All the coefficients of $n_p(s)$ and $d_p(s)$ are unknown a priori, but the order of $P(s)$ and its relative degree are known to be n and ρ, respectively. Without loss of generality, we will assume that the order of $M(s)$ and its relative degree are also n and ρ, respectively.

(A2) The value of high frequency gain k_p is unknown, but its sign should be known. Without loss of generality, we will assume k_p, and hence k_m, are positive.

(A3) $P(s)$ is minimum phase, i.e. all its zeros lie in the open left half complex plane.

For the unmodelled part of the plant:

(A4) The unmodelled dynamics $P_u(s-k_1)$ is a strictly proper and stable transfer matrix such that $|D| < a_1, \|(P_u(s-k_1)s - D)(s+a_2)\|_\infty < a_1$, for some constants $a_1, a_2 > 0$, where $D = \lim_{s\to\infty} P_u(s)s$ and $\|X(s)\|_\infty \equiv \sup_{w\in R}|X(jw)|$ [15].

(A5) The output disturbance is differentiable and the upper bounds on $|d_o(t)|, \left|\frac{d}{dt}d_o(t)\right|$ exist.

Remark 3.1

- Minimum-phase assumption (A3) on the nominal plant $P(s)$ is to guarantee the internal stability since the model reference control involves the cancellation of the plant zeros. However, as commented by [15], this assumption does not imply that the overall plant (3.1) possesses the minimum-phase property.
- The latter part of assumption (A4) is simply to emphasize the fact that $P_u(s)$ are uncorrelated with μ in any case [16]. The reasons for assumption (A5) will be clear in the proof of Theorem 3.1 and that of Theorem 4.1.

3.2.2 MRAC based error model

Since the plant parameters are assumed to be unknown, a basic strategy from the traditional MRAC [17], [19] is now used to construct the error model between $y_p(t)$ and $y_m(t)$. Instead of applying the traditional MRAC technique, a new adaptive variable structure control will be given here in order to pursue better robustness and tracking performance. Let (3.1) be rewritten as

$$y_p(t) - d_o(t) = P(s)\Big[u_p + \mu P_u(s)[u_p]\Big](t) \triangleq P(s)[u_p + \bar{u}](t) \tag{3.4}$$

then from the traditional model reference control strategy [19], it can be shown that there exists $\Theta^* = [\theta_1^*, \ldots, \theta_{2n}^*]^\top \in R^{2n}$ such that if

$$D_b^*(s) = [\theta_1^*, \theta_2^*, \ldots, \theta_{n-1}^*]\frac{a(s)}{\lambda(s)}$$

$$D_f^*(s) = [\theta_n^*, \theta_{n+1}^*, \ldots, \theta_{2n-2}^*]\frac{a(s)}{\lambda(s)} + \theta_{2n-1}^*$$

where $a(s) = [1, s, \ldots, s^{n-2}]^\top$ and $\lambda(s)$ is an nth order monic Hurwitz polynomial, we have

$$1 - D_b^*(s) - D_f^*(s)P(s) = \theta_{2n}^* M^{-1}(s)P(s) \tag{3.5}$$

Applying both sides of (3.5) on $u_p + \bar{u}$, we have

$$u_p(t) + \bar{u}(t) - D_b^*(s)[u_p + \bar{u}](t) - D_f^*(s)[y_p - d_o](t) = \theta_{2n}^* M^{-1}(s)[y_p - d_o](t) \tag{3.6}$$

so that

$$y_p(t) - d_o(t) = M(s)\theta_{2n}^{*-1}\Big[u_p + \bar{u} - D_b^*(s)[u_p + \bar{u}] - D_f^*(s)[y_p - d_o]\Big](t) \tag{3.7}$$

Since

$$\begin{aligned}
&D_b^*(s)[u_p + \bar{u}](t) + D_f^*(s)[y_p - d_o](t) + \theta_{2n}^* r_m(t) \\
&= \Theta^{*\top} \begin{bmatrix} \frac{a(s)}{\lambda(s)}[u_p + \bar{u}](t) \\ \frac{a(s)}{\lambda(s)}[y_p - d_o](t) \\ y_p(t) - d_o(t) \\ r_m(t) \end{bmatrix} \\
&= \Theta^{*\top} \begin{bmatrix} \frac{a(s)}{\lambda(s)}[u_p](t) \\ \frac{a(s)}{\lambda(s)}[y_p](t) \\ y_p(t) \\ r_m(t) \end{bmatrix} - \Theta^{*\top} \begin{bmatrix} 0 \\ \frac{a(s)}{\lambda(s)}[d_o](t) \\ d_o(t) \\ 0 \end{bmatrix} + D_b^*(s)[\bar{u}](t) \\
&\triangleq \Theta^{*\top} w(t) - \Theta^{*\top} w_{d_o}(t) + D_b^*(s)[\bar{u}](t)
\end{aligned} \tag{3.8}$$

we have

$$\begin{aligned}
y_p(t) - d_o(t) &= M(s)\theta_{2n}^{*-1}[u_p - \Theta^{*\top} w + \Theta^{*\top} w_{d_o} + (1 - D_b^*(s))[\bar{u}] + \theta_{2n}^* r_m](t) \\
&= M(s)\theta_{2n}^{*-1}[u_p - \Theta^{*\top} w + \Theta^{*\top} w_{d_o} + \mu\Delta(s)[u_p] + \theta_{2n}^* r_m](t)
\end{aligned} \tag{3.9}$$

where $\Delta(s) = (1 - D_b^*(s))P_u(s) = \left(1 - \frac{\theta_1^* + \ldots + \theta_{n-1}^* s^{n-2}}{\lambda(s)}\right)P_u(s)$. If we define the tracking error $e_0(t)$ as $y_p(t) - y_m(t)$, then the error model due to the unknown parameters, unmodelled dynamics and output disturbances can be readily found from (3.3) and (3.9) as follows:

$$e_0(t) \;=\; M(s)\theta_{2n}^{*-1}\left[u_p - \Theta^{*\top}w + \Theta^{*\top}w_{d_o} + \mu\Delta(s)[u_p]\right](t) + d_o(t) \qquad (3.10)$$

In the following sections, the new adaptive variable structure scheme is proposed for plants with arbitrary relative degree. However, the control structure is much simpler for relative degree-one plant, and hence in Section 3.3 we will first give a discussion for this class of plants. Based on the analysis for relative degree-one plants, the general case can then be presented in a more straightforward manner in Section 3.4.

3.3 The case of relative degree one

When $P(s)$ is relative degree one, the reference model $M(s)$ can be chosen to be strictly positive real (SPR) (Narendra and Annaswamy, 1988). The error model (3.10) can now be rewritten as

$$e_0(t) \;=\; M(s)\theta_{2n}^{*-1}\left[u_p - \Theta^{*\top}w + \Theta^{*\top}w_{d_o} + \theta_{2n}^{*}M^{-1}(s)[d_o] + \mu\Delta(s)[u_p]\right](t) \qquad (3.11)$$

In the error model (3.11), the terms $\Theta^{*\top}w, \Theta^{*\top}w_{d_o} + \theta_{2n}^{*}M^{-1}(s)[d_o]$ and $\mu\Delta(s)[u_p]$ are the uncertainties due to the unknown plant parameters, output disturbance, and unmodelled dynamics, respectively. Let (A_m, B_m, C_m) be any minimal realization of $M(s)\theta_{2n}^{*-1}$ which is SPR, then we can get the following state space representation of (3.11) as:

$$\dot{e}(t) = A_m e(t) + B_m(u_p(t) - \Theta^{*\top}w(t) + \Theta^{*\top}w_{d_o}(t) + \theta_{2n}^{*}M^{-1}(s)[d_o](t) + \mu\Delta(s)[u_p](t))$$
$$e_0(t) = C_m e(t) \qquad (3.12)$$

where the triplet (A_m, B_m, C_m) satisfies

$$P_m A_m + A_m^{\top}P_m \;=\; -2Q_m; \quad P_m B_m \;=\; C_m^{\top} \qquad (3.13)$$

for some $P_m = P_m^{\top} > 0$ and $Q_m = Q_m^{\top} > 0$.

The adaptive variable structure controller for relative degree-one plants is now summarized as follows:

(1) Define the regressor signal

$$w(t) = \left[\frac{a(s)}{\lambda(s)}[u_p](t), \frac{a(s)}{\lambda(s)}[y_p](t), y_p(t), r_m(t)\right]^{\top} = [w_1(t), w_2(t), \ldots, w_{2n}(t)]^{\top} \qquad (3.14)$$

and construct the normalization signal $m(t)$ [15] as the state of the following system:

$$\dot{m}(t) = -\delta_0 m(t) + \delta_1(|u_p(t)| + 1), \; m(0) > \frac{\delta_1}{\delta_0} \tag{3.15}$$

where $\delta_0, \delta_1 > 0$ and $\delta_0 + \delta_2 < \min(k_1, k_2)$ for some $\delta_2 > 0$. The parameter $k_2 > 0$ is selected such that the roots of $\lambda(s - k_2)$ lie in the open left half complex plane, which is always achievable.

(2) Design the control signal $u_p(t)$ as

$$u_p(t) = \sum_{j=1}^{2n}(-\text{sgn}\,(e_0 w_j)\theta_j(t)w_j(t)) - \text{sgn}\,(e_0)\beta_1(t) - \text{sgn}\,(e_0)\beta_2(t)m(t) \tag{3.16}$$

$$\text{sgn}\,(e_0) = \begin{cases} 1 & \text{if } e_0 > 0 \\ 0 & \text{if } e_0 = 0 \\ -1 & \text{if } e_0 < 0 \end{cases}$$

(3) The adaptation law for the control parameters is given as

$$\begin{aligned} \dot{\theta}_j(t) &= \gamma_j |e_0(t) w_j(t)|, \quad j = 1, \ldots, 2n \\ \dot{\beta}_1(t) &= g_1 |e_0(t)| \\ \dot{\beta}_2(t) &= g_2 |e_0(t)| m(t) \end{aligned} \tag{3.17}$$

where $\gamma_j, g_1, g_2 > 0$ are the adaptation gains and $\theta_j(0), \beta_1(0), \beta_2(0) > 0$ (in general, as large as possible) $j = 1, \ldots, 2n$.

The design concept of the adaptive variable structure controller (3.15) and (3.16) is simply to construct some feedback signals to compensate for the uncertainties because of the following reasons:

- By assumption (A5), it can be easily found that $|\Theta^{*\top} w_{d_o}(t) + \theta^*_{2n} M(s)^{-1}[d_o](t)| \leq \beta^*_1$ for some $\beta^*_1 > 0$.
- With the construction of m, it can be shown [15] that $\mu\Delta(s)[u_p](t) \leq \beta^*_2 m(t)$, $\forall t \geq 0$ and for some constant $\beta^*_2 > 0$.

Now, we are ready to state our results concerning the properties of global stability, robust property, and tracking performance of our new adaptive variable structure scheme with relative degree-one system.

Theorem 3.1 (Global Stability, Robustness and Asymptotic Zero Tracking Performance) Consider the system (3.1) satisfying assumptions (A1)–(A5) with relative degree being one. If the control input is designed as in (3.15), (3.16) and the adaptation law is chosen as in (3.17), then there exists $\mu^* > 0$ such that for $\mu \in [0, \mu^*]$ all signals inside the closed loop system are bounded and the tracking error will converge to zero asymptotically.

Proof: Consider the Lyapunov function

$$V_a = \frac{1}{2} e^\top P_m e + \sum_{j=1}^{2n} \frac{1}{2\gamma_j} (\theta_j - |\theta_j^*|)^2 + \sum_{j=1}^{2} \frac{1}{2g_j} (\beta_j - \beta_j^*)^2$$

where P_m satisfies (3.13). Then, the time derivative of V_a along the trajectory (3.12) (3.17) will be

$$\begin{aligned}
\dot{V}_a &= -e^\top Q_m e + e_0 \big(u_p - \Theta^{*\top} w + \Theta^{*\top} w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o] + \mu\Delta(s)[u_p]\big) \\
&\quad + \sum_{j=1}^{2n} \frac{1}{\gamma_j} (\theta_j - |\theta_j^*|)\dot{\theta}_j + \sum_{j=1}^{2} \frac{1}{g_j} (\beta_j - \beta_j^*)\dot{\beta}_j \\
&\le -e^\top Q_m e - \sum_{j=1}^{2n} |e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m \\
&\quad + \sum_{j=1}^{2n} \frac{1}{\gamma_j} (\theta_j - |\theta_j^*|)\dot{\theta}_j + \sum_{j=1}^{2} \frac{1}{g_j} (\beta_j - \beta_j^*)\dot{\beta}_j \\
&\le -q_m |e|^2
\end{aligned}$$

for some constant $q_m > 0$. This implies that $e \in L_2 \cap L_\infty$ and $\theta_j, j = 1, \ldots, 2n, \beta_1, \beta_2, e_0 \in L_\infty$ and, hence, all signals inside the closed loop system are bounded owing to Lemma A in the Appendix. On the other hand, it can be concluded that $\dot{e} \in L_\infty$ by (3.12). Hence, $e \in L_2 \cap L_\infty$ and $\dot{e} \in L_\infty$ readily imply that e and e_0 will at least converge to zero asymptotically by Barbalat's lemma [19]. Q.E.D.

In Theorem 3.1, suitable integral adaptation laws are given to compensate for the unavailable knowledge of the bounds on $|\theta_j^*|$ and β_j^*. Theoretically, the adaptive variable structure controller will stabilize the closed loop system with guaranteed robustness and asymptotic zero tracking performance no matter what $\theta_j(0)$'s and $\beta_j(0)$'s are. However, according to the following Theorem 3.2, we will expect that positive and large values of $\theta_j(0), \beta_j(0)$ should result in better transient response and tracking performance, especially when $\theta_j(0) > |\theta_j^*|, \beta_j(0) > \beta_j^*$.

Theorem 3.2 (Finite-Time Zero Tracking Performance with High Gain Design) Consider the system set-up in Theorem 3.1. If $\theta_j(0) \ge |\theta_j^*|, \beta_j(0) \ge \beta_j^*$, then the output tracking error will converge to zero in finite time with all signals inside the closed loop system remaining bounded.

Proof Consider the Lyapunov function $V_b = \frac{1}{2} e^\top P_m e$ where P_m satisfies

(3.13). The time derivative of V_b along the trajectory (3.12) becomes

$$\begin{aligned}\dot{V}_b &= -e^\top Q_m e - \sum_{j=1}^{2n} |e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m \\ &\leq -e^\top Q_m e \\ &\leq -k_3 V_b\end{aligned}$$

for some $k_3 > 0$ since $\theta_j(t) \geq |\theta_j^*|, \beta_j(t) \geq \beta_j^*, \forall t \geq 0$. This implies that e approaches zero at least exponentially fast. Furthermore, by the fact that

$$\begin{aligned}e_0\dot{e}_0 &= e_0\{C_m A_m e + C_m B_m(u_p - \Theta^{*\top} w + \Theta^{*\top} w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o] + \mu\Delta(s)[u_p])\} \\ &\leq k_4|e_0||e| - \sum_{j=1}^{2n} |e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m \\ &\leq k_4|e_0||e| - |e_0|\left(\sum_{j=1}^{2n} |w_j|(\theta_j - |\theta_j^*|) + (\beta_1 - \beta_1^*) + (\beta_2 - \beta_2^*)m\right)\end{aligned}$$

where $k_4 = |C_m A_m|$, and that $|e|$ approaches zero at least exponentially fast, there exists a finite time $T_1 > 0$ such that $e_0\dot{e}_0 \leq -k_5|e_0|$ for all $t > T_1$ and for some $k_5 > 0$. This implies that the sliding surface $e_0 = 0$ is guaranteed to be reached in some finite time $T_2 > T_1 > 0$. Q.E.D.

Remark 3.2: Although theoretically only asymptotic zero tracking performance is achieved when the initial control parameters are arbitrarily chosen, it is encouraged to set the adaptation gains γ_j and g_j in (3.17) as large as possible. This is because the large adaptation gains will provide high adaptation speed and, hence, increase the control parameters to a suitable level of magnitude so as to achieve a satisfactory performance as quickly as possible. These expected results can be observed in the simulation examples.

3.4 The case of arbitrary relative degree

When the relative degree of (3.1) is greater than one, the controller design becomes more complicated than that given in Section 3.3. The main difference between the controller design of a relative degree-one system and a system with relative degree greater than one can be described as follows. When (3.1) is relative degree-one, the reference model can be chosen to be strictly positive real (SPR) [19]. Moreover, the control structure and its subsequent analysis of global stability, robustness and tracking performance are much simpler. On the contrary, when the relative degree of (3.1) is greater than one, the reference model $M(s)$ is no longer SPR so that the controller and the analysis technique in relative degree-one systems cannot be directly applied. In order to use the similar techniques given in Section 3.3, the adaptive variable structure controller is now designed systematically as follows:

(1) Choose an operator $L_1(s) = \ell_1(s) \dots \ell_{\rho-1}(s) = (s + \alpha_1) \dots (s + \alpha_{\rho-1})$ such that $M(s)L_1(s)$ is SPR and denote $L_i(s) = \ell_i(s) \dots \ell_{\rho-1}(s), i = 2, \dots, \rho - 1, L_\rho(s) = 1$.

(2) Define augmented signal

$$y_a(t) = M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_p]\right](t)$$

and auxiliary errors

$$e_{a1}(t) = e_0(t) + y_a(t) \tag{3.18}$$

$$e_{a2}(t) = \frac{1}{\ell_1(s)}[u_2](t) - \frac{1}{F(\tau s)}[u_1](t) \tag{3.19}$$

$$e_{a3}(t) = \frac{1}{\ell_2(s)}[u_3](t) - \frac{1}{F(\tau s)}[u_2](t) \tag{3.20}$$

$$\vdots$$

$$e_{a\rho}(t) = \frac{1}{\ell_{\rho-1}(s)}[u_p](t) - \frac{1}{F(\tau s)}[u_{\rho-1}](t) \tag{3.21}$$

where $\frac{1}{F(\tau s)}[u_i](t)$ is the average control of $u_i(t)$ with $F(\tau s) = (\tau s + 1)^2$, τ being small enough. In fact, $F(\tau s)$ can be any Hurwitz polynomial in τs with degree at least two and $F(0) = 1$. In the literature, $\frac{1}{F(\tau s)}$ is referred to as an *averaging filter*, which is obviously a low-pass filter whose bandwidth can be arbitrarily enlarged as $\tau \rightarrow 0$. In other words, if τ is smaller and smaller, the filter $\frac{1}{F(\tau s)}$ is flatter and flatter.

(3) Design the control signals u_p, u_i, and the bounding function m as follows:

$$u_1(t) = \sum_{j=1}^{2n}(-\text{sgn }(e_{a1}\xi_j)\theta_j(t)\xi_j(t)) - \text{sgn }(e_{a1})\beta_1(t) - \text{sgn }(e_{a1})\beta_2(t)m(t) \tag{3.22}$$

$$u_i(t) = -\text{sgn }(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}](t)\right| + \eta\right), \ i = 2, \dots, \rho \tag{3.23}$$

$$u_p(t) = u_\rho(t) \tag{3.24}$$

with $\eta > 0$ and

$$\xi(t) = \frac{1}{\ell_1(s)} \cdots \frac{1}{\ell_{\rho-1}(s)}[w](t) \;=\; \frac{1}{L_1(s)}[w](t)$$

The bounding function $m(t)$ is designed as the state of the system

$$\dot{m}(t) = -\delta_0 m(t) + \delta_1(|u_p(t)| + 1),\ m(0) > \frac{\delta_1}{\delta_0} \tag{3.25}$$

with $\delta_0, \delta_1 > 0$ and $\delta_0 + \delta_2 < \min(k_1, k_2, \alpha_1, \dots, \alpha_{\rho-1})$ for some $\delta_2 > 0$.

(4) Finally, the adaptation law for the control parameters $\theta_j, j = 1, \dots, 2n$ and β_1, β_2 are given as follows:

$$\dot{\theta}_j(t) = \gamma_j|e_{a1}(t)\xi_j(t)|, \quad j = 1, \dots, 2n \tag{3.26}$$

$$\dot{\beta}_1(t) = g_1|e_{a1}(t)| \tag{3.27}$$

$$\dot{\beta}_2(t) = g_2|e_{a1}(t)|m(t) \tag{3.28}$$

with $\theta_j(0) > 0, \beta_j(0) > 0$ and $\gamma_j > 0, g_j > 0$.

In the following discussions, the construction of feedback signals $\xi(t), m(t)$ and the controller (3.22) (3.23) will be clear.

In order to analyse the proposed adaptive variable structure controller, we first rewrite the error model (3.10) as follows:

$$\begin{aligned} e_0(t) &= M(s)[u_p - \theta_{2n}^{*-1}\Theta^{*\top}w + \theta_{2n}^{*-1}\Theta^{*\top}w_{d_o} + \theta_{2n}^{*-1}\mu\Delta(s)[u_p] \\ &\quad + (\theta_{2n}^{*-1} - 1)u_p](t) + d_o(t) \\ &= M(s)L_1(s)\left[\frac{1}{L_1(s)}[u_p] - -\theta_{2n}^{*-1}\Theta^{*\top}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\top}w_{do} + \theta_{2n}^{*}M^{-1}(s)[d_o]]\right. \\ &\quad \left. + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1 - \theta_{2n}^{*})u_p]\right](t) \end{aligned} \tag{3.29}$$

Now, according to the design of the above auxiliary error (3.18) and error model (3.29), we can readily find that e_{a1} always satisfies

$$\begin{aligned} e_{a1}(t) = M(s)L_1(s)&\left[u_1 - \theta_{2n}^{*-1}\Theta^{*\top}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\top}w_{do} + \theta_{2n}^{*}M^{-1}(s)[d_o]]\right. \\ &\left. + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1 - \theta_{2n}^{*})u_p]\right](t) \end{aligned} \tag{3.30}$$

It is noted that the auxiliary error e_{a1} is now explicitly expressed as the output term of a linear system with SPR transfer function $M(s)L_1(s)$ driven by some uncertain signals due to unknown parameters, output disturbances, unmodelled dynamics and unknown high frequency gain sign.

Remark 4.1 The construction of the adaptive variable structure controller (3.22) is now clear since the following facts hold:

- Since $\dfrac{\theta_{2n}^{*-1}}{L_1(s)}\Big[\Theta^{*\top}w_{do}+\theta_{2n}^{*}M^{-1}(s)[d_o]\Big](t)$ is uniformly bounded due to (A5), we have

$$\left|\frac{\theta_{2n}^{*-1}}{L_1(s)}\Big[\Theta^{*\top}w_{do}+\theta_{2n}^{*}M^{-1}(s)[d_o]\Big](t)\right|\le\beta_1^{*} \tag{3.31}$$

  for some β_1^*.
- With the design of the bounding function $m(t)$ (3.25), it can be shown that

$$\left|\frac{\theta_{2n}^{*-1}}{L_1(s)}\Big[\mu\Delta(s)[u_p]+(1-\theta_{2n}^{*})u_p\Big](t)\right|\le\beta_2^{*}m(t) \tag{3.32}$$

  for some $\beta_2^*>0$.

The results described in Remark 4.1 show that the similar techniques for the controller design of a relative degree-one system can now be used for auxiliary error e_{a1}. But what happens to the other auxiliary errors $e_{a2},\ldots,e_{a\rho}$, especially the real output error e_0 as concerned? In Theorem 4.1, we summarize the main results of the systematically designed adaptive variable structure controller for plants with relative degree greater than one.

Theorem 4.1 (Global Stability, Robustness and Asymptotic Tracking Performance) Consider the nonlinear time-varying system (3.1) with relative degree $\rho>1$ satisfying (A1)–(A5). If the controller is designed as in (3.18)–(3.25) and parameter update laws are chosen as in (3.26)–(3.28), then there exists $\tau^*>0$ and $\mu^*>0$ such that for all $\tau\in(0,\tau^*)$ and $\mu\in(0,\mu^*)$, the following facts hold:

(i) all signals inside the closed-loop system remain uniformly bounded;
(ii) the auxiliary error e_{a1} converges to zero asymptotically;
(iii) the auxiliary errors $e_{ai}, i=2,\ldots,\rho$, converge to zero at some finite time;
(iv) the output tracking error e_0 will converge to a residual set asymptotically whose size is a class K function of the design parameter τ.

Proof The proof consists of three parts.

Part I Prove the boundedness of e_{ai} and $\theta_1,\ldots,\theta_{2n},\beta_1,\beta_2$.

<u>Step 1</u> First, consider the auxiliary error e_{a1} which satisfies (3.30). Since

$M(s)L_1(s)$ is SPR, we have the following realization of (3.20)

$$\dot{e}_1 = A_1 e_1 + B_1 \bigg(u_1 - \theta_{2n}^{*-1} \Theta^{*\top} \xi + \frac{\theta_{2n}^{*-1}}{L_1(s)} [\Theta^{*\top} w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o]] + \frac{\theta_{2n}^{*-1}}{L_1(s)} [\mu \Delta(s)[u_p] + (1 - \theta_{2n}^*) u_p] \bigg)$$

$$e_{a1} = C_1 e_1 \tag{3.33}$$

with $P_1 A_1 + A_1^\top P_1 = -2Q_1$, $P_1 B_1 = C_1^\top$ for some $P_1 = P_1^\top > 0$ and $Q_1 = Q_1^\top > 0$. Given a Lyapunov function as follows:

$$V_1 = \tfrac{1}{2} e_1^\top P_1 e_1 + \sum_{j=1}^{2n} \frac{1}{2\gamma_j} \left(\theta_j - \left| \frac{\theta_j^*}{\theta_{2n}^*} \right| \right)^2 + \sum_{j=1}^{2} \frac{1}{2g_j} (\beta_j - \beta_j^*)^2 \tag{3.34}$$

it can be shown by using (3.32) and (3.31) that

$$\begin{aligned}
\dot{V}_1 &= -e_1^\top Q_1 e_1 + e_{a1} \bigg(u_1 - \theta_{2n}^{*-1} \Theta^{*\top} \xi + \frac{\theta_{2n}^{*-1}}{L_1(s)} [\Theta^{*\top} w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o]] \\
&\quad + \frac{\theta_{2n}^{*-1}}{L_1(s)} [\mu \Delta(s)[u_p] + (1 - \theta_{2n}^*) u_p] \bigg) \\
&\quad + \sum_{j=1}^{2n} \frac{1}{\gamma_j} \left(\theta_j - \left| \frac{\theta_j^*}{\theta_{2n}^*} \right| \right) \dot{\theta}_j + \sum_{j=1}^{2} \frac{1}{g_j} (\beta_j - \beta_j^*) \dot{\beta}_j \\
&\le -e_1^\top Q_1 e_1 - \sum_{j=1}^{2n} |e_{a1} \xi_j| \left(\theta_j - \left| \frac{\theta_j^*}{\theta_{2n}^*} \right| \right) - |e_{a1}| (\beta_1 - \beta_1^*) - |e_{a1}| (\beta_2 - \beta_2^*) m \\
&\quad + \sum_{j=1}^{2n} \frac{1}{\gamma_j} \left(\theta_j - \left| \frac{\theta_j^*}{\theta_{2n}^*} \right| \right) \dot{\theta}_j + \sum_{j=1}^{2} \frac{1}{g_j} (\beta_j - \beta_j^*) \dot{\beta}_j \\
&= -e_1^\top Q_1 e_1 \\
&\le -q_1 |e_1|^2
\end{aligned}$$

for some $q_1 > 0$ if the controller in (3.22) and update laws in (3.26)–(3.28) are given. This implies that $e_1, \theta_1, \ldots, \theta_{2n}, \beta_1, \beta_2 \in L_\infty$ and $e_{a1} \in L_2 \cap L_\infty$.

Step 2 From (3.19)–(3.21), we can find that $e_{a2}, \ldots, e_{a\rho}$ satisfy

$$\begin{aligned}
\dot{e}_{a2} &= -\alpha_1 e_{a2} + u_2 - \frac{\ell_1(s)}{F(\tau s)} [u_1] \\
&\vdots \\
\dot{e}_{a\rho} &= -\alpha_{\rho-1} e_{a\rho} + u_\rho - \frac{\ell_{\rho-1}(s)}{F(\tau s)} [u_{\rho-1}]
\end{aligned}$$

Now by the following facts that for $i = 2, \ldots, \rho$:

$$\begin{aligned}
\frac{d}{dt}\left(\tfrac{1}{2}e_{ai}^2\right) &= e_{ai}\dot{e}_{ai} \\
&= e_{ai}\left(-\alpha_{i-1}e_{ai} + u_i - \frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right) \\
&= -\alpha_{i-1}e_{ai}^2 + e_{ai}\left\{-sgn(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right| + \eta\right) - \frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right\}
\end{aligned}$$

or

$$\frac{d}{dt}|e_{ai}| \leq -\alpha_{i-1}|e_{ai}| - \eta \tag{3.35}$$

when $|e_{ai}| \neq 0$. This implies that e_{ai} will converge to zero after some finite time $T > 0$.

Part II Prove the boundedness of all signals inside the closed loop system.

Define $\bar{e}_{ai} = M(s)L_{i-1}(s)[e_{ai}], i = 2, \ldots, \rho$ and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{a\rho}$ which is uniformly bounded due to the boundedness of e_{ai}. Then, from (3.18)–(3.21), we can derive that

$$\begin{aligned}
E_a = e_0 &+ M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_p]\right] \\
&+ M(s)L_1(s)\left[\frac{1}{\ell_1(s)}[u_2] - \frac{1}{F(\tau s)}[u_1]\right] \\
&+ M(s)L_2(s)\left[\frac{1}{\ell_2(s)}[u_3] - \frac{1}{F(\tau s)}[u_2]\right] \\
&\vdots \\
&+ M(s)L_{\rho-1}(s)\left[\frac{1}{\ell_{\rho-1}(s)}[u_p] - \frac{1}{F(\tau s)}[u_{\rho-1}]\right] \\
= e_0 &+ \left(1 - \frac{1}{F(\tau s)}\right)M(s)L_1(s)\left[u_1 + \frac{1}{\ell_1(s)}[u_2] + \cdots + \frac{1}{\ell_1(s)\ldots\ell_{\rho-2}(s)}[u_{\rho-1}]\right] \\
\triangleq e_0 &+ R
\end{aligned} \tag{3.36}$$

Now, since $\|(u_i)_t\|_\infty \leq k_6\|(e_0)_t\|_\infty + k_6, i = 1, \ldots, \rho - 1$ for some $k_6 > 0$ by Lemma A in the appendix, it can be easily found that

$$\left\|\left(u_1 + \frac{1}{\ell_1(s)}[u_2] + \cdots + \frac{1}{\ell_1(s)\ldots\ell_{\rho-2}(s)}[u_{\rho-1}]\right)_t\right\|_\infty \leq k_7\|(e_0)_t\|_\infty + k_7$$

for some $k_7 > 0$. Furthermore, since the H_∞ norm of $\|\frac{1}{s}(1 - \frac{1}{F(\tau s)})\|_\infty = 2\tau$ and $\|sM(s)L_1(s)\|_\infty = k_8$ for some $k_8 > 0$, we can conclude that

$$\|(R)_t\|_\infty \le \left\|\frac{1}{s}\left(1 - \frac{1}{F(\tau s)}\right)\right\|_\infty \left\|sM(s)L_1(s)\right\|_\infty (k_7\|(e_0)_t\|_\infty + k_7)$$
$$\le \tau(k_9\|(e_0)_t\|_\infty + k_9)$$

for some $k_9 > 0$. Now from (3.36) we have

$$\|(e_0)_t\|_\infty \ \le \ \|(E_a)_t\|_\infty + \|(R)_t\|_\infty \ \le \ \|(E_a)_t\|_\infty + \tau(k_9\|(e_0)_t\|_\infty + k_9)$$

which implies that there exists a $\tau^* > 0$ such that $1 - \tau^* k_9 > 0$ and for all $\tau \in (0, \tau^*)$:

$$\|(e_0)_t\|_\infty \ \le \ \frac{\|(E_a)_t\|_\infty + \tau k_9}{1 - \tau k_9} \tag{3.37}$$

Combining Lemma A and (3.37), we readily conclude that all signals inside the closed loop system remain uniformly bounded.

Part III: Investigate the tracking performance of e_{a1} and e_0.

Since all signals inside the closed loop system are uniformly bounded, we have

$$e_{a1} \in L_2 \cap L_\infty, \quad \dot{e}_{a1} \in L_\infty$$

Hence, by Barbalat's lemma, e_{a1} approaches zero asymptotically and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{a\rho}$ also approaches zero asymptotically. Now, from the fact of (3.37) and E_a approaching zero, it is clear that e_0 will converge to a small residual set whose size depends on the design parameter τ. Q.E.D.

As discussed in Theorem 3.2, if the initial choices of control parameters $\theta_j(0), \beta_j(0)$ satisfy the high gain conditions $\theta_j(0) \ge \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|$ and $\beta_j(0) \ge \beta_j^*$, then, by using the same argument given in the proof of Theorem 3.2, we can guarantee the exponential convergent behaviour and finite-time tracking performance of all the auxiliary errors e_{ai}. Since e_{ai} reaches zero in some finite time and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{a\rho}$, it can be concluded that E_a converges to zero exponentially and e_0 converges to a small residual set whose size depends on the design parameter τ. We now summarize the results in the following Theorem 4.2.

Theorem 4.2: (Exponential Tracking Performance with High Gain Design) Consider the system set-up in Theorem 4.1. If the initial value of control parameters satisfy the high gain conditions $\theta_j(0) \ge \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|$ and $\beta_j(0) \ge \beta_j^*$, then there exists a τ^* and μ^* such that for all $\tau \in (0, \tau^*]$ and $\mu \in (0, \mu^*]$, the following facts hold:

(i) all signals inside the closed loop system remain bounded;

(ii) the auxiliary errors $e_{ai}, i = 1, \ldots, \rho$, converge to zero in finite time;
(iii) the output tracking errors e_0 will converge to a residual set exponentially whose size depends on the design parameter τ.

Remark 4.2: It is well known that the chattering behaviour will be observed in the input channel due to variable structure control, which causes the implementation problem in practical design. A remedy to the undesirable phenomenon is to introduce the boundary layer concept. Take the case of relative degree one, for example, the practical redesign of the proposed adaptive variable structure controller by using boundary layer design is now stated as follows:

$$u_p(t) = \sum_{j=1}^{2n}\left(-\pi(e_0 w_j)\theta_j(t) w_j(t)\right) - \pi(e_0)\beta_1(t) - \pi(e_0)\beta_2(t)m(t) \qquad (3.38)$$

$$\pi(e_0) = \begin{cases} \text{sgn }(e_0) & \text{if } |e_0| > \varepsilon \\ \dfrac{e_0}{\varepsilon} & \text{if } |e_0| \leq \varepsilon \end{cases}$$

for some small $\epsilon > 0$. Note that $\pi(e_0)$ is now a continuous function. However, one can expect that the boundary layer design will result in bounded tracking error, i.e. e_0 cannot be guaranteed to converge to zero. This causes the parameter drift in parameter adaptation law. Hence, a leakage term is added into the adaptation law as follows:

$$\begin{aligned} \dot{\theta}_j(t) &= \gamma_j|e_0(t)w_j(t)| - \sigma\theta_j(t), \; j = 1, \ldots, 2n \\ \dot{\beta}_1(t) &= g_1|e_0(t)| - \sigma\beta_1(t) \\ \dot{\beta}_2(t) &= g_2|e_0(t)|m(t) - \sigma\beta_2(t) \end{aligned} \qquad (3.39)$$

for some $\sigma > 0$.

3.5 Computer simulations

The adaptive variable structure scheme is now applied to the following unstable plant with unmodelled dynamics and output disturbances:

$$y_p(t) = \frac{8}{s^3 + s^2 + s - 2}\left(1 + 0.01\frac{1}{s+10}\right)[u_p](t) + 0.05\sin(5t)$$

Since the nominal plant is relative degree three, we choose the following steps to design the adaptive variable structure controller:

- reference model and reference input:

$$M(s) = \frac{8}{(s+2)^3}$$

$$r_m(t) = \begin{cases} 2 & \text{if } t < 5 \\ -2 & \text{if } 5 \le t < 10 \end{cases}$$

- design parameters:

$$L_1(s) = \ell_1(s)\ell_2(s), \ell_1(s) = s+1, \ell_2(s) = s+2$$

$$\lambda(s) = (s+1)^2$$

$$F(\tau s) = \left(\frac{1}{60}s + 1\right)^2$$

- augmented signal and auxiliary errors:

$$y_a(t) = M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_p]\right](t)$$

$$e_{a1}(t) = e_0(t) + y_a(t)$$

$$e_{a2}(t) = \frac{1}{\ell_1(s)}[u_2](t) - \frac{1}{F(\tau s)}[u_1](t)$$

$$e_{a3}(t) = \frac{1}{\ell_2(s)}[u_3](t) - \frac{1}{F(\tau s)}[u_2](t)$$

- controller:

$$u_1(t) = \sum_{j=1}^{6}\left(-\text{sgn}\,(e_{a1}\xi_j)\theta_j(t)\xi_j(t)\right) - \text{sgn}\,(e_{a1})\beta_1(t) - \text{sgn}\,(e_{a1})\beta_2(t)m(t)$$

$$u_i(t) = -\text{sgn}\,(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}](t)\right| + 1\right), \quad i = 2, 3$$

$$u_p(t) = u_\rho(t)$$

$$\dot{m}(t) = -m(t) + 0.005(|u_p(t)| + 1), \quad m(0) = 0.2$$

- adaptation law:

$$\dot{\theta}_j(t) = \gamma_j|e_{a1}(t)\xi_j(t)|, \quad j = 1, \ldots, 6$$

$$\dot{\beta}_1(t) = g_1|e_{a1}(t)|$$

$$\dot{\beta}_2(t) = g_2|e_{a1}(t)|m(t)$$

Three simulation cases are studied extensively in this example in order to verify

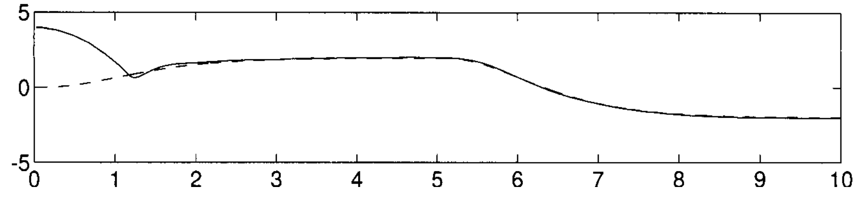

Figure 3.1 $y_p(-), y_m(--)$, *time (sec)*

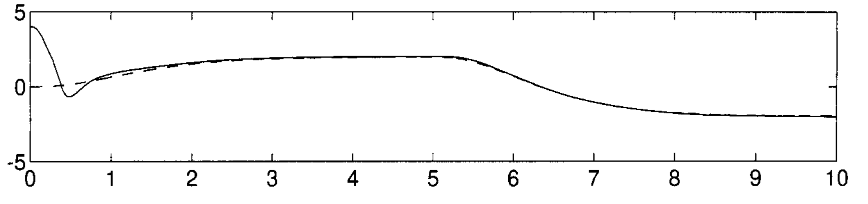

Figure 3.2 $y_p\ (-), ym\ (--)$, *time (sec)*

all the theoretical results and corresponding claims. All the cases will assume that there are initial output error $y_p(0) - y_m(0) = 4$.

(1) In the first case, we arbitrarily choose the initial control parameters as

$$\theta_j(0) = 0.1, \quad j = 1, \ldots, 6$$
$$\beta_j(0) = 0.1, \quad j = 1, 2$$

and set all the adaptation gains $\gamma_j = g_j = 0.1$. As shown in Figure 3.1 (the time trajectories of y_p and y_m), the global stability, robustness, and asymptotic tracking performance are achieved.

(2) In the second case, we want to demonstrate the effectiveness of a proper choice of $\theta_j(0)$ and $\beta_j(0)$ and repeat the previous simulation case by increasing the values of the controller parameters to be

$$\theta_j(0) = 1, \quad j = 1, \ldots, 6$$
$$\beta_j(0) = 1, \quad j = 1, 2$$

The better transient and tracking performance between y_p and y_m can now be observed in Figure 3.2.

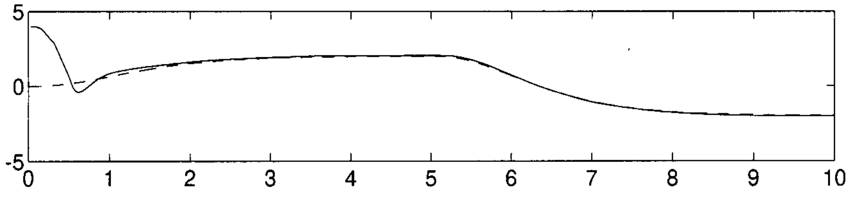

Figure 3.3 y_p $(-), y_m; (--)$, *time (sec)*

(3) As commented in Remark 3.2, if there is no easy way to estimate the suitable initial control parameters $\theta_j(0)$ and $\beta_j(0)$ like those in the second simulation case, it is suggested to use large adaptation gains in order to increase the adaptation rate of control parameters such that the nice transient and tracking performance as described in case 2 can be retained to some extent. Hence, in this case, we use the initial control parameters as in case 1 but set all the adaptation gains to $\gamma_j = g_j = 1$. The expected results are now shown in Figure 3.3, where rapid increase of control parameters do lead to satisfactory transient and tracking performance.

3.6 Conclusion

In this chapter, a new adaptive variable structure scheme is proposed for model reference adaptive control problems for plants with unmodelled dynamic and output disturbance. The main contribution of the chapter is the complete version of adaptive variable structure design for solving the robustness and performance of the traditional MRAC problem with arbitrary relative degree. A detailed analysis of the closed-loop stability and tracking performance is given. It is shown that without any persistent excitation the output tracking error can be driven to zero for relative degree-one plants and driven to a small residual set asymptotically for plants with any higher relative degree. Furthermore, under suitable choice of initial conditions on control parameters, the tracking performance can be improved, which are hardly achievable by the traditional MRAC schemes, especially for plants with uncertainties.

Appendix

Lemma A Consider the controller design in Theorem 3.1 or 4.1. If the control parameters $\theta_j(t), j = 1, \ldots, 2n, \beta_1(t)$ and $\beta_2(t)$ are uniformly bounded $\forall t$, then there exists $\mu^* > 0$ such that $u_p(t)$ satisfies

$$\|(u_p)_t\|_\infty \le \kappa\|(e_0)_t\|_\infty + \kappa \tag{A.1}$$

with some positive constant $\kappa > 0$.

Proof Consider the plant (3.1) which is rewritten as follows:

$$y(t) - d_o(t) = P(s)(1 + \mu P_u(s))[u_p](t) \tag{A.2}$$

Let $f(s)$ be the Hurwitz polynomial with degree $n - \rho$ such that $f(s)P(s)$ is proper, and hence, $f^{-1}(s)P^{-1}(s)$ is proper stable since $P(s)$ is minimum phase by assumption (A3). Then

$$y(t) - d_o(t) = P(s)f(s)f^{-1}(s)(1 + \mu P_u(s))[u_p](t) \tag{A.3}$$

which implies that

$$f^{-1}(s)P^{-1}(s)[y - d_o](t) - \mu f^{-1}(s)P_u(s)[u_p](t) = f^{-1}(s)[u_p](t) \stackrel{\triangle}{=} u^*(t) \tag{A.4}$$

Since $f^{-1}(s)P^{-1}(s)$ and $f^{-1}(s)P_u(s)$ are proper or strictly proper stable, we can find by small gain theorem [7] that there exists $\mu^* > 0$ such that

$$\|(u^*)_t\|_\infty \le \kappa\|(y_p)_t\|_\infty + \kappa \le \kappa\|(e_0)_t\|_\infty + \kappa \tag{A.5}$$

for some suitably defined $\kappa > 0$ and for all $\mu \in [0, \mu^*]$. Now if we can show that

$$\|(u_p)_t\|_\infty \le \kappa\|(u^*)_t\|_\infty + \kappa \tag{A.6}$$

for some $\kappa > 0$, then (A.1) is achieved. By using Lemma 2.8 in [19], the key point to show the boundedness between u_p and u^* in (A.6) is the growing behaviour of signal u_p. The above statement can be stated more precisely as follows: if u_p satisfies the following requirement

$$|u_p(t_1)| \ge c|u_p(t_1 + T)| \tag{A.7}$$

where t_1 and $t_1 + T$ are the time instants defined as

$$[t_1, t_1 + T] \subset \Omega = \{t \mid |u_p| = \|(u_p)_t\|_\infty\} \tag{A.8}$$

and c is a constant $\in (0, 1)$, then u_p will be bounded by u^*, i.e. (A.6) is achieved. Now in order to establish (A.7) and (A.8), let (A_p, B_p, C_p) and (Λ, B) be the state space realizations of $P(s)(1 + \mu P_u(s))$ and $\dfrac{a(s)}{\lambda(s)}$ respectively. Also define $S = [x_p^\top, w_1^\top, w_2^\top, m]^\top$. Then, using the augmented system

$$\begin{bmatrix} \dot{x}_p \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{m} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ BC_p & 0 & \Lambda & 0 \\ 0 & 0 & 0 & -\delta_0 \end{bmatrix} \begin{bmatrix} x_p \\ w_1 \\ w_2 \\ m \end{bmatrix} + \begin{bmatrix} B_p u_p \\ Bu_p \\ Bd_o \\ \delta_1 |u_p| + 1 \end{bmatrix}$$

Since d_o is uniformly bounded, we can easily show according to the control design (3.16) or (3.24) that there exists κ such that

$$|\dot{S}| \leq \kappa \|(S)_t\|_\infty + \kappa$$

This means that S is regular [21] so that x_p, w_1, w_2, m, y_p and u_p will grow at most exponentially fast (if unbounded), which in turn guarantees (A.7) and (A.8) by Lemma 2.8 in [19]. This completes our proof. Q.E.D.

References

[1] Chien, C. J. and Fu, L. C., (1992) 'A New Approach to Model Reference Control for a Class of Arbitrarily Fast Time-varying Unknown Plants', *Automatica*, Vol. 28, No. 2, 437–440.

[2] Chien, C. J. and Fu, L. C., (1992) 'A New Robust Model Reference Control with Improved Performance for a Class of Multivariable Unknown plants', *Int. J. of Adaptive Control and Signal Processing*, Vol. 6, 69–93.

[3] Chien, C. J. and Fu, L. C., (1993) 'An Adaptive Variable Structure Control for a Class of Nonlinear Systems', *Syst. Contr. Lett.*, Vol. 21, No. 1, 49–57.

[4] Chien, C. J. and Fu, L. C., (1994) 'An Adaptive Variable Structure Control of Fast Time-varying Plants', *Control Theory and Advanced Technology*, Vol. 10, No. 4, part I, 593–620.

[5] Chien, C. J., Sun, K. S., Wu, A. C. and Fu, L. C., (1996) 'A Robust MRAC Using Variable Structure Design for Multivariable Plants', *Automatica*, Vol. 32, No. 6, 833–848.

[6] Datta, A. and Ioannou, P. A., (1991) 'Performance Improvement versus Robust Stability in Model Reference Adaptive Control', *Proc. CDC*, 748–753.

[7] Desoer, C. A. and Vidyasagar, M., (1975) *Feedback Systems: Input–Output Properties*, Academic Press, NY.

[8] Filippov, A. F., (1964) 'Differential Equations with Discontinuous Right-hand Side', *Amer. Math. Soc. Transl.,* Vol. 42, 199–231.

[9] Fu, L. C., (1991) 'A Robust Model Reference Adaptive Control Using Variable Structure Adaptation for a Class of Plants', *Int. J. Control*, Vol. 53, 1359–1375.

[10] Fu, L. C., (1992) 'A New Robust Model Reference Adaptive Control Using Variable Structure Design for Plants with Relative Degree Two', *Automatica*, Vol. 28, No. 5, 911–926.

[11] Hsu, L. and Costa, R. R., (1989) 'Variable Structure Model Reference Adaptive Control Using Only Input and Output Measurement: Part 1', *Int. J. Control*, Vol. 49, 339–419.

[12] Hsu, L., (1990) 'Variable Structure Model-Reference Adaptive Control (VS-MRAC) Using Only Input Output Measurements: the General Case', *IEEE Trans. Automatic Control*, Vol. 35, 1238–1243.

[13] Hsu, L. and Lizarralde, F., (1992) 'Redesign and Stability Analysis of I/O VS-MRAC Systems', *Proc. American Control Conference*, 2725–2729.

[14] Hsu, L., de Araujo A. D. and Costa, R. R., (1994) 'Analysis and Design of I/O Based Variable Structure Adaptive Control', *IEEE Trans. Automatic Control*, Vol. 39, No. 1.

[15] Ioannou, P. A. and Tsakalis, K. S., (1986) 'A Robust Direct Adaptive Control', *IEEE Trans. Automatic Control*, Vol. 31, 1033–1043.

[16] Ioannou, P. A. and Tsakalis, K. S., (1988) 'The Class of Unmodeled Dynamics in Robust Adaptive Control', *Proc. of American Control Conference*, 337–342.

[17] Narendra, K. S. and Valavani, L., (1978) 'Stable Adaptive Controller Design – Direct Control', *IEEE Trans. Automatic Control*, Vol. 23, 570–583.

[18] Narendra, K. S. and Annaswamy, A. M., (1987) 'A New Adaptive Law for Robust Adaptation Without Persistent Excitation', *IEEE Trans. Automatic Control*, Vol. 32, 134–145.

[19] Narendra, K. S. and Valavani, L., (1989) *Stable Adaptive Systems*, Prentice-Hall.

[20] Narendra, K. S. and Bŏsković, J. D., (1992) 'A Combined Direct, Indirect and Variable Structure Method for Robust Adaptive Control', *IEEE Trans. Automatic Control*, Vol. 37, 262–268.

[21] Sastry, S. S. and Bodson, M., (1989) *Adaptive Control: Stability, Convergence, and Robustness*, Prentice-Hall, Englewood Cliffs, NJ.

[22] Sun, J., (1991) 'A Modified Model Reference Adaptive Control Scheme for Improved Transient Performance', *Proc. American Control Conference*, 150–155.

[23] Wu, A. C., Fu, L. C. and Hsu, C. F., (1992) 'Robust MRAC for Plants with Arbitrary Relative Degree Using a Variable Structure Design', *Proc. American Control Conference*, 2735–2739.

[24] Wu, A. C. and Fu, L. C., (1994) 'New Decentralized MRAC Algorithms for Large-Scale Uncertain Dynamic Systems', *Proc. American Control Conference*, 3389–3393.

4

Indirect adaptive periodic control

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Abstract

In this chapter a new indirect adaptive control method is presented. This method is based on a lifted representation of the plant which can be stabilized using a simple performant periodic control scheme. The controller parameter's computation involves the inverse of the controllability/observability matrix. Potential singularities of this matrix are avoided by means of an appropriate estimates modification. This estimates transformation is linked to the covariance matrix properties and hence it preserves the convergence properties of the original estimates. This modification involves the singular value decomposition of the controllability/observability matrix's estimate. As compared to previous studies in the subject the controller proposed here does not require the frequent introduction of periodic n-length sequences of zero inputs. Therefore the new controller is such that the system is almost always operating in closed loop which should lead to better performance characteristics.

4.1 Introduction

The problem of adaptive control of possibly nonminimum phase systems has received several solutions over the past decade. These solutions can be divided into several different categories depending on the a priori knowledge on the plant, and on whether persistent excitation can be added into the system or not.

Schemes based on persistent excitation were proposed in [1], [11] among others. This approach has been thoroughly studied and is based on the fact that convergence of the estimates to the true plant parameter values is guaranteed when the plant input and output are rich enough. Stability of the closed loop system follows from the unbiased convergence of the estimates. The external persistent excitation signal is then required to be always present, therefore the plant output cannot exactly converge to its desired value because of the external dither signal. This difficulty has been removed in [2] using a self-excitation technique. In this approach excitation is introduced periodically during some pre-specified intervals as long as the plant state and/or output have not reached their desired values. Once the control objectives are accomplished the excitation is automatically removed. Stability of these type of schemes is guaranteed in spite of the fact that convergence of the parameter estimates to their true values is not assured. This technique has also been extended to the case of systems for which only an upper bound on the plant order is known in [12] for the discrete-time case and in [13] for the continuous-time case.

Since adding extra perturbations into the system is not always feasible or desirable, other adaptive techniques not resorting to persistent exciting signals have been developed. Different strategies have been proposed depending on the available information on the system.

When the parameters are known to belong to given intervals or convex sets inside the controllable regions in the parameter space, the schemes in [9] or [10] can be used, respectively. These controllers require a priori knowledge of such controllable regions. An alternative method proposes the use of a pre-specified number of different controllers together with a switching strategy to commute among them (see [8]). This method offers an interesting solution for the cases when the number of possible controllers in the set is finite and available. In the general case the required number of controllers may be large so as to guarantee that the set contains a stabilizing controller.

In general, very little is known about the structure of the admissible regions in the parameter space. This explains the difficulties encountered in the search of adaptive controllers not relying on exciting signals and using the order of the plant as the only a priori information. In this line of research a different approach to avoid singularities in adaptive control has been proposed in [6] which only requires the order of the plant as available information. The method consists of an appropriate modification to the parameter estimates so that, while retaining all their convergence properties, they are brought to the admissible area. The extension of this scheme to the stochastic case has been carried out in [7]. The extension of this technique to the minimum phase multivariable case can be found in [4]. This method does not require any a priori knowledge on the structure of the controllable region. It can also be viewed as the solution of a least-squares parameter estimation problem subject to the constraint that the estimates belong to the admissible area. The admissible area is defined here as those points in the parameter space whose corresponding Sylvester resultant matrix is nonsingular. The main drawback of the scheme presented in [6] is that the number of directions to be explored in the search for an appropriate modification becomes very large as the order of the system increases. This is due essentially to the fact that the determinant of the Sylvester resultant matrix is a very complex function of the parameters.

The method based on parameter modification has also been previously used in [3] for a particular lifting plant representation. The plant description proposed in [3] has more parameters than the original plant, but has the very appealing feature of explicitly depending on the system's controllability matrix. Indeed, one of the matrix coefficients in the new parametrization turns out to be precisely the controllability matrix times the observability matrix. Therefore, the estimates modification can actually be computed straightforwardly without having to explore a large number of possible directions as is the case in [6]. It actually requires one on-line computation involving a *polar decomposition* of the estimate of the controllability matrix. However, no indication was given in [3] on how this computation can be effectively carried out. Recently, [14] presented an interesting *direct* adaptive control scheme for the same class of liftings proposed in [3]. As pointed out in [14] the polar decomposition can be written in terms of a *singular value decomposition* (SVD) which is more widely known. Methods to perform SVD are readily available. This puts the adaptive periodic controllers in [3] and [14] into a much better perspective. At this point it should be highlighted that even if persistent excitation is allowed into the system, the presented adaptive control schemes offer a better performance during the transient period by avoiding singularities.

The adaptive controller proposed in [3] and [14] is based on a periodic controller. A dead-beat controller is used in one half of the cycle and the input is identically zero during the other half of the cycle. Therefore a weakness of this type of controllers is that the system is left in open loop half of the time. In this chapter we propose a solution to this problem.

As compared to [3] and [14] the controller proposed here does not require the frequent introduction of periodic n-length sequences of zero inputs. The new control strategy is a periodic controller calculated every n-steps, n being the order of the system. For technical reasons we still have to introduce a periodic sequence of zero inputs but the periodicity can be arbitrarily large. As a result the new controller is such that the system is almost always operating in closed loop which should lead to better performance characteristics.

4.2 Problem formulation

Consider a discrete-time system, described by the following state-space representation:

$$\begin{aligned} x_{t+1} &= Ax_t + bu_t + b'v'_t \\ y_t &= c^T x_t + v''_t \end{aligned} \tag{4.1}$$

where x is the $(n \times 1)$ state vector, u, y are respectively the plant's input and output and A, b, c^T are matrices of appropriate dimensions. Signals v' v'' can be identified as perturbations belonging to a class, which will be discussed later.

The state part of equation (4.1) is iterated n times:

$$x_{t+n} = A^n x_t + \left[A^{n-1}b \ldots b\right] U_t + \left[A^{n-1}b' \ldots b'\right] V'_t \tag{4.2}$$

Matrix $\left[A^{n-1}b \ldots b\right]$ is the system's (4.1) controllability matrix $\mathcal{C}(n \times n)$ and similarly $\mathcal{C} = \left[A^{n-1}b' \ldots b'\right]$. Vectors U_t, V'_t are defined as:

$$U_t = \left[u(t) \ldots u(t+n-1)\right] \quad U \in \mathcal{R}^n \quad V'_t = [v'(t) \ldots v'(t+n-1)] \quad V' \in \mathcal{R}^n$$

Working in a similar way as in (4.2), an alternative expression for x_t can be obtained:

$$x_t = A^n x_{t-n} + \mathcal{C}U_{t-n} + \mathcal{C}' V'_{t-n} \tag{4.3}$$

introducing (4.3) into (4.2) becomes:

$$x_{t+n} = A^{2n} x_{t-n} + A^n \mathcal{C} U_{t-n} + A^n \mathcal{C}' V'_{t-n} + \mathcal{C} U_t + \mathcal{C}' V'_t \tag{4.4}$$

The next step will be to express the system's output y for a time interval $[t, t+n-1]$, using (4.1):

$$\begin{aligned} y_t &= c^T x_t + v''_t \\ y_{t+1} &= c^T A x_t + c^T b u_t + c^T b' v'_t + v''_{t+1} \\ &\vdots \\ y_{t+n-1} &= c^T A^{n-1} x_t + c^T A^{n-2} b u_t + \ldots + c^T b u_{t+n-2} + c^T A^{n-2} b' v'_t + \ldots \\ &\quad + c^T b' v'_{t+n-2} + v''_{t+n-1} \end{aligned}$$

which can be written as the following expression:

$$Y_{t+n} = \mathcal{O} x_t + G U_t + G' V'_t + I V''_t \tag{4.5}$$

where

$$\mathcal{O} = [c^T \quad c^T A \ldots c^T A^{n-1}]^T, \quad \mathcal{O} \in \mathcal{R}^{n \times n} \tag{4.6}$$

is the system's observability matrix

$$G = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ c^T b & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ c^T A^{n-2} b & \cdots & c^T b & 0 \end{bmatrix} \quad G \in \mathcal{R}^{n \times n} \quad Y_t = \begin{bmatrix} y_{t-n} \\ \vdots \\ y_{t-1} \end{bmatrix} \quad Y_t \in \mathcal{R}^n \tag{4.7}$$

and V''_t is similarly defined as V'_t but with v' replaced by v''. Identically, G' is defined as G in (4.7) with b' instead of b. From (4.5) we get:

$$Y_{t+2n} = \mathcal{O} x_{t+n} + G U_{t+n} + G' V'_{t+n} + I V''_{t+n} \tag{4.8}$$

Since the state is not supposed to be measurable, we will obtain an expression that depends only on the system's output and input. Introducing (4.4) into the above it follows

$$Y_{t+2n} = \mathcal{O}A^{2n}x_{t-n} + \mathcal{O}A^{n}\mathcal{C}U_{t-n} + \mathcal{O}A^{n}\mathcal{C}'V'_{t-n} + \mathcal{O}\mathcal{C}U_t + \mathcal{O}\mathcal{C}'V'_t + GU_{t+n} + G'V'_{t+n} + IV''_{t+n} \tag{4.9}$$

whereas, by (4.5)

$$Y_t = \mathcal{O}x_{t-n} + GU_{t-n} + G'V'_{t-n} + IV''_{t-n} \tag{4.10}$$

and

$$x_{t-n} = \mathcal{O}^{-1}\left[Y_t - GU_{t-n} - G'V'_{t-n} - IV''_{t-n}\right] \tag{4.11}$$

So finally, by substituting (4.11) into (4.9), one has:

$$Y_{t+2n} = DY_t + BU_t + B'U_{t-n} + GU_{t+n} + N_{t+2n} \tag{4.12}$$

where $t = 0, n, 2n, \ldots$ and

$$D = \mathcal{O}A^{2n}\mathcal{O}^{-1}, \quad B = \mathcal{O}\mathcal{C}, \quad B' = \left[\mathcal{O}A^{n}\mathcal{C} - \mathcal{O}A^{2n}\mathcal{O}^{-1}G\right], \quad D, B, B' \in \mathcal{R}^{n\times n} \tag{4.13}$$

$$N = \mathcal{O}A^{n}\mathcal{C}'V'_{t-n} - \mathcal{O}^{-1}A^{2n}\mathcal{O}^{-1}G'V'_{t-n} + \mathcal{O}\mathcal{C}'V'_t + G'V'_{t+n} + IV''_{t+n} - \mathcal{O}A^{2n}\mathcal{O}^{-1}V''_{t-n} \tag{4.14}$$

The following control law is proposed for the plant:

$$BU_t = -DY_t - B'U_{t-n} \tag{4.15}$$

The closed loop system becomes:

$$Y_{t+2n} - GU_{t+n} = N_{t+2n} \tag{4.16}$$

whereas from (4.11) and (4.16), one has:

$$x_{t+n} = \mathcal{O}^{-1}\left[N_{t+2n} - G'V'_{t+n} + IV''_{t+n}\right] = \mathcal{O}^{-1}\left[N_{t+2n} - N^1_{t+2n}\right] \tag{4.17}$$

where N^1_{t+2n} is another noise term. Therefore the state $x_{t+n},\ t = 0, n, 2n, \ldots$ is bounded by the (bounded) noise. In the ideal case x_{t+n} will be identically zero. From (4.3), it is clear that U_t is bounded and becomes zero in the ideal case.

Remark 2.1 Dead-beat control can induce large control inputs. For a smoother performance, one can use the following control strategy:

$$BU_t = -DY_t - B'U_{t-n} + C[Y_t - GU_{t-n}] \tag{4.18}$$

where C has all its eigenvalues inside the unit disc. The closed loop system is in this case:

$$Y_{t+2n} - GU_{t+n} = C[Y_t - GU_{t-n}] + N_{t+2n} \tag{4.19}$$

In the ideal case the LHS of the above converges to zero and so does the state.

4.3 Adaptive control scheme

In this section the adaptive control scheme based on the strategy previously developed is presented. For sake of simplicity an index k will be associated to t as follows: $t = kn$, $k = 0, 1, 2, \ldots$. The plant representation (4.12) can be rewritten as:

$$Y_{k+1} = \theta\phi_k + N_{k+1} \quad Y_{k+1} \in \mathcal{R}^n \tag{4.20}$$

where:

$$\theta = \left[B \vdots B' \vdots G \vdots D\right] \quad \theta \in \mathcal{R}^{n\times 4n}, \;\; \phi_k = \left[U_{k-1}\; U_{k-2}\; U_k\; Y_{k-1}\right]^T \quad \phi \in \mathcal{R}^{4n} \tag{4.21}$$

for $k = 0, 1, 2, \ldots$. The class of disturbances satisfies:

$$\|N(k)\| \leq \eta \tag{4.22}$$

The following a priori knowledge on the plant is required:

Assumption 1: n and an upper bound η in (4.22) is known

Assumption 2: The existence of a lower bound b_0 is required, so that $b_0 I \leq B^T B$. The value of this bound, however, may not be known.

The equations of the adaptive scheme are given in the order they appear in the control algorithm. Note that signals Y_k, ϕ_{k-1} are measurable.The prediction error:

$$E_k = \bar{Y}_k - \theta_{k-1}\bar{\phi}_{k-1} \quad E_k \in \mathcal{R}^n \tag{4.23}$$

where the variables marked with a bar are the normalized versions of the original ones:

$$\bar{Y}_k = Y_k/[1 + \|\phi_{k-1}\|], \quad \bar{\phi}_{k-1} = \phi_{k-1}/[1 + \|\phi_{k-1}\|] \tag{4.24}$$

Least squares with dead zone:

$$w_k^2 = E_k^T E_k + \bar{\phi}_{k-1}^T P_{k-1}^2 \bar{\phi}_{k-1} \tag{4.25}$$

$$\delta_k = \eta/[1 + \|\phi_{k-1}\|] \tag{4.26}$$

$$\lambda_k = \begin{cases} 0 & \text{if } w_k^2 \leq \delta_k^2(1+\alpha)n \\ \dfrac{\alpha[|w_k| - \delta_k\sqrt{(1+\alpha)n}]}{(1 + \bar{\phi}_{k-1}^T P_{k-}^2 \bar{\phi}_{k-1}|w_k|} \quad \alpha > 0 & \text{otherwise} \end{cases} \tag{4.27}$$

$$P_k = P_{k-1} - \frac{\lambda_k P_{k-1}\bar{\phi}_{k-1}\bar{\phi}_{k-1}^T P_{k-1}}{1 + \lambda_k \bar{\phi}_{k-1}^T P_{k-1}\bar{\phi}_{k-1}} \quad P_k \in \mathcal{R}^{4n\times 4n} \tag{4.28}$$

$$\theta_k = \theta_{k-1} + \lambda_k E_k \bar{\phi}_{k-1}^T P_k = [B_k \vdots B'_k \vdots G_k \vdots D_k] \quad \theta_k \in \mathcal{R}^{n\times 4n} \tag{4.29}$$

The forgetting factor λ can also be chosen to commute between 0 and 1 using the same scheduling variable as in [6].

Modification of the parameters estimates

$$P_k = L_k L_k^T \geq 0, \quad L = \left[\frac{L^1(n \times 4n)}{L^2(3n \times 4n)}\right] \tag{4.30}$$

$$B_k = Q_k S_k; \quad S_k \geq 0; \quad Q_k Q_k^T = I \quad Q, S \in \mathcal{R}^{n\times n} \tag{4.31}$$

$$\Theta_k = \theta_k + \underbrace{Q_k L_k^1}_{\beta_k} L_k^T = [\underline{B}_k \vdots \underline{B'}_k \vdots \underline{G}_k \vdots \underline{D}_k] \tag{4.32}$$

(4.30) is the so-called *Cholesky* decomposition which gives a triangular positive semi-definite matrix L and (4.31) is a *polar decomposition*. The first one is often found in software packages used for control purposes, as it is a special case of the *LU decomposition*. The second one can be calculated in many ways. The way that is described here consists of calculating the *singular value decomposition* of B: $B_k = \mathcal{U}_k \mathcal{S}_k \mathcal{V}_k{}^T$.

Then the matrices Q_k and S_k can be calculated as follows:

$$Q_k = \mathcal{U}_k \mathcal{V}_k^T, \quad S_k = \mathcal{V}_k \mathcal{S}_k \mathcal{V}_k^T \tag{4.33}$$

4.4 Adaptive control law

4.4.1 Properties of the identification scheme

In this section we will present the convergence properties of the parameter identification algorithm proposed in the previous section. These properties will be essential in the stability of the adaptive control closed loop system.

Let us define the parametric distance at instant k as: $\tilde{\theta}_k = \theta - \theta_k$ and matrix H_k as:

$$H_k = H_{k-1} - \frac{\lambda_k E_k E_k^T}{1 + \lambda_k \bar{\phi}_{k-1}^T P_{k-1}\bar{\phi}_{k-1}} \quad H_k \in \mathcal{R}^{4n\times 4n} \tag{4.34}$$

The following properties hold:

(1) The forgetting factor λ in (4.27) satisfies:

$$0 \leq \lambda_k \leq \alpha, \quad \lambda_k \bar{\phi}_{k-1}^T P_{k-1}^2 \bar{\phi}_{k-1} \leq \alpha \tag{4.35}$$

(2) There is a positive definite function V_k defined as: $V_k = \text{tr}\,(P_k + H_k)$ that satisfies: $V_k \leq V_{k-1}$

(3) The augmented error w_k in (4.25) is bounded as follows:

$$\lim_{k\to\infty} \sup\,(w_k^2 - \delta_k^2(1+\alpha)n) \leq 0 \tag{4.36}$$

(4) The covariance matrix converges.

(5) The forgetting factor λ is such that: $\sum_{k=0}^{\infty} \lambda_k \delta_k^2 < \infty$.

(6) The modified parameter estimate $\underline{B}_k$ satisfies the following:

$$\underline{B}_k \underline{B}_k^T \geq \frac{b_0^2 I}{4V_0^2} \tag{4.37}$$

(7) θ_k converges and: $\theta_k \theta_k^T \leq V_0^2 I$

(8) The *modified prediction error* $\hat{E}_k$ defined as:

$$\hat{E}_k = \bar{Y}_k - \Theta_{k-1}\bar{\phi}_{k-1} \quad \hat{E}_k \in \mathcal{R}^n \tag{4.38}$$

is bounded as follows:

$$\hat{E}_k^T \hat{E}_k \leq 2w_k^2 \tag{4.39}$$

Proofs of the above properties are given in [3].

4.4.2 Adaptive control strategy

The expression (4.38) can be written as follows (see also (4.24)):

$$\begin{aligned}\hat{E}_k &= \frac{Y_k - \Theta_{k-1}\phi_{k-1} \pm \Theta_{k-2}\phi_{k-1}}{1 + \|\phi_{k-1}\|} \\ &= \frac{Y_k - \Theta_{k-2}\phi_{k-1} + (\Theta_{k-2} - \Theta_{k-1})\phi_{k-1}}{1 + \|\phi_{k-1}\|}\end{aligned}$$

From (4.21) and (4.32):

$$\begin{aligned}\hat{E}_k = {} & \frac{Y_k - \underline{B}_{k-2}U_{k-2} - \underline{B}'_{k-2}U_{k-3} - \underline{G}_{k-2}U_{k-1} - \underline{D}_{k-2}Y_{k-2}}{1 + \|U_{k-2}U_{k-3}U_{k-1}Y_{k-2}\|} \\ & + \frac{(\Theta_{k-2} - \Theta_{k-1})\phi_{k-1}}{1 + \|\phi_{k-1}\|}\end{aligned} \tag{4.40}$$

Using (4.32), we obtain:

$$\frac{(\Theta_{k-2}-\Theta_{k-1})\phi_{k-1}}{1+\|\phi_{k-1}\|} = (\Theta_{k-2}-\Theta_{k-1})\bar{\phi}_{k-1}$$
$$= (\theta_{k-2}-\theta_{k-1})\bar{\phi}_{k-1} + (\beta_{k-2}L^T{}_{k-2} - \beta_{k-1}L^T{}_{k-1})\bar{\phi}_{k-1} \tag{4.41}$$

Note that property (7) tells us that: $\|\theta_{k-2}-\theta_{k-1}\| \to 0$ as $k \to \infty$.
Note also that the last term in (4.41) can be rewritten as:

$$(\beta_{k-2}L^T{}_{k-2} - \beta_{k-1}L^T{}_{k-1} \pm \beta_{k-2}L^T{}_{k-1})\bar{\phi}_{k-1}$$
$$= \beta_{k-2}(L^T{}_{k-2} - L^T{}_{k-1})\bar{\phi}_{k-1} + (\beta_{k-2}-\beta_{k-1})L^T{}_{k-1}\bar{\phi}_{k-1} \tag{4.42}$$

But $\left\|L^T{}_{k-2} - L^T{}_{k-1}\right\| \to 0$ because matrix L_k results from the *Cholesky decomposition* of the covariance matrix P in (4.30), which converges. On the other hand, using (4.25), term $L^T{}_{k-1}\bar{\phi}_{k-1}$ in (4.42) can be written as:

$$\begin{aligned}(L^T{}_{k-1}\bar{\phi}_{k-1})^T(L^T{}_{k-1}\bar{\phi}_{k-1}) &= \bar{\phi}^T_{k-1}P_{k-1}\bar{\phi}_{k-1}\\ &\le \|\bar{\phi}^T_{k-1}\|\,\|P_{k-1}\bar{\phi}_{k-1}\|\\ &\le \sqrt{\bar{\phi}^T_{k-1}P^T{}_{k-1}P_{k-1}\bar{\phi}_{k-1}} \text{ (using (4.24))}\\ &\le \sqrt{E^T{}_kE_k + \bar{\phi}^T_{k-1}P^T{}_{k-1}P_{k-1}\bar{\phi}_{k-1}}\\ &= \sqrt{w^2{}_k}\end{aligned} \tag{4.43}$$

Using (4.41), (4.42), (4.43) and the facts that as $k \to \infty$, $(\theta_{k-2}-\theta_{k-1}) \to 0$ and $\|(\beta_{k-2}-\beta_{k-1})\| \le \infty$, we can conclude that:

$$\|(\Theta_{k-2}-\Theta_{k-1})\bar{\phi}_{k-1}\| \le \|\beta_{k-2}-\beta_{k-1}\|\sqrt{|w_k|} + \|\xi_k\| \tag{4.44}$$

where

$$\xi_k = (\theta_{k-2}-\theta_{k-1})\bar{\phi}_{k-1} + \beta_{k-2}(L^T{}_{k-2} - L^T{}_{k-1})\bar{\phi}_{k-1} \tag{4.45}$$

and $\|\xi_k\| \to 0$ as $k \to \infty$.

From (4.40) and using (4.39) and the above we have:

$$\frac{\|Y_k - \underline{B}_{k-2}U_{k-2} - \underline{B}'_{k-2}U_{k-3} - \underline{G}_{k-2}U_{k-1} - \underline{D}_{k-2}Y_{k-2}\|}{1+\|U_{k-2}\mathrm{U}_{k-3}\mathrm{U}_{k-1}\mathrm{Y}_{k-2}\|}$$
$$\le \left\|\hat{E}_k\right\| + \|(\Theta_{k-2}-\Theta_{k-1})\bar{\phi}_{k-1}\|$$
$$\le \sqrt{2}|w_k| + \|\beta_{k-2}-\beta_{k-1}\|\sqrt{|w_k|} + \|\xi_k\| \tag{4.46}$$

or equivalently

$$\frac{\|Y_{k+1} - \underline{B}_{k-1}U_{k-1} - \underline{B}'_{k-1}U_{k-2} - \underline{G}_{k-1}U_k - \underline{D}_{k-1}Y_{k-1}\|}{1+\|U_{k-1}U_{k-2}U_kY_{k-1}\|}$$

$$\le \sqrt{2}|w_{k+1}| + \|\beta_{k-1}-\beta_k\|\sqrt{|w_{k+1}|} + \|\xi_k\| \tag{4.47}$$

Using the certainty equivalence principle the adaptive control strategy could simply be obtained from (4.15) by replacing the true parameters by their estimates. However, for technical reasons that will appear clear later, we will have to introduce some sequences of inputs equal to zero. These sequences are introduced periodically after a time interval of arbitrary length L. We will assume that $k \geq k^*$ where k^* is a time instant large enough such that the RHS of (4.47) is small enough. This will be made clear in (4.50). Since some sequences of inputs equal to zero are periodically introduced let us assume that $\bar{k} \geq k^*$ is the first time instant after k^* that we apply the zero sequences defined below:

$$\begin{aligned}
U_{\bar{k}-2} &= 0 \\
U_{\bar{k}-1} &= -\underline{B}_{\bar{k}-1}^{-1}(\underline{D}_{\bar{k}-1} Y_{\bar{k}-1} + \underline{B}'_{\bar{k}-1} U_{\bar{k}-2}) \\
U_{\bar{k}} &= 0 \\
U_{\bar{k}+1} &= -\underline{B}_{\bar{k}+1}^{-1}\left(\underline{D}_{\bar{k}+1} Y_{\bar{k}+1} + \underline{B}'_{\bar{k}+1} U_{\bar{k}}\right) \\
U_{\bar{k}+2} &= -\underline{B}_{\bar{k}+2}^{-1}\left(\underline{D}_{\bar{k}+2} Y_{\bar{k}+2} + \underline{B}'_{\bar{k}+2} U_{\bar{k}+1}\right) \\
U_{\bar{k}+3} &= \text{etc.} \\
&\vdots \qquad \vdots \\
U_{\bar{k}+L-2} &= 0 \\
U_{\bar{k}+L-1} &= -\underline{B}_{\bar{k}+L-1}^{-1}\left(\underline{D}_{\bar{k}+L-1} Y_{\bar{k}+L-1} + \underline{B}'_{\bar{k}+L-1} U_{\bar{k}+L-2}\right) \\
U_{\bar{k}+L} &= 0 \\
U_{\bar{k}+L+1} &= -\underline{B}_{\bar{k}+L+1}^{-1}\left(\underline{D}_{\bar{k}+L+1} Y_{\bar{k}+L+1} + \underline{B}'_{\bar{k}+L+1} U_{\bar{k}+L}\right) \\
U_{\bar{k}+L+2} &= -\underline{B}_{\bar{k}+L+2}^{-1}\left(\underline{D}_{\bar{k}+L+2} Y_{\bar{k}+L+2} + \underline{B}'_{\bar{k}+L+2} U_{\bar{k}+L+1}\right) \\
U_{\bar{k}+L+3} &= \text{etc.}
\end{aligned} \tag{4.48}$$

The convergence analysis will be carried out by contradiction. We will assume that the regressor vector $\phi_{\bar{k}}$ goes to infinity and prove that this leads to a contradiction. Note that if $\|\phi_{\bar{k}}\| \to \infty$ then from (4.26) and (4.36) it follows that $\|w_{\bar{k}}\| \to 0$.

Introducing the control strategy (4.48) into (4.47) we obtain

$$\frac{\| Y_{\bar{k}+1} \|}{1 + \| U_{\bar{k}-1} \; 0 \; 0 \; Y_{\bar{k}-1} \|} \to 0 \tag{4.49}$$

Note from (4.48) that $U_{\bar{k}-1} = -\underline{B}_{\bar{k}-1}^{-1}(\underline{D}_{\bar{k}-1} Y_{\bar{k}-1})$. Therefore $U_{\bar{k}-1}$ is bounded by $Y_{\bar{k}-1}$. Introducing this fact into (4.49)

$$\frac{\| Y_{\bar{k}+1}\|}{1 + \| Y_{\bar{k}-1}\|} \le \varepsilon \tag{4.50}$$

or equivalently

$$\| Y_{\bar{k}+1}\| \le \varepsilon(1 + \| Y_{\bar{k}-1}\|) \tag{4.51}$$

where $\varepsilon > 0$ is arbitrarily small.

Using equation (4.11), (4.51) and the fact that $U_{\bar{k}} = 0$, one has

$$\|x_{\bar{k}}\| \le \|\mathcal{O}^{-1}\| \| Y_{\bar{k}+1}\| + \zeta \le \varepsilon\zeta\| Y_{\bar{k}-1}\| + \zeta \tag{4.52}$$

Hereafter to simplify notation, ζ will generically represent bounded terms. ζ may denote the bound for the system noises or the bound for different matrix norms. We will not give the exact expression for ζ because it is irrelevant for the analysis purposes. Combining (4.52) and (4.3), we have

$$\left\|x_{\bar{k}+1}\right\| \le \varepsilon\zeta\| Y_{\bar{k}-1}\| + \zeta \tag{4.53}$$

From (4.3) we get

$$x_{\bar{k}+2} = A^n x_{\bar{k}+1} + \mathcal{C} U_{\bar{k}+1} + \mathcal{C}' V'_{\bar{k}+1} \tag{4.54}$$

where $U_{\bar{k}+1}$ is given in (4.48)

$$\begin{aligned}\left\|U_{\bar{k}+1}\right\| &= \left\|-\underline{B}_{\bar{k}+1}^{-1}(\underline{B}'_{\bar{k}+1} U_{\bar{k}} + \underline{D}_{\bar{k}+1} Y_{\bar{k}+1})\right\| \\ &= \left\|-\underline{B}_{\bar{k}+1}^{-1}\underline{D}_{\bar{k}+1} Y_{\bar{k}+1}\right\| \\ &\le \varepsilon\left\|\underline{B}_{\bar{k}+1}^{-1}\underline{D}_{\bar{k}+1}\right\|(1 + \| Y_{\bar{k}-1}\|) \text{ (see(4.51))}\end{aligned} \tag{4.55}$$

Consequently one has (see (4.53), (4.54), (4.55)):

$$\left\|x_{\bar{k}+2}\right\| \le \varepsilon\zeta\| Y_{\bar{k}-1}\| + \zeta \tag{4.56}$$

We may proceed in the same way to obtain an expression for $\left\|x_{\bar{k}+3}\right\|$. Consider the following expressions from (4.3), (4.48), (4.10) respectively:

$$x_{\bar{k}+3} = A^n x_{\bar{k}+2} + \mathcal{C} U_{\bar{k}+2} + \mathcal{C}' V'_{\bar{k}+2} \tag{4.57}$$

$$U_{\bar{k}+2} = -\underline{B}_{\bar{k}+2}^{-1}\left(\underline{B}'_{\bar{k}+2} U_{\bar{k}+1} + \underline{D}_{\bar{k}+2} Y_{\bar{k}+2}\right) \tag{4.58}$$

$$Y_{\bar{k}+2} = \mathcal{O} x_{\bar{k}+1} + G U_{\bar{k}+1} + G' V'_{\bar{k}+1} + I V''_{\bar{k}+1} \tag{4.59}$$

From the above and using (4.53) and (4.55), we obtain:

$$\begin{aligned}\|Y_{\bar{k}+2}\| &\leq \|\mathcal{O}x_{\bar{k}+1} + GU_{\bar{k}+1}\| + \zeta \\ &\leq \varepsilon\zeta\| \ Y_{\bar{k}-1}\| + \zeta \end{aligned} \tag{4.60}$$

From the above and (4.55)–(4.58) one has:

$$\|x_{\bar{k}+3}\| \leq \varepsilon b_3\| \ Y_{\bar{k}-1}\| + \zeta \tag{4.61}$$

Introducing (4.48) and (4.59), one has

$$\|x_{\bar{k}+3}\| \leq \varepsilon b_3\| \ x_{\bar{k}-2}\| + \zeta \tag{4.62}$$

It should be noted that b_3 is a finite quantity which, for a given system, depends *only* on the length of the time interval $[\bar{k}-2, \bar{k}+3]$. We may extend this procedure over the interval $[\bar{k}-2, \bar{k}-2+L]$. Then (4.62) becomes:

$$\|x_{\bar{k}-2+L}\| \leq \varepsilon b_L\| \ x_{\bar{k}-2}\| + \zeta \tag{4.63}$$

where b_L is a quantity similar to b_3 in (4.62). Finally we may repeat the same procedure for $iL, \ i = 1, 2, \ldots,$ in order to obtain an expression for $\|x_{\bar{k}-2+iL}\|$

$$\|x_{\bar{k}-2+iL}\| \leq \varepsilon b_L \left\|x_{\bar{k}-2+(i-1)L}\right\| + \zeta \tag{4.64}$$

Expressions (4.63) and (4.64) can be combined to obtain:

$$\|x_{\bar{k}-2+iL}\| \leq (\varepsilon b_L)^i \|x_{\bar{k}-2}\| + \left[(\varepsilon b_L)^{i-1}\zeta + (\varepsilon b_L)^{i-2}\zeta + \ldots + \varepsilon b_L\zeta + \zeta\right] \tag{4.65}$$

Factor $[(\varepsilon b_L)^{i-1}\zeta + (\varepsilon b_L)^{i-2}\zeta + \ldots + \varepsilon b_L\zeta + \zeta]$ will be a sum of an infinite number of terms. But as $\varepsilon b_L \ll 1$ this sum will be bounded. Hence we will have a bounded decreasing state $\|x_{\bar{k}-2+iL}\|$ as follows:

$$\|x_{\bar{k}-2+iL}\| \leq (\varepsilon b_L)^i \|x_{\bar{k}-2}\| + \zeta \quad \forall i = 1, \ldots \tag{4.66}$$

From (4.66) we conclude that the state $x_{\bar{k}-2+iL}$ for $i = 1, \ldots, \infty$ is bounded in the limit by a term that depends on the noise. The states at other time instants inside the intervals $[k-2+(i-1)L, k-2+iL]$ can be proven to be bounded too, by using the procedure described in (4.56) up to (4.62). This clearly contradicts our assumption that ϕ diverges. Therefore all the signals remain bounded. Furthermore in view of the previous analysis, the smaller the noise upper bound, the smaller the state becomes in the limit.

4.5 Simulations

4.5.1 System without noise

The performance of the proposed adaptive control scheme is illustrated in this section through a series of simulations. The following non-minimum phase system is considered:

$$x_{t+1} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} u_t + b' v'_t$$
$$y_t = [1 \quad 0] x_t + v''_t \tag{4.67}$$

The system's poles and zeroes are respectively (1, 2) and 1.5. In this way matrices B, B', G, D are:

$$B = \begin{bmatrix} 1.5 & 1 \\ 2.5 & 1.5 \end{bmatrix}, \quad B' = \begin{bmatrix} -10.5 & 2.5 \\ -22.5 & 4.5 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -14 & 15 \\ -30 & 31 \end{bmatrix} \tag{4.68}$$

The initial parameter estimates B_0, B'_0, G_0, D_0 are chosen in such a way that the signs of the determinants of B_0 and B are opposite. This will lead to important conclusions concerning the utility of the modification of the parameters' estimates. So the initial parameter estimates are:

$$\theta_0 = \begin{bmatrix} 2 & 1 & -10.5 & 2.5 & 0 & 0 & -14 & 15 \\ 2.5 & 1.5 & -22.5 & 4.5 & 1 & 0 & -30 & 31 \end{bmatrix} \tag{4.69}$$

and initial conditions are: zero control input and state $x_0 = [100, 100]^T$ for $t \in [0,\ 8]$. The parameters' modification was used only when the determinant of the estimate of B was below a threshold of 0.1.

It is clear that β_k in (4.32) can be multiplied by a scalar without losing the convergence properties. Such a coefficient can be used to reduce the abrupt changes in the modified parameter estimates and therefore improve the transient behaviour. In our case β in (4.32) was multiplied by a coefficient equal to 0.001. The initial value of the covariance matrix is chosen as $P = 220I_8$, whereas there is no noise corrupting the system. On the other hand sequences of zero inputs have not been used in the simulations that follow. Figures that are referred hereafter are given in the appendix.

Figures 4.1 and 4.2 show that modifying the parameter estimates helps to avoid unboundedness of the plant signals $y(t)$ and $u(t)$. The results that are presented in Figure 4.2 clearly denote that at around $k = 22$ or $t = 44$ there is a value of det(B) smaller than 0.1 in the case of unmodified parameters. The values of det(B) that follow are all very close to zero, a fact that leads to the behaviour shown in Figure 4.1. On the other hand, when the modification of the paprameters estimates is applied, the critical value of det(B) at $t = 44$ is avoided and the evolution of det(B) thereafter is clearly away from zero.

4.5.2 System with noise

Simulation results are now presented in the case when the system is corrupted

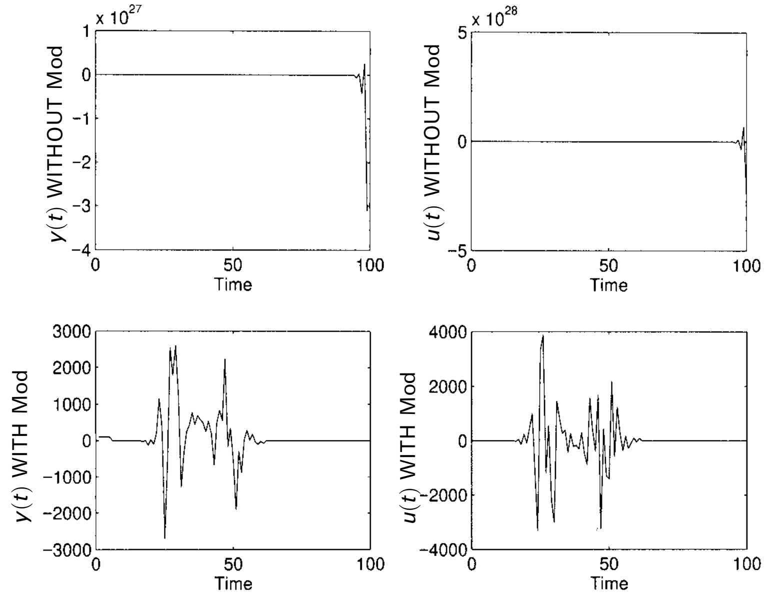

Figure 4.1 *Up: WITHOUT modification, Down: WITH modification of parameter estimates*

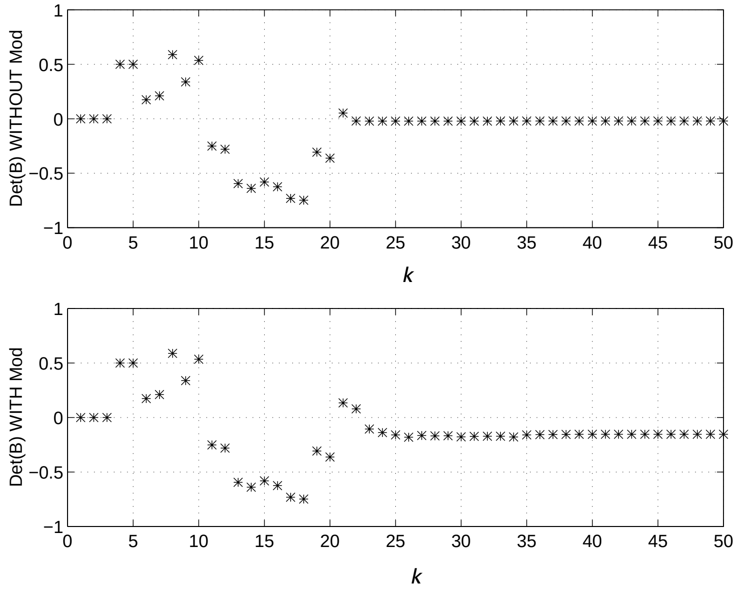

Figure 4.2 *Up: WITHOUT modification, Down: WITH modification of parameter estimates. Note* $t = 2k$

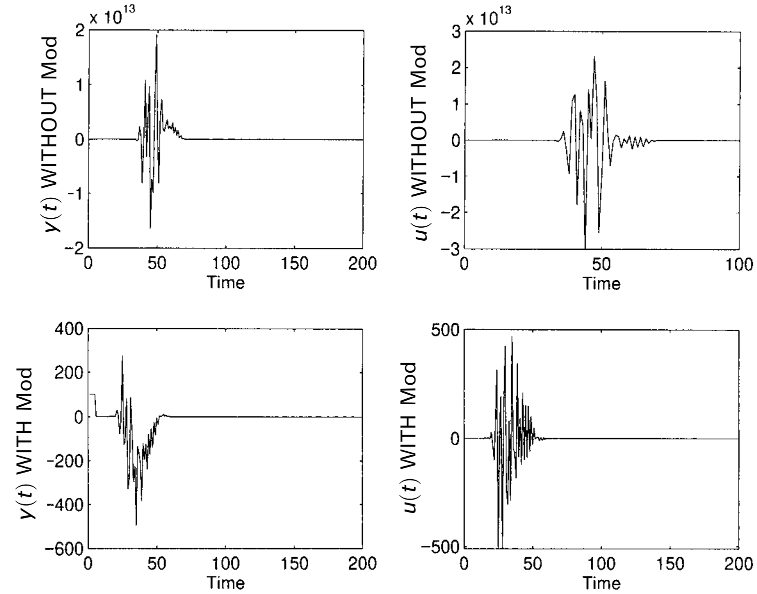

Figure 4.3 *Up: WITHOUT modification, Down: WITH modification of parameter estimates*

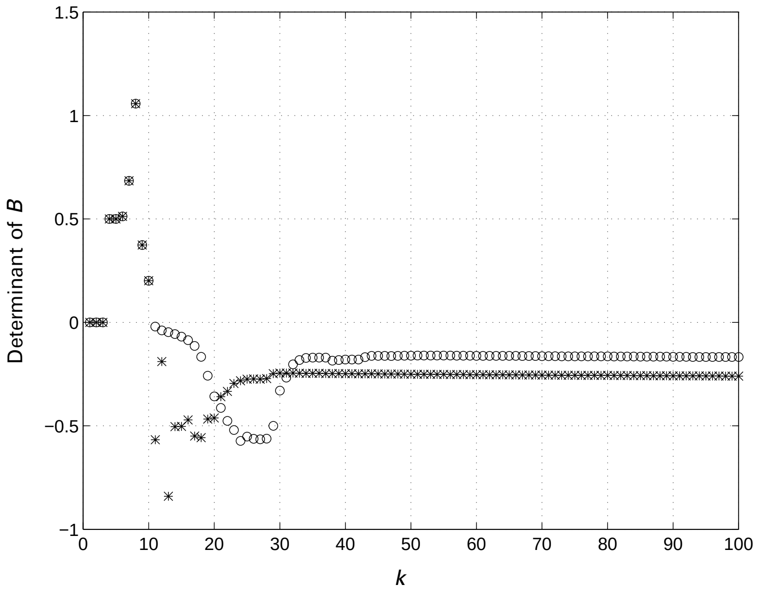

Figure 4.4 ○*: WITHOUT modification,* $*$*: WITH modification of parameter* B *estimate. Note:* $t = 2k$

with noise. System (4.67) remains the same with the exception of $b' = b = [1\ \ -1.5]^T$ in (Figure 4.1) and of a sinusoidal noise described by:

$$v'_t = 0.0004 \sin(2k) + 0.0003 \cos(2k) \tag{4.70}$$

$$v''_t = 0.0004 \sin(2k) - 0.0003 \cos(2k) \tag{4.71}$$

Initial conditions are the same as in the ideal case (i.e. zero control input and state $x_0 = [100\ 100]^T$ until $t = 8$) and $P = 200I_8$. Figure 4.3 shows the evolution of output and input signals and Figure 4.4 explains how values of determinant of B estimates close to zero can lead to huge output and input signals.

4.6 Conclusions

This chapter has presented an indirect adaptive periodic control scheme based on the lifted representation of the plant proposed in [3]. New arguments have been presented so that this technique appears in a better position as a solution to the long standing problem of singularities in adaptive control of non-minimum phase plants. Contrary to other techniques, the only a priori knowledge required on the plant is its order besides the standard controll-ability/observability assumption. As compared to previous studies in the subject [3], [14], the proposed controller does not require the frequent introduction of periodic n-length sequences of zero inputs. Furthermore simulations have shown that in practice there is no need to introduce sequences of inputs equal to zero. Therefore the new controller is such that the system always operates in closed loop which leads to better performance characteristics. Simulation results have also shown that the use of the proposed estimate modification can significantly reduce signal peaks during the transient period even in the case when signals $y(t)$ and $u(t)$ of the original unmodified system do not become unbounded.

References

[1] Kreisselmeier, G. and Smith, M. C., (1986) 'Stable Adaptive Regulation of Arbitrary n-th Order Plants', *IEEE Trans. Contr.*, Vol. AC-31, 299–305.

[2] Kreisselmeier, G. (1989) 'An Indirect Adaptive Controller with a Self-excitation Capability', *IEEE Trans. Automat. Contr.*, Vol. AC-34, 524–528.

[3] Lozano, R. (1989) 'Robust Adaptive Regulation Without Persistent Excitation', *IEEE Trans. Autom. Contr.*, Vol. AC-34, 1260–1267.

[4] de Mathelin, M. and Bodson. M., (1994) 'Multivariable Adaptive Control: Identifiable Parametrizations and Parameter Convergence', *IEEE Transactions on Automatic Control.*, Vol. 39, No. 8, 1612–1617.

[5] Weyer, E., Mareels, I. M. and Polderman, J. W., (1994) 'Limitations of Robust Adaptive Pole Placement Control', *IEEE Transactions on Automatic Control.*, Vol. 39, No 8, 1665–1671.

[6] Lozano, R. and Zhao, X. H., (1994) 'Adaptive Pole Placement Without Excitation Probing Signal', *IEEE Transactions on Automatic Control.*, Vol. 39, No 1, 47–58.

[7] Guo. L., (1996) 'Self-convergence of Weighted Least Squares with Application to Stochastic Adaptive Control', *IEEE Transactions on Automatic Control.*, Vol. 41, No 1, 79–89, January.

[8] Morse, A.S. and Pait, F. M., (1994) 'MIMO Design Models and Internal Regulators for Cyclicly Switched Parameter-Adaptive Control Systems', *IEEE Trans. Autom. Contr.*, Vol. 39, No 9, 1809–1818, September.

[9] Ossman K. A. and Kamen, E. W., (1987) 'Adaptive Regulation of MIMO Linear Discrete-time Systems without Requiring a Persistent Excitation', *IEEE Trans. Autom. Contr.*, Vol. AC-32, No 5, 397–404.

[10] Middleton, R. H., Goodwin, G. C., Hill, D. J. and Mayne, D. Q., (1988) 'Design Issues in Adaptive Control', *IEEE Trans. Autom. Contr.*, Vol. 33, No 1, 50–58.

[11] Giri, F., M'Saad, M., Dugard, L. and Dion, J. M., (1992) 'Robust Adaptive Regulation with Minimal Prior Knowledge', *IEEE Trans. Autom. Contr.*, Vol. AC-37, 305–315.

[12] Kreisselmeier, G., (1994) Parameter Adaptive Control: A Solution to the Overmodeling Problem', *IEEE Trans. Autom. Contr.*, Vol. 39, No. 9, 1819–1826, September.

[13] Kreisselmeier, G. and Lozano, R., (1996) 'Adaptive Control of Continuous-time Overmodeled Plants', *IEEE Trans. on Autom. Contr.*, Vol. 41, No. 12, 1779–1794, December.

[14] Bayard, D., (1996) 'Stable Direct Adaptive Periodic Control Using Only Plant Order Knowledge', *International Journal of Adaptive Control and Signal Processing*, Vol. 10, No. 6, 551–570, November–December.

[15] Lancaster, P. and Tismenetsky M., (1985) *The Theory of Matrices*. 2nd ed. Academic Press, NY.

5

Adaptive stabilization of uncertain discrete-time systems via switching control: the method of localization

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Abstract

In this chapter a new systematic switching control approach to adaptive stabilization of uncertain discrete-time systems with parametric uncertainty is presented. Our approach is based on a localization method which is conceptually different from supervisory adaptive control schemes and other existing switching adaptive schemes. Our approach allows for slow parameter drifting, infrequent large parameter jumps and unknown bound on exogenous disturbance. The unique feature of localization based switching adaptive control distinguishing it from conventional adaptive switching schemes is its rapid model falsification capabilities. In the LTI case this is manifested in the ability of the switching controller to quickly converge to a suitable stabilizing controller. We believe that the approach presented in this chapter is the first design of a switching controller which is applicable to a wide class of linear time-invariant and time-varying systems and which exhibits good transient performance. The performance of the proposed switching controllers is illustrated by many simulation examples.

5.1 Introduction

Control design for both linear and nonlinear dynamic systems with unknown parameters has been extensively studied over the last three decades. Despite significant advances in adaptive and robust control in recent years, control of systems with large-size uncertainty remains a difficult task. Not only are the control problems complicated, so is the analysis of stability and performance.

It is well known [12,14] that classical adaptive algorithms prior to 1980 were all based on the following set of standard assumptions or variations of them:

(i) An upper bound on the plant order is known.
(ii) The plant is minimum phase.
(iii) The sign of high frequency gain is known.
(iv) The uncertain parameters are constant, and the closed loop system is free from measurement noise and input/output disturbances.

Classical adaptive algorithms are known to suffer from various robustness problems [34]. A number of attempts have been made since 1980 to relax the assumptions above. A major breakthrough occurred in the mid-1980s [17, 21, 35] for adaptive control of LTV plants with sufficiently small in the mean parameter variations. Later attempts were made for a broader class of systems. Fast varying continuous-time plants were treated in [36], assuming knowledge of the structure of the parameter variations. By using the concept of polynomial differential (integral) operators the problem of model reference adaptive control was dealt with in [32] for a certain class of continuous-time plants with fast time-varying parameters. An interesting approach based on some internal self-excitation mechanism was considered in [7] for a general class of LTV discrete-time systems. The global boundedness of the state was proved. However, it must be noted that the presence of such self-excitation signals in a closed loop system is often undesirable.

In another research line, a number of switching control algorithms have been proposed recently by several authors [2, 6, 8, 20, 23, 24, 31], thus significantly weakening the assumptions in (i)–(iv). Both continuous and discrete linear time-invariant systems were considered. Research in this direction was originated by the pioneering works of Nussbaum [31] and Martensson [20]. Nussbaum considered the problem of finding a smooth stabilizing controller

$$\begin{cases} \dot{z}(t) = f(g(t), z(t)) \\ u(t) = g(y(t), z(t)) \end{cases} \tag{5.1}$$

for the one-dimensional system

$$\begin{cases} \dot{x}(t) = ax(t) + qu(t) \\ y(t) = x(t) \end{cases} \tag{5.2}$$

with both $q \neq 0$ and $a > 0$ unknown. In [31] Nussbaum describes a whole family of controllers of the form (5.1) which achieve the desired stabilization of the system (5.2). For example, it was shown that every solution $(x(t), z(t))$ of

$$\begin{cases} \dot{x} = ax + qx(z^2+1)\cos(\pi z/2)\exp z^2 \\ \dot{z} = x(z^2+1) \end{cases} \tag{5.3}$$

has the property that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t)$ exists and is finite. We note that the structure of the adaptive controller is explicitly seen from (5.3). Another important result proved in [31] is that there exists no stabilizing controller for the plant (5.2) expressed in terms of polynomial or rational functions. A more general result was presented by Martensson [20]. In particular, it was shown in [20] that the only a priori information which is needed for adaptive stabilization of a minimal linear time-invariant plant is the order of a stabilizing controller. This assumption can even be removed if a slightly more complicated controller is used. Consider the following dynamic feedback problem. Given the plant

$$\begin{cases} \dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \\ y = Cx, \ y \in \mathbf{R}^r \end{cases} \tag{5.4}$$

and the controller

$$\begin{cases} \dot{z} = Fz + Gy, \quad z \in \mathbf{R}^l \\ u = Hz + Ky \end{cases} \tag{5.5}$$

where m, r are known and fixed, and n is allowed to be arbitrary. It is easy to see that this is equivalent to the static feedback problem

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u} \\ \hat{y} = \hat{C}\hat{x}, \ \hat{u} = \hat{K}\hat{y} \end{cases} \tag{5.6}$$

where $\hat{x} = (x^T z^T)^T$, $\hat{u} = (u^T z^T)^T$, $\hat{y} = (y^T z^T)^T$ and $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{K}$ are matrices of appropriate dimensions. Let the regulator be

$$\begin{cases} \hat{u} = g(h(k))N(h(k))\hat{y} \\ \dot{k} = ||\hat{y}||^2 + ||\hat{u}||^2 \end{cases} \tag{5.7}$$

where $N(h)$ is an 'almost periodic' dense function and h and g are continuous, scalar functions satisfying a set of four assumptions (see [20] for more details). Martensson's result reads: 'Assume that l is known so that there exists a fixed stabilizing controller of the form (5.5), and that the augmentation to the form (5.6) has been done. Then the controller (5.7) will stabilize the system in the sense that

$$(x(t),\ z(t),\ k(t)) \to (0,\ 0,\ k_\infty) \quad as \quad t \to \infty \tag{5.8}$$

where $k_\infty < \infty$'.

One such set of functions given by Martensson is

$$h(k) = (\log k)^{1/2},\ k \geq 1,\ g(h) = (\sin h^{1/2} + 1)h^{1/2} \tag{5.9}$$

Martensson's method is based on a 'dense' search over the control parameter space, allows for no measurement noise, and guarantees only asymptotic stability rather than exponential stability. These weaknesses were overcome in [8] where a finite switching control method was proposed for LTI systems with uncertain parameters satisfying some mild compactness assumptions. Different modifications of Martensson's controller aimed at achieving Lyapunov stability, avoiding dense search procedures, as well as extending this approach to discrete-time systems have been reported recently (see, e.g., [2, 8, 19, 23]). However, the lack of exponential stability might result in poor transient performance as pointed out by many researchers (see, e.g., [8,19] for simulation examples). Below we present a simple example of a controller based on a dense search over the parameter space. This controller is a simplified version of that presented in [19].

Example 1.1 The second order plant

$$x(t+1) = a_1 x(t) + a_2 x(t-1) + bu(t) + \xi(t), \quad x, u \in \mathbf{R} \tag{5.10}$$

with $a_{1,2} \in \mathbf{R},\ b \neq 0$ being arbitrary unknown constants and $\sup_{t \geq t_0} |\xi(t)| < \infty$ has to be controlled by the switching controller

$$u(t) = k(t)x(t) \tag{5.11}$$

where $k(0) = h(1)$ and $k(t) = h(i)$, $t \in (t_i,\ t_{i+1}]$ and $h(i)$ is a function dense in $\mathbf{R}$ defined so that it successively looks at each interval $[-p,\ p],\ p \in \mathbf{N}$ and tries points $1/2^p$ apart, namely,

$$\begin{array}{lll} h(1) = 1 & h(4) = -0.5 & h(7) = 1.75 \\ h(2) = 0.5 & h(5) = -1 & \text{etc.} \\ h(3) = 0 & h(6) = 2 & \end{array}$$

The system performance is monitored using a function

$$\Gamma(t) = M(t_{i-1})\beta(t_{i-1})^{(t-t_{i-1})}|x(t_{i-1})| + \nu(t_{i-1}) \tag{5.12}$$

For each $i > 1$ such that $t_{i-1} \neq \infty$, the switching instant is defined as

$$t_i = \begin{cases} \min\{t : t > t_{i-1}, |x(t)| > M(t_{i-1})\beta(t_{i-1})^{(t-t_i)}|x(t_{i-1})| + \nu(t_{i-1})\} & \text{if this exists} \\ \infty \quad \text{otherwise} \end{cases} \tag{5.13}$$

where $0 < M(t),\ 0 < \beta(t) < 1$ and $0 < \nu(t)$ are strictly positive increasing functions satisfying the following conditions $\lim_{t\to\infty} M(t) = +\infty$,

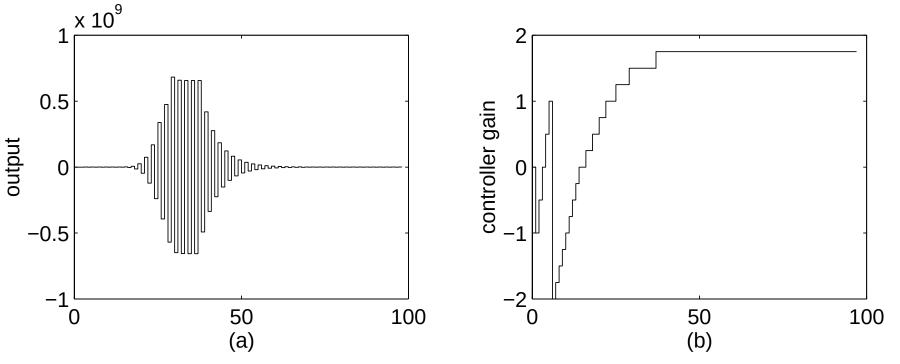

Figure 5.1 *Example of a dense search*

$\lim_{t\to\infty}\beta(t) = 1$, $\lim_{t\to\infty}\nu(t) = +\infty$. The behaviour of the closed loop system with $a_1 = -2.2$, $a_2 = 0.3$ and $b = 1$ is illustrated in Figure 5.1(a)–(b).

A different switching control approach, called hysteresis switching, was reported in a number of papers [22, 27, 37] in the context of adaptive control. In these papers, the hysteresis switching is used to swap between a number of 'standard' adaptive controllers operating in regimes of the parameter space. The switching, in these cases, is used to avoid the 'stabilizability' problem in adaptive controllers.

Conventional switching control techniques are all based on some mechanism of an exhaustive search over the entire set of potential controllers (either a continuum set [20] or a finite set [8]). A major drawback is that the search may converge very slowly, resulting in excessive transients which renders the system 'unstable' in a practical sense. This phenomenon can take place even if the closed loop system is exponentially stable. To alleviate this problem, several new switching control schemes have been proposed recently. The so-called supervisory control of LTI systems for adaptive set-point tracking is proposed by Morse [25, 26] to improve the transient response. A further extension of Morse's approach is given in [13]. A very similar, in spirit, supervisory control scheme for model reference adaptive control is analysed in [29]. The main idea of supervisory control is to orchestrate the process of switching into feedback controllers from a pre-computed finite (continuum) set of fixed controllers based on certain on-line estimation. This represents a significant departure from traditional estimator based tuning algorithms which usually employ recursive or dynamic parameter tuning schemes. This approach has apparently significantly improved the quality of regulation, thus demonstrating that switching control if properly performed is no longer just a nice theoretical toy but a powerful tool for high performance control systems design. However, several issues still remain unresolved. For example:

(i) a finite convergence of switching is not guaranteed. This aspect is especially important in situations when convergence of switching is achievable. It seems intuitively that in adaptive control of a linear time-invariant system it is desirable that the adaptive controller 'converges' to a linear time-invariant controller;
(ii) the analysis of the closed loop stability is quite complicated and often dependent on the system architecture. Without a simpler proof and better understanding of the 'hidden' mechanisms of supervisory switching control its design will remain primarily a matter of trial and error.

In this chapter, we present a new approach to switching adaptive control for uncertain discrete-time systems. This approach is based on a localization method, and is conceptually different from the supervisory control schemes and other switching schemes. The localization method was initially proposed by the authors for LTI systems [39]. This method has the unique feature of fast convergence for switching. That is, it can localize a suitable stabilizing controller very quickly, hence the name of localization. Later this method was extended to LTV plants in [40]. By utilizing the high speed of localization and the rate of admissible parameter variations exponential stability of the closed loop system was proved. The main contribution of this chapter is a unified description of the method of localization. We show that this method is also easy to implement, has no bursting phenomenon, and can be modified to work with or without a known bound on the exogenous disturbance.

To highlight the principal differences between the proposed framework and existing switching control schemes, in particular, supervisory switching control, we outline potential advantages of localization based switching control:

(i) The switching controller is finitely convergent provided that the system is time invariant. Depending on how the switching controller is practically implemented the absence of this property could potentially have far reaching implications.
(ii) Unlike conventional switching control based on an exhaustive search over the parameter space, the switching converges rapidly thus guaranteeing a high quality of regulation.
(iii) The closed loop stability analysis is comparatively simple even in the case of linear time-varying plants. This is in sharp contrast to supervisory switching control where the stability analysis is quite complicated and depends on the system architecture.
(iv) Localization based switching control is directly applicable to both linear time-invariant and time-varying systems.
(v) The localization technique provides a clear understanding of the control mechanism which is important in applications.

The rest of this chapter is organized as follows. Section 5.2 introduces the class of LTI systems to be controlled and states the switching adaptive stabilization problem. Two different localization principles are studied in Sections 5.3 and 5.4. We also study a problem of optimal localization, which allows us to obtain guaranteed lower bounds on the number of controllers discarded at each switching instant and adaptive stabilization in the presence of unknown exogenous disturbance. Simulation examples are given in Section 5.5 to demonstrate the fast switching capability of the localization method. Conclusions are reached in Section 5.6.

5.2 Problem statement

We consider a general class of LTI discrete-time plants in the following form:

$$D(z^{-1})y(t) = N(z^{-1})u(t) + \xi(t-1) + \eta(t-1) \tag{5.14}$$

where $u(t)$ is the input, $y(t)$ is the output, $\xi(t)$ is the exogenous disturbance, $\eta(t)$ represents the unmodelled dynamics (to be specified later), z^{-1} is the unit delay operator:

$$N(z^{-1}) = n_1 z^{-1} + n_2 z^{-2} + \ldots + n_n z^{-n} \tag{5.15}$$

$$D(z^{-1}) = 1 + d_1 z^{-1} + \ldots + d_n z^{-n} \tag{5.16}$$

Remark 2.1 By using simple algebraic manipulations, measurement noise and input disturbance are easily incorporated into the model (5.14). In this case, $y(t)$, $u(t)$, and $\xi(t)$ represent the measured output, computed input and (generalized) exogenous disturbance, respectively. For example, if a linear time-invariant discrete-time plant is described by

$$y(z) = \frac{N(z^{-1})}{D(z^{-1})}(u(z) + d(z)) + q(z)$$

where $d(z)$ and $q(z)$ are the input disturbance and plant noise, respectively, the plant can be rewritten as

$$D(z^{-1})y(z) = N(z^{-1})u(z) + (N(z^{-1})d(z) + D(z^{-1})q(z^{-1}))$$

Consequently, the exogenous input $\xi(z)$ is $N(z^{-1})d(z) + D(z^{-1})q(z^{-1})$.

We will denote by θ the vector of unknown parameters, i.e.

$$\theta = (n_n, \ldots, n_2, -d_n, \ldots, -d_1, n_1)^T \tag{5.17}$$

Throughout the chapter, we will use the following nonminimal state-space description of the plant (5.14):

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + E(\xi(t) + \eta(t)) \tag{5.18}$$

where

$$x(t) = [u(t-n+1)\cdots u(t-1) \mid y(t-n+1)\cdots y(t))]^T \tag{5.19}$$

and the matrices $A(\theta)$, $B(\theta)$ and E are constructed in a standard way

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ n_n & n_{n-1} & \cdots & n_2 & -d_n & \cdots & -d_2 & -d_1 \end{bmatrix} \tag{5.20}$$

$$B(\theta) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \text{---} \\ 0 \\ \vdots \\ 0 \\ n_1 \end{bmatrix}; \quad E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \text{---} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5.21}$$

We also define the regressor vector

$$\phi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{5.22}$$

Then, (5.14) can be rewritten as

$$y(t) = \theta^T \phi(t-1) + \xi(t-1) + \eta(t-1) \tag{5.23}$$

The following assumptions are used throughout this section:

(A1) The order n of the nominal plant (excluding the unmodelled dynamics) is known.

(A2) A compact set $\Omega \in \mathbf{R}^{2n}$, is known such that $\theta \in \Omega$.

(A3) The plant (5.14) without unmodelled dynamics (i.e. $\eta(t) \equiv 0$) is stabilizable over Ω. That is, for any $\theta \in \Omega$, there exists a linear time-invariant controller $C(z^{-1})$ such that the closed loop system is exponentially stable.

(A4) The exogenous disturbance ξ is uniformly bounded, i.e. for all $t_0 \in \mathbf{N}$

$$\sup_{t \geq t_0} |\xi(t)| \leq \bar{\xi} \tag{5.24}$$

for some known constant $\bar{\xi}$.

(A5) The unmodelled dynamics is arbitrary subject to

$$|\eta(t)| \leq \bar{\eta}(t) = \varepsilon \sup_{0 \leq k \leq t} \sigma^{t-k} \|x(k)\| \tag{5.25}$$

for some constants $\varepsilon > 0$ and $0 \leq \sigma < 1$ which represent the 'size' and 'decay rate' of the unmodelled dynamics, respectively.

Remark 2.2 Assumption (A1) can be relaxed so that only an upper bound $n_{\max}$ is known. Assumption (A4) will be used in Sections 5.3–5.4 and will be relaxed to allow $\bar{\xi}$ to be unknown in Sections 5.3.2 and 5.4.1 where an estimation scheme is given for $\bar{\xi}$.

Remark 2.3 We note that the assumptions outlined above are quite standard and have been used in adaptive control to derive stability results for systems with unmodelled dynamics (see, e.g., [7, 16, 21, 30] for more details).

The switching controller to be designed will be of the following form:

$$u(t) = K_{i(t)} x(t) \tag{5.26}$$

where $K_{i(t)}$ is the control gain applied at time t, and $i(t)$ is the switching index at time t, taking value in a finite index set I. The objective of the control design is to determine the set of control gains

$$K_I = \{K_i, i \in I\} \tag{5.27}$$

and an on-line switching algorithm for $i(t)$ so that the closed loop system will be 'stable' in some sense.

We note that switching controllers can be classified according to the logic governing the process of switching. Here are some typical examples.

1. Conventional switching control

The switching index is defined as

$$i(t) = \begin{cases} i(t-1) & \text{if } \mathbf{G}_t \leq 0 \\ i(t-1) + 1 & \text{otherwise} \end{cases} \tag{5.28}$$

where $\mathbf{G}_t$ is some appropriately chosen performance index. This type of switching control is finitely convergent and based on an exhaustive search over the parameter space (see, e.g., [8], [9]).

2 Supervisory switching control
The switching index is defined as

$$i(t) = \begin{cases} i(t-1) & \text{if } t - s(t) < t_d \\ \arg\min_{i \in I} |e_i(t)| & \text{otherwise} \end{cases} \tag{5.29}$$

where $s(t)$ is the time of the most recent switching, t_d is a positive dwell time, and $e_i(t)$, $\forall i \in I$ is a weighted prediction error computed for the ith nominal system. This type of switching control has been extensively studied recently by a number of researchers (see, e.g., [25, 26]). The proof of the closed loop stability in this case is not dependent on finite convergence of the switching process, furthermore, supervisory switching control is not finitely convergent in general.

5.3 Direct localization principle

The switching algorithms to be used in this section are based on a *localization* technique. This technique, originally used in [39] for LTI plants, allows us to falsify incorrect controllers very rapidly while guaranteeing exponential stability of the closed loop system. In this section, we describe a direct localization principle (see, e.g., [40]) for LTI plants which is slightly different from [39] but is readily extended to LTV plants. The main idea behind this principle consists of simultaneous falsification of potentially stabilizing controllers based explicitly on the model of the controlled plant. That implies the use of some effective mechanism of discarding controllers inconsistent with the measurements.

The specific notion of stability to be used in this section is described below.

Definition 3.1 The system (5.14) satisfying (A1)–(A5) is said to be globally $\bar{\xi}$-exponentially stabilized by the controller (5.26) if there exist constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : R_+ \to R_+$ with $M_2(0) = 0$ such that

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi}) \tag{5.30}$$

holds for all $t_0 \geq 0$, $x(t_0)$, $\bar{\xi} \geq 0$, and $\xi(\cdot)$ and $\eta(\cdot)$ satisfying (A4)–(A5), respectively.

The definition above yields exponential stability of the closed loop system provided that $\bar{\xi} = 0$ and exponential attraction of the states to an origin centred ball whose radius is related to the magnitude of the exogenous disturbance.

First, we decompose the parameter set Ω to obtain a *finite cover* $\{\Omega_i\}_{i=1}^{L}$ which satisfies the following conditions:

(C1) $\Omega_i \subset \Omega$, $\Omega_i \neq \{\ \}$, $i = 1, \ldots, L$.
(C2) $\bigcup_{i=1}^{L} \Omega_i = \Omega$.
(C3) For each $i = 1, \ldots, L$, let θ_i and $r_i > 0$ denote the 'centre' and 'radius' of Ω_i, i.e. $\theta_i \in \Omega_i$ and $||\theta - \theta_i|| \leq r_i$ for all $\theta \in \Omega_i$. Then, there exist K_i, $i = 1, \ldots, L$, such that

$$|\lambda_{\max}(A(\theta) + B(\theta)K_i)| < 1, \quad \forall \ \|\theta - \theta_i\| \leq r_i, \quad i = 1, \ldots, L. \tag{5.31}$$

Conditions (C1)–(C2) basically say that the uncertainty set Ω is presented as a finite union of non-empty subsets while condition (C3) defines each subset Ω_i as being stabilizable by a single LTI controller K_i. It is well known that such a finite cover can be found under assumptions (A1)–(A3) (see, e.g., [8, 24, 25] for technical details and examples). More specifically, there exist (sufficiently large) L, (sufficiently small) r_i, and suitable K_i, $i = 1, \ldots, L$, such that (C1)–(C3) hold. Leaving apart the computational aspects of decomposing the uncertainty set satisfying conditions (C1)–(C3) we just note that decomposition can be conducted off-line, moreover, some additional technical assumptions (see, e.g., (C3′) below) make the process of decomposing pretty trivial. The computational complexity of decomposing the uncertainty set, in general, depends on many factors including the 'size' of the set, its dimension and 'stabilizability' properties, and has to be evaluated on a case-by-case basis.

The key observation used in the localization technique is the following fact: given any parameter vector $\theta \in \Omega_j$ and a control gain $K_{i(t)}$ for some $i(t), j = 1, \ldots, L$. If $i(t) = j$, then it follows from

$$y(t) = \theta^T \phi(t-1) + \xi(t-1) + \eta(t-1) \tag{5.32}$$

that

$$|\theta_j^T \phi(t-1) - y(t)| \leq r_j ||\phi(t-1)|| + \bar{\xi} + \bar{\eta}(t-1) \tag{5.33}$$

This observation leads to a simple localization scheme by elimination: If the above inequality is violated at any time instant, we know that the switching index $i(t)$ is wrong (i.e. $i(t) \neq j$), so it can be eliminated. In identification theory this concept is sometimes referred to as falsification; see, e.g., a survey [15] and references therein. The unique feature of the localization technique comes from the fact that violation of (5.33) allows us not only to eliminate $i(t)$ from the set of possible controller indices, but many others. This is the key point! As a result, a correct controller can be found very quickly.

We now describe the localization algorithm. Let $I(t)$ denote the set of 'admissible' control gain indices at time t and initialize it to be

$$I(t_0) = \{1, 2, \ldots, L\} \tag{5.34}$$

Choose any initial switching index $i(t_0) \in I(t_0)$. For $t > t_0$, define

$$\hat{I}(t) = \{j : (5.33) \text{ holds}, \quad j = 1, \ldots, L\} \tag{5.35}$$

Then, the localization algorithm is simply given by

$$I(t) = I(t-1) \cap \hat{I}(t), \quad \forall t > t_0 \tag{5.36}$$

The switching index is updated by taking[1]

$$i(t) = \begin{cases} i(t-1) & \text{if } t > t_0 \text{ and } i(t-1) \in I(t) \\ \text{any member of } I(t) & \text{otherwise} \end{cases} \tag{5.37}$$

A simple geometrical interpretation of the localization algorithm (5.36) is given in Figure 5.2. One possible way to view the localization technique is to interpret it as family set identification of a special type, that is, family set identification conducted on a finite set of elements. Interpreted in this way the localization technique represents a significant departure from traditional family set identification ideas. Either strip depicted in Figure 5.2 contains only those elements which are consistent with the measurement of the input/output pair $\{y(t), u(t-1)\}$. The high falsifying capability of the proposed algorithm observed in simulations can informally be explained in the following way. Let the index $i(t)$ be falsified, then the discrete set of elements $\{\theta_i : i \in I(t)\}$ consistent with all the past measurements is separated from the point $\theta_{i(t)}$ by one of the hyperplanes

$$\theta_j^T \phi(t-1) = y(t) + r_j||\phi(t-1)|| + \bar{\xi} + \bar{\eta}(t-1) \tag{5.38}$$

or

$$\theta_j^T \phi(t-1) = y(t) - r_j||\phi(t-1)|| - \bar{\xi} - \bar{\eta}(t-1) \tag{5.39}$$

dividing the parameter space into two half-spaces. It is also clear that every element belonging to the half-space containing the point $\theta_{i(t)}$ is falsified by the algorithm of localization (5.36) at the switching instant t. We note that the rigorous analysis of the problem of optimal localization conducted in Section 5.3.1 allows us to derive a guaranteed lower bound on the number of controllers falsified at an arbitrary switching instant. A different non-identification based interpretation of localization can be given in terms of the prediction errors $e_j = |\theta_j^T \phi(t-1) - y(t)|$, $j = 1, 2, \dots L$ computed for the entire set of 'nominal' models. Thus, any model giving a large prediction error is falsified. The following technical lemma describes the main properties of the algorithm of localization (5.36).

Lemma 3.1 Given the uncertain system (5.14) satisfying assumptions (A1)–(A5), suppose the finite cover $\{\Omega_i\}_{i=1}^L$ of Ω satisfies conditions (C1)–(C3). Then, the localization algorithm given in (5.34)–(5.37) applied to an LTI plant (5.14) possesses the following properties:

(i) $I(t) \neq \{\ \}, \quad \forall t \geq t_0$.

[1] In fact, we will see in Section 5.3.1 that there may be 'clever' ways of selecting $i(t)$ when $i(t-1)$ is falsified.

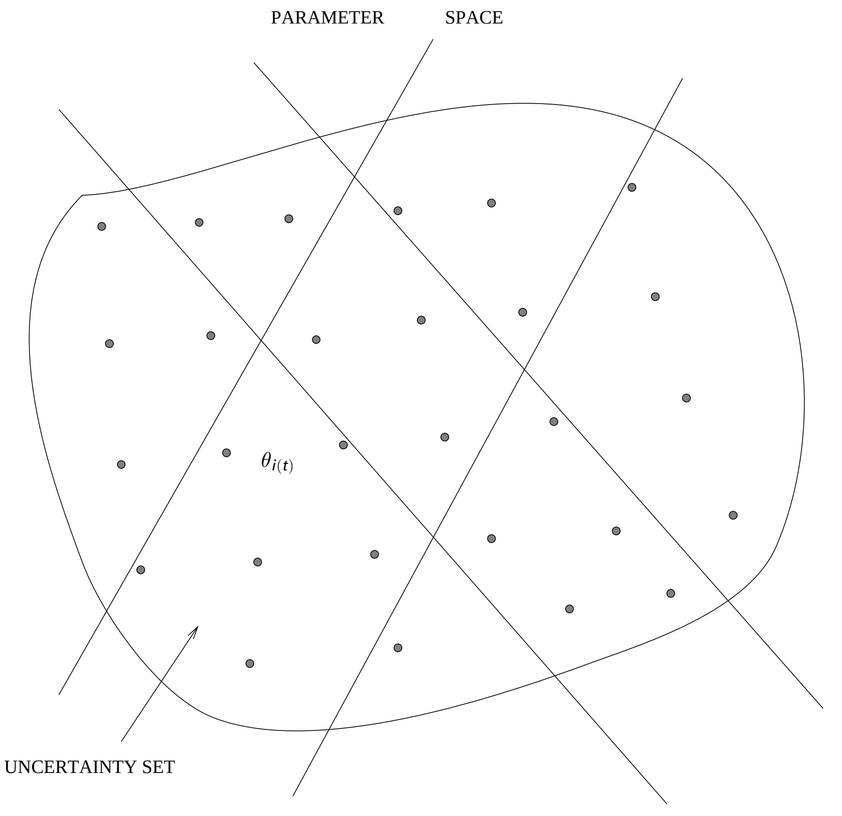

Figure 5.2 *Localization*

(ii) There exists a switching index $j \in I(t)$ for all $t \geq t_0$ such that the closed loop system with $u(t) = K_j x(t)$ is globally exponentially stable.

Proof The proof is trivial: suppose the parameter vector θ for the true plant is in Ω_j for some $j \in \{1, \ldots, L\}$. Then, the localization algorithm guarantees that $j \in \hat{I}(t)$ for all t. Hence, both (i) and (ii) hold.

To guarantee exponential stability of the closed loop system, we need a further property of the finite cover of Ω. To explain this, we first introduce the notion of quadratic stability [3].

Definition 3.2 A given set of matrices $\{A(\theta) : \theta \in \Omega\}$ is called quadratically stable if there exist symmetric positive-definite matrices H, Q such that

$$A^T(\theta) H A(\theta) - H \leq -Q, \quad \forall\, \theta \in \Omega \tag{5.40}$$

It is obvious that the finite cover $\{\Omega_i\}_{i=1}^{L}$ of Ω can always be made such that each Ω_i is 'small' enough for the corresponding family of the 'closed-loop' matrices $\{A(\theta) + B(\theta)K_i : \theta \in \Omega_i\}$ to be quadratically stable with some K_i.

In view of the observation above, we replace the condition (C3) with the following:

(C3′) For each $i = 1, \ldots, L$, let θ_i and $r_i > 0$ denote the 'centre' and 'radius' of Ω_i, i.e. $\theta_i \in \Omega_i$ and $||\theta - \theta_i|| \leq r_i$ for all $\theta \in \Omega_i$. Then, there exist control gain matrices K_i, symmetric positive-definite matrices H_i and Q_i, $i = 1, \ldots, L$, and a positive number q such that

$$(A(\theta) + B(\theta)K_i)^T H_i (A(\theta) + B(\theta)K_i) - H_i \leq -Q_i, \quad \forall\ \|\theta - \theta_i\| \leq (r_i + q), \quad i = 1, \ldots, L \tag{5.41}$$

Remark 3.1 Condition (C3′) requires that every subset Ω_i obtained as a result of decomposition be quadratically stabilized by a single LTI controller. We also note that a finite cover which satisfies (C1)–(C2) and (C3′) is guaranteed to exist. Moreover, Condition (C3′) translated as one requiring the existence of a common quadratic Lyapunov function for any subset Ω_i further facilitates the process of decomposition.

The following theorem contains the main result for the LTI case:

Theorem 3.1 Given an LTI plant (5.14) satisfying assumptions (A1)–(A5). Let $\{\Omega_i\}_{i=1}^{L}$ be a finite cover of Ω satisfying conditions (C1)–(C2) and (C3′). Then, the localization algorithm given in (5.34)–(5.37) will guarantee the following properties when ε (i.e. the 'size' of unmodelled dynamics) is sufficiently small:

(i) The closed loop system is globally $\bar{\xi}$-exponentially stable, i.e., there exist constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$, $M_2(0) = 0$ such that

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi}) \tag{5.42}$$

holds for all $t \geq t_0$ and $x(t_0)$.

(ii) The switching sequence $\{i(t_0), i(t_0 + 1), \ldots\}$ is finitely convergent, i.e. $i(t) = \text{const}$, $\forall\ t \geq t^*$ for some t^*.

Proof See Appendix A.

The proof of the theorem presented in Appendix A is based on the observation that between any two consecutive switchings the closed loop system behaves as an exponentially stable LTI system subject to small parametric perturbations and bounded exogenous disturbance. This is the key point offering a clear understanding of the control mechanisms.

It follows from the proof of Theorem 3.1 that the constant M_1 in the bound (5.42) is proportional to the total number of switchings made by the controller while the parameter ρ is dependent on the 'stabilizability' property of the uncertainty set Ω. This further emphasizes the importance of fast switching capabilities of the controller for achieving good transient performance.

5.3.1 Optimal localization

The localization scheme described above allows an arbitrary new switching index in $I(t)$ to be used when a switching occurs. That is, when the previous switching index $i(t-1)$ is eliminated from the current index set $I(t)$, any member of $I(t)$ can be used for $i(t)$. The problem of optimal localization addresses the issue of optimal selection of the new switching index at each switching instant so that the set of admissible switching indices $I(t)$ is guaranteed to be pruned down as rapidly as possible. The problem of optimal localization is solved in this section in terms of the indices of localization defined below. In the following for notational convenience we drop when possible the index t from the description of the set of indices $I(t)$. Also we make the technical assumption

(A7) $r_i = r_j = r, \ \forall i,j = 1,2,\ldots,L,$ and some $r > 0$.

For any set $I \subset \{1,2,\ldots,L\}$, $\Theta = \{\theta_i : i \in I\}$, a fixed $j \in I$ and any $z \neq 0$, $z \in \mathbf{R}^{2n}$, define the function

$$\lambda(z,j,\Theta) = |\{\theta_i : (\theta_i - \theta_j)^T z \geq 0, \quad i \in I\}| \tag{5.43}$$

where $|\cdot|$ denotes the cardinal number of a set. Then

$$\mathrm{ind}(\theta_j,\Theta) = \min_{||z||=1} \lambda(z,j,\Theta) \tag{5.44}$$

will be referred to as the index of localization of the element θ_j with respect to the set Θ.

Lemma 3.2 The index of localization $\mathrm{ind}(\theta_j,\Theta)$ represents a guaranteed lower bound on the number of indices discarded from the localization set $I(t)$ at the next switching instant provided that $u(t) = K_j x(t)$.

Proof Without loss of generality we assume that $(t+1)$ is the next switching instant, and controller K_j is discarded. From (5.35) we have $j \notin \hat{I}(t+1)$, equivalently

$$\theta_j^T \phi(t) > y(t+1) + (r_j + q)||\phi(t)|| + \bar{\xi} + \bar{\eta}(t+1) \tag{5.45}$$

or

$$\theta_j^T \phi(t) < y(t+1) - (r_j + q)||\phi(t)|| - \bar{\xi} - \bar{\eta}(t+1) \tag{5.46}$$

Taking $z = -\phi(t)/||\phi(t)||$ for (5.45), or $z = \phi(t)/||\phi(t)||$ for (5.46) and using

(5.43) we see that there are $\lambda(z, j, \Theta)$ number of controller indices which do not belong to $\hat{I}(t+1)$. We note that $\phi(t) \neq 0$, because otherwise it is easy to see from (5.23) that there exists no element $\theta_j \in \Theta$ satisfying (5.45) or (5.46), and, consequently, switching is not possible. Since $\text{ind}(\theta_j, \Theta) \leq \lambda(z, j, \Theta)$, we conclude that there are at least $\text{ind}(\theta_j, \Theta)$ number of controllers to be discarded at the switching instant $(t+1)$.

In terms of (5.44) the index of localization of the discrete set Θ is defined as

$$\text{ind}\,\Theta = \max_j \{\text{ind}(\theta_j, \Theta) : \; j \in I\} \tag{5.47}$$

That is, $\text{ind}\,\Theta$, is the largest attainable lower bound on the number of controllers eliminated at the time of switching, assuming that the regressor vector can take any value. The structure of an optimal switching controller is described by

$$u(t) = K_{i(t)} x(t) \tag{5.48}$$

$$i(t) = \begin{cases} i(t-1) & \text{if } i(t-1) \in I(t); \\ i_{\text{opt}}(t) = \;\text{arg max}_j \{\text{ind}(\theta_j, \Theta(t)) : j \in I(t)\} & \text{otherwise} \end{cases} \tag{5.49}$$

The problem of optimal localization reduces to determining the optimal control law, that is, specifying the switching index $i_{opt}(t)$ at each time instant when switching has to be made. To solve this problem we introduce the notion of *separable sets*.

Definition 3.3 Given a finite set $\Theta \subset \mathbf{R}^n$ and a subset $J \subset \Theta$; J is called a separable set of order k if

(i) $|J| = k$.
(ii) co $\{J\} \bigcap$ co $\{\Theta - J\} = \{\}$ where co $\{\cdot\}$ stands for the convex hull of a set.

The main properties of separable sets are listed below:

(a) A vertex of co $\{\Theta\}$ is a separable set of order 1.
(b) The order of a separable set $k \leq |\Theta|$.
(c) For each separable set J of order k, $k > 1$, there exists a set $J' \subset J$ such that J' is a separable set of order $(k-1)$.

Proof (a), (b) are obvious. To prove (c), we note that for each separable set J, there exists a hyperplane $\mathbf{P}$ separating J and $\Theta - J$. Let $\vec{n}$ be the normal direction of $\mathbf{P}$. Move $\mathbf{P}$ along $\vec{n}$ towards J until it hits J. Two cases are possible.

Case 1: One vertex is in contact. In this case move $\mathbf{P}$ a bit further to pass the vertex. The remaining points in J form J'.

Case 2: Multiple vertices are in contact. One can always change $\vec{n}$ slightly so that $\mathbf{P}$ still separates J and $\Theta - J$, but there is only one vertex in contact with $\mathbf{P}$, and we are back to Case 1.

Lemma 3.3 Let $\mathbf{\Theta}^k$ be the set of all separable sets of order k and $\mathbf{\Xi}^k = \bigcup_{J_k \in \mathbf{\Theta}^k} J_k$. Then,

$$\text{ind}\ \Theta = 1 + \arg\ \max_k \{k : \mathbf{\Xi}^k \neq \Theta\} \tag{5.50}$$

Proof Follows immediately from Definition 3.3 and the property of separable sets (c). Indeed, suppose that the index of localization satisfies the relation

$$\text{ind}\ \Theta \ = m > 1 + \arg\ \max_k \{k : \mathbf{\Xi}^k \neq \Theta\} \tag{5.51}$$

then there must exist an element $\theta_j \in \Theta$, such that $\text{ind}(\theta_j, \Theta) = m$, moreover

$$\theta_j \notin \mathbf{\Xi}^{m-1}, \quad \theta_j \in \Theta - \mathbf{\Xi}^{m-1} \tag{5.52}$$

since otherwise, by definition of separable sets $\text{ind}(\theta_j, \Theta) \leq m - 1$. But it follows from (5.51) that $\Theta - \mathbf{\Xi}^{m-1} = \{\}$. On the other hand by Definition 3.3 and the properties of separable sets (b), (c) the index of localization of the set Θ cannot be smaller than that given by (5.50). This concludes the proof.

Denote by $V(\cdot)$ the set of vertices of co $(\cdot)$. The complete solution to the problem is given by the following iterative algorithm.

Algorithm A

Step 1 Initialize $k = 1$. Compute $\mathbf{\Theta}^1 = \{\{\theta\} : \theta \in V(\Theta)\}$.

Step 2 Set $k = k + 1$. Compute

$$\mathbf{\Theta}^k = \{J_{k-1} \cup \theta_i : J_{k-1} \subset \mathbf{\Theta}^{k-1}, \quad \theta_i \in V(\Theta - J_{k-1}), \quad J_{k-1} \cup \theta_i \text{ is separable}\}$$

Step 3 If $\mathbf{\Xi}^k = \Theta$, then ind $\Theta = k$, and stop, otherwise go to Step 2.

The properties of localization based switching control are summarized in the following theorem. Let sub$\{\cdot\}$ denote the set of subscripts of all the elements in $\{\cdot\}$.

Theorem 3.2

(i) The solution to the problem of optimal localization may not be unique and is given by the set

$$\mathbf{I}_{\text{opt}} = \text{sub}\{\Theta - \mathbf{\Xi}^{m-1}\} \tag{5.53}$$

where

$$m = \text{ind}\ \Theta = 1 + \arg\ \max_k \{k : \mathbf{\Xi}^k \neq \Theta\} \tag{5.54}$$

(ii) For any $\bar{\xi} \geq 0$, $\varepsilon \geq 0$, the total number of switchings l made by the optimal switching controller (5.48), (5.49) applied to an LTI plant (5.14) satisfies the relation

$$\sum_{p=0}^{l-1} \text{ind } \Theta(t_p) - 2 \leqslant L - 1 \tag{5.55}$$

where $t_p,\ p = 0, 1, \ldots, l-1$ denote the switching instants.

Proof The proof of (i) follows directly from Lemma 3.3. To prove (ii) we note that

$$|\Theta(t_1)| \leq L - \text{ind } \Theta(t_0);$$

$$|\Theta(t_2)| \leq |\Theta(t_1)| - \text{ind } \Theta(t_1) \leq L - \text{ind } \Theta(t_0) - \text{ind } \Theta(t_1)$$

$$\ldots$$

then

$$|\Theta(t_l)| \leq \nu_l = L - \sum_{i=0}^{l-1} \text{ind } \Theta(t_i)$$

Since $\nu_l \geq 1$ the result follows.

Algorithm A applied to an arbitrary localization set Θ indicates that except for a very special case, namely, $\{\theta_j\}_{j \in I} = V(\Theta)$, localization with any choice of the switching index $i(t)$ such that $\theta_{i(t)} \notin V(\Theta)$ will always result in elimination of more than one controller at any switching instant. This is a remarkable feature distinguishing localization based switching controllers from conventional switching controllers. Moreover, a simple geometrical analysis (see, e.g., Figure 5.2) indicates that for 'nicely' shaped uncertainty sets (for example, a convex Ω) and large L the index of localization is typically large, that is, $\text{ind}(\Theta) >> 1$. Theorem 3.2 gives a complete theoretical solution to the problem of optimal localization formulated above in terms of indices of localization. However, it must be pointed out that the search for optimality in general is involved and may be computationally demanding. To alleviate potential computational difficulties we propose one possible way of constructing a suboptimal switching controller.

Algorithm B

Step 1 Initialize $k = 1$. Compute $\mathbf{\Gamma}^1 = V(\Theta)$.

Step 2 Set $k = k + 1$. Compute

$$\mathbf{\Gamma}^k = \mathbf{\Gamma}^{k-1} \cup V(\Theta - \mathbf{\Gamma}^{k-1})$$

Step 3 If $\mathbf{\Gamma}^k = \Theta$, then ind $\Theta \geq k$, and stop, otherwise go to Step 2.

Algorithm B allows for a simple geometrical interpretation, namely, at each step a new set $\mathbf{\Gamma}^k$ is obtained recursively by adding the set of vertices of

$(\Theta - \mathbf{\Gamma}^{k-1})$. The simplicity of the proposed algorithm is explained by the fact that we no longer need to check the property of separability (see Step 2 in Algorithm A).

The main property of the Algorithm B is presented in the following proposition.

Proposition 3.3 The index of localization ind Θ satisfies the inequality

$$\text{ind } \Theta \geq 1 + \arg \max_k \{k : \mathbf{\Gamma}^k \neq \Theta\} \tag{5.56}$$

Proof The proof is very simple and follows from the fact that for any $\theta \in \Theta$, such that $\theta \notin V(\Theta)$ it is true that $\text{ind}(\theta, \Theta) \geq 2$. By applying this rule recursively we obtain (5.56).

Example 3.1 To illustrate the idea of optimal (suboptimal) localization we consider a simple localization set $\Theta = \{\theta_j\}_{j=1}^5$ depicted in Figure 5.3.

We note that the point θ_5 is located exactly in the centre of the square $(\theta_1,\ \theta_2,\ \theta_4,\ \theta_3)$. Applying Algorithm A to the set Θ we obtain

$$\begin{aligned}
\mathbf{\Theta}^1 &= \{\{\theta_1\},\ \{\theta_2\},\ \{\theta_3\},\ \{\theta_4\}\},\\
\mathbf{\Theta}^2 &= \{\{\theta_1,\ \theta_2\}, \{\theta_1, \theta_3\}, \{\theta_2,\ \theta_4\}, \{\theta_3,\ \theta_4\}\},\\
\mathbf{\Theta}^3 &= \{\{\theta_1,\ \theta_2,\ \theta_5\}, \{\theta_1, \theta_3,\ \theta_5\}, \{\theta_3,\ \theta_4,\ \theta_5\}, \{\theta_2, \theta_4,\ \theta_5\}\}
\end{aligned}$$

Since $\cup_{J \in \mathbf{\Theta}^3} J = \Theta$ we conclude that ind $\Theta = 3$ and the optimal switching index is given by $i(t) = 5$. To compute a guaranteed lower bound on the index of

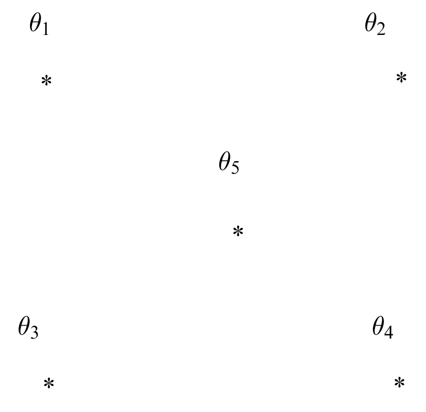

Figure 5.3 *Example of optimal localization*

localizaton ind Θ Algorithm B is used. We have

$$\mathbf{\Gamma}^1 = \{\theta_1,\ \theta_2,\ \theta_3,\ \theta_4\},$$

$$\mathbf{\Gamma}^2 = \{\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5\} = \Theta$$

therefore, ind $\Theta \geq 2$. We note that in this particular example the optimal solution, that is, $i(t) = 5$, coincides with the suboptimal one.

Remark 3.2 To deal with the problem of optimal (suboptimal) localization different simple heuristic procedures can be envisioned. For example, the following 'geometric mean' algorithm of computing a new switching index is likely to perform well in practice, though it is quite difficult in general to obtain any guaranteed lower bounds on the indices of localization. At any switching instant t we choose

$$i(t) = \arg \min_j \ ||\theta_j - \sum_{i \in I(t)} \theta_i / \Theta(t)|| \tag{5.57}$$

5.3.2 Localization in the presence of unknown disturbance bound

In this section we further relax assumption (A4) to allow the disturbance bound $\bar{\xi}$ to be unknown. That is, we replace (A4) with

(A4′) The exogenous disturbance ξ is uniformly bounded

$$\sup_{t \geq t_0} |\xi(t)| \leq \bar{\xi} \tag{5.58}$$

for some *unknown* constant $\bar{\xi}$.

We further relax assumptions (A1)–(A5) by allowing parameters to be slowly varying. To this end we introduce the following additional assumption

(A7) The uncertain parameters are allowed to have slow drifting described by

$$||\theta(t) - \theta(t-1)|| \leq \alpha, \quad \forall\ t > t_0 \tag{5.59}$$

for some constant $\alpha > 0$.

Following the results presented in previous sections, we introduce a generalized localization algorithm to tackle the new difficulty. The key feature of the algorithm is the use of an on-line estimate of $\bar{\xi}$. This estimate starts with a small (or zero) initial value, and is gradually increased when it is invalidated by the observations of the output. With the trade-off between a larger number of switchings and a higher complexity, the new localization algorithm guarantees qualitatively similar properties for the closed loop system as for the case of known disturbance bound.

Let $\bar{\xi}(t)$ be the estimate for $\bar{\xi}$ at time t. Define

$$\begin{aligned}\hat{I}(t,\bar{\xi}(t)) = \{j : |\theta_j^T\phi(t-1) - y(t)| \le (r_i + q)\|\phi(t-1)\| \\ + \bar{\xi}(t-1) + \bar{\eta}(t-1), \quad j = 1,\ldots,L\}\end{aligned} \tag{5.60}$$

That is, $\hat{I}(t,\bar{\xi}(t))$ is the index set of parameter subsets which cannot be falsified by any exogenous disturbance $\sup_{t\ge t_0}|\xi(t)| \le \bar{\xi}(t-1)$.

Denote the most recent switching instant by $s(t)$. We define $s(t)$ and $\bar{\xi}(t)$ as follows:

$$s(t_0) = t_0 \tag{5.61}$$

$$\bar{\xi}(t_0) = 0 \tag{5.62}$$

$$s(t) = \begin{cases} t & \text{if } \bigcap_{k=s(t-1)}^{t}\hat{I}(k,\bar{\xi}(k)) = \{\ \} \quad \text{and} \quad t - s(t) \ge t_d \\ s(t-1) & \text{otherwise}\end{cases} \tag{5.63}$$

$$\bar{\xi}(t) = \begin{cases} \bar{\xi}(t-1) + \delta(t)\mu & \text{if } \bigcap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)) = \{\ \} \quad \text{and} \quad t - s(t) < t_d \\ \bar{\xi}(t-1) & \text{otherwise}\end{cases} \tag{5.64}$$

where t_d is some positive integer representing a length of a moving time interval over which validation of a new estimate $\bar{\xi}(t)$ is conducted, μ is a small positive constant representing a steady state residual (to be clarified later), and $\delta(t)$ is an integer function defined as follows:

$$\delta(t) = \min\left\{\delta : \bigcap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)+\delta\mu) \ne \{\ \}, \delta \in \mathbf{N}\right\} \tag{5.65}$$

The main idea behind the estimation scheme presented above is as follows. At each time instant when the estimate $\bar{\xi}(t-1)$ is invalidated, that is, $\bigcap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)) = \{\}$ we determine the least possible value $\delta \in \mathbf{N}$ which guarantees that no exogenous disturbance $\sup_{t\ge t_0}|\xi(t)| \le (\bar{\xi}(t-1)+\delta\mu)$ would have caused the falsification of all the indices in the current localization set. This is done by recomputing the sequence of localization sets over the finite period of time $[s(t), t]$ whose length is bounded from above by t_d. Since the total number of switchings caused by the 'wrong' estimate $\bar{\xi}(t)$ is finite and for every sufficiently large interval of time the number of switchings due to slow parameter drifting can be made arbitrarily small by decreasing the rate of slow parameter drifting α it is always possible to choose a sufficiently large t_d which would guarantee global stability of the system.

The algorithm of localization is modified as follows:

$$I(t) = \cap_{k=s(t-1)}^{t}\hat{I}(k,\bar{\xi}(k)) \tag{5.66}$$

But the switching index $i(t)$ is still defined as in (5.37).

The key properties of the algorithm above are given as follows:

Theorem 3.4 For any constant $\mu > 0$, there exist a parameter drifting bound $\alpha > 0$, a 'size' of unmodelled dynamics $\varepsilon > 0$ (both sufficiently small), and an integer t_d (sufficiently large), such that the localization algorithm described above, when applied to the plant (5.14) with assumptions (A1)–(A3), (A4′) and (A7), possesses the following properties:

(1) $I(t) \neq \{\}$ for all $t \geq t_0$.
(2) $\sup_{t \geq t_0} \bar{\xi}(t) \leq \bar{\xi} + \mu$.

Subsequently, the following properties hold:

(3) The closed loop system is globally $(\bar{\xi} + \mu)$-exponentially stable, i.e. there exist constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ with $M_2(0) = 0$ such that

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi} + \mu) \tag{5.67}$$

holds for all $t \geq t_0$ and $x(t_0)$.

(4) The switching sequence $\{i(t_0), i(t_0 + 1), \ldots\}$ is finitely convergent, i.e. $i(t)$ =const, $\forall\ t \geq t^*$ for some t^* if the uncertain parameters are constant.

Proof See Appendix B.

We note that even though the value μ can be arbitrarily chosen, the estimate of the disturbance bound, $\bar{\xi}(t)$, can theoretically be larger than $\bar{\xi}$ by the margin μ. Consequently, the state is only guaranteed to converge to a residual set slightly larger than what is given in Theorem 3.4. Our simulation results indicate that $\bar{\xi}(t)$ very likely converges to a value substantially smaller than $\bar{\xi}$. Nevertheless, there are cases where $\bar{\xi}(t)$ exceeds $\bar{\xi}$. One possible solution is to reduce the value of μ. However, a small μ may imply a large number of potential switchings.

5.4 Indirect localization principle

The idea of indirect localization was first proposed in [39] and is based on the use of a specially constructed performance criterion as opposed to direct localization considered in the previous section. To this end the output of the plant is replaced by some auxiliary output observation which is subsequently used for the purpose of model falsification. The notion of 'stabilizing sets' introduced below is central in the proposed indirect localization scheme. We first define an auxiliary output, $z(t)$, as

$$z(t) = Cx(t), \quad C^T \in \mathbf{R}^{2n-1} \tag{5.68}$$

and the inclusion:

$$\mathcal{I}_t : |z(t)| \leq \Delta\|x(t-1)\| + c_0 \tag{5.69}$$

Definition 4.1 $\mathcal{I}_t$ is said to be a stabilizing inclusion of the system (5.18) if $\mathcal{I}_t$ being satisfied for all $t > t_0$ and boundedness of $\xi(t), (\xi(t) \in \ell_\infty)$, implies boundedness of the state, $x(t)$, and in particular, there exist α_0, β_0 and $\sigma \in (0, 1)$ such that:

$$\|x(t)\| \leq \alpha_0 \sigma^{t-t_0} \|x(t_0)\| + \beta_0 \|\xi_t\|_{\ell_\infty}$$

Remark 4.1 Note that the inclusion, $\mathcal{I}_t$ is transformed into a discrete-time sliding hyperplane [11] as $\Delta \to 0, c_0 \to 0$. In contrast with conventional discrete-time sliding mode control we explicitly define an admissible vicinity around the sliding hyperplane by specifying the values $\Delta \geq 0$ and $c_0 \geq 0$.

Definition 4.2 The uncertain system (5.18) is said to be globally (C, Δ)-stabilizable if

(1) $\mathcal{I}_t$ is a stabilizing inclusion of the system (5.18), and
(2) there exists a control, $u(t) = -Kx(t)$, such that after a finite time, $\mathcal{I}_t$ is satisfied.

We will show below that stabilizing sets can be effectively used in the process of localization. Before we proceed further we need some preliminary results. Assume for simplicity that $\eta(t) \equiv 0$. The case $\eta(t) \neq 0$ is analysed similarly, provided ε is sufficiently small.

Lemma 4.1 Let $\sup_{t \geq t_0} |\xi(t)| < \infty,\ CB > 0$. Then there exists a c_0 such that the system (5.18) is globally $(C, 0)$-stabilizable if and only if

$$|\lambda_{\max}(PA)| < 1 \tag{5.70}$$

where

$$P = \left(I - (CB)^{-1}BC\right) \tag{5.71}$$

Proof First, suppose that (5.70) is violated, that is

$$|\lambda_{\max}(PA)| \geq 1 \tag{5.72}$$

We now show that $\mathcal{I}_t$ is not a stabilizing inclusion for any $c_0 > 0$. To do this, we take $\xi(t) \equiv 0$, and $u(t) = -(CB)^{-1}CAx(t)$. With this control we note from (5.18) that $z(t) = 0$ for $t > 0$, and so for any $c_0 > 0, z(t)$ satisfies (5.69). The equation for the closed loop system takes the form

$$x(t+1) = Ax(t) + Bu(t) = PAx(t) \tag{5.73}$$

which is not exponentially stable. Therefore, (5.72) implies that there is no c_0 such that $\mathcal{I}_t$ is a stabilizing inclusion. We now establish the converse. Suppose

(5.70) is satisfied. Then we can rewrite (5.18) as:

$$\begin{aligned} x(t+1) &= PAx(t) + Bu(t) + \frac{1}{CB}(BCA)x(t) + E\xi(t) \\ &= PAx(t) + \frac{B}{CB}(z(t+1) - CE\xi(t)) + E\xi(t) \end{aligned} \tag{5.74}$$

From (5.74) it is clear that if z and ξ are bounded, then in view of (5.70), $x(t)$ is bounded. Therefore, $\mathcal{I}_t$ is a stabilizing inclusion for any c_0. Finally, we take the control

$$u(t) = -\frac{1}{CB}(CA)x(t) \tag{5.75}$$

which gives

$$z(t+1) = CE\xi(t) \tag{5.76}$$

Therefore, for $c_0 \geq |CE| \sup_t |\xi(t)|, \mathcal{I}_t$ is satisfied for all $t > 0$, and the proof is complete.

Remark 4.2 The control, (5.75), is a 'one step ahead' control on the auxiliary output, $z(t)$. It then follows that the stability condition (5.70), (5.71) is equivalent to the condition that $C(zI - A)^{-1}B$ be relative degree one, and minimum phase.

Remark 4.3 If the original plant transfer function, (5.14), is known to be minimum phase, and relative degree one then it suffices to take $C = E^T$, and the system is then c_0 stabilizable for any $c_0 \geq 0$.

If the original plant transfer function is non-minimum phase, then let:

$$C = [f_0, f_1 \ \dots f_{n-2}, g_0, g_1 \ \dots \ g_{n-1}] \tag{5.77}$$

The transfer function from $u(t)$, via (5.14) to $z(t)$ is then:

$$\begin{aligned} z(t) &= F(q)u(t) + G(q)y(t) \\ &= \left(\frac{D(q)F(q) + G(q)N(q)}{D(q)}\right)u(t) \end{aligned} \tag{5.78}$$

where $F(q) = \left(f_0 + f_1 q + \cdots + f_{n-2}q^{n-2}\right)$ and $G(q) = (g_0 + g_1 q + \ \dots + g_{n-1}q^{n-1})$.

Therefore, for a non minimum phase plant, knowledge of a C such that $\mathcal{I}_t$ is a stabilizing inclusion is equivalent to knowledge of a (possible improper) controller $\{u(t) = -G(q)/F(q)y(t)\}$ which stabilizes the system. Because we are dealing with discrete-time systems, it is not clear whether this corresponds to knowledge of a proper, stabilizing controller for the set.

Remark 4.4 Because of the robustness properties of exponentially stable linear time invariant systems, Lemma 4.1 can easily be generalized to include nonzero, but sufficiently small Δ.

Lemma 4.2 Any Ω which satisfies assumptions (A2) and (A3) has a finite decomposition into compact sets:

$$\Omega = \bigcup_{\ell=1}^{L} \Omega_\ell \tag{5.79}$$

such that for each ℓ, there exists a C_ℓ, Δ_ℓ and $c_{0,\ell}$ such that, for all $(A, B) \in \Omega^\ell, \mathcal{I}_t$ is a stabilizing inclusion, and $C_\ell B$ has constant sign.

Proof (Outline)
It is well known that (see, e.g., [8]) that Ω has a finite decomposition into sets stabilized by a fixed controller. From Remark 4.3, the requirements for knowledge of a C_ℓ such that $\mathcal{I}(C_\ell, \cdot, \cdot)$ is a stabilizing set on Ω are less stringent than knowledge of a stabilizing controller for the set Ω^ℓ.

We now introduce our control method, including the method of localization for determining which controller to use. The first case we consider is the simplest case where there is a single set to consider.

Case 1: $L = 1$ (sign of CB known)
This case covers a class of minimum phase plants, plus also certain classes of nonminimum phase plants.

For $L = 1$ we have:

$$\Omega = \Omega^1 = \bigcup_{i=1}^{s} \Omega_i \tag{5.80}$$

For $i = 1 \dots s$ we define a control law:

$$u(t)^i = -K_i x(t) \triangleq -\frac{1}{CB_i} CA_i x(t) \tag{5.81}$$

where the plant model, A_i, B_i is in the set Ω_i. We require knowledge of a Δ such that:

$$\|C(A - A_i\left(\frac{CB}{CB_i}\right))\| \leq \Delta; \quad \forall i, \forall (A, B) \in \Omega_i \tag{5.82}$$

and $\mathcal{I}_t$ is a stabilizing inclusion on Ω_i for all i. Note that for any bounded Ω, for which we can find a single C which gives $C(zI - A)^{-1}B$ minimum phase and relative degree one we can always find, for s large enough, a Δ with the required properties (see, e.g., [8]).

At any time $t > 0$, the auxiliary output $z(t+1)^i$ which would have resulted if we applied $u(t)^i = -K_i x(t)$ to the true plant is, using (5.18).

$$\begin{aligned} z(t+1)^i &\triangleq CAx(t) + CBu(t)^i + CE\xi(t) \\ &= z(t+1) - CB\left(u(t) - u(t)^i\right) \end{aligned} \tag{5.83}$$

Note that if the true plant is in the set Ω_i, then from (5.83) and (5.81)

$$z(t+1)^i = C\left(A - A_i\left(\frac{CB}{CB_i}\right)\right)x(t) + CE\xi(t) \tag{5.84}$$

and, therefore, if the true plant is in Ω_i, then from (5.82), and with $c_0 = |CE\bar{\xi}|$

$$|z(t+1)^i| \leq \Delta\|x(t)\| + c_0 \tag{5.85}$$

Our proposed control algorithm for Case 1 is as follows (where, without loss of generality, we take $CB > 0$).

Algorithm C

1.1 Initialization
Define

$$S_0 = \{1, 2, \ldots, s\} \tag{5.86}$$

1.2 If $t > 0$,
If $z(t) > \Delta|x(t-1)| + c_0$ then set $S_t = S_{t-1} - \{k,\ \ldots, j_{s-1},\ j_s\}$
If $-z(t) > \Delta|x(t-1)| + c_0$ then set $S_t = S_{t-1} - \{j_1, j_2, \ldots, k\}$
otherwise, $S_t = S_{t-1}$.
where $k, j_1\ \ldots\ j_s$ and s are integers from the previous time instant (see 1.4, 1.5).

1.3 If $t > 0$,
For all $i \in S_t$, compute $u(t)^i$ as in (5.81).

1.4 Order $u(t)^i,\ i \in S_t$ such that:

$$u(t)^{j_1} \leq u(t)^{j_2} \leq\ \ldots\ \leq u(t)^{j_s} \tag{5.87}$$

1.5 Apply the 'median' control:

$$u(t) = u(t)^k \tag{5.88}$$

where $k = j_{\lfloor s/2 \rfloor}$,

1.6 Then wait for the next sample and return to 1.2.

We then have the following stability result for this control algorithm.

Theorem 4.1 The control algorithm, (5.86)–(5.88), applied to a plant where C is known, and where the decomposition (5.80) has the properties that (5.82) is satisfied and $\mathcal{I}_t$ is a stabilizing inclusion, has the following properties:

(a) The inclusion:

$$\mathcal{I}_t : |z(t)| \leq \Delta\|x(t-1)\| + c_0 \tag{5.89}$$

is violated no more than $N = \lfloor \log_2(s) \rfloor$ times, and

(b) All signals in the closed loop system are bounded. In particular, there exist constants $\alpha, \beta < \infty, \sigma \in (0,1)$ such that all trajectories satisfy, for any $t_0, T > 0$

$$\|x(t_0 + T)\| \leq \alpha\sigma^T\|x(t_0)\| + \beta \tag{5.90}$$

Proof

(a) Suppose at time $(t+1)$, (5.89) is violated. This can occur in one of two ways which we consider separately:

(i)

$$z(t+1) = z(t+1)^{i_t} > \Delta\|x(t)\| + c_0 \tag{5.91}$$

In this case, because of the ordering of $u(t)^i$ in (5.87), and the definition of $z(t+1)^i$ in (5.83), then

$$z(t+1)^i > \Delta\|x(t)\| + c_0 \tag{5.92}$$

for all $i \in \{k, \ldots, j_{s-1}, j_s\}$

(ii)

$$z(t+1) = z(t+1)^{i_t} < -(\Delta\|x(t)\| + c_0) \tag{5.93}$$

In this case

$$z(t+1)^i < -(\Delta\|x(t)\| + c_0) \tag{5.94}$$

for all $i = \{j_1, j_2, \ldots, k\}$. In either case, we see that if (5.91) is violated at time t, then

$$s_{t+1} \le \frac{1}{2} s_t \tag{5.95}$$

from which the result follows.

(b) First, we note that the control is well defined, that is, S_t is never empty. This follows since there is at least one index, namely the index of the set Ω_i which contains the true plant, which is always an element of S_t.

Next, we note that although we cannot guarantee that we converge to the correct control, from (a) we know (5.85) is satisfied all but a finite number of times.

Since $\mathcal{I}_T$ is a stabilizing inclusion, then by definition the states and all signals will be bounded.

Furthermore, since $\mathcal{I}_t$ is a stabilizing inclusion, there exist α_0, β_0 and $\sigma \in (0,1)$ such that if the inclusion (5.89) is satisfied, for $t \in [t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \le \alpha_0 \sigma^T \|x(t_0)\| + \beta_0 \tag{5.96}$$

(Note that if this is not the case, then from the definition, $\mathcal{I}_t$ is not a stabilizing inclusion.) Also, there exist $\bar{\alpha}$ and $\bar{\beta}$ such that when (5.89) is violated:

$$\|x(t+1)\| \le \bar{\alpha}\|x(t)\| + \bar{\beta} \tag{5.97}$$

If we define $\alpha_1 = \frac{\alpha_0^2 \bar{\alpha}}{\sigma}$ and $\beta_1 = (\alpha_0 \bar{\alpha} \beta_0 + \beta_0 + \alpha_0 \bar{\beta})$, then after some algebraic manipulations we can show that for any $t_0, T > 0$ such that (5.89) is violated not more than once in the interval, $(t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \le \alpha_1 \sigma^T \|x(t_0)\| + \beta_1 \tag{5.98}$$

Also, we can show that with $\alpha_2 = \dfrac{\bar{\alpha}\alpha_0\alpha_1}{\sigma} = \dfrac{\bar{\alpha}\alpha_0^3}{\sigma^2}$, and $\beta_2 = \beta_0 + \bar{\beta} + \alpha_0\bar{\alpha}\beta_1 =$

$(1 + \alpha_0\bar{\alpha} + (\alpha_0\bar{\alpha})^2)\beta_0(1 + \alpha_0\bar{\alpha})\bar{\beta}$, provided (5.89) is not violated more than twice in the interval $[t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \le \alpha_2\sigma^T\|x(t_0)\| + \beta_2 \tag{5.99}$$

Repeating this style of argument leads to the conclusion that with

$$\alpha_N = (\bar{\alpha}\alpha_0)\left(\frac{\alpha_0}{\sigma}\right)^N, \quad \beta_N = (\alpha_0\bar{\alpha})^N\beta_0 + \left[\frac{(\alpha_0\bar{\alpha})^N - 1}{(\alpha_0\bar{\alpha}) - 1}\right](\beta_0 + \bar{\beta})$$

then if there are not more than N switches in $[t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \le \alpha_N\sigma^T\|x(t_0)\| + \beta_N \tag{5.100}$$

The desired result follows from (a) since we know that there are at most $N = [\log_2(s)]$ times at which (5.89) is violated.

Case 2: $L > 1$
Suppose that we do not know a single C such that I_t is a stabilizing inclusion, and CB is of known sign, then using finite covering ideas [8], as in Remark 4.3 let

$$\Omega = \bigcup_{\ell=1}^{L}\Omega^\ell = \bigcup_{\ell=1}^{L}\bigcup_{m=1}^{s^\ell}\Omega_m^\ell \tag{5.101}$$

where for each ℓ, we know $C_\ell, \Delta_\ell, c_0^\ell$ such that $\mathcal{I}_t$ is a stabilizing inclusion on Ω^ℓ and the sign of $(C_\ell B)$ is constant for all plants in Ω^ℓ.

At this point, one might be tempted to apply localization, as previously defined, on the sets Ω^ℓ individually and switch from Ω^ℓ should the set of valid indices, S^ℓ, become empty. Unfortunately, this procedure cannot be guaranteed to work. In particular, if Ω^ℓ does not contain the true plant, $\mathcal{I}_t$ need not be a stabilizing inclusion, and so divergence of the states may occur without violating (5.89). To alleviate this problem, we use the exponential stability result, (5.90), in our subsequent development.

Algorithm D
We initialize $t(i) = 0$, $R_0 = \{1, 2, \ldots, L\}$ and take any $\ell_0 \in R_0$.

We then perform localization on Ω^ℓ, with the following additional[2] steps: If at any time

$$\|x(t)\| > \alpha\sigma^{t-t(i)}\|x(t(i))\| + \beta \tag{5.102}$$

(where α, σ, β are the appropriate constants for Ω^ℓ from Theorem 4.1), then we set $S^\ell = \{\}$. If at any time t, S^ℓ becomes empty, we set $R_t = R_{t-1} - \{\ell\}$, $t(i) = t$, and we take a new ℓ from R_t.

[2] In fact, we can localize simultaneously within other $\Omega^i, i \ne \ell$: however, for simplicity and brevity we analyse only the case where we localize in one set at a time.

With these modifications, it is clear that Theorem 4.1 can be extended to cover this case as well:

Corollary 4.1 The control algorithm (5.86)–(5.88) with the above modifications applied to a plant with decomposition as in (5.101) satisfies:

(a) There are no more than: $L - 1 + \sum_{\ell=1}^{L} \lfloor \log_2(s_\ell) \rfloor$ instants such that

$$|z(t+1)^{\ell_t}| \geq \Delta_{\ell_t} \|x(t)\| + c_{0,\ell_t} \tag{5.103}$$

(where ℓ_t denotes the value of ℓ at time t).

(b) All signals in the closed loop are bounded. In particular, there exist constants $\bar{\alpha}, \bar{\beta} < \infty, \bar{\sigma} \in (0,1)$ such that for any $t_0, T > 0$

$$\|x(t_0 + T)\| \leq \bar{\alpha}\bar{\sigma}^T \|x(t_0)\| + \bar{\beta} \tag{5.104}$$

Proof Follows from Theorem 4.1.

5.4.1 Localization in the presence of unknown disturbance

In the previous section the problem of indirect localization based switching control for linear uncertain plants was considered assuming that the level of the generalized exogenous disturbance $\xi(t)$ was known. This is equivalent to knowing some upper bound on $\xi(t)$. The flexibility of the proposed adaptive scheme allows for simple extension covering the case of exogenous disturbances of unknown magnitude. This can be done in the way similar to that considered in Section 5.3.2. Omitting the details we just make the following useful observation. The control law described by Algorithms C and D is well defined, that is, $R_t \neq \{\}$ for all $t \geq t_0$ if $c_0^\ell \geq \sup_{t \geq t_0} |C_\ell E \xi(t)|, \ \forall \ell = 1, \ldots, L$. This is the key point allowing us to construct an algorithm of on-line identification of the parameters $c_0^\ell, \ell = 1, \ldots, L$.

5.5 Simulation examples

Extensive simulations conducted for a wide range of LTI, LTV and nonlinear systems demonstrate the rapid falsification capabilities of the proposed method. We summarize some interesting features of the localization technique observed in simulations which are of great practical importance.

(i) Falsification capabilities of the algorithm of localization do not appear to be sensitive to the switching index update rule. One potential implication of this observation is as follows. If not otherwise specified any choice of a new switching index is admissible and will most likely lead to good transient performance;

(ii) The speed of localization does not appear to be closely related to the total

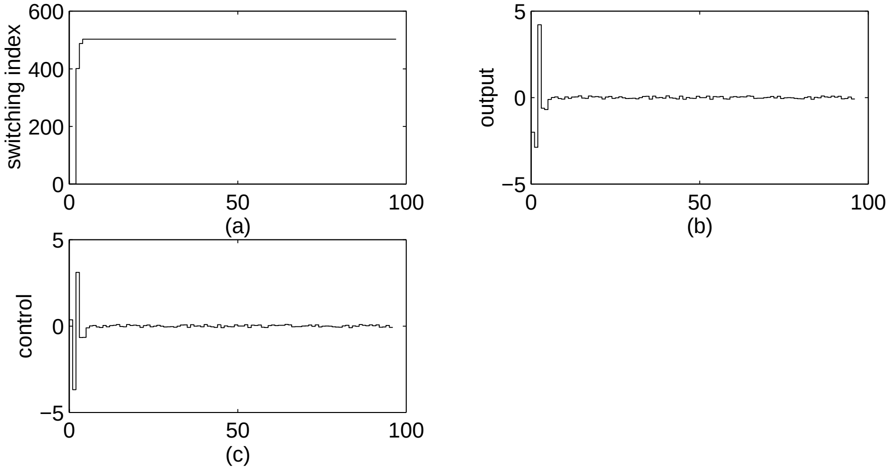

Figure 5.4 *Example of localization: constant parameters*

number of fixed controllers obtained as a result of decomposition. The practical implication of this observation (combined with the quadratic stability assumption) is that decomposition of the uncertainty set Ω can be conducted in a straightforward way employing, for example, a uniform lattice which produces subsets Ω_i, $i = 1, 2, \ldots, L$ of an equal size.

Example 5.1 Consider the following family of unstable (possibly nonminimum phase) LTV plants:

$$y(t) = 1.2y(t-1) - 1.22y(t-2) + b_1(t)u(t-1) + b_2(t)u(t-2) + \xi(t) \tag{5.105}$$

where the exogenous disturbance $\xi(t)$ is uniformly distributed on the interval $[-0.1,\ 0.1]$, and $b_1(t)$ and $b_2(t)$ are uncertain parameters. We deal with two cases which correspond to constant parameters and large-size jumps in the values of the parameters.

Case 1: Constant parameters
The a priori uncertainty bounds are given by

$$b_1(t) \in [-1.6,\ -0.15] \cup [0.15,\ 1.6], \quad b_2(t) \in [-2,\ -1] \cup [1,\ 2] \tag{5.106}$$

i.e. $\Omega = \{[-1.6,\ -0.15] \cup [0.15,\ 1.6] \times [-2,\ -1] \cup [1,\ 2]\}$. To meet the requirements of the localization technique, we decompose Ω into 600 nonintersecting subsets with their centres $\theta_i = (b_{1i}, b_{2i}), i = 1, \ldots, 600$ corresponding to

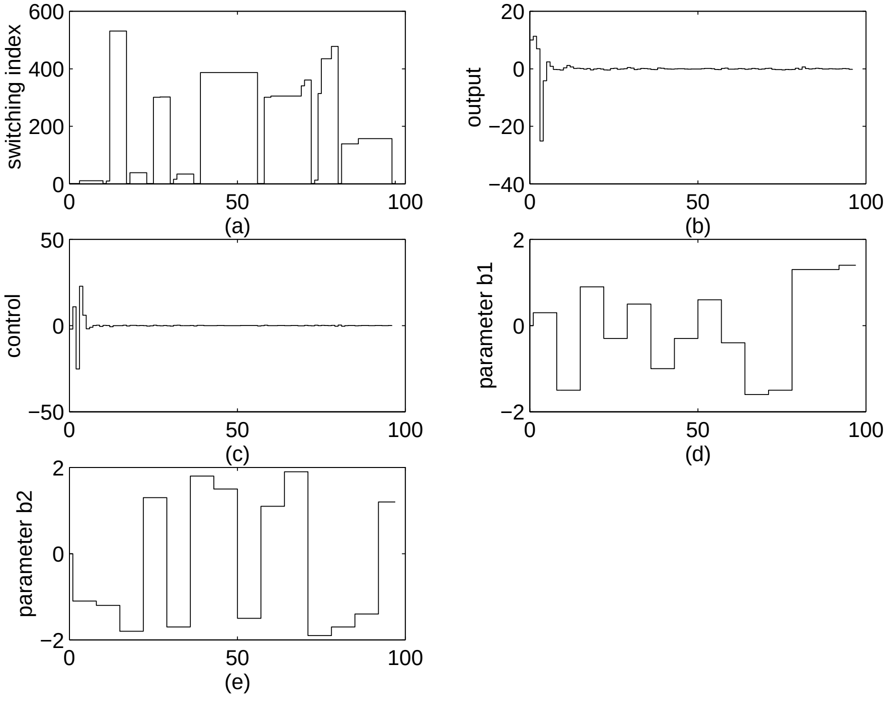

Figure 5.5 *Example of localization: parameters jump every 7 steps*

$$b_{1i} \in \{-1.6,\ -1.5, \ldots, -0.3,\ -0.2,\ 0.2,\ 0.3, \ldots, 1.5,\ 1.6\}$$
$$b_{2i} \in \{-2,\ -1.9, \ldots, -1.1,\ -1,\ 1,\ 1.1, \ldots, 1.9,\ 2\}$$

respectively.

Figures 5.4(a)–(c) illustrate the case where θ is constant. The switching sequence $\{i(1), i(2), \ldots\}$ depicted in Figure 5.4(a) indicates a remarkable speed of localization.

Case 2: Parameter jumps
The results of localization on the finite set $\{\theta_i\}_{i=1}^{600}$ are presented in Figures 5.5(a)–(e). Random abrupt changes in the values of the plant parameters occur every 7 steps. In both cases above the algorithm of localization in Section 5.2 is used. However, in the latter case the algorithm of localization is appropriately modified. Namely, $I(t)$ is updated as follows

$$I(t) = \begin{cases} I(t-1) \cap \hat{I}(t) & \text{if } I(t-1) \cap \hat{I}(t) \neq \{\ \} \\ \hat{I}(t) & \text{otherwise} \end{cases} \tag{5.107}$$

Once (or if) the switching controller, based on (5.107) has falsified every index in the localization set it disregards all the previous measurements, and the process of localization continues (see [40] for details). In the example above a pole placement technique was used to compute the set of the controller gains $\{K_i\}_{i=1}^{600}$. The poles of the nominal closed loop system were chosen to be (0, 0.07, 0.1).

Example 5.2 Here we present an example of indirect localization considered in Section 5.4. The model of a third order unstable discrete-time system is given by

$$y(t+1) = a_1 y(t) + a_2 y(t-1) + a_3 y(t-2) + u(t) + \xi(t) \tag{5.108}$$

where a_1, a_2, a_3 are unknown constant parameters, and $\xi(t) = \xi_0 \sin(0.9t)$ represents exogenous disturbance. The a priori uncertainty bounds are given by

$$a_1 \in [-1.6, -0.1] \cup [0.1, 1.6], \quad b_2 \in [-1.6, -0.1] \cup [0.1, 1.6], \quad a_3 \in [0.1, 1.6] \tag{5.109}$$

i.e. $\Omega = \{[-1.6, -0.1] \cup [0.1, 1.6] \times [-1.6, -0.1] \cup [0.1,\ 1.6] \times [0.1, 1.6]\}$. Choosing

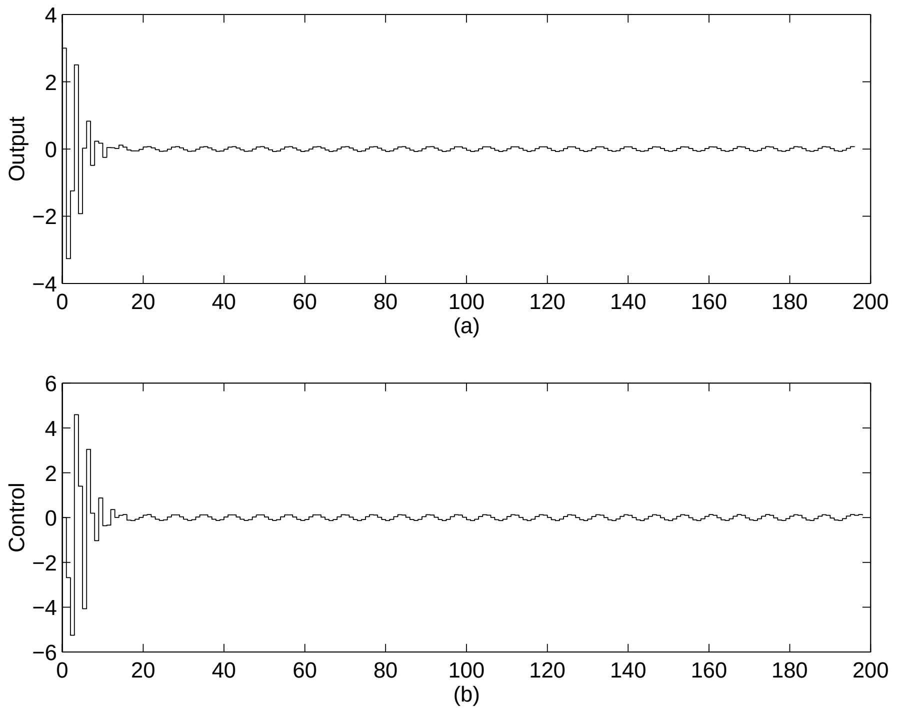

Figure 5.6 *Example of indirect localization*

the vector C and the stabilizing set $\mathcal{I}$ as prescribed in Section 5.4, we obtain

$$\mathcal{I} : |z(t+1)| \leq \Delta||x(t)|| + c_0 \tag{5.110}$$

where $C = (0,\ 0,\ 1)$ and $\Delta = 0.6$. We decompose Ω into 256 nonintersecting subsets with their centres $\theta_i = (a_{1i}, a_{2i}, a_{3i}),\ i = 1, \ldots, 256$ corresponding to

$$a_{1i} \in \{-0.3, -0.7, -1.1, -1.5, 0.3, 0.7, 1.1, 1.5\} \tag{5.111}$$

$$a_{2i} \in \{-0.3, -0.7, -1.1, -1.5, 0.3, 0.7, 1.1, 1.5\} \tag{5.112}$$

$$a_{3i} \in \{0.3, 0.7, 1.1, 1.5\} \tag{5.113}$$

respectively. This allows us to compute the set of controller gains $\{K_i\}_{i=1}^{256}$, $K_i = (k_{1i}, k_{2i}, k_{3i})$. Each element of the gain vector $k_{ij},\ i \in \{1, 2, 3\}$, $j \in \{1, \ldots, 256\}$ takes values in the sets (5.111), (5.112), (5.113), respectively. The results of simulation with $\xi_0 = 0.1,\ a_1 = -1.1,\ a_2 = -0.7,\ a_3 = 1.4$, are presented in Figure 5.6(a)–(b). Algorithm C has been used for this study.

5.6 Conclusions

In this chapter we have presented a new unified switching control based approach to adaptive stabilization of parametrically uncertain discrete-time systems. Our approach is based on a localization method which is conceptually different from the existing switching adaptive schemes and relies on on-line simultaneous falsification of incorrect controllers. It allows slow parameter drifting, infrequent large parameter jumps and unknown bound on exogenous disturbance. The unique feature of localization based switching adaptive control distinguishing it from conventional adaptive switching controllers is its rapid model falsification capabilities. In the LTI case this is manifested in the ability of the switching controller to quickly converge to a suitable stabilizing controller. We believe that the approach presented in this chapter is the first design of a falsification based switching controller which is applicable to a wide class of linear time-invariant and time-varying systems and which exhibits good transient performance.

Appendix A

Proof of Theorem 3.1 First we note that it follows from Lemma 3.1 and the switching index update rule (5.37) that the total number of switchings made by the controller is finite. Let $\{t_1, t_2, \ldots, t_l\}$ be a finite set of switching instants. By virtue of (5.31)–(5.33) the behaviour of the closed loop system between any two consecutive switching instants $t_s, t_j, 1 \leq s, j \leq l, t_j \geq t_s$ is described by

$$\begin{aligned} x(t+1) &= (A(\theta) + B(\theta)K_{i(t_s)})x(t) + E(\xi(t) + \eta(t)) \\ &= (A(\theta_{i(t_s)}) + B(\theta_{i(t_s)})K_{i(t_s)})x(t) + E\psi(t) \end{aligned} \quad (5.114)$$

where $|\psi(t)| \leq r_{i(t_s)}||\phi(t)|| + \bar{\xi} + \bar{\eta}(t)$.

Therefore, taking into account the structure of the parameter dependent matrices $A(\theta)$ and $B(\theta)$, namely the fact that only the last rows of $A(\theta)$ and $B(\theta)$ depend on θ the last equation can be rewritten as

$$x(t+1) = (A(\theta_{i(t_s)} + \Delta\theta(t)) + B(\theta_{i(t_s)} + \Delta\theta(t))K_{i(t_s)})x(t) + E\hat{\xi}(t) \quad (5.115)$$

for some $\Delta\theta(t) : ||\Delta\theta(t)|| \leq r_{i(t_s)} + q$ and $|\hat{\xi}(t)| \leq \bar{\xi} + \bar{\eta}(t)$. This is a direct consequence of the fact that the last equation in (5.114) can be rewritten as $y(t+1) = \theta^T_{i(t_s)}\phi(t) + \psi(t)$ and that $\max_{||\Delta\theta|| \leq 1} ||\Delta\theta^T \phi(t)|| = ||\phi(t)||$ holds for any $\phi(t)$. By Definition 3.2 and condition (C3′) the system (5.115) is quadratically stable with $\hat{\xi}(t) \equiv 0$ and t_s being fixed; moreover, there exists a positive definite matrix $H^T_{t_s} = H_{t_s}$ such that

$$P_{t_s} = \max_{||\Delta\theta(t)|| \leq r_{i(t_s)} + q} ||A(\theta_{i(t_s)} + \Delta\theta(t)) + B(\theta_{i(t_s)} + \Delta\theta(t))K_{i(t_s)}||_{H_{t_s}} < 1 \quad (5.116)$$

Here $||x||_H = (x^T Hx)^{1/2}$ and for any matrix $A \in \mathbf{R}^{n \times n}$, $||A||_H$ denotes the corresponding induced matrix norm. The equation (5.115) along with the property of quadratic stability guarantee that between any two consecutive switchings the closed loop system behaves as an exponentially stable LTI system subject to small parametric perturbations $\Delta\theta(t)$ and bounded disturbance $\hat{\xi}(t)$ and this property holds regardless of the possible evolution of the plant parameters. This is the key point making the rest of the proof transparent.

Assume temporarily that $\bar{\eta}(t) \equiv 0$, then it follows from (5.115), (5.116) that

$$||x(t_s+1)||_{H_{t_s}} \leq P_{t_s}||x(t_s)||_{H_{t_s}} + \hat{\xi}_{t_s} \quad (5.117)$$

$$||x(t_s+2)||_{H_{t_s}} \leq P^2_{t_s}||x(t_s)||_{H_{t_s}} + (P_{t_s}+1)\hat{\xi}_{t_s} \quad (5.118)$$

$$\cdots$$

$$||x(t_s+k)||_{H_{t_s}} \leq P^k_{t_s}||x(t_s)||_{H_{t_s}} + \hat{\xi}\sum_{i=1}^{k} P^{i-1}_{t_s} \quad (5.119)$$

$$||x(t_s+k)|| \leq (\lambda_{\max}(H_{t_s})/\lambda_{\min}(H_{t_s}))^{1/2} P^k_{t_s}||x(t_s)|| + \hat{\xi}_{t_s}\sum_{i=1}^{k} P^{i-1}_{t_s}/\lambda_{\min}(H_{t_s})^{1/2} \quad (5.120)$$

where $\hat{\xi}_{t_s} = \max_{|\xi| \leq \bar{\xi}} ||E\xi||_{H_{t_s}}$.

Denote

$$M = \max_{t_1 \le i \le t_l} (\lambda_{\max}(H_{t_s})/\lambda_{\min}(H_{t_s}))^{1/2}, \quad \rho = \max_{t_1 \le i \le t_l} P_i < 1, \tag{5.121}$$

$$M(\bar{\xi}) = \max_{t_1 \le i \le t_l} \hat{\xi}_i/(\lambda_{\min}(H_i))^{1/2} \sum_{j=1}^{\infty} P_i^{j-1} < \infty \tag{5.122}$$

Since $\theta_{i(t)} \in \theta(I_{t_0})$, $K_{i(t)} \in \{K_i\}_{i=1}^{L}$ for all $t \in \mathbf{N}$, $i(t) \in I_{t_0}$ there exist constants $0 < M_0 < \infty$, $\gamma_0 = \max_{|\xi| \le \bar{\xi}} ||E\xi|| < \infty$ such that

$$||x(t_s)|| \le M_0||x(t_s - 1)|| + \gamma_0 \tag{5.123}$$

for any switching instant $t_1 \le t_s \le t_l$.

Hence,

$$[t_0, t_l): \quad ||x(t_1)|| \le M_0||x(t_1 - 1)|| + \gamma_0 \le M_0 M \rho^{t_1 - t_0 - 1}||x(t_0)|| + M_0 M(\bar{\xi}) + \gamma_0 \tag{5.124}$$

$$||x(t_2)|| \le M_0||x(t_2 - 1)|| + \gamma_0 \le M_0^2 M^2 \rho^{t_2 - t_0 - 2}||x(t_0)|| + \hat{M}_2(\bar{\xi}) \tag{5.125}$$

where $\hat{M}_2(\bar{\xi}) = M_0(M(M_0 M(\bar{\xi}) + \gamma_0) + M(\bar{\xi})) + \gamma_0$;

. . .

$$[t_l, \infty): \quad ||x(t)|| \le M_0^l M^l \rho^{t - t_0 - l}||x(t_0)|| + \hat{M}_l(\bar{\xi}) \tag{5.126}$$

Having denoted $M_1 = (M_0 M/\rho)^l$, $M_2(\bar{\xi}) = \hat{M}_l(\bar{\xi}) < \infty$ we obtain (5.42). To conclude the proof we note that the result above can be easily extended to the case $\bar{\eta}(t) \ne 0$, provided that the 'size' of unmodelled dynamics ε is sufficiently small. Indeed, let $\eta(t) \ne 0$. First, we note that due to the term $\bar{\eta}(t)$ in the algorithm of localization (5.33)–(5.37) the process of localization cannot be disrupted by the presence of small unmodelled dynamics. In view of (A5), (5.117)–(5.126) it is easy to show that provided that ε is sufficiently small

$$[t_l, \infty): \ ||x(t)|| \le M_0^l M^l \rho^{t - t_0 - l}||x(t_0)|| + \hat{M}_l(\bar{\xi}) + M_\eta \varepsilon ||x(t_0)|| \tag{5.127}$$

with M_η being a positive constant independent of $x(t_0)$. Therefore

$$||x(t)|| \le (M_1 \rho^{t - t_0} + M_\eta \varepsilon)||x(t_0)|| + \hat{M}_l(\bar{\xi}) \tag{5.128}$$

is valid for all $t_0 \in \mathbf{N}$, $t \ge t_l$. From (5.128) and assumption (A5) exponential stability of the closed loop system (if $\hat{M}_l(\bar{\xi}) = 0$) or exponential convergence of the states to the residual set (if $\hat{M}_l(\bar{\xi}) > 0$) can be easily established. Indeed, in this case it is always possible to specify a sufficiently large integer T such that $(M_1 \rho^T + M_\eta \varepsilon) < 1$. This, in turn, trivially implies stability. The finite number of the controller switchings follows directly from the switching index update rule (5.37). This also implies the finite convergence of switching; however, it is quite difficult, in general, to put an upper bound on t_l. This obviously does not affect the stability properties of the closed loop.

Appendix B

Proof of Theorem 3.4 First we note that the property (1) follows directly from the structure of the algorithm of localization (5.66). It is straightforward to verify that relations (5.60)–(5.65) guarantee that the sequence of localization sets $I(t)$ is well defined.

To prove (2) consider first the case $\alpha = 0$. It is clear that

$$\bigcap_{k=s(t)}^{t} \hat{I}(k, \bar{\xi}(k-1)) \neq \{\} \tag{5.129}$$

if $\min_{k\in[s(t),t]}\{\bar{\xi}(k+1)\} \geq \bar{\xi}$ for all $t > t_0$. Since, according to (5.64), the estimate $\bar{\xi}(t)$ is updated only if (5.129) does not hold, and taking into account the discrete nature of updating expressed by (5.65) we conclude that

$$\sup_{t\geq t_0} \bar{\xi}(t) \leq \bar{\xi} + \mu \tag{5.130}$$

Let $\alpha > 0$. Then it is easy to see that the arguments above remain valid for any finite interval of time $[s(t), s(t) + t_d)$, provided that the rate of parameter variations is sufficiently small, namely, $\alpha \leq q/t_d$. To conclude the proof of (5.130) it suffices to note that the estimate $\bar{\xi}(t)$ in (5.64) does not change if $t - s(t) \geq t_d$.

Proof of statements $(5.3), (5.4)$ follows closely those of Theorem 3.1. Here we present a brief sketch of the proof. Consider a finite time interval $T = [s(t), s(t) + t_d), l < t_d < \infty$. Let $\bar{\xi}(s(t)) \geq \bar{\xi}$, then the total number of switchings s made by the controller over T satisfies the condition $s \leq l$ if $\alpha \leq q/t_d$. Therefore, the states are bounded by (5.126) with t_0 replaced by $s(t)$. Moreover, (5.126) is valid for any time interval $\bar{T} = [s(t), s(t) + \bar{t}),\ \bar{t} > t_d$ such that

$$\bigcap_{k=s(t)}^{\bar{t}} \hat{I}(k, \bar{\xi}(s(t))) \neq \{\} \tag{5.131}$$

Relying on (5.126) and taking into account the fact that the index $s(t)$ is reset every time when (5.131) is violated for $t - s(t) \geq t_d$ it is always possible to choose sufficiently large integer t_d such as to guarantee exponential stability of the closed loop system. Let $\bar{\xi}$ be unknown, then for any $\bar{\xi}(t_0) > 0$ the inequality (5.126) can be possibly violated no more than $([\bar{\xi}/\mu] + 1)$ times. Relying on this fact and using standard arguments exponential stability of the closed loop system is easily established.

References

[1] Agarwal, M. and Seborg, D. E. (1987) 'Self-tuning Controllers For Nonlinear Systems', *Automatica*, **23**, 209–214.

[2] Bai, E. W. (1988) 'Adaptive Regulation of Discrete-time Systems by Switching Control', *Systems and Control Letters*, **11**, 129–133.

[3] Barmish, B. R. (1985) 'Necessary and Sufficient Conditions for Quadratic Stabilizability of an Uncertain System', *J. Optimiz. Theory Appl.*, **46**, 4, 399–408.

[4] Byrnes, C. and Willems, J. (1984) 'Adaptive Stabilization of Multivariable Linear Systems', *in Proceedings of the 23rd IEEE Conference on Decision and Control*, Las Vegas, 1574–1577.

[5] Byrnes, C. I., Lin, W. and Ghosh, B. K. (1993) 'Stabilization of Discrete-Time Non-linear Systems by Smooth State Feedback', *Systems and Control Letters*, **21**, 255–263.

[6] Chang, M. and Davison, E. J. (1995) 'Robust Adaptive Stabilization of Unknown MIMO Systems using Switching Control', *Proceedings of the 34th Conference on Decision Control*, 1732–1737.

[7] Etxebarria, V. and De La Sen, M. (1995) 'Adaptive Control Based on Special Compensation Methods for Time-varying Systems Subject to Bounded Disturbances', *Int. J. Control*, **61**, 3, 667–699.

[8] Fu, M. and Barmish, B. R. (1986) 'Adaptive Stabilization of Linear Systems via Switching Control', *IEEE Trans. Auto. Contr.*, **AC-31**, 12, 1097–1103.

[9] Fu, M. and Barmish, B. R. (1988) 'Adaptive Stabilization of Linear Systems with Singular Perturbations', *Proc. IFAC Workshop on Robust Adaptive Control*, Newcastle, Australia.

[10] Fu, M. (1996) 'Minimum Switching Control for Adaptive Tracking', *Proceedings 25th IEEE Conference on Decision and Control*, Kobe, Japan, 3749–3754.

[11] Furuta, K. (1990) 'Sliding Mode Control of a Discrete System', *Systems and Control Letters*, **14**, 145–152.

[12] Goodwin, G. C. and Sin, K. S. (1984) *Adaptive Filtering Prediction and Control*. Prentice-Hall, Englewood Cliffs, N.J.

[13] Hocherman-Frommer, J., Kulkarni, S. R. and Ramadge, P. (1995) 'Supervised Switched Control Based on Output Prediction Errors', *Proc. 34th Conference on Decision and Control*, 2316–2317.

[14] Hocherman-Frommer, S. K. J. and Ramadge, P. (1993) 'Controller Switching Based on Output Prediction Errors, preprint, Department of Electrical Engineering, Princeton University.

[15] Haber, R. and Unbehauen, H. (1990) 'Structure Identification of Nonlinear Dynamic Systems – A Survey on Input/Output Approaches', *Automatica*, **26**, 4, 651-677.

[16] Ioannou, P. A. and Sun, J. (1996) *Robust Adaptive Control*. Prentice-Hall.

[17] Kreisselmeier, G. (1986) 'Adaptive Control of a Class of Slowly Time-varying Plants', *Systems and Control Letters*, **8**, 97–103.

[18] Kung, M. C. and Womack, B. F. (1983) 'Stability Analysis of a Discrete-Time Adaptive Control Algorithm Having a Polynomial Input', *IEEE Trans. Auto. Contr.*, **28**, 1110–1112.

[19] Lai, W. and Cook, P. (1995) 'A Discrete-time Universal Regulator', *Int. J. of Control*, **62**, 17–32.

[20] Martensson, B. (1985) 'The Order of Any Stabilizing Regulator is Sufficient a priori Information for Adaptive Stabilizing', *Systems and Control Letters*, **6**, 2, 87–91.

[21] Middleton, R. H. and Goodwin, G. C. (1988) 'Adaptive Control of Time-varying Linear Systems', *IEEE Trans. Auto. Contr.*, **33**, 1, 150–155.

[22] Middleton, R. H., Goodwin, G. C., Hill, D. J. and Mayne, D. Q. (1988) 'Design Issues in Adaptive Control', *IEEE Trans. Auto. Contr.*, **33**, 1, 50–80.

[23] Miller, D. and Davison, E. J. (1989) 'An Adaptive Controller Which Provides Lyapunov Stability', *IEEE Trans. Auto. Contr.*, **34**, 599-609.

[24] Morse, A. S. (1982) 'Recent Problems in Parametric Adaptive Control', *Proc. CNRS Colloquium on Development and Utilization of Mathematical Models in Automatic. Control.*, Belle-Isle, France, 733–740.

[25] Morse, A. S. (1993) 'Supervisory Control of Families of Linear Set-point Controllers', *Proceedings of the 32nd Conference on Decision and Control*, 1055–1060.

[26] Morse, A. S. (1995) 'Supervisory Control of Families of Linear Set-point Controllers – Part 2: Robustness', *Proceedings of the 34th Conference on Decision Control*, 1750–1760.

[27] Morse, A. S., Mayne, D. Q. and Goodwin, G. C. (1992) 'Applications of Hysteresis Switching in Parameter Adaptive Control', *IEEE Trans. Auto. Contr.*, **37**, 9, 1343–1354.

[28] Mudgett, D. and Morse, A. (1985) 'Adaptive Stabilization of Linear Systems with Unknown High-frequency Gains', *IEEE Trans. Auto. Contr.*, **30**, 549–554.

[29] Narendra, K. S. and Balakrishnan, J. (1994) 'Intelligent Control Using Fixed and Adaptive Models', *Proceedings of the 33th Conference on Decision Control.*

[30] Narendra, K. S. and Annaswamy, A. M. (1989) *Stable Adaptive Systems*. Prentice-Hall.

[31] Nussbaum, R. D. (1983) 'Some Remarks on a Conjecture in Parameter Adaptive Control', *Systems and Control Letters*, **3**, 243–246.

[32] Peng, H. J. and Chen, B. S. (1995) 'Adaptive Control of Linear Unstructured Time-varying Systems', *Int. J. Control.*, **62**, 3, 527–555.

[33] Praly, L., Marino, R. and Kanellakopoulos, I. (eds) (1992) 'Special Issue on Adaptive Nonlinear Control', *International Journal of Adaptive Control and Signal Processing*, **6**.

[34] Rohrs, C. E., Valavani, E., Athans, M. and Stein, G. (1985) 'Robustness of Continuous-time Adaptive Control Algorithms in the Presence of Unmodelled Dynamics', *IEEE Trans. Auto. Contr.*, **AC-30**, 9, 881–889.

[35] Tsakalis, K. S. and Ioannou, P. A. (1987) 'Adaptive Control of Linear Time-varying Plants', *Automatica*, **23**, 459–468.

[36] Tsakalis, K. S. and Ioannou, P. A. (1989) 'Adaptive Control of Linear Time-varying Plants: A New Model Reference Controller Structure', *IEEE Trans. Auto. Contr.*, **34**, 10, 1038–1046.

[37] Weller, S. R. and Goodwin, G. C. (1992) 'Hysteresis Switching Adaptive Control of Linear Multivariable Systems', *Proceedings of the 31st Conference on Decision and Control*, 1731–1736.

[38] Willems, J. and Byrnes, C. (1989) 'Global Adaptive Stabilization in the Absence of Information on the Sign of the High Frequency Gain', in *Proceedings of the INRIA Conference on Analysis and Optimization of Systems, Springer Lecture Notes in Springer-Verlag*, **62**, 49–57.
[39] Zhivoglyadov, P. V., Middleton, R. H. and Fu, M. (1997) 'Localization Based Switching Adaptive Controllers', *Proceedings of European Control Conference*. Brussels, Belgium.
[40] Zhivoglyadov, P. V., Middleton, R. H. and Fu, M. (1997) 'Localization Based Switching Adaptive Control for Time Varying Discrete Time Systems', *Proceedings of the 36th Conference on Decision Control*, San Diego, 4151–4157.

6

Adaptive nonlinear control: passivation and small gain techniques

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Abstract

In this chapter, passivation and small gain techniques are used as two fundamental tools to systematically design stabilizing adaptive controllers for new classes of nonlinear systems. We first show that, for a class of linearly parametrized nonlinear systems with only unknown parameters, the concept of adaptive passivation can be used to unify and extend most of the known adaptive nonlinear control algorithms based on Lyapunov methods. Then, we consider the global robust adaptive control problem for a broader class of nonlinear systems with time-varying and dynamic uncertainties in addition to parametric uncertainties. Small gain arguments are used to provide a robustification methodology for prior backstepping-based adaptive controllers.

6.1 Introduction

The area of adaptive nonlinear control has moved on quickly since the early 1990s – see the survey paper [37] and two more recent textbooks [26, 32]. We observe that the development of most adaptive nonlinear controller designs was based on Lyapunov methods. In this chapter, we shall approach this field from the somewhat different input/output viewpoint using the concepts of passivation and small gain. Our contributions are twofold: (1) we exploit recent advances on the feedback stabilization of nonlinear systems via passive systems theory and apply some of these useful results to formulate a passivation framework for adaptive nonlinear stabilization; (2) we employ small gain techniques as a means to study robustness issues in adaptive systems with unmodelled dynamics. The latter topic has received less attention in the literature and the results presented here are a substantial development beyond earlier results.

In the first part, after reviewing some needed definitions and properties of passivity and passive systems, we briefly state the breakthrough made by Byrnes, Isidori and Willems [2] on the feedback equivalence of nonlinear systems to passive systems (or simply, passivation). Roughly speaking, a nonlinear control system can be transformed into a passive system via a change of feedback if and only if it is minimum phase and of relative degree one. As shown in [2], this theorem together with other passivity tools allow unification of early global stabilization results. Using this as a starting point, we first establish an adaptive version of a basic result in [2], that is, a linearly parametrized nonlinear system is feedback equivalent to a passive system if and only if it is adaptively stabilizable. Then, we show this property can be propagated each time we add a feedback passive system with linearly appearing unknown parameters. This recursive passivation design procedure is different to the currently popular *adaptive backstepping with tuning functions* introduced in [26] in several respects. We were motivated by a passivity-aimed adaptive design strategy. Some elementary examples illustrate this point in the case of output feedback passivation [15]. Our passivation-based adaptive controller design reduces to the above-mentioned Lyapunov-type adaptive scheme in the case of strict-feedback structures, but the construction turns out to be simpler and easier to understand by means of feedback passivation. To be more precise, we first augment the system (with unknown parameters) under consideration by adding a new parameter update system with adaptive law as input. Then, we only need to render the augmented system passive by a change of feedback at the levels of both control input and adaptive law. The immediate benefit is that we can do it recursively and shed light on how overparametrization is avoided.

The second part of the chapter collects a series of nonlinear small gain techniques recently presented in our papers [14, 16, 20, 17, 21, 36] on the basis of Sontag's input-to-state stability (ISS) concept [42, 44]. As demonstrated in these papers and other references therein, nonlinear small gain theorems have proved to be powerful design tools for interconnected nonlinear systems with complex structure. For a class of feedback linearizable systems with various disturbances including parametric uncertainty and unbounded nonlinear unmeasured dynamics, we show that these techniques are of paramount importance in designing a robust adaptive controller in the presence of unbounded unmodelled dynamics. Adaptive feedback designs without and with dynamic normalization will be proposed and compared in two elementary examples including a physical example of a simple pendulum.

The rest of the chapter is organized as follows: Section 6.2 states needed definitions and some known basic results. Section 6.3 is devoted to the adaptive passivation development for interconnected nonlinear systems. Section 6.4

presents a novel small-gain based adaptive scheme for nonlinear systems with unbounded unmodeled dynamics. Section 6.5 offers some brief concluding remarks.

6.2 Mathematical preliminaries

6.2.1 Notation and basic definitions

The notation used in this article is quite standard: $|\cdot|$ denotes the usual Euclidean norm for vectors and $\|\cdot\|$ denotes the L_∞ norm for time functions. For a real-valued differentiable function η, η' stands for its derivative. For a vector-valued function z of time t, $\dot{z}$ denotes its time derivative while $z_{[t_1,t_2]}$ denotes the truncated function of z over the interval $[t_1, t_2]$. x^T is the transposition of the vector $x \in \mathbb{R}^n$. For a symmetric and positive definite matrix P in $\mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ is the maximal eigenvalue of P.

The concepts of positive definite, proper, class $K-$, $K_\infty-$ and $KL-$ functions are widely used in the literature of Lyapunov stability theory [7], [24]. A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *positive definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. It is said to be *proper*, or *radially unbounded* if $V(x) \to \infty$ as $|x| \to \infty$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class K if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class K_∞ if, in addition, it is proper. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity.

6.2.2 Passivity and feedback passivation

We begin with a brief review of basic definitions and properties related to passive systems. Then we present several more recent results about feedback equivalence of a nonlinear system to a passive system via static state feedback. The interested reader is referred to [50, 12, 2] and references therein for the details.

6.2.2.1 Passivity and passive systems

Consider a nonlinear control system with outputs:

$$\dot{x} = f(x) + G(x)u \tag{6.1}$$

$$y = h(x) \tag{6.2}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. Assume that these functions are locally Lipschitz with $f(0) = 0$ and $h(0) = 0$.

Definition 2.1 (Passivity) A system (6.1)–(6.2) is said to be C^r-*passive* if there exists a C^r storage function $V : \mathbb{R}^n \to \mathbb{R}_+$, with $V(0) = 0$, such that, for all admissible inputs u, all initial conditions x^o and all $t \geq 0$

$$V(x(t)) - V(x^o) \leq \int_0^t y^T(s)u(s)\, \mathrm{d}s \tag{6.3}$$

Definition 2.2 (Strict passivity) A system (6.1)–(6.2) is said to be *output strictly* C^r-*passive* if there exists a C^r storage function $V : \mathbb{R}^n \to \mathbb{R}_+$, with $V(0) = 0$, and a positive definite function $S_1(y)$ such that, for all admissible inputs u, all initial conditions x^o and all $t \geq 0$

$$V(x(t)) - V(x^o) \leq \int_0^t y^T(s)u(s)\, \mathrm{d}s \; - \; \int_0^t S_1(y(s))\, \mathrm{d}s \tag{6.4}$$

If (6.4) holds with a positive definite function $S_2(x)$, i.e.

$$V(x(t)) - V(x^o) \leq \int_0^t y^T(s)u(s)\, \mathrm{d}s \; - \; \int_0^t S_2(x(s))\, \mathrm{d}s \tag{6.5}$$

then the system is said to be *(state) strictly* C^r-*passive*.

The passivity and strict passivity properties of a nonlinear system can be tested by the following ubiquitous nonlinear KYP lemma.

Lemma 2.1 [10] A system (6.1)–(6.2) is (resp. strictly) C^1-passive if and only if there exists a C^1 non-negative function $V(x)$, with $V(0) = 0$, such that $L_f V(x)$ is negative semi-definite (resp. negative definite) and $L_g V(x) = h(x)$ for all $x \in \mathbb{R}^n$.

A fundamental property of a passive system is the Lyapunov stability of its unforced system (i.e. $u = 0$ in (6.1)–(6.2)) and its reduced system with zero-output (i.e. $y = 0$ in (6.1)). More interestingly, under a zero-state detectability condition, passive systems are asymptotically stabilizable by static output feedback. We first recall the definition of the zero-state detectability.

Definition 2.3 (Zero-state detectability) A system of the form (6.1)–(6.2) is *locally zero-state detectable* if there exists a neighbourhood $\mathcal{N}$ of $x = 0$ such that for any initial condition $x(0) \in \mathcal{N}$,

$$u \equiv 0 \quad \text{and} \quad y \equiv 0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0 \tag{6.6}$$

It is *(globally) zero-state detectable* if the property (6.6) holds with $\mathcal{N} = \mathbb{R}^n$.

Theorem 2.1 [11], [2] Assume that system (6.1)–(6.2) is C^0-passive with a positive definite storage function V and that it is locally zero-state detectable. Then, for any 'first-sector third-sector' function ϕ, that is, a continuous function ϕ such that $y^T\phi(y) > 0$ if $y \neq 0$

$$u = -\phi(y) \tag{6.7}$$

is an asymptotically stabilizing controller for the system (6.1)–(6.2). Furthermore, if V is proper and (6.1)–(6.2) is (globally) zero-state detectable, the equilibrium $x = 0$ of the closed loop system (6.1)–(6.7) is globally asymptotically stable (GAS).

6.2.2.2 Feedback passivation

As seen in the preceding subsection and demonstrated by numerous researchers, passive systems enjoy many desirable properties which turn out to be very useful for practical control systems design. Naturally, this motivated people to address the feedback passivation issue. That is, when can a nonlinear control system in the form (6.1)–(6.2) be rendered passive via a state feedback transformation? This question had remained open until [2] where Byrnes, Isidori and Willems nicely provided a rather complete answer by means of differential geometric systems theory.

Definition 2.4 (Passivation) A system (6.1)–(6.2) is said to be *feedback (strictly) C^r-passive* if there exists a change of feedback law

$$u = \mu_1(x) + \mu_2(x)v \tag{6.8}$$

such that the system (6.1)–(6.2)-(6.8) with new input v is (strictly) C^r-passive.

Theorem 2.2 [2] Consider a nonlinear system (6.1)–(6.2) having a global normal form

$$\begin{aligned} \dot{z} &= q(z, y) \\ \dot{y} &= b(z, y) + a(z, y)u \end{aligned} \tag{6.9}$$

where $a(\cdot)$ is globally invertible. Then system (6.1)–(6.2), or (6.9) is globally feedback equivalent to a (resp. strictly) C^2-passive system with a positive definite storage function, if and only if it is globally weakly minimum phase (resp. globally minimum phase).

As an application of this important result to the global stabilization of cascaded nonlinear systems, it was shown in [2] that several previous stabilization schemes via different approaches can be unified. In particular, combining Theorems 2.1 and 2.2, the following general result can be established.

Theorem 2.3 [2, 34] Consider a cascaded nonlinear system of the form

$$\begin{aligned} \dot{\zeta} &= f_0(\zeta) + f_1(\zeta, x)y \\ \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{6.10}$$

Suppose $\dot{\zeta} = f_0(\zeta)$ is GAS at $\zeta = 0$. It is also assumed that $\{f, g, h\}$ is zero-state detectable and C^r-passive with a positive definite and proper storage function ($r \geq 1$). Then system (6.10) is GAS by smooth state feedback.

It should be noted that corresponding local results also hold.

6.2.3 Nonlinear gain

The concept of nonlinear gain has recently been brought to the literature from two apparently different routes: the state-space approach [42, 44, 45, 46] and the input–output approach [39, 9, 30]. The special case of linear or affine gain had previously been extensively used in the stability theory of interconnected systems [5, 11, 33]. In Section 6.4, this notion will play a key role in addressing the robust adaptive control problem for parametric-strict-feedback systems in the presence of nonlinear dynamic uncertainties.

In what follows, we limit ourselves to state-space systems. We begin with the introduction of Sontag's input-to-state stability (ISS) concept in which the role of initial conditions are made explicit. Then, we give an *output* version of this notion to allow for a broader set of controlled dynamical systems. Recent applications of these notions to establish nonlinear small gain theorems for interconnected feedback systems are recalled in the subsection 6.2.3.2.

6.2.3.1 Input-to-state stable systems

Consider the general time-varying dynamical system with outputs

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(t, x, u) \end{aligned} \tag{6.11}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{m_1}$ is the control input and $y \in \mathbb{R}^{m_2}$ is the system output. Notice that many dynamical systems subject to exogenous disturbances can be described by a differential equation of the form (6.11).

Roughly speaking, the property of input-to-state stability (ISS) says that the ultimate bound of the system trajectories depends only on the magnitude of the control input u and that the zero-input ISS system is globally uniformly asymptotically stable (GUAS) at the origin. More precisely,

Definition 2.5 A system of the form (6.11) is said to be *input-to-state stable (ISS)* if for any initial conditions $t_0 \geq 0$ and $x(t_0)$ and for any measurable essentially bounded control function u, the corresponding solution $x(t)$ exists for each $t \geq t_0$ and satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u_{[t_0,t]}\|) \tag{6.12}$$

where β is a class KL-function and γ is a class K-function.

An output version of the ISS property was given in [21, 16] and is recalled as follows.

Definition 2.6 A system of the form (6.11) is said to be *input-to-output practically stable (IOpS)* if for any initial conditions $t_0 \geq 0$ and $x(t_0)$ and for any measurable essentially bounded control function u, the corresponding solution $x(t)$ exists for each $t \geq t_0$ and satisfies

$$|y(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u_{[t_0,t]}\|) + d \tag{6.13}$$

where β is a class KL-function, γ is a class K-function and d is a non-negative constant.

If $d = 0$ in (6.13), the IOpS property becomes *IOS (input-to-output stability)*. Moreover, the system (6.11) is *input-to-state practically stable (ISpS)* if (6.13) holds with $y = x$.

An obvious property of an IOpS system of form (6.11) is that the system is bounded-input bounded-output (BIBO) stable. In addition, for any IOS system (6.11), a converging input yields a converging output.

Among various characterizations of the ISS property is the notion of an ISS-Lyapunov function which was introduced by means of some differential dissipation inequality [46], [28]. In Section 6.4, we consider a class of uncertain systems with persistently exciting disturbances. As a consequence, an extension of the ISS-Lyapunov function notion turns out to be necessary.

Definition 2.7 A smooth function V is said to be an *ISpS-Lyapunov function* for system (6.11) if

- V is proper and positive definite, that is, there exist functions ϕ_1, ϕ_2 of class K_∞ such that
$$\phi_1(|x|) \leq V(x) \leq \phi_2(|x|), \quad \forall x \in \mathbb{R}^n \tag{6.14}$$
- there exist a positive-definite function α, a class K-function χ and a non-negative constant c such that the following implication holds:
$$\{|x| \geq \chi(|u|) + c\} \implies \frac{\partial V}{\partial x}(x) f(t, x, u) \leq -\alpha(|x|) \tag{6.15}$$

When (6.15) holds with $c = 0$, V is called an *ISS-Lyapunov function* for system (6.11) as in [28].

With the help of arguments in [46, 47, 28], it is not hard to prove the following fact.

Fact: If a system of form (6.11) has an ISpS- (resp. ISS-) Lyapunov function V, then the system is ISpS (resp. ISS).

6.2.3.2 Nonlinear small gain theorems

In this subsection, we recall two nonlinear versions of the classical small (finite-) gain theorem [5] which were recently proposed in the work [21, 20, 16]. They will be used to construct robust adaptive controllers and develop the stability analysis in Section 6.4. The interested reader is referred to [4, 6, 20, 21, 49] and references therein for applications of the nonlinear small gain theorems to several feedback control problems.

Consider the general interconnected time-varying nonlinear system

$$\dot{x}_1 = f_1(t, x_1, y_2, u_1), \quad y_1 = h_1(t, x_1, y_2, u_1) \tag{6.16}$$

$$\dot{x}_2 = f_2(t, x_2, y_1, u_2), \quad y_2 = h_2(t, x_2, y_1, u_2) \tag{6.17}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_{1i}}$ and $y_i \in \mathbb{R}^{m_{2i}}$, and f_i, h_i are C^1 in all their arguments. It is assumed that there is a unique C^1 solution to the algebraic loop introduced by the interconnection functions h_1, h_2.

Theorem 2.4 [21, 16] Assume that the subsystems (6.16) and (6.17) are IOpS in the sense that, for all $0 \le t_0 \le t$,

$$\begin{aligned} |y_1(t)| &\le \beta_1(|x_1(t_0)|, t - t_0) + \gamma_1^y(\|y_{2[t_0,t]}\|) + \gamma_1^u(\|u_{1[t_0,t]}\|) + d_1 \\ |y_2(t)| &\le \beta_2(|x_2(t_0)|, t - t_0) + \gamma_2^y(\|y_{1[t_0,t]}\|) + \gamma_2^u(\|u_{2[t_0,t]}\|) + d_2 \end{aligned} \tag{6.18}$$

Assume further that the subsystems (6.16) and (6.17) are also ISpS (resp. ISS). If there exist two class K_∞-functions ρ_1 and ρ_2 and a nonnegative real number s_ℓ satisfying:

$$\left.\begin{aligned} (I + \rho_2) \circ \gamma_2^y \circ (I + \rho_1) \circ \gamma_1^y(s) \le s \\ (I + \rho_1) \circ \gamma_1^y \circ (I + \rho_2) \circ \gamma_2^y(s) \le s \end{aligned}\right\} \quad \forall s \ge s_\ell \tag{6.19}$$

then the interconnected system (6.16)–(6.17) with (y_1, y_2) as output is ISpS (resp. ISS) and IOpS (resp. IOS when $s_\ell = d_i = 0$ for $i = 1, 2$).

Motivated by the above Fact, we consider properties of the feedback system derived from existence of ISpS-Lyapunov functions on each subsystem. Following step by step the proof of Theorem 3.1 in [20], we can prove the following result.

Theorem 2.5 Consider the interconnected system (6.16)–(6.17) with $y_1 = x_1$ and $y_2 = x_2$. Assume that, for $i = 1, 2$, the x_i-subsystem has an ISpS-Lyapunov function V_i satisfying the properties

(1) there exist class K_∞-functions ϕ_{i1} and ϕ_{i2} such that

$$\phi_{i1}(|x_i|) \le V_i(x_i) \le \phi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i} \tag{6.20}$$

(2) there exist class K_∞-functions α_i, class K-functions χ_i, γ_i and some constant $c_i \ge 0$ so that $V_1(x_1) \ge \max\{\chi_1(V_2(x_2)), \gamma_1(|u_1|) + c_1\}$ implies

$$\frac{\partial V_1}{\partial x_1}(x_1) f_1(t, x_1, x_2, u_1) \le -\alpha_1(V_1(x_1)) \tag{6.21}$$

and $V_2(x_2) \geq \max\{\chi_2(V_1(x_1)), \gamma_2(|u_2|) + c_2\}$ implies

$$\frac{\partial V_2}{\partial x_2}(x_2)f_2(t, x_1, x_2, u_2) \leq -\alpha_2(V_2(x_2)) \tag{6.22}$$

If there exists some $c_0 \geq 0$ such that

$$\chi_1 \circ \chi_2(r) < r, \quad \forall r > c_0, \tag{6.23}$$

then the interconnected system (6.16)–(6.17) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$, then the system is ISS.

Remark 2.1 As noticed in the Fact, it follows from (6.21) that the x_1-system is ISpS. Moreover, the following ISpS property can be proved:

$$V_1(x_1(t)) \leq \beta_1(|x_1(t_0)|, t - t_0) + \chi_1(\|V_2(x_2)_{[t_0,t]}\|) + \gamma_1(\|u_{1[t_0,t]}\|) + c_1, \quad t \geq t_0 \geq 0 \tag{6.24}$$

where β_1 is a class *KL*-function.

Similarly, (6.22) implies that the x_2-system is ISpS and enjoys the following IOpS property:

$$V_2(x_1(t)) \leq \beta_2(|x_2(t_0)|, t - t_0) + \chi_2(\|V_1(x_1)_{[t_0,t]}\|) + \gamma_2(\|u_{2[t_0,t]}\|) + c_2, \quad t \geq t_0 \geq 0 \tag{6.25}$$

In order to invoke Theorem 2.4 to conclude the ISpS and IOpS properties for the feedback system, we shall require the small gain condition (6.19) to hold between the gain functions χ_1 and χ_2. This obviously leads to a stronger gain condition than (6.23).

6.3 Adaptive passivation

The main purpose of this section is to extend passivation tools in [2] to a nonlinear system with unknown parameters θ described by:

$$\dot{x} = \varphi_0(x) + \varphi(x)\theta + (\psi_0(x) + \psi(x)\theta)u \tag{6.26}$$

Then, we show that these adaptive passivation tools can be recursively used to design an adaptive controller for a class of interconnected nonlinear systems with unknown parameters. Our results can be regarded as a passivity interpretation of the popular *adaptive backstepping with tuning functions* algorithm as advocated in [26].

6.3.1 Definitions and basic results

We will introduce notions of adaptive stabilizability and adaptive passivation for a system of the form (6.26). Our definition of adaptive stabilizability was motivated by a related notion in [26, p. 132] but is applicable to a broader class of systems – see Section 6.3.2 below. First, we define two useful properties.

Definition 3.1 (COCS- and UO-functions) Consider a control system of the general form $\dot{x} = f(x, u)$.

(i) Assume a dynamic controller $u = \mu(x, \chi)$ where $\dot{\chi} = \nu(x, \chi)$. A continuous function $y = \eta(x, \chi)$ is *converging-output converging-state (COCS)* for the closed loop system if all the bounded solutions $(x(t), \chi(t))$ satisfy the following implication

$$\{\eta(x(t), \chi(t)) \longrightarrow 0\} \quad \Longrightarrow \quad \{x(t) \longrightarrow 0\} \tag{6.27}$$

(ii) A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *unboundedness observable (UO)* if for any initial condition $x(0)$ and any control input $u : [0, T) \to \mathbb{R}^m$, with $0 < T \leq +\infty$, the following holds

$$\{|x(t)| \quad \stackrel{t \to T^*}{\longrightarrow} \quad +\infty\} \quad \Longrightarrow \quad \{V(x(t)) \quad \stackrel{t \to T^*}{\longrightarrow} \quad +\infty\} \tag{6.28}$$

for some $0 < T^* \leq T$.

Note that any proper function $V(x)$ is a UO-function for system $\dot{x} = f(x, u)$, but the converse is not true. Similarly, a sufficient condition for η being a COCS-function is that η is positive definite in x for each χ.

Throughout the chapter, we use $\hat{\theta}$ to denote the update estimate of the unknown parameter θ and restrict ourselves to the case where $\dim \hat{\theta} = \dim \theta$. In other words, overparametrization is avoided in that the number of parameter estimates equals to the number of unknown parameters.

Definition 3.2 (Adaptive stabilizability) The system (6.26) is said to be *globally adaptively (quadratically) stabilizable* if there exist a smooth UO-function $V(x, \theta)$ for each θ, an adaptive law $\dot{\hat{\theta}} = \tau(x, \hat{\theta})$ and an adaptive controller $u = \vartheta(x, \hat{\theta})$ such that the time derivative of the augmented Lyapunov function candidate

$$\overline{V}(x, \hat{\theta}) = V(x, \hat{\theta}) + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1} (\hat{\theta} - \theta) \tag{6.29}$$

satisfies

$$\dot{\overline{V}} \leq -\eta_1(x, \hat{\theta}) \tag{6.30}$$

where $\eta_1(x, \hat{\theta})$ is a COCS-function for the closed loop system (6.26).

Remark 3.1 A system which satisfies this property is stabilizable in the sense that all the solutions of the closed loop system are bounded and that $x(t)$ tends to zero as $t \to \infty$. Indeed, the boundedness property follows from (6.29) and (6.30). In addition, by application of Barbalat's lemma [24], $\eta_1(x(t), \hat{\theta}(t))$ goes to zero as $t \to \infty$. Since η_1 is a COCS-function, the claim follows readily.

Definition 3.3 (Adaptive passivation) The system (6.26) is said to be *adaptively (quadratically) feedback passive (AFP)* if there exist smooth functions

$V(x, \theta) \geq 0$, ϑ and τ, with $V(0, \theta) = 0 \, \forall \theta$, and a C^0 function h such that the resulting system with adaptive feedback

$$\dot{\hat{\theta}} = \tau(x, \hat{\theta}) + \bar{\tau}, \quad u = \vartheta(x, \hat{\theta}) + v \tag{6.31}$$

is passive with respect to input $(v, \bar{\tau})$ and output h and the storage function $\overline{V}$ of the form (6.29). That is

$$\dot{\overline{V}} \leq -\eta_2(x, \hat{\theta}) + h^T(x, \hat{\theta}, \theta) \begin{pmatrix} v \\ \bar{\tau} \end{pmatrix} \tag{6.32}$$

where η_2 is a nonnegative function.

If $\eta_2(x, \hat{\theta})$ is a COCS-function for the zero-input closed loop system (6.26), then system (6.26) is said to be *strongly AFP*.

The following result, which is an adaptive version of [2, Proposition 4.14], shows that the above notions of adaptive stabilizability and adaptive passivation are equivalent.

Proposition 3.1 A system of the form (6.26) is strongly AFP with a UO-function V for (6.26) if and only if it is globally adaptively stabilizable.

Proof The necessity is obvious. To prove the sufficiency, define h by

$$h^T(x, \hat{\theta}, \theta) = \left(\frac{\partial V}{\partial x}(x, \hat{\theta})(\psi_0(x) + \psi(x)\theta), (\hat{\theta} - \theta)^T \Gamma^{-1} \right)$$

which completes the proof by Lemma 2.1.

Let us point out that the problem of adaptive passivation cannot be trivially solved using the existing (known parameter) passivation tools. To this end, consider a scalar nonlinear system

$$\begin{aligned} \dot{x} &= x^2 + \theta x + u \\ y &= x \end{aligned} \tag{6.33}$$

where θ is an unknown constant parameter.

If θ were known, system (6.33) would be made passive via the change of feedback law $u = -x - x^2 - \theta x + v$. However, in the present case where θ is unavailable for feedback design, it is not clear how to find a feedback transformation of the form (6.8) so that the resulting system is passive for every θ in $\mathbb{R}$. In fact, it can be shown that no memoryless continuous state-feedback law of the form (6.8) achieves the passivation goal regardless of the value of θ.

We now give an adaptive passivation result for cascaded nonlinear systems in the form of (6.26) but without unknown parameters in the input space. This result was motivated by Theorem 6.3 in which all parameters are known.

Theorem 3.1 Consider a cascaded nonlinear system of the form

$$\begin{aligned}\dot{\zeta} &= f_0(\zeta) + \sum_{i=1}^{l} f_{1i}(\zeta, x) y \theta_i \\ \dot{x} &= f(x) + \Delta f(x)\theta + g(x)(u + \Delta g(x)\theta) \\ y &= h(x)\end{aligned} \tag{6.34}$$

where θ is a vector of unknown constant parameters.

Suppose $\dot{\zeta} = f_0(\zeta)$ is GAS at $\zeta = 0$. It is also assumed that $\{f, g, h\}$ is strictly C^r-passive with a positive definite and proper storage function ($r \geq 1$). If there exists a matrix g_1 of appropriate dimension such that

$$\Delta f(x) = g(x) g_1(x), \quad \forall x \in \mathbb{R}^n \tag{6.35}$$

then system (6.34) is strongly adaptively feedback passive with a proper storage function $\overline{V}$.

Proof By a Converse Lyapunov Theorem [27], there exists a C^∞ Lyapunov function U which is positive definite and proper satisfying

$$\frac{\partial U}{\partial \zeta}(\zeta) f_0(\zeta) \leq -\eta(\zeta) \tag{6.36}$$

where η is a positive definite function.

Let W be a C^r storage function associated with $\{f, g, h\}$ and consider the augmented Lyapunov function candidate

$$\overline{V}(\zeta, x, \hat{\theta}) = U(\zeta) + W(x) + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1} (\hat{\theta} - \theta) \tag{6.37}$$

By hypotheses and Lemma 2.1, it follows that

$$\begin{aligned}\dot{\overline{V}} \leq &-\eta(\zeta) + \sum_{i=1}^{l} \frac{\partial U}{\partial \zeta}(\zeta) f_{1i}(\zeta, x) y \theta_i + \frac{\partial W}{\partial x}(x) f(x) \\ &+ y[u + (g_1(x) + \Delta g(x))\theta] + (\hat{\theta} - \theta)^T \Gamma^{-1} \dot{\hat{\theta}}\end{aligned} \tag{6.38}$$

Notice that $\frac{\partial W}{\partial x}(x) f(x)$ is negative definite in x.

For each (ζ, x), denote the $m \times l$ matrix $c(\zeta, x)$ as

$$c(\zeta, x) = \left(f_{11}(\zeta, x)^T \frac{\partial U}{\partial \zeta}(\zeta)^T, \ldots, f_{1l}(\zeta, x)^T \frac{\partial U}{\partial \zeta}(\zeta)^T\right) \tag{6.39}$$

Then, (6.38) yields

$$\dot{\overline{V}} \leq -\eta(\zeta) + \frac{\partial W}{\partial x}(x)f(x) + y^T[u + c(\zeta, x)\theta + (g_1(x) + \Delta g(x))\theta] + (\hat{\theta} - \theta)^T \Gamma^{-1} \dot{\hat{\theta}} \tag{6.40}$$

Thus, by choosing the following parameter update law and adaptive controller

$$\dot{\hat{\theta}} = \Gamma(c(\zeta, x) + g_1(x) + \Delta g(x))^T y^T \tag{6.41}$$

$$u = -\phi(y) - (c(\zeta, x) + g_1(x) + \Delta g(x))\hat{\theta} \tag{6.42}$$

with ϕ a 'first-sector third-sector' function, (6.40) implies

$$\dot{\overline{V}} \leq -\eta(\zeta) + \frac{\partial W}{\partial x}(x)f(x) - y^T \phi(y) \tag{6.43}$$

Since $\overline{V}$ is proper in its argument $(\zeta, x, \hat{\theta})$, it follows from (6.43) that all the solutions $(\zeta(t), x(t), \hat{\theta}(t))$ of the closed loop system (6.34), (6.41) and (6.42) are well-defined and uniformly bounded on $[0, \infty)$.

Furthermore, a direct application of LaSalle's invariance principle ensures that all the trajectories $(\zeta(t), x(t), \hat{\theta}(t))$ converge to the largest invariant set E contained in the manifold $\{(\zeta, x, \hat{\theta}) \,|\, (\zeta, x) = (0, 0)\}$. Therefore, $x(t)$ tends to zero as t goes to ∞. In other words, the cascade system (6.34) is globally adaptively stabilized by (6.41) and (6.42). Finally, Proposition 3.1 ends the proof of Theorem 3.1

Remark 3.2 It is of interest to note that, if rank $\{g(0)(g_1(0) + \Delta g(0))\} = l = \dim \theta$, then

$$\lim_{t \to +\infty} |\hat{\theta}(t) - \theta| = 0$$

Indeed, on the set E, we have $g(0)(g_1(0) + \Delta g(0))(\theta - \hat{\theta}(t)) = 0$ which, in turn, implies that $\hat{\theta}(t) = \theta$. So, $E = \{(0, 0, \theta)\}$.

The following corollary is an immediate consequence of Theorem 3.1 where the ζ-system in (6.34) is void.

Corollary 3.1 Consider a linearly parametrized nonlinear system in the form

$$\begin{aligned} \dot{x} &= f(x) + \Delta f(x)\theta + g(x)(u + \Delta g(x)\theta) \\ y &= h(x) \end{aligned} \tag{6.44}$$

If $\{f, g, h\}$ is strictly C^∞-passive with a positive definite and proper storage function, and if (6.35) holds, then system (6.44) is strongly adaptively C^∞-feedback passive with a proper storage function $\overline{V}$.

6.3.2 Recursive adaptive passivation

In this section, we show that the adaptive passivation property can be propagated via adding a feedback passive system with linearly appearing parameters. This is indeed the design ingredient which was used in [19]. More precisely, consider a multi-input multi-output nonlinear system of the form (6.26) with linear parametrization:

$$\begin{aligned}\dot{\xi} &= f_{10}(\xi) + f_1(\xi)\theta + (G_{10}(\xi) + \Delta G_1(\xi)\theta)y \\ \dot{z} &= q(\xi, z, y) \\ \dot{y} &= f_{20}(x) + f_2(x)\theta + (G_{20}(x) + \Delta G_2(x)\theta)u\end{aligned} \tag{6.45}$$

with $x = (\xi^T, z^T, y^T)^T$, $\xi \in \mathbb{R}^{n_0}$, $z \in \mathbb{R}^{n-m}$ and y, $u \in \mathbb{R}^m$. Denote $G_2(x, \theta) = G_{20}(x) + \Delta G_2(x)\theta$.

Proposition 3.2 If the ξ-subsystem of (6.45) with y considered as input is AFP, if G_2 is globally invertible for each θ, then the interconnected system (6.45) is also AFP. Furthermore, under the additional condition that the z-system is BIBS (bounded-input bounded-state) stable and is GAS at $z = 0$ whenever $(\xi, y) = (0, 0)$, if the ξ-system has a UO-function V_1 and a COCS-function η_1 associated with its AFP property, then the whole composite system (6.45) also possesses a UO-function V_2 and a COCS-function η_2 associated with its AFP property.

Remark 3.3 Under the conditions of Proposition 3.2, it follows from Theorem 2.2 that the (z, y)-system in (6.45) is feedback passive for every frozen ξ and each θ.

Proof Introduce the extra integrator $\dot{\hat{\theta}} = \tau_0$ where τ_0 is a new input to be built recursively. By assumption, there exist a smooth positive semidefinite function $V_1(\xi, \hat{\theta})$, smooth functions ϑ_1 and τ_1 as well as a nonnegative function η_1 and a continuous function h_1 such that the time derivative of the function

$$\overline{V}_1 = V_1(\xi, \hat{\theta}) + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1} (\hat{\theta} - \theta) \tag{6.46}$$

satisfies

$$\dot{\overline{V}}_1 \leq -\eta_1(\xi, \hat{\theta}) + h_1^T(\xi, \hat{\theta}, \theta) \begin{pmatrix} \bar{y} \\ \bar{\tau}_1 \end{pmatrix} \tag{6.47}$$

where $\bar{y} := y - \vartheta_1(\xi, \hat{\theta})$ and $\bar{\tau}_1 := \tau_0 - \tau_1(\xi, \hat{\theta})$.

Letting $h_1 = (h_{11}^T, h_{12}^T)^T$, it follows from Lemma 2.1 that

$$h_{11}^T = \frac{\partial V_1}{\partial \xi}(\xi, \hat{\theta})(G_1(\xi) + \Delta G_1(\xi)\theta), \quad h_{12}^T = \frac{\partial V_1}{\partial \hat{\theta}}(\xi, \hat{\theta}) + (\hat{\theta} - \theta)^T \Gamma^{-1} \tag{6.48}$$

which implies $h_{11}(\xi, \hat{\theta}, \theta)$ is affine in θ. Then, there exist smooth functions $\hat{h}_{11}$ and Δh_{11} such that

$$h_{11}(\xi, \hat{\theta}, \theta) = \hat{h}_{11}(\xi, \hat{\theta}) + \Delta h_{11}(\xi, \hat{\theta})(\theta - \hat{\theta}) \tag{6.49}$$

Consider the nonnegative functions

$$V_2 = V_1(\xi, \hat{\theta}) + \tfrac{1}{2}|y - \vartheta_1(\xi, \hat{\theta})|^2 \tag{6.50}$$

$$\overline{V}_2 = \overline{V}_1 + \tfrac{1}{2}|y - \vartheta_1(\xi, \hat{\theta})|^2 \tag{6.51}$$

In view of (6.47) and the definition of $\bar{y}$ and τ_1, the time derivative of $\overline{V}_2$ along the solutions of (6.45) and $\dot{\hat{\theta}} = \tau_0$ satisfies

$$\begin{aligned}\dot{\overline{V}}_2 \le& -\eta_1(\xi, \hat{\theta}) + h_{11}^T(\xi, \hat{\theta}, \theta)\bar{y} + \left(\frac{\partial V_1}{\partial \hat{\theta}}(\xi, \hat{\theta}) + (\hat{\theta} - \theta)^T \Gamma^{-1}\right)\bar{\tau}_1 \\ &+ \bar{y}^T \Bigg[f_{20}(x) + f_2(x)\theta + G_2(x, \theta)u \\ &- \frac{\partial \vartheta_1}{\partial \hat{\theta}}(\tau_1 + \bar{\tau}_1) - \frac{\partial \vartheta_1}{\partial \xi}\left(f_{10}(\xi) + f_1(\xi)\theta + (G_{10}(\xi) + \Delta G_1(\xi)\theta)y_{j+1}\right)\Bigg]\end{aligned} \tag{6.52}$$

We wish to find changes of feedback laws which are independent of θ

$$\begin{aligned} u &= \vartheta_2(x, \hat{\theta}) + v \\ \tau_0 &= \tau_1 + \bar{\tau}_1 = \tau_2 + \bar{\tau}_2 \end{aligned} \tag{6.53}$$

such that $\overline{V}_2$ satisfies a differential dissipation inequality like (6.47).

To this end, set

$$\Delta\tau_2 = \Gamma(\Omega_2^T + \Upsilon_2^T)\bar{y} \tag{6.54}$$

$$\bar{\tau}_2 = \Delta\tau_2 + \bar{\tau}_2 \tag{6.55}$$

where Ω_2, Υ_2 and $\Psi_2 \in \mathbb{R}^{m \times l}$ are defined by

$$\Psi_2 = \left(\frac{\partial \vartheta_1}{\partial \xi}(\xi, \hat{\theta})\Delta G_{11}(\xi)y, \ldots, \frac{\partial \vartheta_1}{\partial \xi}(\xi, \hat{\theta})\Delta G_{1p}(\xi)y\right) \tag{6.56}$$

$$\Omega_2 = f_2(x) + \Delta h_{11}(\xi, \hat{\theta}) - \frac{\partial \vartheta_1}{\partial \xi} f_1(\xi) - \Psi_2(x, \hat{\theta}) \tag{6.57}$$

$$\Upsilon_2 = (\Delta G_{21}(x)\vartheta_2(x, \hat{\theta}), \ldots, \Delta G_{2p}(x)\vartheta_2(x, \hat{\theta})) \tag{6.58}$$

Noticing that

$$\begin{aligned}\left(\frac{\partial V_1}{\partial \hat{\theta}}(\xi, \hat{\theta}) + (\hat{\theta} - \theta)^T \Gamma^{-1}\right)\bar{\tau}_1 =& \frac{\partial V_1}{\partial \hat{\theta}}(\xi, \hat{\theta})\Gamma(\Omega_2^T + \Upsilon_2^T)\bar{y} \\ &+ (\hat{\theta} - \theta)^T(\Omega_2^T + \Upsilon_2^T)\bar{y} \\ &+ \left(\frac{\partial V_1}{\partial \hat{\theta}}(\xi, \hat{\theta}) + (\hat{\theta} - \theta)^T \Gamma^{-1}\right)\bar{\tau}_2\end{aligned} \tag{6.59}$$

and that

$$\frac{\partial V_2}{\partial \hat{\theta}}(x,\hat{\theta}) = \frac{\partial V_1}{\partial \hat{\theta}}(\xi,\hat{\theta}) - \bar{y}^T \frac{\partial \vartheta_1}{\partial \hat{\theta}}(\xi,\hat{\theta}), \tag{6.60}$$

with (6.52), (6.53) and (6.55), simple computation yields

$$\begin{aligned}\dot{V}_2 \le & -\eta_1(\xi,\hat{\theta}) + \left(\frac{\partial V_2}{\partial \hat{\theta}}(x,\hat{\theta}) + (\hat{\theta}-\theta)^T\Gamma^{-1}\right)\bar{\tau}_2 \\ & + \bar{y}^T\Bigg[\hat{h}_{11}(\xi,\hat{\theta}) + f_{20}(x) + f_2(x)\hat{\theta} - \frac{\partial \vartheta_1}{\partial \hat{\theta}}(\tau_1 + \Delta\tau_2) + G_2(x,\theta)v + G_2(x,\hat{\theta})\vartheta_2 \\ & + (\Omega_2 + \Upsilon_2)\Gamma\frac{\partial^T V_1}{\partial \hat{\theta}} - \frac{\partial \vartheta_1}{\partial \xi}(f_{10}(\xi) + f_1(\xi)\hat{\theta} + (G_{10}(\xi) + \Delta G_1(\xi)\hat{\theta})y)\Bigg] \\ := & -\eta_1(\xi,\hat{\theta}) + \left(\frac{\partial V_2}{\partial \hat{\theta}}(x,\hat{\theta}) + (\hat{\theta}-\theta)^T\Gamma^{-1}\right)\bar{\tau}_2 + \bar{y}^T G_2(x,\theta)v \\ & + \bar{y}^T\left(\kappa_2(x,\hat{\theta}) + G_2(x,\hat{\theta})\vartheta_2(x,\hat{\theta}) + \Upsilon_2\Gamma\frac{\partial^T V_1}{\partial \hat{\theta}}(\xi,\hat{\theta})\right)\end{aligned} \tag{6.61}$$

Observing that Υ_2 depends on ϑ_2, the following variable is introduced to split this dependence

$$\rho = \Gamma\frac{\partial^T V_1}{\partial \hat{\theta}}(\xi,\hat{\theta}) \in \mathbb{R}^l$$

With (6.58)

$$G_2(x,\hat{\theta})\vartheta_2(x,\hat{\theta}) + \Upsilon_2\Gamma\frac{\partial^T V_1}{\partial \hat{\theta}}(\xi,\hat{\theta}) = G_2(x,\hat{\theta}+\rho)\vartheta_2(x,\hat{\theta}) \tag{6.62}$$

Since $G_2(x,\theta)$ is globally invertible, by choosing ϑ_2 as

$$\vartheta_2(x,\hat{\theta}) = G_2(x,\hat{\theta}+\rho)^{-1}(-\kappa_2(x,\hat{\theta}) - \bar{y}) \tag{6.63}$$

and defining $h_2 = (h_{21}^T, h_{22}^T)^T$ by

$$h_{21} = G_2^T(x,\theta)\bar{y}, \quad h_{22} = \frac{\partial^T V_2}{\partial \hat{\theta}}(x,\hat{\theta}) + \Gamma^{-1}(\hat{\theta}-\theta) \tag{6.64}$$

we obtain

$$\dot{V}_2 \le -\eta_2(x,\hat{\theta}) + h_2^T(x,\hat{\theta},\theta)\begin{pmatrix} v \\ \bar{\tau}_2 \end{pmatrix} \tag{6.65}$$

with $\eta_2 = \eta_1(\xi,\hat{\theta}) + |\bar{y}|^2$. The first statement of Proposition 3.2 was proved.

The second part of Proposition 3.2 follows readily from our construction and the main result of Sontag [43].

On the basis of Corollary 3.1 and Proposition 3.1, a repeated application of Proposition 3.2 yields the following result on adaptive backstepping stabilization.

Corollary 3.2 [26] Any system in strict-feedback form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \phi_i(x_1, \ldots, x_i)\theta, \quad 1 \le i \le n-1 \\ \dot{x}_n &= u + \phi_n(x_1, \ldots, x_n)\theta\end{aligned} \tag{6.66}$$

is globally adaptively (quadratically) stabilizable.

6.3.3 Examples and extensions

We close this section by illustrating our adaptive passivation algorithm with the help of cascade-interconnected controlled Duffing equations. Possible extensions to the output-feedback case and nonlinear parametrization are briefly discussed via two elementary examples.

6.3.3.1 Controlled Cuffing equations

Consider an interconnected system which is composed of two (modified) Duffing equations in controlled form, i.e.

$$\begin{aligned}\ddot{x}_1 + \delta_1\dot{x}_1 + \theta_1 x_1 + \theta_2 x_1^3 &= u_1 \\ \ddot{x}_2 + \delta_2\dot{x}_2 + \theta_3 x_2 + \theta_4 x_2^3 &= u_2\end{aligned} \tag{6.67}$$

where δ_1, $\delta_2 > 0$ are known parameters, $\theta = (\theta_1, \theta_2, \theta_3.\theta_4)$ is a fourth-order vector of unknown constant parameters and u_2 is the control input. The interconnection constraint is given by

$$u_1 = \delta_2 x_2 + \dot{x}_2 \tag{6.68}$$

Denoting $z_1 = x_1$ and $z_2 = x_2$, the coupled Duffing equations (6.67) can be transformed into the following state-space model

$$\begin{aligned}\dot{z}_1 &= -\delta_1 z_1 + y_1, \quad \dot{y}_1 = -\theta_1 z_1 - \theta_2 z_1^3 + y_2, \\ \dot{z}_2 &= -\delta_2 z_2 + y_2, \quad \dot{y}_2 = -\theta_3 z_2 - \theta_4 z_2^3 + u_2\end{aligned} \tag{6.69}$$

Obviously, the (z_1, y_1)-system in (6.69) is AFP by means of the change of parameter update law and adaptive controller

$$\begin{aligned}\dot{\hat{\theta}}_1 &= -\lambda_1 y_1 z_1 + \bar{\tau}_{11}, \quad \lambda_1 > 0 \\ \dot{\hat{\theta}}_2 &= -\lambda_2 y_1 z_1^3 + \bar{\tau}_{12}, \quad \lambda_2 > 0 \\ y_2 &= -y_1 + \hat{\theta}_1 z_1 + \hat{\theta}_2 z_1^3 + \bar{y}_2\end{aligned}$$

In addition, $V_1 = \frac{1}{2}y_1^2$ is a UO-function for the (z_1, y_1)-system which satisfies the differential dissipation equality

$$\dot{\overline{V}}_1 = -y_1^2 + (y_1, \frac{\hat{\theta}_1 - \theta_1}{\lambda_1}, \frac{\hat{\theta}_2 - \theta_2}{\lambda_2})(\bar{y}_2, \bar{\tau}_1)^T \tag{6.70}$$

with $\overline{V}_1 = V_1(y_1) + \dfrac{1}{2\lambda_1}(\hat{\theta}_1 - \theta_1)^2 + \dfrac{1}{2\lambda_2}(\hat{\theta}_2 - \theta_2)^2$.

It is easy to check that the conditions of Proposition 3.2 hold. A direct application of our adaptive passivation method in the proof of Proposition 3.2 gives our passivity-aimed adaptive stabilizer for system (6.69), or the original system (6.67):

$$\begin{aligned}
\dot{\hat{\theta}}_1 &= -\lambda_1(2y_1 + y_2 - \hat{\theta}_1 z_1 - \hat{\theta}_2 z_1^3)z_1 \\
\dot{\hat{\theta}}_2 &= -\lambda_2(2y_1 + y_2 - \hat{\theta}_1 z_1 - \hat{\theta}_2 z_1^3)z_1^3 \\
\dot{\hat{\theta}}_3 &= -\lambda_3(y_1 + y_2 - \hat{\theta}_1 z_1 - \hat{\theta}_2 z_1^3)z_2, \quad \lambda_3 > 0 \\
\dot{\hat{\theta}}_4 &= -\lambda_4(y_1 + y_2 - \hat{\theta}_1 z_1 - \hat{\theta}_2 z_1^3)z_2^2, \quad \lambda_4 > 0 \\
u &= -2y_1 - 2y_2 + \hat{\theta}_3 z_2 + \hat{\theta}_4 z_2^3 + \dot{\hat{\theta}}_1 z_1 + \dot{\hat{\theta}}_2 z_1^3 + (\hat{\theta}_1 + 3\hat{\theta}_2 z_1^3)(-\delta_1 z_1 + y_1)
\end{aligned} \tag{6.71}$$

6.3.3.2 Adaptive output feedback passivation

The adaptive passivation results presented in the previous sections rely on full-state feedback (6.31). In many practical situations, we often face systems whose state variables are not accessible by the designer except the information of the measured outputs. Unlike the state feedback case, the minimum-phase and the relative degree-one conditions are not sufficient to achieve adaptive passivation if only the output feedback is allowed. This is the case even in the context of (nonadaptive) output feedback passivation, as demonstrated in [40] using the following example:

$$\begin{aligned}
\dot{z} &= -z^3 + \xi \\
\dot{\xi} &= z + u \\
y &= \xi
\end{aligned} \tag{6.72}$$

It was shown in [40, p. 67] that any linear output feedback $u = -ky + v$, with $k > 0$, cannot render the system (6.72) passive. In fact, Byrnes and Isidori [1] proved that the system (6.72) is not stabilizable under *any* C^1 output feedback law. As a consequence, this system is not feedback passive via any C^1 *output* feedback law though it is feedback passive via a C^∞ state feedback law.

However, system (6.72) can be made passive via the C^0 output-feedback given by

$$u = -ky^{\frac{1}{3}} + v, \quad k > \frac{3}{2^{4/3}} \tag{6.73}$$

Indeed, consider the quadratic storage function

$$V = \tfrac{1}{2}z^2 + \tfrac{1}{2}\xi^2 \tag{6.74}$$

Forming the derivative of V with respect to the solutions of (6.72), using Young's inequality [8] gives

$$\begin{aligned}\dot{V} &= -z^4 + 2z\xi - k\xi^{\frac{4}{3}} + yv \\ &\leq -(1-\epsilon)z^4 - (k - \frac{3}{2^{4/3}\epsilon^{1/3}})\xi^{4/3} + yv \end{aligned} \tag{6.75}$$

where $\dfrac{27}{16k^3} < \epsilon < 1$.

Therefore, system (6.72) in closed loop with output feedback (6.73) is (state) strictly passive.

As seen from this example, the output feedback passivation issue is more involved and requires additional conditions on the system or nonsmooth feedback strategy. Thus, it is not surprising that the problem of adaptive output feedback passivation is also complex and solving it needs extra conditions in addition to minimum phaseness and relative-degree one. As an illustration, let us consider a nonlinearly parametrized system with output-dependent nonlinearity:

$$\begin{aligned}\dot{z} &= -z^3 + \xi \\ \dot{\xi} &= z + u + \varphi(y, \theta) \\ y &= \xi \end{aligned} \tag{6.76}$$

where θ is a vector of unknown constant parameters. Assume that the nonlinear function φ checks the following concavity-like condition.

(C) For any y and any pair of parameters (θ_1, θ_2), we have

$$y\varphi(y, \theta_2) \ - \ y\varphi(y, \theta_1) \ \geq \ y\frac{\partial \varphi}{\partial \theta}(y, \theta_2)(\theta_2 - \theta_1) \tag{6.77}$$

Non-trivial examples of φ verifying such a condition include all linear parametrization (i.e. $\varphi(y, \theta) = \varphi_1(y)\theta$) and some nonlinearly parametrized functions like $\varphi(y, \theta) = \varphi_2(y)(\exp(\theta\varphi_3(y)) + \varphi_1(y)\theta)$ where $y\varphi_2(y) \leq 0$ for all $y \in \mathbb{R}$.

Consider the augmented storage function

$$\overline{V} = V(z, \xi) + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1}(\hat{\theta} - \theta) \tag{6.78}$$

where $\hat{\theta}$ is an update parameter to be précised later.

By virtue of (6.73) and (6.75), we have

$$\dot{\overline{V}} \leq -(1-\epsilon)z^4 - \left(k - \frac{3}{2^{4/3}\epsilon^{1/3}}\right)\xi^{4/3} + y(v + \varphi(y, \theta)) + (\hat{\theta} - \theta)^T \Gamma^{-1}\dot{\hat{\theta}} \tag{6.79}$$

Letting

$$v = -\varphi(y, \hat{\theta}) + \bar{v}, \quad \dot{\hat{\theta}} = \Gamma \frac{\partial \varphi}{\partial \theta}(y, \hat{\theta})^T y + \bar{\tau}, \tag{6.80}$$

it follows from the condition (C) and (6.79) that

$$\dot{\bar{V}} \leq -(1-\epsilon)z^4 - \left(k - \frac{3}{2^{4/3}\epsilon^{1/3}}\right)\xi^{4/3} + y\bar{v} + (\hat{\theta} - \theta)^T \Gamma^{-1} \bar{\tau} \tag{6.81}$$

In other words, the system (6.76) is made passive via adaptive output-feedback law (6.73)–(6.80). In particular, the zero-input closed-loop system (i.e. $(\bar{v}, \bar{\tau}) = (0, 0)$) is globally stable at $(z, \xi, \hat{\theta}) = (0, 0, \theta)$ and, furthermore, the trajectories $(z(t), \xi(t))$ go to zero as t goes to ∞.

6.4 Small gain-based adaptive control

Up to now, we have considered nonlinear systems with parametric uncertainty. The synthesis of global adaptive controllers was approached from an input/output viewpoint using passivation–a notion introduced in the recent literature of nonlinear feedback stabilization. The purpose of this section is to address the global adaptive control problem for a broader class of nonlinear systems with various uncertainties including unknown parameters, time-varying and nonlinear disturbance and unmodelled dynamics. Now, instead of passivation tools, we will invoke nonlinear small gain techniques which were developed in our recent papers [21, 20, 16], see references cited therein for other applications.

6.4.1 Class of uncertain systems

The class of uncertain nonlinear systems to be controlled in this section is described by

$$\begin{aligned}
\dot{z} &= q(t, z, x_1) \\
\dot{x}_i &= x_{i+1} + \theta^T \varphi_i(x_1, \ldots, x_i) + \Delta_i(x, z, u, t), \quad 1 \leq i \leq n-1 \\
\dot{x}_n &= u + \theta^T \varphi_n(x_1, \ldots, x_n) + \Delta_n(x, z, u, t) \\
y &= x_1
\end{aligned} \tag{6.82}$$

where u in $\mathbb{R}$ is the control input, y in $\mathbb{R}$ is the output, $x = (x_1, \ldots, x_n)$ is the measured portion of the state while z in $\mathbb{R}^{n_0}$ is the unmeasured portion of the state. θ in $\mathbb{R}^l$ is a vector of unknown constant parameters. It is assumed that the Δ_i's and q are unknown Lipschitz continuous functions but the φ_i's are known smooth functions which are zero at zero.

The following assumptions are made about the class of systems (6.82).

(A1) For each $1 \le i \le n$, there exist an *unknown* positive constant p_i^* and two known nonnegative smooth functions ψ_{i1}, ψ_{i2} such that, for all (z, x, u, t)

$$|\Delta_i(x, z, u, t)| \le p_i^* \psi_{i1}(|(x_1, \ldots, x_i)|) + p_i^* \psi_{i2}(|z|) \tag{6.83}$$

Without loss of generality, assume that $\psi_{i2}(0) = 0$.

(A2) The z-system with input x_1 has an ISpS-Lyapunov function V_0, that is, there exists a smooth positive definite and proper function $V_0(z)$ such that

$$\frac{\partial V_0}{\partial z}(z) q(t, z, x_1) \le -\alpha_0(|z|) + \gamma_0(|x_1|) + d_0 \quad \forall (z, x_1) \tag{6.84}$$

where α_0 and γ_0 are class K_∞-functions and d_0 is a nonnegative constant.

The nominal model of (6.82) without unmeasured z-dynamics and external disturbances Δ_i was referred to as a *parametric-strict-feedback system* in [26] and has been extensively studied by various authors–see the texts [26, 32] and references cited therein. The robustness analysis has also been developed to a perturbed form of the parametric-strict-feedback system in recent years [48, 22, 31, 35, 51]. Our class of uncertain systems allows the presence of more uncertainties and recovers the uncertain nonlinear systems considered previously within the context of *global* adaptive control.

The theory developed in this section presupposes the knowledge of partial x-state information and the virtual control coefficients. Extensions to the cases of output feedback and unknown virtual control coefficients are possible at the expense of more involved synthesis and analysis – see, for instance, [17, 18]. An illustration is given in subsection 6.4.4 via a simple pendulum example.

6.4.2 Adaptive controller design

6.4.2.1 Initialization

We begin with the simple x_1-subsystem of (6.82), i.e.

$$\dot{x}_1 = x_2 + \theta^T \varphi_1(x_1) + \Delta_1(x, z, u, t) \tag{6.85}$$

where x_2 is considered as a virtual control input and z as a disturbance input.

Consider the Lyapunov function candidate

$$V_1 = \tfrac{1}{2}\eta(x_1^2) + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1}(\hat{\theta} - \theta) + \frac{1}{2\lambda}(\hat{p} - p)^2 \tag{6.86}$$

where $\Gamma > 0, \lambda > 0$ are two adaptation gains, η is a smooth class-K_∞ function to be chosen later, $p \ge \max\{p_i^\star, p_i^{\star 2} | 1 \le i \le n\}$ is an unknown constant and the time-varying variables $\hat{\theta}$, $\hat{p}$ are introduced to diminish the effects of parametric uncertainties.

With the help of Assumption (A1), the time derivative of V_1 along the solutions of (6.82) satisfies:

$$\dot{V}_1 \leq \eta'(x_1^2)x_1\big(x_2 + \theta^T\varphi_1(x_1)\big) + p_1^*\eta'(x_1^2)|x_1|[\psi_{11}(|x_1|) + \psi_{12}(|z|)] \\ + (\hat{\theta} - \theta)^T\Gamma^{-1}\dot{\hat{\theta}} + \frac{1}{\lambda}(\hat{p} - p)\dot{\hat{p}} \tag{6.87}$$

where $\eta'(x_1^2)$ is the value of the derivative of η at x_1^2. In the sequel, η is chosen such that η' is nonzero over $\mathbb{R}_+$.

Since ψ_{11} is a smooth function and $\psi_{11}(|x_1|) = \psi_{11}(0) + |x_1|\int_0^1 \psi'_{11}(s|x_1|)\,\mathrm{d}s$, given any $\varepsilon_1 > 0$, there exists a smooth nonnegative function $\hat{\psi}_1$ such that

$$p_1^*\eta'(x_1^2)|x_1|\psi_{11}(|x_1|) \;\leq\; p\eta'(x_1^2)x_1^2\hat{\psi}_1(x_1) + \varepsilon_1\psi_{11}(0)^2, \quad \forall x_1 \in \mathbb{R} \tag{6.88}$$

By completing the squares, (6.87) and (6.88) yield

$$\dot{V}_1 \leq \eta'(x_1^2)x_1(x_2 + \theta^T\varphi_1(x_1) + px_1\hat{\psi}_1(x_1) + p\tfrac{1}{4}x_1\eta'(x_1^2)) + (\hat{\theta} - \theta)^T\Gamma^{-1}\dot{\hat{\theta}} \\ + \frac{1}{\lambda}(\hat{p} - p)\dot{\hat{p}} + \psi_{12}(|z|)^2 + \varepsilon_1\psi_{11}(0)^2 \tag{6.89}$$

Define

$$\tau_1 = -\Gamma\sigma_\theta\hat{\theta} + \Gamma\eta'(x_1^2)x_1\varphi_1(x_1) \tag{6.90}$$

$$\varpi_1 = -\lambda\sigma_p\hat{p} + \lambda x_1^2(\hat{\psi}_1(x_1) + \tfrac{1}{4}\eta'(x_1^2))\eta'(x_1^2) \tag{6.91}$$

$$\vartheta_1 = -x_1\nu_1(x_1^2) - \hat{\theta}^T\varphi_1(x_1) - \hat{p}(x_1\hat{\psi}_1(x_1) + \tfrac{1}{4}x_1\eta'(x_1^2)) \tag{6.92}$$

$$w_2 = x_2 - \vartheta_1(x_1, \hat{\theta}, \hat{p}) \tag{6.93}$$

where σ_θ, $\sigma_p > 0$ are design parameters, ν_1 is a smooth and nondecreasing function satisfying that $\nu_1(0) > 0$.

Consequently, it follows from (6.89) that

$$\dot{V}_1 \leq -\eta' x_1^2\nu_1(x_1^2) + \eta' x_1 w_2 - \sigma_\theta(\hat{\theta} - \theta)^T\hat{\theta} - \sigma_p(\hat{p} - p)\hat{p} + (\hat{\theta} - \theta)^T\Gamma^{-1}(\dot{\hat{\theta}} - \tau_1) \\ + \frac{1}{\lambda}(\hat{p} - p)(\dot{\hat{p}} - \varpi_1) + \psi_{12}(|z|)^2 + \varepsilon_1\psi_{11}(0)^2 \tag{6.94}$$

It is shown in the next subsection that a similar inequality to (6.94) holds for each $(x_1, \ldots, x_i)$-subsystem of (6.82), with $i = 2, \ldots, n$.

6.4.2.2 Recursive steps

Assume that, for a given $1 \leq k < n$, we have established the following property (6.95) for the $(x_1, \ldots, x_k)$-subsystem of system (6.82). That is, for each

$1 \le i \le k$, there exists a proper function V_i whose time derivative along the solutions of (6.82) satisfies

$$\begin{aligned}\dot{V}_i \le & -\eta' x_1^2(\nu_1(x_1^2) - i + 1) - \sum_{j=2}^{i}(c_j - i + j)w_j^2 + w_i w_{i+1} - \sigma_\theta(\hat{\theta} - \theta)^T\hat{\theta} - \sigma_p(\hat{p} - p)\hat{p} \\ & + \left((\hat{\theta} - \theta)^T\Gamma^{-1} - \sum_{j=2}^{i} w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_i) + \left(\frac{1}{\lambda}(\hat{p} - p) - \sum_{j=2}^{i} w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}}\right)(\dot{\hat{p}} - \varpi_i) \\ & + \sum_{j=1}^{i} j\psi_{(i-j+1)2}(|z|)^2 + \sum_{j=1}^{i} j\varepsilon_{i-j+1}\psi_{(i-j+1)1}(0)^2 \end{aligned} \tag{6.95}$$

In (6.95), $\varepsilon_j > 0\,(1 \le j \le i)$ are arbitrary, $c_j > n - j\,(2 \le j \le i)$ are design parameters, $\vartheta_j\,(1 \le j \le i)$, τ_i and ϖ_i are smooth functions and the variables w_j's are defined by

$$w_1 := x_1\eta'(x_1^2), \quad w_j := x_j - \vartheta_{j-1}(x_1, \dots, x_{j-1}, \hat{\theta}, \hat{p}), \quad 2 \le j \le i + 1 \tag{6.96}$$

It is further assumed that $\vartheta_i(0, \dots, 0, \hat{\theta}, \hat{p}) = 0$ for each pair of $(\hat{\theta}, \hat{p})$ and all $1 \le i \le k$.

The above property was established in the preceding subsection with $k = 1$. In the sequel, we prove that (6.95) holds for $i = k + 1$.

Consider the Lyapunov function candidate

$$V_{k+1} = V_k(x_1, w_2, \dots, w_k, \hat{\theta}, \hat{p}) + \tfrac{1}{2}w_{k+1}^2 \tag{6.97}$$

In view of (6.95), differentiating V_{k+1} along the solutions of system (6.82) gives

$$\begin{aligned}\dot{V}_{k+1} \le & -\eta' x_1^2(\nu_1(x_1^2) - k + 1) - \sum_{j=2}^{k}(c_j - k + j)w_j^2 + w_k w_{k+1} - \sigma_\theta(\hat{\theta} - \theta)^T\hat{\theta} - \sigma_p(\hat{p} - p)\hat{p} \\ & + \left((\hat{\theta} - \theta)^T\Gamma^{-1} - \sum_{j=2}^{k} w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_k) + \left(\frac{1}{\lambda}(\hat{p} - p) - \sum_{j=2}^{k} w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}}\right)(\dot{\hat{p}} - \varpi_k) \\ & + w_{k+1}\left(x_{k+2} + \theta^T\varphi_{k+1} + \Delta_{k+1} - \sum_{j=1}^{k}\frac{\partial \vartheta_k}{\partial x_j}(x_{j+1} + \theta^T\varphi_j + \Delta_j) - \frac{\partial \vartheta_k}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \vartheta_k}{\partial \hat{p}}\dot{\hat{p}}\right) \\ & + \sum_{j=1}^{k} j\psi_{(k-j+1)2}(|z|)^2 + \sum_{j=1}^{k} j\varepsilon_{k-j+1}\psi_{(k-j+1)1}(0)^2 \end{aligned} \tag{6.98}$$

Recalling that $p \ge \max\{p_i^\star, p_i^{\star 2} | 1 \le i \le n\}$, by virtue of assumption (A1), we have

$$w_{k+1}\left(\Delta_{k+1}-\sum_{j=1}^{k}\frac{\partial\vartheta_k}{\partial x_j}\Delta_j\right) \le pw_{k+1}^2\left(\tfrac{1}{4}+\tfrac{1}{4}\sum_{j=1}^{k}\left(\frac{\partial\vartheta_k}{\partial x_j}\right)^2\right)$$
$$+|w_{k+1}|\Bigg(p_{k+1}^*\psi_{(k+1)1}(|(x_1,\ldots,x_{k+1})|)$$
$$+\sum_{j=1}^{k}p_j^*\left|\frac{\partial\vartheta_k}{\partial x_j}\right|\psi_{j1}(|(x_1,\ldots,x_j)|)\Bigg)+\sum_{j=1}^{k+1}\psi_{j2}(|z|)^2 \tag{6.99}$$

From (6.93) and (6.96), it is seen that $(w_1,\ldots,w_k,w_{k+1})=(0,\ldots,0,0)$ if and only if $(x_1,\ldots,x_k,x_{k+1})=(0,\ldots,0,0)$. Recall that $\eta'(x_1^2)\neq 0$ by selection. With this observation in hand, given any $\varepsilon_{k+1}>0$, lengthy but simple calculations imply the existence of a smooth nonnegative function $\hat{\psi}_{k+1}$ such that

$$|w_{k+1}|\left(p_{k+1}^*\psi_{(k+1)1}(|(x_1,\ldots,x_{k+1})|)+\sum_{j=1}^{k}p_j^*\left|\frac{\partial\vartheta_k}{\partial x_j}\right|\psi_{j1}(|(x_1,\ldots,x_j)|)\right)$$
$$\le pw_{k+1}^2\hat{\psi}_{k+1}(w_1,w_2,\ldots,w_{k+1},\hat{\theta},\hat{p})+x_1^2\eta'(x_1^2)+\sum_{j=2}^{k}w_j^2+\sum_{j=1}^{k+1}\varepsilon_j\psi_{j1}(0)^2 \tag{6.100}$$

Combining (6.98), (6.99) and (6.100), we obtain

$$\dot{V}_{k+1}\le -\eta' x_1^2(\nu_1(x_1^2)-k)-\sum_{j=2}^{k}(c_j-k-1+j)w_j^2-\sigma_\theta(\hat{\theta}-\theta)^T\hat{\theta}-\sigma_p(\hat{p}-p)\hat{p}$$
$$+\left((\hat{\theta}-\theta)^T\Gamma^{-1}-\sum_{j=2}^{k}w_j\frac{\partial\vartheta_{j-1}}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}}-\tau_k)+\left(\frac{1}{\lambda}(\hat{p}-p)-\sum_{j=2}^{k}w_j\frac{\partial\vartheta_{j-1}}{\partial\hat{p}}\right)(\dot{\hat{p}}-\varpi_k)$$
$$+w_{k+1}\Bigg[x_{k+2}+w_k+\theta^T\varphi_{k+1}-\sum_{j=1}^{k}\frac{\partial\vartheta_k}{\partial x_j}(x_{j+1}+\theta^T\varphi_j)$$
$$+pw_{k+1}\left(\frac{1}{4}+\frac{1}{4}\sum_{j=1}^{k}(\frac{\partial\vartheta_k}{\partial x_j})^2+\hat{\psi}_{k+1}\right)-\frac{\partial\vartheta_k}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{\partial\vartheta_k}{\partial\hat{p}}\dot{\hat{p}}\Bigg]$$
$$+\sum_{j=1}^{k+1}j\psi_{(k-j+2)2}(|z|)^2+\sum_{j=1}^{k+1}j\varepsilon_{k-j+2}\psi_{(k-j+2)1}(0)^2 \tag{6.101}$$

Inspired by the tuning functions method proposed in [26, Chapter 4] for parametric-strict-feedback systems without unmodelled dynamics, introduce the notation

$$\tau_{k+1} = \tau_k + \Gamma\left(\varphi_{k+1} - \sum_{j=1}^{k} \frac{\partial \vartheta_k}{\partial x_j} \varphi_j\right) w_{k+1} \tag{6.102}$$

$$\varpi_{k+1} = \varpi_k + \lambda\left(\tfrac{1}{4} + \tfrac{1}{4}\sum_{j=1}^{k} \left(\frac{\partial \vartheta_k}{\partial x_j}\right)^2 + \hat{\psi}_{k+1}\right) w_{k+1}^2 \tag{6.103}$$

$$\begin{aligned}\vartheta_{k+1} = & -c_{k+1} w_{k+1}^2 - w_k + \sum_{j=1}^{k} \frac{\partial \vartheta_k}{\partial x_j} x_{j+1} \\ & - \left(\hat{\theta}^T - \sum_{j=2}^{k} w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{\theta}} \Gamma\right)\left(\varphi_{k+1} - \sum_{j=1}^{k} \frac{\partial \vartheta_k}{\partial x_j} \varphi_j)\right) \\ & - w_{k+1}\left(\hat{p} - \sum_{j=2}^{k} \lambda w_j \frac{\partial \vartheta_{j-1}}{\partial \hat{p}}\right)\left(\tfrac{1}{4} + \tfrac{1}{4}\sum_{j=1}^{k} \left(\frac{\partial \vartheta_k}{\partial x_j}\right)^2 + \hat{\psi}_{k+1}\right) \\ & + \frac{\partial \vartheta_k}{\partial \hat{\theta}} \tau_{k+1} + \frac{\partial \vartheta_k}{\partial \hat{p}} \varpi_{k+1} w_{k+2}\end{aligned} \tag{6.104}$$

$$w_{k+2} = x_{k+2} - \vartheta_{k+1}(x_1, \ldots, x_{k+1}, \hat{\theta}, \hat{p}) \tag{6.105}$$

Therefore, inequality (6.95) follows readily after equalities (6.102) to (6.105) are substituted into (6.101).

By induction, at the last step where $k = n$ in (6.95), if we choose the following parameter update laws and adaptive controller

$$\dot{\hat{\theta}} = \tau_n(x_1, \ldots, x_n, \hat{\theta}, \hat{p}), \quad \dot{\hat{p}} = \varpi_n(x_1, \ldots, x_n, \hat{\theta}, \hat{p}) \tag{6.106}$$

$$u = \vartheta_n(x_1, \ldots, x_n, \hat{\theta}, \hat{p}) \tag{6.107}$$

the time derivative of the augmented Lyapunov function

$$V_n = \tfrac{1}{2}\eta(y^2) + \sum_{i=2}^{n}(x_i - \vartheta_{i-1})^2 + \tfrac{1}{2}(\hat{\theta} - \theta)^T \Gamma^{-1}(\hat{\theta} - \theta) + \frac{1}{2\lambda}(\hat{p} - p)^2 \tag{6.108}$$

satisfies

$$\begin{aligned}\dot{V}_n \leq & -\eta'(x_1^2)x_1^2(\nu_1(x_1^2) - n + 1) - \sum_{i=2}^{n}(c_i - n + i)w_i^2 - \sigma_\theta(\hat{\theta} - \theta)^T \hat{\theta} - \sigma_p(\hat{p} - p)\hat{p} \\ & + \sum_{i=1}^{n} i\psi_{(n-i+1)2}(|z|)^2 + \sum_{i=1}^{n} i\varepsilon_{n-i+1}\psi_{(n-i+1)1}(0)^2\end{aligned} \tag{6.109}$$

As a major difference with most common adaptive backstepping design procedures [26, 32], because of the presence of dynamic uncertainties z, we are unable to conclude any significant stability property from the inequality (6.109). Another step is needed to robustify the obtained adaptive backstepping controllers (6.106) and (6.107).

6.4.2.3 Small gain design step

The above design steps were devoted to the x-subsystem of (6.82) with z considered as the disturbance input. The effect of unmeasured z-dynamics has not been taken into account in the synthesis of adaptive controllers (6.106) and (6.107). The goal of this section is to specify a subclass of adaptive controllers in the form of (6.106), (6.107) so that the overall closed loop system is Lagrange stable. Furthermore, the output y can be driven to a small vicinity of the origin if the control design parameters are chosen appropriately.

First of all, the design function ν_1 as introduced in subsection 6.4.2.1 is selected to satisfy

$$\eta'(x_1^2)x_1^2(\nu_1(x_1^2) - n + 1) \geq c_1\eta(x_1^2) \tag{6.110}$$

with $c_1 > 0$ a design parameter. Such a smooth function always exists because $\eta'(x_1^2) \neq 0$ for any x_1.

Then, let η_1 be a smooth class-K_∞ function which satisfies the inequality

$$\sum_{i=1}^{n} i\psi_{(n-i+1)2}(|z|)^2 \leq \eta_1(|z|^2) \tag{6.111}$$

Noticing that

$$-\sigma_\theta(\hat{\theta} - \theta)^T\hat{\theta} \leq -\frac{\sigma_\theta}{2\lambda_{\max}(\Gamma^{-1})}(\hat{\theta} - \theta)^T\Gamma^{-1}(\hat{\theta} - \theta) + \frac{\sigma_\theta}{2}|\theta|^2 \tag{6.112}$$

$$-\sigma_p(\hat{p} - p)\hat{p} \leq -\frac{\sigma_p}{2}(\hat{p} - p)^2 + \frac{\sigma_p}{2}p^2 \tag{6.113}$$

(6.106) yields:

$$\dot{V}_n \leq -cV_n + \eta_1(|z|^2) + \varepsilon_1 \tag{6.114}$$

where

$$c = \min\{2c_1, 2(c_i - n + i), \frac{\sigma_\theta}{\lambda_{\max}(\Gamma^{-1})}, \sigma_p\lambda;\ 2 \leq i \leq n\} \tag{6.115}$$

$$\varepsilon_1 = \frac{\sigma_\theta}{2}|\theta|^2 + \frac{\sigma_p}{2}p^2 + \sum_{i=1}^{n} i\varepsilon_{n-i+1}\psi_{(n-i+1)1}(0)^2 \tag{6.116}$$

Let $\underline{\alpha}_v$ and $\overline{\alpha}_v$ be two class-K_∞ functions such that

$$\underline{\alpha}_v(|z|) \leq V_0(z) \leq \overline{\alpha}_v(|z|) \tag{6.117}$$

Given any $0 < \varepsilon_2 < c$, (6.114) ensures that

$$\dot{V}_n \leq -\varepsilon_2 V_n \tag{6.118}$$

whenever

$$V_n \geq \max\left\{\frac{2}{c-\varepsilon_2}\eta_1(\alpha_v^{-1}(V_0(z))^2),\ \frac{2\varepsilon_1}{c-\epsilon_2}\right\} \tag{6.119}$$

Return to the z-subsystem. According to assumption (A2), we have

$$\frac{\partial \tilde{V}_0}{\partial z}(z)q(z,x_1) \leq -\varepsilon_3\alpha_0(|z|) + \gamma(\gamma^{-1}\circ\varepsilon_3\gamma_0(|x_1|)) + \varepsilon_3 d_0 \tag{6.120}$$

where $\tilde{V}_0 = \varepsilon_3 V_0$, $\varepsilon_3 > 0$ is arbitrary and γ is a class-K_∞ function to be determined later.

Given any $0 < \varepsilon_4 < 1$, we obtain

$$\dot{\tilde{V}}_0 \leq -\varepsilon_3\varepsilon_4\alpha_0(|z|) \tag{6.121}$$

as long as

$$\tilde{V}_0 \geq \max\left\{\varepsilon_3\overline{\alpha}_v\circ\alpha_0^{-1}\circ\frac{2}{1-\epsilon_4}\gamma(\gamma^{-1}\circ\gamma_0(|x_1|)),\ \epsilon_3\overline{\alpha}_v\circ\alpha_0^{-1}\left(\frac{2d_0}{1-\epsilon_4}\right)\right\} \tag{6.122}$$

To check the small gain condition (6.23) in Theorem 2.5, we select any class-K_∞ function γ such that

$$\gamma(s) < \frac{1-\varepsilon_4}{2}\alpha_0\circ\overline{\alpha}_v^{-1}\circ\underline{\alpha}_v\left(\sqrt{\eta_1^{-1}\left(\frac{c-\epsilon_2}{4}s\right)}\right), \quad \forall s > 0 \tag{6.123}$$

Finally, to invoke the Small Gain Theorem 2.5, it is sufficient to choose the function η appropriately so that

$$\gamma^{-1}\circ\gamma_0(|x_1|) \leq \tfrac{1}{2}\eta(x_1^2) + \varepsilon_5 \tag{6.124}$$

where $\varepsilon_5 > 0$ is arbitrary. In other words,

$$\gamma^{-1}\circ\gamma_0(|x_1|) \leq V_n(x_1, x_2, \ldots, x_n, \hat{\theta}, \hat{p}) + \varepsilon_5 \tag{6.125}$$

Clearly, such a choice of the smooth function η is always possible. Consequently,

$$\dot{\tilde{V}}_0 \leq -\varepsilon_3\varepsilon_4\alpha_0(|z|) \tag{6.126}$$

as long as

$$\tilde{V}_0 \geq \max\left\{\varepsilon_3\overline{\alpha}_v\circ\alpha_0^{-1}\circ\frac{2}{1-\epsilon_4}\gamma(2V_n),\ \varepsilon_3\overline{\alpha}_v\circ\alpha_0^{-1}\circ\frac{2}{1-\epsilon_4}\gamma(2\varepsilon_5),\ \varepsilon_3\overline{\alpha}_v\circ\alpha_0^{-1}\left(\frac{2d_0}{1-\varepsilon_4}\right)\right\} \tag{6.127}$$

Under the above choice of the design functions η and ν_1, the stability properties of the closed loop system (6.82), (6.106) and (6.107) will be analysed in the next subsection.

6.4.3 Stability analysis

If we apply the above combined backstepping and small-gain approach to the plant (6.82), the stability properties of the resulting closed loop plant (6.82), (6.106) and (6.107) are summarized in the following theorem.

Theorem 4.1 Under Assumptions (A1) and (A2), the solutions of the closed loop system are uniformly bounded. In addition, if a bound on the unknown parameters p_i^* is available for controller design, the output y can be driven to an arbitrarily small interval around the origin by appropriate choice of the design parameters.

Proof Letting $\tilde{\theta} = \hat{\theta} - \theta$ and $\tilde{p} = \hat{p} - p$, it follows that V_n is a positive definite and proper function in $(x_1, \ldots, x_n, \tilde{\theta}, \tilde{p})$. Also, $\dot{\tilde{\theta}} = \dot{\hat{\theta}}$ and $\dot{\tilde{p}} = \dot{\hat{p}}$. Decompose the closed-loop system (6.82), (6.106) and (6.107) into two interconnected subsystems, one is the $(x_1, \ldots, x_n, \tilde{\theta}, \tilde{p})$-subsystem and the other is the z-subsystem. We will employ the Small Gain Theorem 2.5 to conclude the proof.

Consider first the $(x_1, \ldots, x_n, \tilde{\theta}, \tilde{p})$-subsystem. From (6.118) and (6.119), it follows that a gain for this ISpS system with input $\tilde{V}_0$ and output V_n is given by

$$\chi_1(s) = \frac{2}{c - \epsilon_2}\eta_1\left(\underline{\alpha}_v^{-1}\left(\frac{1}{\varepsilon_3}s\right)^2\right) \tag{6.128}$$

Similarly, with the help of (6.126) and (6.127), a gain for the ISpS z-subsystem with input V_n and output $\tilde{V}_0$ is given by

$$\chi_2(s) = \epsilon_3\overline{\alpha}_v \circ \alpha_0^{-1} \circ \frac{2}{1 - \varepsilon_4}\gamma(2s) \tag{6.129}$$

As it can be directly checked, with the choice of γ as in (6.123), the small gain condition (6.23) as stated in Theorem 2.5 is satisfied between χ_1 and χ_2. Hence, a direct application of Theorem 2.5 concludes that the solutions of the interconnected system are uniformly bounded. The second statement of Theorem 4.1 can be proved by noticing that the drift constants in (6.119) and (6.127) can be made arbitrarily small.

Remark 4.1 It is of interest to note that the adaptive regulation method presented in this section can be easily extended to the tracking case. Roughly speaking, given a desired reference signal $y_r(t)$ whose derivatives $y_r^{(i)}(t)$ of order up to n are bounded, we can design an adaptive state feedback controller so that the system output $y(t)$ remains near the reference trajectory $y_r(t)$ after a considerable period of time.

Remark 4.2 Our control design procedure can be applied *mutatis mutandis* to a broader class of block-strict-feedback systems [26] with nonlinear unmodelled dynamics

$$\begin{aligned}
\dot{z} &= q(t, z, x_1) \\
\dot{x}_i &= x_{i+1} + \theta^T \varphi_i(x_1, \ldots, x_i, \zeta_1, \ldots, \zeta_i) + \Delta_{x_i}(x, \zeta, z, u, t) \\
\dot{\zeta}_i &= \Phi_{i,0}(x_1, \ldots, x_i, \zeta_1, \ldots, \zeta_i) + \theta^T \Phi_i(x_1, \ldots, x_i, \zeta_1, \ldots, \zeta_i) \\
&\quad + \Delta_{\zeta_i}(x, \zeta, z, u, t), \quad 1 \leq i \leq n
\end{aligned} \tag{6.130}$$

where $x_{n+1} = u$, $x = (x_1, \ldots, x_n)^T$ and $\zeta = (\zeta_1, \ldots, \zeta_n)^T$. Assume that all ζ_i-dynamics are measured and satisfy a BIBS stability property when $(x_1, \ldots, x_i, \zeta_1, \ldots, \zeta_{i-1}, z)$ is considered as the input. Similar conditions to (6.83) are required on the disturbances Δ_{x_i} and Δ_{ζ_i}.

6.4.4 Examples and discussions

We demonstrate the effectiveness of our robust adaptive control algorithm by means of a simple pendulum with external disturbances. Along the way, we show that our combined backstepping and small gain control design procedure can be extended to cover systems with *unknown* virtual control coefficients. Moreover, we shall see that the consideration of dynamic uncertainties occurring in our class of systems (6.82) becomes very natural when the output feedback control problem is addressed. Then, in the subsection 6.4.4.2, we compare the above adaptive design method with the *dynamic normalization*-based adaptive scheme proposed in our recent contribution [18] via a second order nonlinear system.

6.4.4.1 Pendulum example

The following simple pendulum model has been used to illustrate several nonlinear feedback designs (see, e.g., [3,24]):

$$ml\ddot{\delta} = -mg \sin \delta - kl\dot{\delta} + \frac{1}{l} u + \Delta_0(t) \tag{6.131}$$

where $u \in \mathbb{R}$ is the torque applied to the pendulum, $\delta \in \mathbb{R}$ is the anticlockwise angle between the vertical axis through the pivot point and the rod, g is the acceleration due to gravity, and the constants k, l and m denote a coefficient of friction, the length of the rod and the mass of the bob, respectively. $\Delta_0(t)$ is a time-varying disturbance such that $|\Delta_0(t)| \leq a_0$ for all $t \geq 0$. It is assumed that these constants k, l, m and a_0 are unknown and that the angular velocity $\dot{\delta}$ is not measured.

Using the adaptive regulation algorithm proposed in subsection 6.4.3, we want to design an adaptive controller using angle-only so that the pendulum is kept around any angle $-\pi < \delta = \delta_0 \leq \pi$.

We first introduce the following coordinates

$$\begin{aligned}\xi_1 &= ml^2(\delta - \delta_0)\\ \xi_2 &= ml^2\left(\dot{\delta} + \frac{k}{m}(\delta - \delta_0)\right)\end{aligned} \tag{6.132}$$

to transform the target point $(\delta, \dot{\delta}) = (\delta_0, 0)$ into the origin $(\xi_1, \xi_2) = (0, 0)$.

It is easy to check that the pendulum model (6.131) is written in ξ-coordinates as

$$\dot{\xi}_1 = \xi_2 - \frac{k}{m}\xi_1 \tag{6.133}$$

$$\dot{\xi}_2 = u \; - \; mgl\sin\left(\delta_0 + \frac{1}{ml^2}\xi_1\right) + l\Delta_0(t) \tag{6.134}$$

Since the parameters k, l and m are unknown and the angular velocity $\dot{\delta}$ is unmeasured, the state $\xi = (\xi_1, \xi_2)$ of the transformed system (6.133)–(6.134) is therefore not available for controller design. We try to overcome this burden with the help of the 'Separation Principle' for output-feedback nonlinear systems used in recent work (see, e.g., [23, 26, 32, 36]).

Here, an observer-like dynamic system is introduced as follows

$$\begin{aligned}\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \ell_1((\delta - \delta_0) - \hat{\xi}_1)\\ \dot{\hat{\xi}}_2 &= u + \ell_2((\delta - \delta_0) - \hat{\xi}_1)\end{aligned} \tag{6.135}$$

where ℓ_1 and ℓ_2 are design parameters. Denote the error dynamics e as

$$e := (\xi_1 - \hat{\xi}_1, \xi_2 - \hat{\xi}_2)^T \tag{6.136}$$

Noticing (6.132), the e-dynamics satisfy

$$\dot{e} = \underbrace{\begin{bmatrix} -\ell_1 & 1\\ \ell_2 & 0\end{bmatrix}}_{A} e + \begin{bmatrix} \left(\ell_1 - \dfrac{\ell_1}{ml^2} - \dfrac{k}{m}\right)\xi_1 \\ \left(\ell_2 - \dfrac{\ell_2}{ml^2}\right)\xi_1 - mgl\sin\left(\delta_0 + \dfrac{1}{ml^2}\xi_1\right) + l\Delta_0(t)\end{bmatrix} \tag{6.137}$$

For the purpose of control law design, let us choose a pair of design parameters ℓ_1 and ℓ_2 so that A is an asymptotically stable matrix.

Letting $x_1 = \delta - \delta_0$, $x_2 = \hat{\xi}_2$ and $z = e/a$ with

$$a = \max\left\{|\ell_1 ml^2 - \ell_1 - kl^2|, |\ell_2 ml^2 - \ell_2|, mgl, la_0\right\} \tag{6.138}$$

we establish the following system to be used for controller design:

$$
\begin{aligned}
\dot{z} &= Az + \begin{bmatrix} x_1(\ell_1 ml^2 - \ell_1 - kl^2)/a \\ x_1(\ell_2 ml^2 - \ell_2)/a - (mgl/a)\sin(x_1 + \delta_0) + (l/a)\Delta_0 \end{bmatrix} \\
\dot{x}_1 &= \frac{1}{ml^2}x_2 - \frac{k}{m}x_1 + \frac{1}{ml^2}az_2 \\
\dot{x}_2 &= u + \ell_2(1 - ml^2)x_1 + \ell_2 az_1 \\
y &= x_1
\end{aligned}
\tag{6.139}
$$

Since the unknown coefficient $\frac{1}{ml^2}$, referred to as a 'virtual control coefficient' [26], occurs before x_2, this system (6.139) is not really in the form (6.82). Nonetheless, we show in the sequel that our control design procedure in subsection 6.4.3 can be easily adapted to this situation.

Let $P > 0$ be the solution of the Lyapunov equation

$$PA + A^T P = -2I \tag{6.140}$$

Then, it is directly checked that along the solutions of the z-system in (6.139) the time derivation of $V_0 = z^T Pz$ satisfies

$$\dot{V}_0 \leq -|z|^2 + 3\lambda_{\max}(P)x_1^2 + 8\lambda_{\max}(P) \tag{6.141}$$

Step 1: Instead of (6.86), consider the proper function

$$V_1 = \frac{1}{2ml^2}x_1^2 + \frac{1}{2\lambda}(\hat{p} - p)^2 \tag{6.142}$$

where

$$p \geq \max\left\{a^2, \frac{1}{ml^2}, \frac{1}{m^2l^4}, \frac{k^2}{m^2}, \frac{a^2}{m^2l^4}\right\} \tag{6.143}$$

It is important to note that we have not introduced the update parameter $\hat{\theta}_1$ for the unknown but negative parameter $-k/m$ because the term $-(k/m)x_1$ is stabilizing in the x_1-subsystem of (6.139).

The time derivative of V_1 along the solutions of (6.139) yields:

$$\dot{V}_1 \leq -\nu_1 x_1^2 + x_1 w_2 - \sigma_p(\hat{p} - p)\hat{p} + \frac{1}{\lambda}(\hat{p} - p)(\dot{\hat{p}} - \varpi_1) + \tfrac{1}{2}z_2^2 \tag{6.144}$$

where $\nu_1 > 0$ is a design parameter, ϖ_1 and w_2 are defined by

$$\varpi_1 = -\lambda\sigma_p\hat{p} + \lambda\frac{x_1^2}{2}, \tag{6.145}$$

$$\vartheta_1 = -\nu_1 x_1 - \hat{p}\frac{x_1}{2}, \tag{6.146}$$

$$w_2 = x_2 - \vartheta_1(x_1, \hat{p}) \tag{6.147}$$

Step 2: Consider the proper function

$$V_2 = V_1 + \tfrac{1}{2}w_2^2 \tag{6.148}$$

Then, with (6.144), the time derivative of V_2 along the solutions of (6.139) satisfies

$$\begin{aligned}\dot{V}_2 \le& -\nu_1 x_1^2 + x_1 w_2 - \sigma_p(\hat{p}-p)\hat{p} + \frac{1}{\lambda}(\hat{p}-p)(\dot{\hat{p}}-\varpi_1) + \tfrac{1}{2}z_2^2\\ &+ w_2\left(u + \ell_2(1-ml^2)x_1 + \ell_2 a z_1 + \left(\nu_1 + \frac{\hat{p}}{2}\right)\left(\frac{1}{ml^2}x_2 - \frac{k}{m}x_1 + \frac{1}{ml^2}az_2\right) + \dot{\hat{p}}\frac{x_1}{2}\right)\end{aligned} \tag{6.149}$$

From the definition of x_2 in (6.147) and p as in (6.143), we ensure that

$$w_2\left(\nu_1 + \frac{\hat{p}}{2}\right)\frac{1}{ml^2}x_2 \le p\left(\left|\nu_1 + \frac{\hat{p}}{2}\right| + \left(\nu_1 + \frac{\hat{p}}{2}\right)^4\right)w_2^2 + \tfrac{1}{4}x_1^2 \tag{6.150}$$

With the choice of p as in (6.143), by completing the squares, it follows from (6.149) and (6.150) that

$$\begin{aligned}\dot{V}_2 \le& -(\nu_1 - 1)x_1^2 - \sigma_p(\hat{p}-p)\hat{p} + \frac{1}{\lambda}(\hat{p}-p)(\dot{\hat{p}}-\varpi_1) + |z|^2\\ &+ w_2\left[u + x_1 + \dot{\hat{p}}\frac{x_1}{2} + p\left(1 + \left|\nu_1 + \frac{\hat{p}}{2}\right| + (\nu_1 + \frac{\hat{p}}{2})^2 + (\nu_1 + \frac{\hat{p}}{2})^4 + \frac{\ell_2^2}{4}\right)w_2\right]\end{aligned} \tag{6.151}$$

Thus, setting

$$\begin{aligned}\dot{\hat{p}} &= \varpi_1 + \lambda\left(1 + \left|\nu_1 + \frac{\hat{p}}{2}\right| + \left(\nu_1 + \frac{\hat{p}}{2}\right)^2 + \left(\nu_1 + \frac{\hat{p}}{2}\right)^4 + \frac{\ell_2^2}{4}\right)w_2^2\\ u &= -\nu_2 w_2 - x_1 - \dot{\hat{p}}\frac{x_1}{2} - \hat{p}\left(1 + \left|\nu_1 + \frac{\hat{p}}{2}\right| + (\nu_1 + \frac{\hat{p}}{2})^2 + (\nu_1 + \frac{\hat{p}}{2})^4 + \frac{\ell_2^2}{4}\right)w_2\end{aligned} \tag{6.152}$$

and substituting these definitions into (6.151), we establish

$$\dot{V}_2 \ \leq\ -(\nu_1 - 1)x_1^2 - \nu_2 w_2^2 - \frac{\sigma_p}{2}(\hat{p} - p)^2 + \frac{\sigma_p}{2}p^2 + |z|^2 \tag{6.153}$$

with $\nu_2 > 0$ a design parameter.

In the present situation, it is easy to compute a storage function for the whole closed loop system from the above differential inequalities (6.141) and (6.153). So, instead of pursuing the small gain design step as in subsection 6.4.2.3 where no storage function for the total system was given, we give such a storage function for the closed loop pendulum system. Indeed, consider the composite storage function

$$V = V_0(z) + \tfrac{1}{2}V_2(x_1, x_2, \hat{p}) \tag{6.154}$$

Clearly, from (6.141) and (6.153), it follows

$$\dot{V} \leq -0.5(\nu_1 - 1)x_1^2 - 0.5\nu_2 w_2^2 - \frac{\sigma_p}{4}(\hat{p} - p)^2 - 0.5|z|^2 + \frac{\sigma_p}{4}p^2 \tag{6.155}$$

In other words

$$\dot{V} \leq -cV + \frac{\sigma_p}{4}p^2 \tag{6.156}$$

with

$$c = \min\{(\nu_1 - 1)ml^2, \nu_2, 0.5\sigma_p\lambda, 0.5\lambda_{\max}(P)^{-1}\}$$

Finally, from (6.156), it is seen that all the solutions of the closed loop system are bounded. In particular, the angle δ eventually stays arbitrarily close to the given angle δ_0 if an a priori bound on the system parameters m, l are known and the design parameters ν_1, ν_2, σ_p and λ are chosen appropriately.

6.4.4.2 Robustification via dynamic normalization

It should be noted that an alternative adaptive control design was recently proposed in [17, 18] for a similar class of uncertain systems (6.82). The adaptive strategy in [17, 18] is a nonlinear generalization of the well-known *dynamic normalization* technique in the adaptive linear control literature [13] in that a dynamic signal was introduced to inform about the size of unmodelled dynamics. The adaptive nonlinear control design presented in this chapter yields a lower order adaptive controller than in [17, 18]. Nevertheless, due to the worse-case nature of this design, the consequence is that the present adaptive scheme may yield a conservative adaptive control law for some systems with parametric and dynamic uncertainties. Therefore, a co-ordinated design which exploits the advantages and avoids the disadvantages of these two adaptive control approaches is certainly desirable and this is left for future investigation.

As an illustration of this important point, let us compare the two methods with the following simple example:

$$\dot{z} = -z + x \tag{6.157}$$

$$\dot{x} = u + \theta x + z^2 \tag{6.158}$$

where z is unmeasured and θ is unknown.

Let us start with Method I: robust adaptive control approach without dynamic normalization as proposed in this chapter.

Considering the z-subsystem with input x, assumption (A2) holds with $V_0 = z^2$ whose time derivative satisfies

$$\dot{V}_0 \leq -|z|^2 + x^2 \tag{6.159}$$

In order to apply the Small Gain Theorem 2.5 we show that the x-system can be made ISpS (input-to-state practically stable) via an adaptive controller. An ISpS-Lyapunov function is obtained for the augmented system.

To this purpose, consider the function

$$W = \tfrac{1}{2}\eta(x^2) + \frac{1}{2\gamma}(\hat{\theta} - \theta)^2 \tag{6.160}$$

where $\gamma > 0$ and η is a smooth function of class K_∞.

A direct computation implies:

$$\dot{W} = x\eta'\left(u + \theta x + z^2\right) + \frac{1}{\gamma}(\hat{\theta} - \theta)\dot{\hat{\theta}} \tag{6.161}$$

$$\leq x\eta'\left(u + \hat{\theta}x + \tfrac{1}{4}x\eta'\right) + z^4 + \frac{1}{\gamma}(\hat{\theta} - \theta)\left(\dot{\hat{\theta}} - \gamma x^2\eta'\right) \tag{6.162}$$

where η' stands for the derivative of η.

By choosing the adaptive law and adaptive controller

$$\dot{\hat{\theta}} = -\sigma_\theta\gamma\hat{\theta} + \gamma x^2\eta'(x^2) \tag{6.163}$$

$$u = -x\nu(x^2) - \hat{\theta}x - \tfrac{1}{4}x\eta'(x^2) \tag{6.164}$$

where $\sigma_\theta > 0$ and $\nu(\cdot) > 0$ is a smooth nondecreasing function, it holds:

$$\dot{W} \leq -x^2\eta'\nu(x^2) - \tfrac{1}{2}\sigma_\theta(\hat{\theta} - \theta)^2 + z^4 + \tfrac{1}{2}\sigma_\theta\theta^2 \tag{6.165}$$

Select ν so that

$$x^2\eta'(x^2)\nu(x^2) \geq \eta(x^2) \tag{6.166}$$

Then (6.165) gives:

$$\dot{W} \leq -\delta W + z^4 + \frac{\sigma_\theta}{2}\theta^2 \tag{6.167}$$

with $\delta := \min\{2, \sigma_\theta\gamma\}$.

Hence, letting $\widetilde{\theta} = \hat{\theta} - \theta$ and noticing $\dot{\widetilde{\theta}} = \dot{\hat{\theta}}$, (6.167) implies that W is an ISpS-Lyapunov function for the x-system augmented with the $\widetilde{\theta}$-system when z is considered as the input.

To complete our small gain argument, we need to choose an appropriate function η so that a small gain condition holds. Following the small gain design step developed in subsection 6.4.2.3, we need to pick a function η such that

$$2x^4 \leq \frac{\delta}{4}\eta(x^2) \leq \frac{\delta}{2}W \tag{6.168}$$

A choice of such a function η to meet (6.168) is:

$$\eta(s) = \frac{8}{\delta}s^2 \tag{6.169}$$

This leads to the following choice for ν

$$\nu(x^2) \geq \tfrac{1}{2} \tag{6.170}$$

and therefore to the following controller

$$\dot{\hat{\theta}} = -\sigma_\theta \gamma \hat{\theta} + \frac{16\gamma}{\delta}x^4 \tag{6.171}$$

$$u = -x - \hat{\theta}x - \frac{4}{\delta}x^3 \tag{6.172}$$

In view of (6.159) and (6.167), a direct application of the Small Gain Theorem 2.5 concludes that all the solutions $(x(t), z(t), \hat{\theta}(t))$ are bounded over $[0, \infty)$.

In the sequel, we concentrate on the Method II: robust adaptive control approach with dynamic normalization as advocated in [17,18].

To derive an adaptive regulator on the basis of the adaptive control algorithm in [17], [18], we notice that, thanks to (6.159), a dynamic signal $r(t)$ is given by:

$$\dot{r} = -0.8r + x^2, \quad r(0) > 0 \tag{6.173}$$

The role of this signal r is to dominate $V_0(z)$ – the output of the unmodelled effects – in finite time. More precisely, there exist a finite $T^o > 0$ and a nonnegative time function $D(t)$ such that $D(t) = 0$ for all $t \geq T^o$ and

$$V_0(z(t)) \leq r(t) + D(t), \quad \forall\, t \geq 0 \tag{6.174}$$

Consider the function

$$V = \tfrac{1}{2}x^2 + \frac{1}{2\gamma}(\hat{\theta} - \theta)^2 + \frac{1}{\lambda_0}r \tag{6.175}$$

where $\lambda_0 > 0$. A direct application of the adaptive scheme in [18] yields the

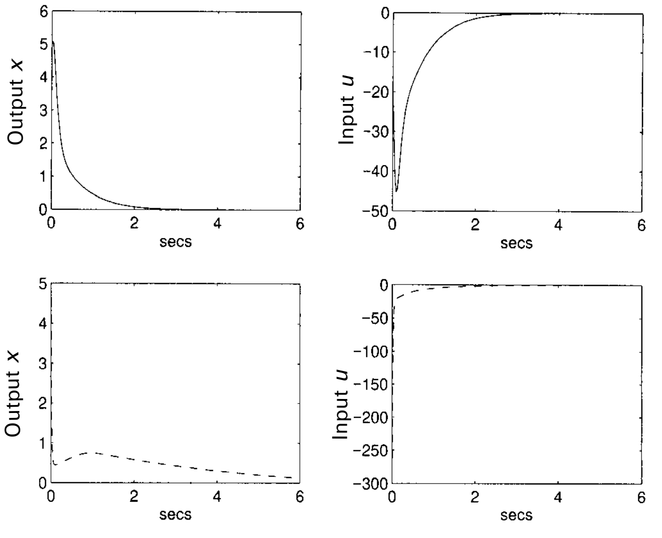

Figure 6.1 *Method II with $r(t)$ versus Method I without $r(t)$: the solid lines refer to Method II while the dashed lines to Method I*

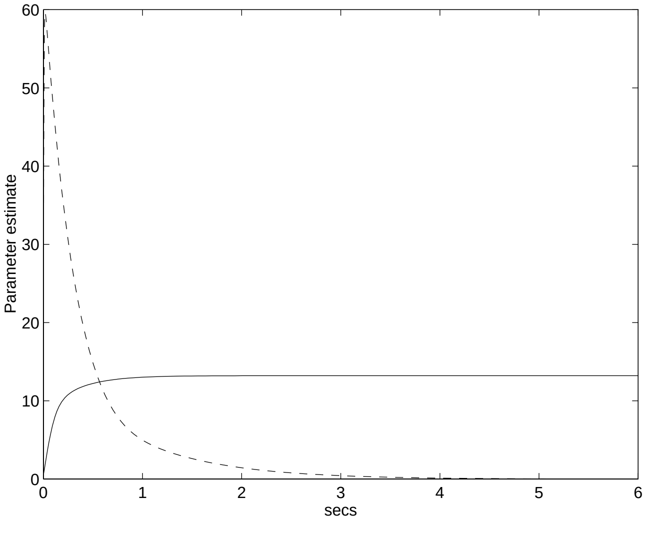

Figure 6.2 *Method II with $r(t)$ versus Method I without $r(t)$: the solid lines refer to Method II while the dashed lines to Method I*

following adaptive regulator:

$$\dot{\hat{\theta}} = \gamma x^2, \quad \gamma > 0 \tag{6.176}$$

$$u = -\left(\frac{5}{4} + \frac{1}{\lambda_0}\right)x - \hat{\theta}x - \frac{\lambda_0}{1.6}xr \tag{6.177}$$

With such a choice, the time derivative of V satisfies:

$$\dot{V} \leq -x^2 - \frac{0.4}{\lambda_0}r \tag{6.178}$$

Therefore, all solutions $x(t)$, $r(t)$ and $z(t)$ converge to zero as t goes to ∞.

Note that the adaptive controller (6.177) contains the dynamic signal r which is a filtered version of x^2 while in the adaptive controller (6.172), we have directly x^2. But, more interestingly, the adaptation law (6.176) is in x^2 whereas (6.171) is in x^4. As seen in our simulation (see Figures 6.1 and 6.2), for larger initial condition for x, this results in a larger estimate $\hat{\theta}$ and consequently a larger control u. Note also that, with the dynamic normalization approach II, the output $x(t)$ is driven to $+0.5\%$ in two seconds.

For simulation use, take $\theta = 0.1$ and design parameters $\gamma = 3$ and $\sigma_\theta = r(0) = \lambda_0 = 1$. The simulations in Figures 6.1 and 6.2 are based on the following choice of initial conditions:

$$z(0) = x(0) = 5, \quad \hat{\theta}(0) = 0.5$$

Summarizing the above, though conservative in some situations, the adaptive nonlinear control design without dynamic normalization presented in this chapter requires less information on unmodelled dynamics and gives simple adaptive control laws. As seen in Example (6.157), the robustification scheme using dynamic normalization may yield better performance at the price of requiring more information on unmodelled dynamics and a more complex controller design procedure. A robust adaptive control design which has the best features of these approaches deserves further study.

6.5 Conclusions

We have revisited the problem of global adaptive nonlinear state-feedback control for a class of block-cascaded nonlinear systems with unknown parameters. It has been shown that adaptive passivation represents an important tool for the systematic design of adaptive nonlinear controllers. Early Lyapunov-type adaptive controllers for systems in parametric-strict-feedback form can be reobtained via an alternative simpler path. While passivation is well suited for systems with a feedback structure, small gain arguments have proved to be more appropriate for systems with unstructured disturbances. From practical considerations, a new class of uncertain nonlinear systems with unmodelled dynamics has been considered in the second part of this chapter. A novel recursive robust adaptive control method by means of backstepping and small gain techniques was proposed to generate a new class of adaptive nonlinear controllers with robustness to nonlinear unmodelled dynamics.

It should be mentioned that passivity and small gain ideas are naturally complementary in stability theory [5]. However, this idea has not been used in nonlinear control design. We hope that the passivation and small gain frameworks presented in this chapter show a possible avenue to approach this goal.

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References

[1] Byrnes, C. I. and Isidori, A. (1991) 'Asymptotic Stabilization of Minimum-Phase Nonlinear Systems', *IEEE Trans. Automat. Control*, **36**, 1122–1137.
[2] Byrnes, C. I., Isidori, A. and Willems, J.C. (1991) 'Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems', *IEEE Trans. Automat. Control*, **36**, 1228–1240.
[3] Corless, M. J. and Leitmann, G. (1981) 'Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems', *IEEE Trans. Automat. Control*, **26**, 1139–1144.
[4] Coron, J. -M., Praly, L. and Teel, A. (1995) 'Feedback Stabilization of Nonlinear Systems: Sufficient Conditions and Lyapunov and Input-Output Techniques'. In: *Trends in Control* (A. Isidori, ed.) 293–348, Springer.
[5] Desoer, C. A. and Vidyasagar, M. (1975) '*Feedback Systems: Input-Output Properties*'. New York: Academic Press.
[6] Fradkov, A. L., Hill, D. J., Jiang, Z. P. and Seron, M. M. (1995) 'Feedback Passification of Interconnected Systems', *Prep. IFAC NOLCOS'95*, 660–665, Tahoe City, California.
[7] Hahn, W. (1967) *Stability of Motion*. Springer-Verlag.
[8] Hardy, G., Littlewood, J. E. and Polya, G. (1989) *Inequalities*. 2nd Edn, Cambridge University Press.
[9] Hill, D. J. (1991) 'A Generalization of the Small-gain Theorem for Nonlinear Feedback Systems', *Automatica*, **27**, 1047–1050.
[10] Hill, D. J. and Moylan, P. (1976) 'The Stability of Nonlinear Dissipative Systems', *IEEE Trans. Automat. Contr.*, **21**, 708–711.
[11] Hill, D. J. and Moylan, P. (1977) 'Stability Results for Nonlinear Feedback Systems', *Automatica*, **13**, 377–382.
[12] Hill, D. J. and Moylan, P. (1980) 'Dissipative Dynamical Systems: Basic Input–Output and State Properties', *J. of The Franklin Institute*, **309**, 327–357.

[13] Ioannou, P. A. and Sun, J. (1996) *Robust Adaptive Control*. Prentice-Hall, Upper Saddle River, NJ.

[14] Jiang, Z. P. and Hill, D. J. (1997) 'Robust Adaptive Nonlinear Regulation with Dynamic Uncertainties', *Proc. 36th IEEE Conf. Dec. Control*, San Diego, USA.

[15] Jiang, Z. P. and Hill, D. J. (1998) 'Passivity and Disturbance Attenuation via Output Feedback for Uncertain Nonlinear Systems', *IEEE Trans. Automat. Control*, **43**, 992–997.

[16] Jiang, Z. P. and Mareels, I. (1997) 'A Small Gain Control Method for Nonlinear Cascaded Systems with Dynamic Uncertainties', *IEEE Trans. Automat. Contr.*, **42**, 292–308.

[17] Jiang, Z. P. and Praly, L. (1996) 'A Self-tuning Robust Nonlinear Controller', *Proc. 13th IFAC World Congress*, Vol. K, 73–78, San Francisco.

[18] Jiang, Z. P. and Praly, L. (1998) 'Design of Robust Adaptive Controllers for Nonlinear Systems with Dynamic Uncertainties', *Automatica*, Vol. 34, No. 7, 825–840.

[19] Jiang, Z. P., Hill, D. J. and Fradkov, A. L. (1996) 'A Passification Approach to Adaptive Nonlinear Stabilization', *Systems & Control Letters*, **28**, 73–84.

[20] Jiang, Z. P., Mareels, I. and Wang, Y. (1996) 'A Lyapunov Formulation of the Nonlinear Small-gain Theorem for Interconnected ISS Systems', *Automatica*, **32**, 1211–1215.

[21] Jiang, Z. P., Teel, A. and Praly, L. (1994) 'Small-gain Theorem for ISS Systems and Applications', *Mathematics of Control, Signals and Systems*, **7**, 95–120.

[22] Kanellakopoulos, I., Kokotović, P. V. and Marino, R. (1991) 'An Extended Direct Scheme for Robust Adaptive Nonlinear Control', *Automatica*, **27**, 247–255.

[23] Kanellakopoulos, I., Kokotović, P. V. and Morse, A. S. (1992) 'A Toolkit for Nonlinear Feedback Design', *Systems & Control Letters*, **18**, 83–92.

[24] Khalil, H. K. (1996) *Nonlinear Systems*. 2nd Edn, Prentice-Hall, Upper Saddle River, NJ.

[25] Kokotović, P. V. and Sussmann, H. J. (1989) 'A Positive Real Condition for Global Stabilization of Nonlinear Systems', *Systems & Control Letters*, **13**, 125–133.

[26] Krstić, M., Kanellakopoulos, I., and Kokotović, P. V. (1995) *Nonlinear and Adaptive Control Design*. New York: John Wiley & Sons.

[27] Kurzweil, J. (1956) 'On the Inversion of Lyapunov's Second Theorem on Stability of Motion', *American Mathematical Society Translations*, Series 2, **24**, 19–77.

[28] Lin, Y. (1996) 'Input-to-state Stability with Respect to Noncompact Sets, *Proc. 13th IFAC World Congress*, Vol. E, 73–78, San Francisco.

[29] Lozano, R., Brogliato, B. and Landau, I. (1992) 'Passivity and Global Stabilization of Cascaded Nonlinear Systems', *IEEE Trans. Autom. Contr.*, **37**, 1386–1388.

[30] Mareels, I. and Hill, D. J. (1992) 'Monotone Stability of Nonlinear Feedback Systems', *J. Math. Systems Estimation Control*, **2**, 275–291.

[31] Marino, R. and Tomei, P. (1993) 'Robust Stabilization of Feedback Linearizable Time-varying Uncertain Nonlinear Systems', *Automatica,* **29**, 181–189.

[32] Marino, R. and Tomei, P. (1995) *Nonlinear Control Design: Geometric, Adaptive and Robust*. Prentice-Hall, Europe.

[33] Moylan, P. and Hill, D. (1978) 'Stability Criteria for Large-scale Systems', *IEEE Trans. Autom. Control*, **23**, 143–149.

[34] Ortega, R. (1991) 'Passivity Properties for the Stabilization of Cascaded Nonlinear Systems', *Automatica*, **27**, 423–424.

[35] Polycarpou, M. M. and Ioannou, P. A. (1995) 'A Robust Adaptive Nonlinear Control Design. *Automatica*, **32**, 423–427.

[36] Praly, L. and Jiang, Z.-P. (1993) 'Stabilization by Output Feedback for Systems with ISS Inverse Dynamics', *Systems & Control Letters*, **21**, 19–33.

[37] Praly, L., Bastin, G., Pomet, J.-B. and Jiang, Z. P. (1991) 'Adaptive Stabilization of Nonlinear Systems, In *Foundations of Adaptive Control* (P.V. Kokotović, ed.), 347–433, Springer-Verlag.

[38] Rodriguez, A. and Ortega, R. (1990) 'Adaptive Stabilization of Nonlinear Systems: the Non-feedback Linearizable Case', in *Prep. of the 11th IFAC World Congress*, 121–124.

[39] Safanov, M. G. (1980) *Stability and Robustness of Multivariable Feedback Systems*. Cambridge, MA: The MIT Press.

[40] Sepulchre, R., Janković, M. and Kokotović, P. V. (1996) '*Constructive Nonlinear Control*. Springer-Verlag.

[41] Seron, M. M., Hill, D. J. and Fradkov, A. L. (1995) 'Adaptive Passification of Nonlinear Systems', *Automatica*, **31**, 1053–1060.

[42] Sontag, E. D. (1989) 'Smooth Stabilization Implies Coprime Factorization', *IEEE Trans. Automat. Contr.*, **34**, 435–443.

[43] Sontag, E. D. (1989) 'Remarks on Stabilization and Input-to-state Stability', *Proc. 28th Conf. Dec. Contr.*, 1376–1378, Tampa.

[44] Sontag, E. D. (1990) 'Further Facts about Input-to-state Stabilization', *IEEE Trans. Automat. Contr.*, **35**, 473–476.

[45] Sontag, E. D. (1995) 'On the Input-to-State Stability Property', *European Journal of Control*, **1**, 24–36.

[46] Sontag, E. D. and Wang, Y. (1995) 'On Characterizations of the Input-to-state Stability Property', *Systems & Control Letters*, **24**, 351–359.

[47] Sontag, E. D. and Wang, Y. (1995) 'On Characterizations of Set Input-to-state Stability', in *Prep. IFAC Nonlinear Control Systems Design Symposium (NOLCOS'95)*, 226–231, Tahoe City, CA.

[48] Taylor, D. G., Kokotović, P. V., Marino, R. and Kanellokopoulos, I. (1989) 'Adaptive Regulation of Nonlinear Systems with Unmodeled Dynamics', *IEEE Trans. Automat. Contr.*, **34**, 405–412.

[49] Teel, A. and Praly, L. (1995) 'Tools for Semiglobal Stabilization by Partial-state and Output Feedback', *SIAM J. Control Optimiz.*, **33**, 1443–1488.

[50] Willems, J. C. (1972) 'Dissipative Dynamical Systems, Part I: General Theory; Part II: Linear Systems with Quadratic Supply Rates', *Archive for Rational Mechanics and Analysis*, **45**, 321–393.

[51] Yao, B. and Tomizuka, M. (1995) 'Robust Adaptive Nonlinear Control with Guaranteed Transient Performance', *Proceedings of the American Control Conference*, Washington, 2500–2504.

7

Active identification for control of discrete-time uncertain nonlinear systems

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Abstract

The problem of controlling nonlinear systems with unknown parameters has received a great deal of attention in the continuous-time case. In contrast, its discrete-time counterpart remains largely unexplored, primarily due to the difficulties associated with utilizing Lyapunov design techniques in a discrete-time framework. Existing results impose restrictive growth conditions on the nonlinearities to yield global stability.

In this chaper we propose a novel approach which removes this obstacle and yields global stability and tracking for systems that can be transformed into an output-feedback, strict-feedback, or partial-feedback canonical form. The main novelties of our design are: (i) the temporal and algorithmic separation of the parameter estimation task from the control task, and (ii) the development of an *active identification* procedure, which uses the control input to actively drive the system state to points in the state space that allow the orthogonalized projection estimator to acquire all the necessary information about the unknown parameters. We prove that our algorithm guarantees complete (for control purposes) identification in a finite time interval, whose maximum length we compute.

Thus, the traditional structure of concurrent on-line estimation and control is replaced by a two-phase control strategy: first use active identification, and then utilize the acquired parameter information to implement *any* control strategy as if the parameters were known.

7.1 Introduction

In recent years, a great deal of progress has been made in the area of adaptive control of continuous-time nonlinear systems [1], [2]. In contrast, adaptive control of discrete-time nonlinear systems remains a largely unsolved problem. The few existing results [3, 4, 5, 6] can only guarantee global stability under restrictive growth conditions on the nonlinearities, because they use techniques from the literature on adaptive control of linear systems [7, 8]. Indeed, it has recently been shown that any discrete-time adaptive nonlinear controller using a least-squares estimator cannot provide global stability in either the deterministic [9] or the stochastic [10] setting. The only available result which guarantees global stability without imposing any such growth restrictions is found in [11], but it only deals with a scalar nonlinear system which contains a single unknown parameter.

The backstepping methodology [1], which provided a crucial ingredient for the development of solutions to many continuous-time adaptive nonlinear problems, has a very simple discrete-time counterpart: one simply 'looks ahead' and chooses the control law to force the states to acquire their desired values after a finite number of time steps. One can debate the merits of such a *deadbeat* control strategy [12], especially for nonlinear systems [13], but it seems that in order to guarantee global stability in the presence of arbitrary nonlinearities, any controller will have to have some form of prediction capability. In the presence of unknown parameters, however, it is impossible to calculate these 'look-ahead' values of the states. Furthermore, since these calculations involve the unknown parameters as arguments of arbitrary nonlinear functions, no known parameter estimation method is applicable, since all of them require a linear parametrization to guarantee global results. This is the biggest obstacle to providing global solutions for any of the more general discrete-time nonlinear problems.

In this chapter we introduce a completely different approach to this problem, which allows us to obtain globally stabilizing controllers for several classes of discrete-time nonlinear systems with unknown parameters, without imposing any growth conditions on the nonlinearities. The major assumptions are that the unknown parameters appear linearly in the system equations, and that the system at hand can be transformed, via a global parameter-independent diffeomorphism, into one of the canonical forms that have been previously considered in the continuous-time adaptive nonlinear control literature [1].

Another major assumption is that our system is free of noise; this allows us to replace the usual least-squares parameter estimator with an *orthogonalized projection* scheme, which is known to converge in finite time, provided the actual values of the regressor vector form a basis for the regressor subspace. The main difficulty with this type of estimator is that in general there is no way to guarantee that this basis will be formed in finite time. The first steps towards removing this obstacle were taken in preliminary versions of this work [14, 15]. In those papers we developed procedures for selecting the value of the control input during the initial identification period in a way that drives the system state towards points in the state space that generate a basis for this subspace in a specified number of time steps. In this chapter we integrate those procedures with the orthogonalized projection estimator to construct a true *active identification* scheme, which produces a parameter estimate in a familiar recursive (and thus computationally efficient) manner, and at each time instant uses the current estimate to compute the appropriate control input. As a result, we guarantee that all the parameter information necessary for control purposes will be available after at most $2nr$ steps for output-feedback systems and $(n+1)r$ steps for strict-feedback systems, where n is the dimension of the system and r is the dimension of the regressor subspace. If the number of unknown parameters p is equal to r, as it would be in any well-posed identification problem, this implies that at the end of the active identification phase the parameters are completely known. If, on the other hand, $p > r$, then we only identify the projection of the parameter vector that is relevant to the system at hand, and that is all that is necessary to implement any control algorithm. In essence, our active identification scheme guarantees that all the conditions for persistent excitation will be satisfied in a finite time interval: in the noise-free case and for the systems we are considering, all the parameter information that could be acquired by *any* identification procedure in *any* amount of time, will in fact be acquired by our scheme in an interval which is made as short as possible, and whose upper bound is computed a priori. The fact that our scheme attempts to minimize the length of this interval is important for transient performance considerations, since this will prevent the state from becoming too large during the identification phase.

Once this active identification phase is over, the acquired parameter information can be used to implement *any* control algorithm as if the parameters were completely known. As an illustration, in this chapter we use a straightforward deadbeat strategy. The fact that discrete-time systems (even nonlinear ones) cannot exhibit the finite escape time phenomenon, makes it possible to delay the control action until after the identification phase and still be able to guarantee global stability.

7.2 Problem formulation

The systems we consider in this section comprise all systems that can be transformed via a global diffeomorphism to the so-called *parametric-output-*

feedback form:

$$\begin{aligned} x_1(t+1) &= x_2(t) + \theta^{\mathrm{T}}\psi_1(x_1(t)) \\ &\vdots \\ x_{n-1}(t+1) &= x_n(t) + \theta^{\mathrm{T}}\psi_{n-1}(x_1(t)) \\ x_n(t+1) &= u(t) + \theta^{\mathrm{T}}\psi_n(x_1(t)) \\ y(t) &= x_1(t) \end{aligned} \tag{7.1}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters and ψ_i, $i = 1, \ldots, n$ are known nonlinear functions. The name 'parametric-output-feedback form' denotes the fact that the nonlinearities ψ_i that are multiplied by unknown parameters depend only on the output $y = x_1$, which is the *only* measured variable; the states $x_2, \ldots, x_n$ are not measured. It is important to note that the functions ψ_i are not restricted by *any* type of growth conditions; in fact, they are not even assumed to be smooth or continuous. The only requirement is that they take on finite values whenever their argument x_1 is finite; this excludes nonlinearities like $\frac{1}{x_1 - 1}$, for example, but it is necessary since we want to obtain global results. This requirement also guarantees that the solutions of (7.1) (with any control law that remains finite for finite values of the state variables) exist on the infinite time interval, i.e. there is *no finite escape time*. Furthermore, no restrictions are placed on the values of the unknown constant parameter vector θ or on the initial conditions. However, the form (7.1) already contains several structural restrictions: the unknown parameters appear linearly, the nonlinearities are not allowed to depend on the unmeasured states, and the system is completely noise free: there is no process noise, no sensor noise, and no actuator noise.

Our control objective consists of the global stabilization of (7.1) and the global tracking of a known reference signal $y_d(t)$ by the output $x_1(t)$.

For notational simplicity, we will denote $\psi_{i,t} = \psi_i(x_1(t))$ for $i = 1, \ldots n$.

7.2.1 A second-order example

To illustrate the difficulties present in this problem, let us consider the case when the system (7.1) is of second order, i.e.

$$\begin{aligned} x_1(t+1) &= x_2(t) + \theta^{\mathrm{T}}\psi_{1,t} \\ x_2(t+1) &= u(t) + \theta^{\mathrm{T}}\psi_{2,t} \\ y(t) &= x_1(t) \end{aligned} \tag{7.2}$$

Even if θ were known, the control $u(t)$ would only be able to affect the output

x_1 at time $t+2$. In other words, given any initial conditions $x_1(0)$ and $x_2(0)$, we have no way of influencing $x_1(1)$ through $u(0)$. The best we can do is to drive $x_1(2)$ to zero and keep it there. The control would simply be a *deadbeat* controller, which utilizes our ability to express future values of x_1 as functions of current and past values of x_1 and u:

$$\begin{aligned} x_1(t+2) &= x_2(t+1) + \theta^{\mathrm{T}}\psi_{1,t+1} \\ &= u(t) + \theta^{\mathrm{T}}\left[\psi_{2,t} + \psi_{1,t+1}\right] \\ &= u(t) + \theta^{\mathrm{T}}\left[\psi_{2,t} + \psi_1(x_2(t) + \theta^{\mathrm{T}}\psi_{1,t})\right] \\ &= u(t) + \theta^{\mathrm{T}}\left[\psi_{2,t} + \psi_1(u(t-1) + \theta^{\mathrm{T}}(\psi_{2,t-1} + \psi_{1,t}))\right] \end{aligned} \tag{7.3}$$

Thus, the choice of control

$$\begin{aligned} u(t) &= y_{\mathrm{d}}(t+2) - \theta^{\mathrm{T}}\left[\psi_{2,t} + \psi_{1,t+1}\right] \\ &= y_{\mathrm{d}}(t+2) - \theta^{\mathrm{T}}\left[\psi_{2,t} + \psi_1\big(u(t-1) + \theta^{\mathrm{T}}(\psi_{2,t-1} + \psi_{1,t})\big)\right], \; t \geq 1 \end{aligned} \tag{7.4}$$

would yield $x_1(t) = y_{\mathrm{d}}(t)$ for all $t \geq 3$ and would achieve the objective of global stabilization.

We emphasize that here we use a deadbeat control law only because it makes the presentation simpler. All the arguments made here are equally applicable to any other discrete-time control strategy, as is the parameter information supplied by our active identification procedure. We hasten to add, however, that, from a strictly technical point of view, deadbeat control is perfectly acceptable in this case, for the following two reasons:

(1) The well-known problems of poor inter-sample behaviour resulting from applying deadbeat control to sampled-data systems do not arise here, since we are dealing with a purely discrete-time problem.
(2) Deadbeat control can result to instability when applied to general polynomial nonlinear systems. As an example, consider the system

$$\begin{aligned} x_1(t+1) &= x_2(t) + x_1(t)x_2^2(t) \\ x_2(t+1) &= u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{7.5}$$

If we implement a deadbeat control strategy to track the reference signal $y_{\mathrm{d}}(t) = 2^{-t}$, one of the two possible closed-form solutions yields

$$x_2(t) = -2^t - \sqrt{1 + 2^{2t}} \tag{7.6}$$

which is clearly unbounded. The computational procedures presented in [13] provide ways of avoiding such problems. However, in the case of systems of the form (7.1) and of all the other forms we deal with in this chapter, such issues do not even arise, owing to the special structure of our systems which guarantees that boundedness of $x_1, \ldots, x_i$ automatically ensures boundedness of x_{i+1}, since $x_{i+1}(t) = x_i(t+1) - \theta^{\mathrm{T}}\psi_i(x_1(t))$.

Of course, when θ is unknown, the controller (7.4) cannot be implemented. Furthermore, it is clear that any attempt to replace the unknown θ with an estimate $\hat{\theta}$ would be stifled by the fact that θ appears inside the nonlinear function ψ_1. Available estimation methods cannot provide global results for such a nonlinearly parametrized problem, except for the case where ψ_1 is restricted by linear growth conditions.

7.2.2 Avoiding the nonlinear parametrization

Our approach to this problem does not solve the nonlinear parametrization problem; instead, it bypasses it altogether. Returning to the control expression (7.4), we see that its implementation relies on the ability to compute the term

$$\theta^{\mathrm{T}}(\psi_{2,t} + \psi_{1,t+1}) \tag{7.7}$$

Since this computation must happen at time t, the argument $x_1(t+1)$ is not yet available, so it must be 'pre-computed' from the expression

$$\begin{aligned} x_1(t+1) &= x_2(t) + \theta^{\mathrm{T}}\psi_{1,t} \\ &= u(t-1) + \theta^{\mathrm{T}}(\psi_{2,t-1} + \psi_{1,t}) \end{aligned} \tag{7.8}$$

Careful examination of the expressions (7.4)–(7.8) reveals that our controller would be implementable if we had the ability to calculate the projection of the unknown parameter vector θ along known vectors of the form

$$\psi_2(x) + \psi_1(\tilde{x}) \tag{7.9}$$

since then we would be able at time t to compute the terms

$$\theta^{\mathrm{T}}(\psi_{2,t-1} + \psi_{1,t}) \tag{7.10}$$

$$\theta^{\mathrm{T}}(\psi_{2,t} + \psi_{1,t+1}) \tag{7.11}$$

and from them the control (7.4).

Hence, our main task is to compute the projection of θ along vectors of the form (7.9). To achieve this, we proceed as follows:

Regressor subspace: First, we define the subspace spanned by all vectors of the form (7.9):

$$S_\psi^0 \triangleq \mathcal{R}\{\psi_1(x) + \psi_2(\tilde{x}), \quad \forall\, x \in \mathbb{R},\ \forall\, \tilde{x} \in \mathbb{R}\} \tag{7.12}$$

Note that the known nonlinear functions ψ_1 and ψ_2 need to be evaluated *independently over all possible values of their arguments*. This is necessary because we are not imposing any smoothness or continuity assumptions on these functions. However, for any reasonable nonlinearities, determining this subspace will be a fairly straightforward task which of course can be performed off-line. The dimension of S_{ψ}^{0}, denoted by r_0, will always be less than or equal to the number of unknown parameters p: $r_0 \leq p$. In fact, in any reasonably posed problem we will have $r_0 = p$, since $r_0 < p$ means that we are considering more parameters than are actually entering the system equations; in that case, complete parameter identification cannot be achieved with any method or input, since the regressor vector cannot acquire the values necessary to identify some of the parameters. Hence, if $r_0 < p$, then the number of unknown parameters can be reduced to r_0 without any loss of information or generality.

Projection measurements Clearly, in order to be able to implement the control (7.4), all we need to know about θ is its projection on the subspace S_{ψ}^{0}. But how do we acquire this projection? From (7.3) we see that at time t, using the measurements $x_1(t), x_1(t-1), x_1(t-2)$ and the known value of the control $u(t-2)$, we can compute the following projection:

$$\theta^{\mathrm{T}}[\psi_2(x_1(t-2)) + \psi_1(x_1(t-1))] = x_1(t) - u(t-2) \qquad (7.13)$$

Hence, if the values of x_1 are such that the corresponding values of the vector $\psi_2(x_1(t-2)) + \psi_1(x_1(t-1))$ eventually form a basis for the subspace S_{ψ}^{0}, we will obtain all the necessary information about θ. But how do we guarantee that this identification phase will be of finite duration?

Active identification Instead of allowing the system state to drift on its own, we use the control input u to *drive* the output x_1 to values which result in linearly independent vectors $\psi_{2,t-2} + \psi_{1,t-1}$ and form a basis for S_{ψ}^{0} in at most $2nr_0$ steps (where n is the dimension of the system state and r_0 the dimension of the nonlinearity subspace). But how can we determine the values of u that will result in such basis vectors in the presence of unknown parameters? This seemingly hopeless dilemma can be resolved by the following observation, which will be clarified further later on:

> The expression (7.4) is not computable *if and only if* at least one of the vectors $\psi_{2,t-1} + \psi_{1,t}$ and $\psi_{2,t} + \psi_{1,t+1}$ is independent of the past values $\psi_{2,j-1} + \psi_{1,j}$, $j \leq t-1$. Thus, inability to compute (7.4) from already measured projections is equivalent to the knowledge that new independent directions are being generated by the system.

In other words, whenever our identification process gets 'stuck', that is, the system does not generate new directions over the next few steps, then the projection information we have already acquired is enough for us to compute a value of control which will get the system 'unstuck' and will generate a new direction after at most $2n$ (in this case 4) steps: this is the time it takes to change the arguments of both ψ_1 and ψ_2 and measure the resulting projection.

Orthogonalized projection estimation All the projection information of θ is automatically incorporated into the parameter estimate $\hat{\theta}$ produced by an orthogonalized projection algorithm. This means that after the active identification phase is complete, all the terms appearing in (7.10) and (7.11) can be computed simply by replacing θ by its estimate $\hat{\theta}$. This allows us to proceed with the implementation of the controller (7.4) *or any other control strategy* as if the parameters were known.

Clearly, this two-stage process depends critically on the fact that, contrary to their continuous-time counterparts, *discrete-time nonlinear systems cannot exhibit finite escape times*, as long as their nonlinearities take on finite values whenever their arguments are finite. This property allows us to postpone closing the loop with a controller until after the finite-duration identification phase has been completed.

7.3 Active identification

Let us now elaborate further on the above outlined approach by presenting in detail its two most challenging ingredients, namely the pre-computation scheme and the input selection for active identification. To do this, we return to the general output-feedback form (7.1) and rewrite it in the following scalar form:

$$\begin{aligned} x_1(t+n) &= x_2(t+n-1) + \theta^{\mathrm{T}}\psi_1(x_1(t+n-1)) \\ &= x_3(t+n-2) + \theta^{\mathrm{T}}\psi_2(x_1(t+n-2)) + \theta^{\mathrm{T}}\psi_1(x_1(t+n-1)) \\ &\vdots \\ &= u(t) + \sum_{k=1}^{n} \theta^{\mathrm{T}}\psi_k(x_1(t+n-k)) \end{aligned} \tag{7.14}$$

Hence, the following choice of a deadbeat control law:

$$u(t) = y_d(t+n) - \sum_{k=1}^{n} \theta^{\mathrm{T}}\psi_{k,t+n-k} \tag{7.15}$$

will globally stabilize the system (7.1) and yield $x_1(t) = y_{\mathrm{d}}(t)\,,\ t \geq n$.

Clearly, the implementation of the control law (7.15) requires us to calculate (at time t) the projection of the unknown θ along the vector $\sum_{k=1}^{n} \psi_{k,t+n-k}$. This means that we need to compute the value of $\sum_{k=1}^{n} \psi_{k,t+n-k}$ at time t. Rewriting $\sum_{k=1}^{n} \psi_{k,t+n-k}$ as

$$\sum_{k=1}^{n} \psi_{k,t+n-k} = \sum_{k=1}^{n} \psi_k(x_1(t+n-k)) \tag{7.16}$$

we can therefore infer that it is necessary for us to be able to calculate the value of the states $x_1(t+1), \ldots, x_1(t+n-1)$ at time t. To see how to calculate these states, let us return to equation (7.14) and express $x_1(t+1), \ldots, x_1(t+n-1)$ as

$$x_1(t+i) = u(t-n+i) + \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_k(x_1(t+i-k)), \quad i = 1, \ldots, n-1 \tag{7.17}$$

Clearly, equation (7.17) shows that the value of $x_1(t+1)$ depends on the values of both $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k}$ and $u(t-n+1)$. Since the values of $u(t-n+1)$ and the vector $\sum_{k=1}^{n} \psi_{k,t+1-k}$ are known at time t, the key to successfully calculating (at time t) the value of $x_1(t+1)$ depends on whether we are able to compute the projection of the unknown θ along the vector $\sum_{k=1}^{n} \psi_{k,t+1-k}$ at time t.

Next, let us examine what we need to calculate the value of $x_1(t+2)$ at time t. From (7.17), the value of $x_1(t+2)$ is equal to the sum of $u(t-n+2)$ and $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+2-k}$. Clearly, if we are able to calculate the values of both $u(t-n+2)$ and $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+2-k}$ at time t, then the value of $x_2(t+2)$ can be acquired (at time t). The value of $u(t-n+2)$ is known at time t, while from the expression

$$\sum_{k=1}^{n} \psi_{k,t+2-k} = \psi_{1,t+1} + \sum_{k=2}^{n} \psi_{k,t+2-k} = \psi_1(x_1(t+1)) + \sum_{k=1}^{n-1} \psi_{k+1}(x_1(t+1-k)) \tag{7.18}$$

we see that the value of $\sum_{k=1}^{n} \psi_{k,t+2-k}$ depends on $x_1(t+1)$. This means that pre-computing the value of $x_2(t+2)$ requires the values of both $x_1(t+1)$ and $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+2-k}$. In view of the discussion of the previous paragraph, the calculation of $x_1(t+1)$ at time t requires us to compute (at time t) the value of $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k}$. Thus, in summary, the calculation of $x_1(t+2)$ requires us to pre-compute (at time t) the values of

$$\begin{bmatrix} \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k} \\ \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+2-k} \end{bmatrix} \tag{7.19}$$

Generalizing the argument of the previous two paragraphs, we can conclude that the pre-computation of the value of $x_1(t+l)$ $(1 \le l \le n-1)$ requires knowledge (at time t) of the vector

$$\begin{bmatrix} \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k} \\ \vdots \\ \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+l-k} \end{bmatrix} \tag{7.20}$$

Hence, the knowledge (at time t) of the above vector with $l = n - 1$

$$\begin{bmatrix} \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k} \\ \vdots \\ \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+n-1-k} \end{bmatrix} \tag{7.21}$$

enables us to determine the values of the states $x_1(t+1), \ldots, x_1(t+n-1)$ at time t. However, the implementation of the control law (7.15) requires the value of $\theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+n-k}$ also. This leads finally to the conclusion that, in order to use the control law (7.15), we need to pre-compute (at time t) the vector

$$\begin{bmatrix} \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+1-k} \\ \vdots \\ \theta^{\mathrm{T}} \sum_{k=1}^{n} \psi_{k,t+n-k} \end{bmatrix} \tag{7.22}$$

The procedure for acquiring (7.22) at time t ($t \geq n$) is given in the next two sections, whose contents can be briefly summarized as follows: first, the section on pre-computation explains how to pre-compute the vector (7.22) *after* the identification phase is complete, that is, for any time t such that $S_{t-1}^{0} = S_{\psi}^{0}$, where S_{t-1}^{0} denotes the subspace that has been identified at time t (with $t > n$):

$$S_{t-1}^{0} \triangleq \mathcal{R}\left\{ \sum_{k=1}^{n} \psi_{k,i-k},\ i = n, \ldots, t \right\} \tag{7.23}$$

while S_{ψ}^{0} denotes the subspace formed by all possible values of the regressor vector:

$$S_{\psi}^{0} \triangleq \mathcal{R}\left\{ \sum_{i=1}^{n} \psi_i(z_i),\ \forall\, (z_1, \ldots, z_n) \in \mathbb{R}^n \right\},\ \ r_0 \triangleq \dim S_{\psi}^{0} \tag{7.24}$$

Then, the section on input selection for identification shows how to guarantee that the identification phase will be completed in finite time, that is, how to ensure the existence of a finite time t_{f} at which $S_{t_{\mathrm{f}}-1}^{0} = S_{\psi}^{0}$. In addition, we will show that $t_{\mathrm{f}} \leq 2nr_0$.

The reason for this seemingly inverted presentation, where we first show what to do with the results of the active identification and then discuss how to obtain these results, is that it makes the procedure easier to understand.

7.3.1 Pre-computation of projections

The pre-computation of (7.22) is implemented through an orthogonalized projection estimator. Therefore, we first review briefly the standard version of this estimation scheme; for more details, the reader is referred to Section 3.3 of [7].

Orthogonalized projection algorithm Consider the problem of estimating an unknown parameter vector θ from a simple model of the following form:

$$y(t) = \phi(t-1)^{\mathrm{T}}\theta \tag{7.25}$$

where $y(t)$ denotes the (scalar) system output at time t, and $\phi(t-1)$ denotes a vector that is a linear or nonlinear function of past measurements

$$\begin{aligned} \mathcal{Y}(t-1) &= \{y(t-1), y(t-2), \ldots\} \\ \mathcal{U}(t-1) &= \{u(t-1), u(t-2), \ldots\} \end{aligned} \tag{7.26}$$

The orthogonalized projection algorithm for (7.25) starts with an initial estimate $\hat{\theta}_0$ and the $p \times p$ identity matrix P_{-1}, and then updates the estimate $\hat{\theta}$ and the covariance matrix P for $t \geq 1$ through the recursive expressions:

$$\hat{\theta}_t = \begin{cases} \hat{\theta}_{t-1} + \dfrac{P_{t-2}\phi_{t-1}}{\phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1}}(y(t) - \phi_{t-1}^{\mathrm{T}}\hat{\theta}_{t-1}) & \text{if } \phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1} \neq 0 \\ \hat{\theta}_{t-1} & \text{if } \phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1} = 0 \end{cases} \tag{7.27}$$

$$P_{t-1} = \begin{cases} P_{t-2} - \dfrac{P_{t-2}\phi_{t-1}\phi_{t-1}^{\mathrm{T}} P_{t-2}}{\phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1}} & \text{if } \phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1} \neq 0 \\ P_{t-2} & \text{if } \phi_{t-1}^{\mathrm{T}} P_{t-2}\phi_{t-1} = 0 \end{cases} \tag{7.28}$$

This algorithm has the following useful properties, which are given here without proof:

(i) $P_{t-1}\phi(t)$ is a linear combination of the vectors $\phi(1), \ldots, \phi(t)$.
(ii) $\mathcal{N}[P_t] = \mathcal{R}\{\phi(1), \ldots, \phi(t)\}$. In other words, $P_t x = 0$ if and only if x is a linear combination of the vectors $\phi(1), \ldots, \phi(t)$.
(iii) $P_{t-1}\phi(t) \perp \mathcal{R}\{\phi(1), \ldots, \phi(t-1)\}$.
(iv) $\tilde{\theta}(t) \perp \mathcal{R}\{\phi(1), \ldots, \phi(t-1)\}$, where $\tilde{\theta}_t \hat{\theta}_t - \theta$ is the parameter estimation error.

It is worth noting that the orthogonalized projection algorithm produces an estimate $\hat{\theta}_t$ that renders

$$J_t(\hat{\theta}) = \sum_{k=1}^{t} (y(k) - \hat{\theta}^{\mathrm{T}}\phi(k-1))^2 = 0 \tag{7.29}$$

and thus minimizes this cost function. This implies that orthogonalized projection is actually an implementable form of the batch least-squares algorithm, which minimizes the same cost function (7.29). However, the batch least-squares algorithm, that is, the least-squares algorithm with infinite initial covariance ($P_{-1}^{-1} = 0$), relies on the necessary conditions for optimality, namely

$$\frac{\partial J_t(\hat{\theta})}{\partial \hat{\theta}} = 0 \;\Rightarrow\; \sum_{k=1}^{t} \phi(k-1)\phi(k-1)^{\mathrm{T}}\hat{\theta} = \sum_{k=1}^{t} y(k)\phi(k-1) \tag{7.30}$$

which cannot be solved to produce a computable parameter estimate before enough linearly independent measurements have been collected to make the matrix $\sum_{k=1}^{t} \phi(k-1)\phi(k-1)^{\mathrm{T}}$ invertible. In contrast, the orthogonalized projection algorithm produces an estimate which, at each time t, incorporates all the information about the unknown parameters that has been acquired up to that time.

Now we are ready to describe how to apply the orthogonalized projection algorithm (7.27)–(7.28) to our output-feedback system (7.1).

Orthogonalized projection for output-feedback systems At each time t, we can only measure the output $x_1(t)$. Utilizing this measurement and (7.14), we can compute the projection of the unknown vector θ along the known vector $\sum_{k=1}^{n} \psi_{k,t-k}$, that is,

$$\bar{x}_t = \sum_{k=1}^{n} \theta^{\mathrm{T}}\psi_k(x_1(t-k)) = \phi_{t-1}^{\mathrm{T}}\theta \tag{7.31}$$

where $\phi_{t-1} \triangleq \sum_{k=1}^{n} \psi_{k,t-k}$ and $\bar{x}_t \triangleq x_1(t) - u(t-n)$ for any t with $t \geq n$.

Since equation (7.31) is in the same form as (7.25), we can use the expressions (7.27)–(7.28) to recursively construct the estimate $\hat{\theta}_t$ and the covariance matrix P_t (with $t \geq n+1$ in order for (7.31) to be valid), starting from initial estimates $\hat{\theta}_n$ and $P_{n-1} = I$:[1]

$$\hat{\theta}_t = \begin{cases} \hat{\theta}_{t-1} + \dfrac{P_{t-2}\phi_{t-1}}{\phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1}}(\bar{x}_t - \phi_{t-1}^{\mathrm{T}}\hat{\theta}_{t-1}) & \text{if } \phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1} \neq 0 \\ \hat{\theta}_{t-1} & \text{if } \phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1} = 0 \end{cases} \tag{7.32}$$

$$P_{t-1} = \begin{cases} P_{t-2} - \dfrac{P_{t-2}\phi_{t-1}\phi_{t-1}^{\mathrm{T}}P_{t-2}}{\phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1}} & \text{if } \phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1} \neq 0 \\ P_{t-2} & \text{if } \phi_{t-1}^{\mathrm{T}}P_{t-2}\phi_{t-1} = 0 \end{cases} \tag{7.33}$$

Lemma 3.1 When the estimation algorithm (7.32)–(7.33) is applied to the system (7.31), the following properties are true for $t \geq n$:

$$\phi_t^{\mathrm{T}}P_{t-1}\phi_t = 0 \;\Leftrightarrow\; \phi_t \in S_{t-1}^0 \tag{7.34}$$

$$\phi_t \in S_{t-1}^0 \;\Rightarrow\; \hat{\theta}_{t+1} = \hat{\theta}_t, \quad P_t = P_{t-1} \tag{7.35}$$

[1] This notation is used in place of the traditional $\hat{\theta}_0$ and P_{-1} to emphasize the fact that for the first n time steps we cannot produce any parameter estimates.

$$v \in S^0_{t-1} \;\Rightarrow\; \hat{\theta}_t^{\mathrm{T}} v = \hat{\theta}_{t+l}^{\mathrm{T}} v = \theta^{\mathrm{T}} v, \quad l = 0, 1, 2, \ldots \tag{7.36}$$

The proof of this lemma is given in the appendix. The properties (7.34)–(7.36) are crucial to our further development, so let us understand what they mean. Properties (7.34) and (7.35) state that whenever the regressor vector ϕ_t is linearly dependent on the past regressor vectors, then our estimator does not change the value of the parameter estimate and the covariance matrix; this is due to the fact that the new measurement provides no new projection information. Property (7.35) states that the estimate $\hat{\theta}_t$ produced at time t is exactly equal to the true parameter vector θ, when both are projected onto the subspace spanned by the regressor vectors used to generate this estimate, namely the subspace $S^0_{t-1} = \mathcal{R}\{\phi_0, \ldots, \phi_{t-1}\}$. This property is one of the cornerstones on which we develop our pre-computing methodology in the following sections, because it implies that at time t we know the projection of the true parameter vector θ along the subspace S^0_{t-1}.

Pre-computation procedure In order to better explain the pre-computation part of our algorithm, we postulate that there exists a finite time instant $t_{\mathrm{f}} \geq n$ at which the regressor vectors that have been measured span the entire subspace generated by the nonlinearities:

$$S^0_{t_{\mathrm{f}}-1} = S^0_{\psi} \tag{7.37}$$

Hence, at time t_{f} the identification procedure is completed, because the orthogonalized projection algorithm has covered all r_0 independent directions of the parameter subspace, and hence has identified the true parameter vector θ. It is important to note (i) that the existence of such a time $t_{\mathrm{f}} \leq 2nr_0$ will be guaranteed through appropriate input selection as part of our active identification procedure in the next section, and (ii) that our ability to compute θ at such a time t_{f}, be it through orthogonalized projection or through batch least squares, is in fact independent of the manner in which the linearly independent regressor vectors were obtained.

The definition (7.24) tells us that

$$\phi_t \in S^0_{\psi} = S^0_{t_{\mathrm{f}}-1}\,, \quad \forall\, t \geq t_{\mathrm{f}} \geq n \tag{7.38}$$

Hence, (7.36) implies that

$$\hat{\theta}_{t_{\mathrm{f}}}^{\mathrm{T}} \phi_t = \theta^{\mathrm{T}} \phi_t\,, \quad \forall\, t \geq t_{\mathrm{f}} \geq n \tag{7.39}$$

In particular, with the help of (7.14) we can then pre-compute (at time t with $t \geq t_{\mathrm{f}}$) the state $x_1(t+1)$ through

$$x_1(t+1) = u(t+1-n) + \theta^{\mathrm{T}} \phi_t = u(t+1-n) + \hat{\theta}_{t_{\mathrm{f}}}^{\mathrm{T}} \phi_t \tag{7.40}$$

Using the pre-computed $x_1(t+1)$, we can also pre-compute (at time t with

$t \geq t_f$) $\psi_{1,t+1}, \ldots, \psi_{n,t+1}$

$$\begin{bmatrix} \psi_{1,t+1} \\ \vdots \\ \psi_{n,t+1} \end{bmatrix} = \begin{bmatrix} \psi_1\Big(u(t+1-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_t\Big) \\ \vdots \\ \psi_n\Big(u(t+1-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_t\Big) \end{bmatrix} \tag{7.41}$$

and the vector

$$\begin{aligned} \phi_{t+1} &= \psi_{1,t+1} + \sum_{i=2}^{n} \psi_{i,t+2-i} \\ &= \psi_1\Big(u(t+1-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_t\Big) + \sum_{i=1}^{n-1} \psi_{i+1,t+1-i} \end{aligned} \tag{7.42}$$

Since $S^0_{t_f-1} = S^0_\psi$, the pre-computed ϕ_{t+1} still belongs to $S^0_{t_f-1}$. Hence, we can repeat the argument from (7.40) to (7.42) to pre-compute (at time $t \geq t_f$) the state $x_1(t+2)$ as

$$\begin{aligned} x_1(t+2) &= u(t+2-n) + \theta^{\mathrm{T}} \phi_{t+1} \\ &= u(t+2-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_{t+1} \end{aligned} \tag{7.43}$$

Then, using the pre-computed $x_1(t+1)$ and $x_1(t+2)$, we can calculate (at time $t \geq t_f$) the vectors $\psi_{1,t+2}, \ldots, \psi_{n,t+2}$ through

$$\begin{bmatrix} \psi_{1,t+2} \\ \vdots \\ \psi_{n,t+2} \end{bmatrix} = \begin{bmatrix} \psi_1\Big(u(t+2-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_{t+1}\Big) \\ \vdots \\ \psi_n\Big(u(t+2-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_{t+1}\Big) \end{bmatrix} \tag{7.44}$$

and also the vector ϕ_{t+2} as follows:

$$\begin{aligned} \phi_{t+2} &= \psi_{1,t+2} + \psi_{2,t+1} + \sum_{i=3}^{n} \psi_{i,t+3-i} \\ &= \sum_{j=1}^{2} \psi_j\Big(u(t+3-j-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_{t+2-j}\Big) + \sum_{i=1}^{n-2} \psi_{i+2,t+1-i} \end{aligned} \tag{7.45}$$

In general, since the pre-computed vector ($1 \leq l \leq n-1$) satisfies

$$\phi_{t+l-1} \in S^0_\psi = S^0_{t_f-1} \tag{7.46}$$

we can pre-compute (at time $t \geq t_f$) the state $x_1(t+l)$ as

$$x_1(t+l) = u(t+l-n) + \theta^{\mathrm{T}} \phi_{t+l-1} = u(t+l-n) + \hat{\theta}_{t_f}^{\mathrm{T}} \phi_{t+l-1} \tag{7.47}$$

Then, using the pre-computed $x_1(t+1), \ldots, x_1(t+l)$, we can pre-compute (still at time $t \geq t_f$) the vectors:

$$\begin{bmatrix} \psi_{1,t+l} \\ \vdots \\ \psi_{n,t+l} \end{bmatrix} = \begin{bmatrix} \psi_1\Big(u(t+l-n) + \hat{\theta}_{t_f}^{\mathrm{T}}\phi_{t+l-1}\Big) \\ \vdots \\ \psi_n\Big(u(t+l-n) + \hat{\theta}_{t_f}^{\mathrm{T}}\phi_{t+l-1}\Big) \end{bmatrix} \tag{7.48}$$

and also $\phi_{t+l} = \sum_{i=1}^{n} \psi_{i,t+l+1-i}$.

In summary, for any t with $t \geq t_f$, using the procedure from (7.46)–(7.48), we can pre-compute the vectors $\psi_{i,t+l}$, with $i = 1, \ldots, n$ and $l = 1, \ldots, n-1$ and, thus, the vectors $\phi_{t+1}, \ldots, \phi_{t+n-1} \in S_{t_f-1}^0$. Combining this with (7.36), we can pre-compute the vector (7.22) as

$$\begin{bmatrix} \theta^{\mathrm{T}}\phi_t \\ \vdots \\ \theta^{\mathrm{T}}\phi_{t+n-1} \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{t_f}^{\mathrm{T}}\phi_t \\ \vdots \\ \hat{\theta}_{t_f}^{\mathrm{T}}\phi_{t+n-1} \end{bmatrix} \tag{7.49}$$

which implies that after time t_f we can implement any control algorithm as if the parameter vector θ were known.

7.3.2 Input selection for identification

So far, we have shown how to pre-compute the values of the future states and the vectors associated with these future states, provided that we can ensure the existence of a finite time instant $t_f \geq n$ at which $S_{t_f-1}^0 = S_\psi^0$. Now we show how to guarantee the existence of such a time t_f; this is achieved by using the control input u to drive the output x_1 to values that yield linearly independent directions for the vectors ϕ_i. This input selection takes place whenever necessary during the identification phase, that is, whenever we see that the system will not produce any new directions on its own. The main idea behind our input selection procedure is the following:

> At time t, we can determine whether any of the regressor vectors $\phi_t, \phi_{t+1}, \ldots, \phi_{t+n-1}$ will be linearly independent of the vectors we have already measured. If they are not, then we can use our current estimate $\hat{\theta}_t$ and the equation (7.15) to select a control input $u(t)$ to drive $x_1(t+n)$ to a value that will generate a linearly independent vector ϕ_{t+n}. In the worst-case scenario, we will have to use $u(t), u(t+1), \ldots, u(t+n-1)$ to specify the values $x_1(t+n), x_1(t+n+1), \ldots, x_1(t+2n-1)$ in order to generate a linearly independent vector ϕ_{t+2n-1}.

Proposition 3.1 As long as there are still directions in S_ψ^0 along which the projection of θ is unknown, it is always possible to choose the input u so that a new direction is generated after at most $2n$ steps:

$$\dim S^0_{t-1} < \dim S^0_\psi = r_0 \;\Rightarrow\; \dim S^0_{t+2n-1} \geq \dim S^0_{t-1} + 1\,, \quad \forall\, t \geq n \tag{7.50}$$

Proof The proof of this proposition actually constructs the input selection algorithm. Let us first note that (7.23) yields (for $t \geq n$)

$$S^0_{n-1} \subseteq S^0_n \subseteq \cdots \subseteq S^0_{t-1} \subseteq S^0_\psi \tag{7.51}$$

which implies

$$\dim S^0_{n-1} \leq \dim S^0_n \leq \cdots \leq \dim S^0_{t-1} \leq \dim S^0_\psi = r_0 \tag{7.52}$$

The algorithm that guarantees $\dim S^0_{t+2n-1} \geq \dim S^0_{t-1} + 1$ is implemented as follows:

Step 1 At time t, measure $x_1(t)$ and compute $\psi_{1,t}$ and ϕ_t.

Case 1.1 If $\phi_t^{\mathrm{T}} P_{t-1} \phi_t \neq 0$, then by (7.34) we have $\phi_t \notin S^0_{t-1}$, and therefore $\dim S^0_t = \dim S^0_{t-1} + 1$. No input selection is needed; return to Step 1 and wait for the measurement of $x_1(t+1)$.

Case 1.2 If $\phi_t^{\mathrm{T}} P_{t-1} \phi_t = 0$, then $\phi_t \in S^0_{t-1}$ and $S^0_t = S^0_{t-1}$. Go to Step 2.

Step 2 Since $\phi_t \in S^0_{t-1}$, use the procedure (7.38)–(7.42) to calculate all of the following quantities, whose values are independent of $u(t)$ (since $u(t)$ affects only $x_1(t+n)$):

$$\{x_1(t+1)\,, \psi_{1,t+1}\,, \ldots\,, \psi_{n,t+1}\,, \phi_{t+1}\} \tag{7.53}$$

Case 2.1 If $\phi_{t+1}^{\mathrm{T}} P_{t-1} \phi_{t+1} \neq 0$, then $\phi_{t+1} \notin S^0_{t-1}$ and $\dim S^0_{t+1} = \dim S^0_{t-1} + 1$. No input selection is needed; return to Step 1 and wait for the measurement of $x_1(t+2)$.

Case 2.2 If $\phi_{t+1}^{\mathrm{T}} P_{t-1} \phi_{t+1} = 0$ then $\phi_{t+1} \in S^0_{t-1}$ and $S^0_{t+1} = S^0_{t-1}$. Go to Step 3.

Step i ($3 \leq i \leq n$) Since $\phi_{t+i-1} \in S^0_{t-1}$, use the procedure (7.38)–(7.42) to calculate all of the following quantities (whose values are also independent of $u(t)$):

$$\{x_1(t+i-1)\,, \psi_{1,t+i-1}\,, \ldots\,, \psi_{n,t+i-1}\,, \phi_{t+i-1}\} \tag{7.54}$$

Case i.1 If $\phi_{t+i-1}^{\mathrm{T}} P_{t-1} \phi_{t+i-1} \neq 0$, then $\phi_{t+i-1} \notin S^0_{t-1}$ and $\dim S^0_{t+i-1} = \dim S^0_{t-1} + 1$. No input selection is needed; return to Step 1 and wait for the measurement of $x_1(t+i)$.

Case i.2 If $\phi_{t+i-1}^{\mathrm{T}} P_{t-1} \phi_{t+i-1} \neq 0$, then $\phi_{t+i-1} \in S^0_{t-1}$ and $S^0_{t+i-1} = S^0_{t-1}$. Go to Step $i+1$.

Step $n+1$ At this step, we have pre-computed all of the following quantities:

$$\left\{\begin{array}{ccc} x_1(t+1), & \ldots, & x_1(t+n-1) \\ \phi_{t+1}, & \ldots, & \phi_{t+n-1} \\ \psi_{1,t+1}, & \ldots, & \psi_{1,t+n-1} \\ \vdots, & \ldots, & \vdots \\ \psi_{n,t+1}, & \ldots, & \psi_{n,t+n-1} \end{array}\right\} \tag{7.55}$$

and we know that the pre-computed vectors satisfy

$$\phi_t \in S^0_{t-1}, \ \ldots, \phi_{t+n-1} \in S^0_{t-1} \tag{7.56}$$

Case $(n+1).1$ If there exists a real number a_{11} such that $\phi^{\mathrm{T}}_{a_{11}} P_{t-1}\phi_{a_{11}} \neq 0$, that is, $\phi_{a_{11}} \notin S^0_{t-1}$,where

$$\phi_{a_{11}} \stackrel{\Delta}{=} \psi_1(a_{11}) + \sum_{i=2}^{n} \psi_{i,t+n+1-i} \tag{7.57}$$

then choose the control input $u(t)$ to be

$$u(t) = a_{11} - \hat{\theta}^{\mathrm{T}}_t \phi_{t+n-1} \tag{7.58}$$

This choice yields

$$\begin{aligned} x_1(t+n) &= u(t) + \theta^{\mathrm{T}}\phi_{t+n-1} \\ &= a_{11} - \hat{\theta}^{\mathrm{T}}_t \phi_{t+n-1} + \theta^{\mathrm{T}}\phi_{t+n-1} \\ &= a_{11} \end{aligned} \tag{7.59}$$

where the last equality follows from (7.56) and (7.36). Therefore, we have $\phi_{t+n} = \phi_{a_{11}} \notin S^0_{t-1}$ and, hence, $\dim S^0_{t+n} = \dim S^0_{t-1} + 1$. Return to Step 1 and wait for the measurement of $x_1(t+n+1)$.

Case $(n+1).2$ If there exists no a_{11} that renders $\phi^{\mathrm{T}}_{a_{11}} P_{t-1}\phi_{a_{11}} \neq 0$, that is, if

$$\psi_1(a) + \sum_{j=2}^{n} \psi_{j,t+n+1-j} = \psi_1(a) + \sum_{j=2}^{n} \psi_j(x_1(t+n+1-j)) \in S^0_{t-1} \quad \forall a \in \mathbb{R} \tag{7.60}$$

then go to Step $n+2$.

Step $n+2$ We have pre-computed all the quantities in (7.55), and we also know from (7.60) that any choice of $u(t)$ will result in $\phi_{t+n} \in S^0_{t-1}$.

Case $(n+2).1$ If there exist two real numbers a_{21}, a_{22} such that

$\phi^{\mathrm{T}}_{a_{21},a_{22}} P_{t-1} \phi_{a_{21},a_{22}} \neq 0$, that is, $\phi_{a_{21},a_{22}} \notin S^0_{t-1}$, where

$$\phi_{a_{21},a_{22}} \triangleq \psi_1(a_{21}) + \psi_2(a_{22}) + \sum_{i=3}^{n} \psi_{i,t+n+2-i} \tag{7.61}$$

then choose the control inputs $u(t)$ and $u(t+1)$ as

$$u(t) = a_{21} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n-1} \tag{7.62}$$

$$u(t+1) = a_{22} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n} \tag{7.63}$$

In view of (7.14), these choices yield

$$\begin{aligned} x_1(t+n) &= u(t) + \theta^{\mathrm{T}} \phi_{t+n-1} \\ &= a_{11} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n-1} + \theta^{\mathrm{T}} \phi_{t+n-1} \\ &= a_{21}, \\ x_1(t+n+1) &= u(t+1) + \theta^{\mathrm{T}} \phi_{t+n} \\ &= a_{22} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n} + \theta^{\mathrm{T}} \phi_{t+n} \\ &= a_{22} \end{aligned} \tag{7.64}$$

where we have used (7.36) and the fact that $\phi_{t+n-1}, \phi_{t+n} \in S^0_{t-1}$. Therefore, we have $\phi_{t+n+1} = \phi_{a_{21},a_{22}} \notin S^0_{t-1}$ and, hence, $\dim S^0_{t+n+1} = \dim S^0_{t-1} + 1$. Return to Step 1 and wait for the measurement of $x_1(t+n+2)$.

Case $(n+2).2$ If no such $\phi_{a_{21},a_{22}}$ exist, that is, if

$$\sum_{j=1}^{2} \psi_j(a_j) + \sum_{j=3}^{n} \psi_{j,t+n+2-j} = \sum_{j=1}^{2} \psi_j(a_j) + \sum_{j=3}^{n} \psi_j(x_1(t+n+2-j)) \in S^0_{t-1} \quad \forall\, (a_1, a_2) \in \mathbb{R}^2 \tag{7.65}$$

then go to Step $n+3$.

<u>Step $n+i$ $(3 \leq i \leq n-1)$</u> We have pre-computed all the quantities in (7.55), and we also know from the previous steps that any choice of $u(t), u(t+1), \ldots, u(t+i-2)$ will result in $\phi_{t+n}, \phi_{t+n+1}, \ldots, \phi_{t+n+i-2} \in S^0_{t-1}$.

Case $(n+i).1$ If there exist i real numbers $a_{i1}, a_{i2}, \ldots, a_{ii}$ such that $\phi^{\mathrm{T}}_{a_{i1},\ldots,a_{ii}} P_{t-1} \phi_{a_{i1},\ldots,a_{ii}} \neq 0$, that is, $\phi_{a_{i1},\ldots,a_{ii}} \notin S^0_{t-1}$, where

$$\phi_{a_{i1},\ldots,a_{ii}} \sum_{j=1}^{i} \psi_j(a_{ij}) + \sum_{j=i+1}^{n} \psi_{j,t+n+i-j} \tag{7.66}$$

then choose the control inputs $u(t), \ldots, u(t+i-1)$ as

$$u(t+j-1) = a_{ij} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n+j-2}, \quad 1 \le j \le i \tag{7.67}$$

In view of (7.14), these choices yield

$$\begin{aligned} x_1(t+n+j-1) &= u(t+j-1) + \theta^{\mathrm{T}} \phi_{t+n+j-2} \\ &= a_{ij} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n+j-2} + \theta^{\mathrm{T}} \phi_{t+n+j-2} \\ &= a_{ij}, \ 1 \le j \le i \end{aligned} \tag{7.68}$$

where we have used (7.36) and the fact that $\phi_{t+n-1}, \ldots, \phi_{t+n+i-2} \in S_{t-1}^0$. Therefore, we have $\phi_{t+n+i-1} = \phi_{a_{i1},\ldots,a_{ii}} \notin S_{t-1}^0$ and, hence, $\dim S_{t+n+i-1}^0 = \dim S_{t-1}^0 + 1$. Return to Step 1 and wait for the measurement of $x_1(t+n+i)$.

Case $(n+i).2$ If no such $a_{i1}, a_{i2}, \ldots, a_{ii}$ exist, that is, if

$$\sum_{j=1}^{i} \psi_j(a_j) + \sum_{j=i+1}^{n} \psi_{j,t+n+i-j} = \sum_{j=1}^{i} \psi_j(a_j) + \sum_{j=i+1}^{n} \psi_j(x_1(t+n+i-j)) \in S_{t-1}^0 \tag{7.69}$$

for all $(a_1, a_2, \ldots, a_i) \in \mathbb{R}^i$, then go to Step $n+i+1$.

Step $2n$ We have pre-computed all the quantities in (7.55), and we also know from Step $2n-1$ that any choice of $u(t), u(t+1), \ldots, u(t+n-2)$ will result in $\phi_{t+n}, \phi_{t+n+1}, \ldots, \phi_{t+2n-2} \in S_{t-1}^0$. Since from (7.50) we know that $\dim S_{t-1}^0 < \dim S_\psi^0$, we conclude that there is at least one vector of the form

$$\phi_{a_{n1},\ldots,a_{nn}} \triangleq \sum_{j=1}^{n} \psi_j(a_{nj}) \tag{7.70}$$

that does *not* belong to S_{t-1}^0, that is, such that $\phi_{a_{n1},\ldots,a_{nn}}^{\mathrm{T}} P_{t-2} \phi_{a_{n1},\ldots,a_{nn}} \neq 0$. Therefore, choose the control inputs $u(t), \ldots, u(t+n-1)$ as

$$u(t+j-1) = a_{nj} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n+j-2}, \quad 1 \le j \le n \tag{7.71}$$

In view of (7.14), these choices yield

$$\begin{aligned} x_1(t+n+j-1) &= u(t+j-1) + \theta^{\mathrm{T}} \phi_{t+n+j-2} \\ &= a_{nj} - \hat{\theta}_t^{\mathrm{T}} \phi_{t+n+j-2} + \theta^{\mathrm{T}} \phi_{t+n+j-2} \\ &= a_{nj}, \ 1 \le j \le n \end{aligned} \tag{7.72}$$

where we have used (7.36) and the fact that $\phi_{t+n-1}, \ldots, \phi_{t+2n-2} \in S_{t-1}^0$. Therefore, we have $\phi_{t+2n-1} = \phi_{a_{n1},\ldots,a_{nn}} \notin S_{t-1}^0$ and, hence, $\dim S_{t+2n-1}^0 = \dim S_{t-1}^0 + 1$. Return to Step 1 and wait for the measurement of $x_1(t+2n)$.

This completes the input selection procedure as well as the proof. The input selection algorithm is summarized in Figure 7.1.

Figure 7.2 provides a graphic description of the relationships between the control inputs, the vectors ψ and ϕ, and the output x_1. This graph illustrates the fact that in order to compute the value of $x_1(t+2)$ (at time t) we must know the values of $\psi_{3,t-1}$, $\psi_{2,t}$, $\psi_{1,t+1}$ and $u(t-1)$. The pre-computed vectors $\psi_{3,t-1}$, $\psi_{2,t}$ and $\psi_{1,t+1}$ further enable us to calculate the vector ϕ_{t+2}. On the other hand, if we want to compute $x_1(t+3)$ (at time t), then from Figure 7.1 we see that we need (1) the pre-computed $\phi_{t+2} \in S^0_{t-1}$, (2) the pre-computation of $\psi_{3,t}$, $\psi_{2,t+1}$, $\psi_{1,t+2}$, and (3) the value of $u(t)$.

7.4 Finite duration

Using the computational procedure developed in the proof of Proposition 3.1, we can now guarantee that our active identification procedure will have a finite duration:

Theorem 4.1 The active identification procedure completely identifies the projection of the unknown parameter vector θ along the subspace S^0_ψ at time

$$t_f \le 2nr_0 = \boxed{2n \dim S^0_\psi} \tag{7.73}$$

Proof Proposition 3.1 shows that each independent direction in S^0_ψ takes at most $2n$ time steps to identify. Since the dimension r_0 of the subspace S^0_ψ is equal to the number of such independent directions, the active identification procedure will be completed in at most $2nr_0$ time steps.

Once this procedure is completed, we can proceed with the implementation of any control algorithm as if the parameter vector θ were known.

7.5 Concluding remarks

In this chaper, we have developed a systematic method to achieve global stabilization and tracking for discrete-time output-feedback nonlinear systems with unknown parameters. Our two-phase control strategy bears some resemblance to dual control [16], which not only stabilizes and regulates the system, but also improves the parameter estimates and the future value of the control. First, in the active identification phase, we systematically use the control to drive the states to desired points so that useful projection information about the unknown parameters is obtained. This process of

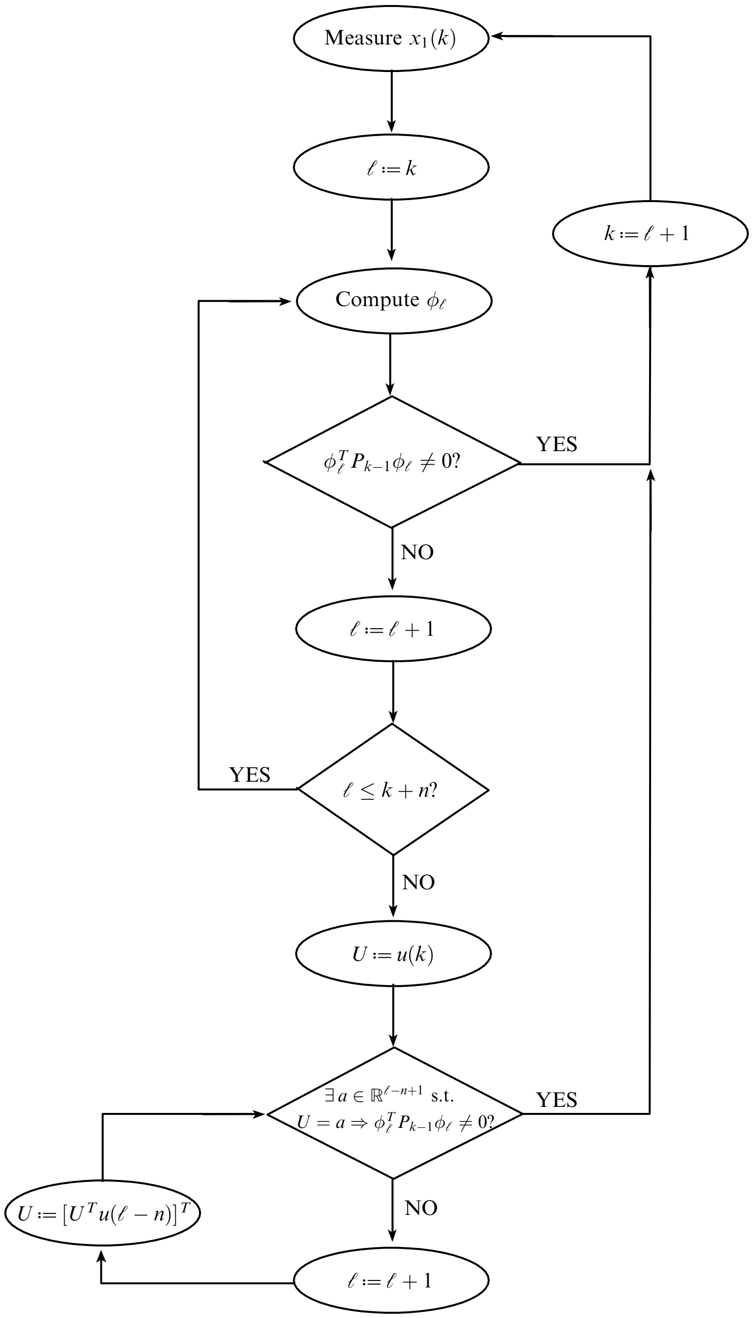

Figure 7.1 *The input selection algorithm*

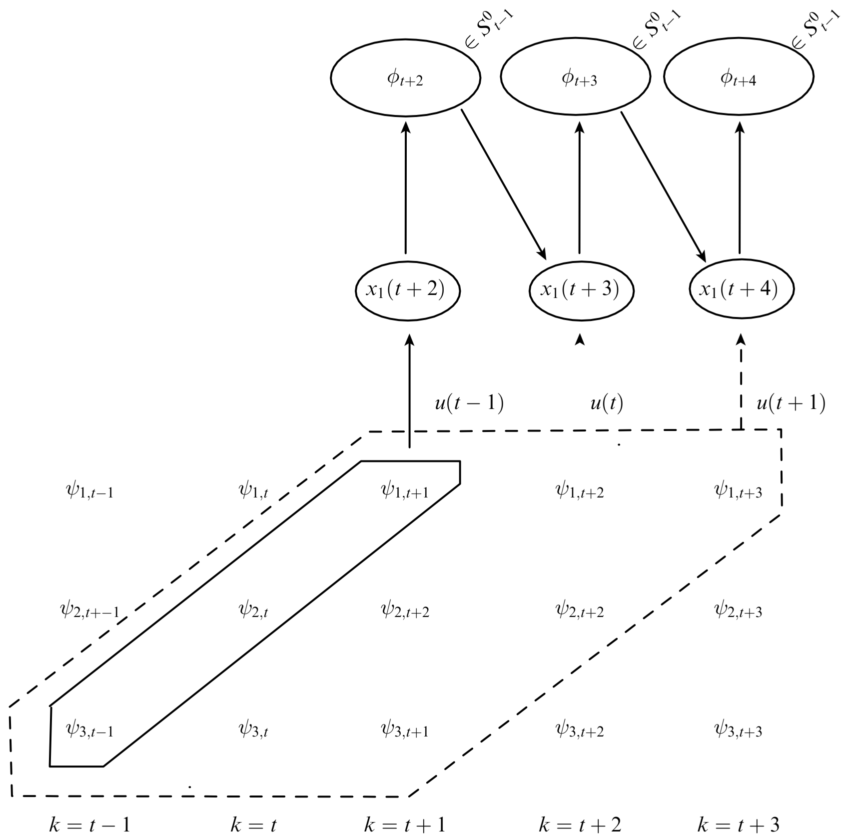

Figure 7.2 *A graphic representation of the pre-computation procedure*

active identification is finite. Once all the necessary projection information is obtained, we are able to systematically pre-compute future states and the associated projections. Then, in the subsequent control phase, we use this prediction capability to treat the system as completely known; this means that one can apply any control algorithm (the simplest being ‘deadbeat’ control) that globally stabilizes the system and tracks any given bounded reference signal when the parameters are known.

The input selection procedure that we proposed here guarantees that the active identification interval will be of finite duration. However, it does not provide any guarantees on the transient behaviour of the states during this phase. Clearly, one may be able to exploit the freedom of choice of $u(t)$ in order to make this phase shorter and smoother. This issue is a topic of current research.

Appendix

Proof of Lemma 3.1 First, let us prove (7.34).

($\Leftarrow$) Since $\mathcal{N}[P_{t-1}] = \mathcal{R}\{\phi_{n-1}, \ldots, \phi_{t-1}\} = S_{t-1}^0$, $\phi_t \in S_{t-1}^0$ implies that $P_{t-1}\phi_t = 0$. Thus, we have

$$\phi_t \in S_{t-1}^0 \Rightarrow \phi_t^{\mathrm{T}} P_{t-1}\phi_t = \phi_t^{\mathrm{T}} 0 = 0 \tag{A.1}$$

($\Rightarrow$) Assume that $\phi_t^{\mathrm{T}} P_{t-1}\phi_t = 0$. Then, the fact that $P_{t-1}\phi_t$ is a linear combination of the vectors $\phi_{n-1}, \ldots, \phi_t$ implies that $P_{t-1}\phi_t \in S_t^0$, that is, there exist constants $c_{n-1}, \ldots, c_t$ such that

$$P_{t-1}\phi_t = \sum_{i=n-1}^{t} c_i \phi_i \tag{A.2}$$

Hence, using the definition (3.23), we can infer

$$P_{t-1}\phi_t - c_t\phi_t = \sum_{i=n-1}^{t-1} c_i\phi_i \in S_{t-1}^0 \tag{A.3}$$

Since $\phi_t \in S_{t-1}^0 \subseteq S_t^0$, we can decompose the vector ϕ_t into

$$\phi_t = v_\Sigma + v_\Sigma^\perp \tag{A.4}$$

where v_Σ denotes the component of ϕ_t which belongs to S_{t-1}^0 and $v_\Sigma^\perp \in S_t^0$ is the component of ϕ_t which is orthogonal to S_{t-1}^0. Then (A.3) can be reorganized as

$$P_{t-1}\phi_t - c_t v_\Sigma^\perp = c_t v_\Sigma + \sum_{i=n-1}^{t-1} c_i\phi_i \in S_{t-1}^0 \tag{A.5}$$

We know that $P_{t-1}\phi_t \perp \mathcal{R}\{\phi_{n-1}, \ldots, \phi_{t-1}\} = S_{t-1}^0$. Hence, we conclude that $P_{t-1}\phi_t - c_t v_\Sigma^\perp \perp S_{t-1}^0$. Combining this with the fact that $P_{t-1}\phi_t - c_{t-n+2} v_\Sigma^\perp \in S_{t-1}^0$ (from (A.5)) yields

$$P_{t-1}\phi_t - c_t v_\Sigma^\perp = 0$$

that is

$$P_{t-1}\phi_t = c_t v_\Sigma^\perp \tag{A.7}$$

Multiplying both sides of equation (A.7) by ϕ_t^{T}, we obtain

$$\phi_t^{\mathrm{T}} P_{t-1}\phi_t = c_t \phi_t^{\mathrm{T}} v_\Sigma^\perp \tag{A.8}$$

In view of the decomposition (A.4), we can simplify (A.8) and obtain

$$\begin{aligned} \phi_t^{\mathrm{T}} P_{t-1}\phi_t &= c_t \phi_t^{\mathrm{T}} v_\Sigma^\perp \\ &= c_t (v_\Sigma + v_\Sigma^\perp)^{\mathrm{T}} v_\Sigma^\perp \\ &= c_t \|v_\Sigma^\perp\|^2 \end{aligned} \tag{A.9}$$

On the other hand, from the assumption we have $\phi_t^{\mathrm{T}} P_{t-1}\phi_t = 0$. Thus, equation (A.9) implies

$$c_t \|v_{\Sigma}^{\perp}\|^2 = 0 \tag{A.10}$$

from which we can conclude that

$$c_{t-n+2} v_{\Sigma}^{\perp} = 0 \tag{A.11}$$

Substituting (A.11) into (A.7) gives $P_{t-1}\phi_t = 0$, that is

$$\phi_t \in \mathcal{N}(P_{t-1}) = S_{t-1}^0 \tag{A.12}$$

So, we have shown that

$$\phi_t^{\mathrm{T}} P_{t-1}\phi_t = 0 \Rightarrow \phi_t \in S_{t-1}^0 \tag{A.13}$$

To prove (7.35), we first use (7.34):

$$\phi_t \in S_{t-1}^0 \;\; \Rightarrow \;\; \phi_t^{\mathrm{T}} P_{t-1}\phi_t = 0 \tag{A.14}$$

But, when $\phi_t^{\mathrm{T}} P_{t-1}\phi_t = 0$ the update laws (7.32)–(7.33) yield $\hat{\theta}_{t+1} = \hat{\theta}_t$ and $P_t = P_{t-1}$.

Finally, the proof of (7.36) is as follows: We know that $\tilde{\theta}_t \perp \mathcal{R}\{\phi_{n-1}, \ldots, \phi_{t-1}\}$. Thus, $v \in S_{t-1}^0$ implies that $\tilde{\theta}_t^{\mathrm{T}} v = 0$, which can be rewritten as $\hat{\theta}_t^{\mathrm{T}} v = \theta^{\mathrm{T}} v$. On the other hand, $v \in S_{t-1}^0$ implies that $v \in S_{t+l-1}^0$, since $S_{t-1}^0 \subseteq S_{t+l-1}^0$. Hence, we also conclude that $\tilde{\theta}_{t+l}^{\mathrm{T}} v = 0$, that is, $\hat{\theta}_{t+l}^{\mathrm{T}} v = \theta^{\mathrm{T}} v$. Combining these, we obtain

$$\hat{\theta}_t^{\mathrm{T}} v = \theta^{\mathrm{T}} v = \hat{\theta}_{t+l}^{\mathrm{T}} v, \quad \forall\, l = 0, 1, 2, \ldots \tag{A.15}$$

References

[1] Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1995). *Nonlinear and Adaptive Control Design*, Wiley-Interscience, NY.

[2] Marino, R. and Tomei, P. (1995). *Nonlinear Control Design: Geometric, Adaptive and Robust*, Prentice-Hall, London.

[3] Chen, F.-C. and Khalil, H. K. (1995). 'Adaptive Control of a Class of Nonlinear Discrete-time Systems Using Neural Networks', *IEEE Transactions on Automatic Control*, Vol. 40, 791–801.

[4] Song, Y. and Grizzle, J. W. (1993). 'Adaptive Output-feedback Control of a Class of Discrete-time Nonlinear Systems', *Proceedings of the 1993 American Control Conference*, San Francisco, CA, 1359–1364.

[5] Yeh, P.-C. and Kokotović, P. V. (1995). 'Adaptive Control of a Class of Nonlinear Discrete-time Systems', *International Journal of Control*, Vol. 62, 302–324.

[6] Yeh, P.-C. and Kokotović, P. V. (1995). 'Adaptive Output-feedback Design for a Class of Nonlinear discrete-time Systems', *IEEE Transactions on Automatic Control*, vol. 40, 1663–1668.

[7] Goodwin, G. C. and Sin, K. S. (1984). *Adaptive Filtering, Prediction and Control*, Prentice-Hall, Englewood Cliffs, NJ.

[8] Chen, H. F. and Guo, L. (1991). *Identification and Stochastic Adaptive Control*, Birkhäuser, Boston, MA.

[9] Guo, L. and Wei, C. (1996). 'Global Stability/Instability of LS-based Discrete-time Adaptive Nonlinear Control', *Preprints of the 13th IFAC World Congress*, San Francisco, CA, July, Vol. K, 277–282.

[10] Guo, L. (1997). 'On Critical Stability of Discrete-time Adaptive Nonlinear Control', *IEEE Transactions on Automatic Control*, Vol. 42, 1488–1499.

[11] Kanellakopoulos, I. (1994). 'A Discrete-time Adaptive Nonlinear System', *IEEE Transactions on Automatic Control*, Vol. 39, 2362–2364.

[12] Åström, K. J. and Wittenmark, B. (1984). *Computer Controlled Systems*, Prentice-Hall, Englewood Cliffs, NJ.

[13] Nešić, D. and Mareels, I. M. Y. (1998). 'Dead Beat Controllability of Polynomial Systems: Symbolic Computation Approaches', *IEEE Transactions on Automatic Control*, Vol. 43, 162–175.

[14] Zhao, J. and Kanellakopoulos, I. (1997). 'Adaptive Control of Discrete-time Strict-feedback Nonlinear Systems', *Proceedings of the 1997 American Control Conference*, Albuquerque, NM, June, 828–832.

[15] Zhao, J. and Kanellakopoulos, I. (1997). 'Adaptive Control of Discrete-time Output-feedback Nonlinear Systems', *Proceedings of the 36th Conference on Decision and Control*, San Diego, CA, December, 4326–4331.

[16] Feldbaum, A. A. (1965). *Optimal Control Systems*, Academic Press, NY.

8

Optimal adaptive tracking for nonlinear systems

M. Krstić and Z.-H. Li

Abstract

We pose and solve an 'inverse optimal' adaptive tracking problem for nonlinear systems with unknown parameters. A controller is said to be inverse optimal when it minimizes a meaningful cost functional that incorporates integral penalty on the tracking error state and the control, as well as a terminal penalty on the parameter estimation error. The basis of our method is an *adaptive tracking control Lyapunov function (atclf)* whose existence guarantees the solvability of the inverse optimal problem. The controllers designed in this chapter are not of certainty equivalence type. Even in the linear case they would not be a result of solving a Riccati equation for a given value of the parameter estimate. Our abandoning of the CE approach is motivated by the fact that, in general, this approach does not lead to optimality of the controller with respect to the overall plant-estimator system, even though both the estimator and the controller may be optimal as separate entities. Our controllers, instead, compensate for the effect of parameter adaptation transients in order to achieve optimality of the overall system.

We combine inverse optimality with backstepping to design a new class of adaptive controllers for strict-feedback systems. These controllers solve a problem left open in the previous adaptive backstepping designs – getting transient performance bounds that include an estimate of control effort, which is the first such result in the adaptive control literature.

8.1 Introduction

Because of the burden that the Hamilton–Jacobi–Bellman (HJB) pde's impose on the problem of optimal control of nonlinear systems, the efforts made over the last few years in control of nonlinear systems with uncertainties (adaptive and robust, see, e.g., Krstić *et al.*, 1995; Marino and Tomei, 1995; and the references therein) have been focused on achieving *stability* rather than optimality. Recently, Freeman and Kokotović (1996a, b) revived the interest in the optimal control problem by showing that the solvability of the (robust) *stabilization* problem implies the solvability of the (robust) *inverse optimal* control problem. Further extensive results on inverse optimal nonlinear stabilization were presented by Sepulchre *et al.* (1997).

The difference between the *direct* and the *inverse* optimal control problems is that the former seeks a controller that minimizes a *given* cost, while the latter is concerned with finding a controller that minimizes *some* 'meaningful' cost. In the inverse optimal approach, a controller is designed by using a control Lyapunov function (clf) obtained from solving the stabilization problem. The clf employed in the inverse optimal design is, in fact, a solution to the HJB pde with a meaningful cost.

In this chapter we formulate and solve the inverse optimal *adaptive tracking* problem for nonlinear systems. We focus on the tracking rather than the (set-point) regulation problem because, even when a bound on the parametric uncertainty is known, tracking cannot (in general) be achieved using robust techniques – adaptation is necessary to achieve tracking. The cost functional in our inverse optimal problem includes integral penalty on both the tracking error state and control, as well as a penalty on the terminal value of the parameter estimation error. To solve the inverse optimal adaptive tracking problem we expand upon the concept of *adaptive control Lyapunov functions (aclf)* introduced in our earlier paper (Krstić and Kokotović, 1995) and used to solve the adaptive stabilization problem.

Previous efforts to design adaptive 'linear-quadratic' controllers (see, e.g., Ioannou and Sun, 1995) were based on the certainty equivalence principle: a parameter estimate computed on the basis of a gradient or least-squares update law is substituted into a control law based on a Riccati equation solved for that value of the parameter estimate. Even though both the estimator and the controller independently possess optimality properties, when combined, they fail to exhibit optimality (and even stability becomes difficult to prove) because the controller 'ignores' the time-varying effect of adaptation. In contrast, the Lyapunov-based approach presented in this chapter results in controllers that compensate for the effect of adaptation.

A special class of systems for which we constructively solve the inverse optimal adaptive tracking problem in this chapter are the *parametric strict-feedback* systems, a representative member of a broader class of systems dealt with in Krstić *et al.* (1995), which includes feedback linearizable systems and, in particular, linear systems. A number of adaptive designs for parametric strict-feedback systems are available, however, none of them is optimal. In this chapter we present a new design which is optimal with respect to a meaningful cost. We also improve upon the existing transient performance results. The transient performance results achieved with the *tuning functions* design in Krstić *et al.* (1995), even though the strongest such results in the adaptive control literature, still provide only performance estimates on the tracking error but not on control effort (the control is allowed to be large to achieve good tracking performance). The inverse optimal design in this chapter solves the open problem of incorporating control effort in the performance bounds.

The optimal adaptive control problem posed here is not entirely dissimilar from the problem posed in the award-winning paper of Didinsky and Başar (1994) and solved using their cost-to-come method. The difference is twofold: (a) our approach does not require the inclusion of a noise term in the plant model in order to be able to design a parameter estimator, (b) while Didinsky and Başar (1994) only go as far as to derive a Hamilton–Jacobi–Isaacs equation whose solution would yield an optimal controller, we actually solve our HJB equation and obtain inverse optimal controllers for strict-feedback systems. A nice marriage of the work of Didinsky and Başar (1994) and the backstepping design in Krstić *et al.* (1995) was brought out in the paper by Pan and Başar (1996) who solved an adaptive disturbance attenuation problem for strict-feedback systems. Their cost, however, does not impose a penalty on control effort.

This chapter is organized as follows. In Section 8.2, we pose the *adaptive* tracking problem (without optimality). The solution to this problem is given in Sections 8.3 and 8.4 which generalize the results of Krstić and Kokotović (1995). Then in Section 8.5 we pose and solve the *inverse optimal* problem for general nonlinear systems assuming the existence of an *adaptive tracking Lyapunov function (atclf)*. A constructive method for designing atclf's based on backstepping is presented in Section 8.6, and then used to solve the inverse optimal adaptive tracking problem for strict-feedback systems in Section 8.7. A summary of the transient performance analysis is given in Section 8.8.

8.2 Problem statement: adaptive tracking

We consider the problem of global tracking for systems of the form

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)u \\ y &= h(x) \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, the mappings $f(x)$, $F(x)$, $g(x)$ and $h(x)$ are smooth, and θ is a constant *unknown* parameter vector which can take *any* value in $\mathbb{R}^p$. To make tracking possible in the presence of an unknown parameter, we make the following key assumption.

(A1) For a given smooth function $y_r(t)$, there exist functions $\rho(t,\theta)$ and $\alpha_r(t,\theta)$ such that

$$\begin{aligned}\frac{d\rho(t,\theta)}{dt} &= f(\rho(t,\theta)) + F(\rho(t,\theta))\theta + g(\rho(t,\theta))\alpha_r(t,\theta)\\ y_r(t) &= h(\rho(t,\theta)), \quad \forall t \geq 0, \quad \forall \theta \in \mathbb{R}^p\end{aligned} \tag{8.2}$$

Note that this implies that

$$\frac{\partial}{\partial\theta} h \circ \rho(t,\theta) = 0, \quad \forall t \geq 0,\ \forall \theta \in \mathbb{R}^p \tag{8.3}$$

For this reason, we can replace the objective of tracking the signal $y_r(t) = h \circ \rho(t,\theta)$ by the objective of tracking $y_r(t) = h \circ \rho(t,\hat{\theta}(t))$, where $\hat{\theta}(t)$ is a time function — an estimate of θ customary in adaptive control.

Consider the signal $x_r(t) = \rho(t,\hat{\theta}(t))$ which is governed by

$$\dot{x}_r = \frac{\partial\rho(t,\hat{\theta})}{\partial t} + \frac{\partial\rho(t,\hat{\theta})}{\partial\hat{\theta}}\dot{\hat{\theta}} = f(x_r) + F(x_r)\hat{\theta} + g(x_r)\alpha_r(t,\hat{\theta}) + \frac{\partial\rho(t,\hat{\theta})}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{8.4}$$

We define the tracking error $e = x - x_r = x - \rho(t,\hat{\theta})$ and compute its derivative:

$$\begin{aligned}\dot{e} &= f(x) - f(x_r) + [g(x) - g(x_r)]\alpha_r(t,\hat{\theta})\\ &\quad + F(x)\theta - F(x_r)\hat{\theta} - \frac{\partial\rho(t,\hat{\theta})}{\partial\hat{\theta}}\dot{\hat{\theta}} + g(x)[u - \alpha_r(t,\hat{\theta})]\\ &= \tilde{f} + \tilde{F}\theta + F_r\tilde{\theta} - \frac{\partial\rho}{\partial\hat{\theta}}\dot{\hat{\theta}} + g\tilde{u}\end{aligned} \tag{8.5}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and

$$\begin{aligned}\tilde{f}(t,e,\hat{\theta}) &:= f(x) - f(x_r) + [g(x) - g(x_r)]\alpha_r(t,\hat{\theta})\\ \tilde{F}(t,e,\hat{\theta}) &:= F(x) - F(x_r)\\ F_r(t,\hat{\theta}) &:= F(x_r)\\ \tilde{u} &:= u - \alpha_r(t,\hat{\theta})\end{aligned} \tag{8.6}$$

(With a slight abuse of notation, we will write $g(x)$ also as $g(t,e,\hat{\theta})$.) The global tracking problem is then transformed into the problem of global stabilization of the error system (8.5). This problem is, in general, not solvable with *static* feedback. This is obvious in the scalar case $n = p = 1$ where, even in the case $y_r(t) = x_r(t) \equiv 0$, a control law $u = \alpha(x)$ independent of θ would have the impossible task to satisfy $x[f(x) + F(x)\theta + g(x)\alpha(x)] < 0$ for all $x \neq 0$ and all $\theta \in \mathbb{R}$. Therefore, we seek *dynamic* feedback controllers to stabilize system (8.5) for all θ.

Definition 2.1 The *adaptive tracking* problem for system (8.1) is solvable if (A1) is satisfied and there exist a function $\tilde{\alpha}(t, e, \hat{\theta})$ smooth on $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$ with $\tilde{\alpha}(t, 0, \hat{\theta}) \equiv 0$, a smooth function $\tau(t, e, \hat{\theta})$, and a positive definite symmetric $p \times p$ matrix Γ, such that the dynamic controller

$$\tilde{u} = \tilde{\alpha}(t, e, \hat{\theta}) \tag{8.7}$$

$$\dot{\hat{\theta}} = \Gamma \tau(t, e, \hat{\theta}) \tag{8.8}$$

guarantees that the equilibrium $e = 0$, $\tilde{\theta} = 0$ of the system (8.5) is globally stable and $e(t) \to 0$ as $t \to \infty$ for any value of the unknown parameter $\theta \in \mathbb{R}^p$.

8.3 Adaptive tracking and atclf's

Our approach is to replace the problem of adaptive stabilization of (8.5) by a problem of nonadaptive stabilization of a modified system. This allows us to study adaptive stabilization in the Sontag–Artstein framework of *control Lyapunov functions (clf)* (Sontag, 1983; Artstein, 1983; Sontag, 1989).

Definition 3.1 A smooth function $V_a : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite, decrescent, and proper (radially unbounded) in e (uniformly in t) for each θ, is called an *adaptive tracking control Lyapunov function (atclf)* for (8.1) (or alternatively, an adaptive control Lyapunov function (aclf) for (8.5)), if (A1) is satisfied and there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{p \times p}$ such that for each $\theta \in \mathbb{R}^p$, $V_a(t, e, \theta)$ is a *clf* for the modified nonadaptive system

$$\dot{e} = \tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u} \tag{8.9}$$

that is, V_a satisfies

$$\inf_{\tilde{u} \in} IR\left\{\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u}\right]\right\} < 0 \tag{8.10}$$

In the sequel we will show that in order to achieve adaptive stabilization of (8.5) it is necessary and sufficient to achieve nonadaptive stabilization of (8.9). Noting that for $\tilde{\theta}(t) \equiv 0$ the system (8.5) reduces to the nonadaptive system

$$\dot{e} = \tilde{f} + \tilde{F}\theta + g\tilde{u} \tag{8.11}$$

we see that the modification in (8.9) is

$$F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} \tag{8.12}$$

Since these terms are present only when Γ is nonzero, the role of these terms is to account for the effect of adaptation. Since $V_a(t, e, \theta)$ has a minimum at $e = 0$ for all t and θ, the modification terms vanish at the $e = 0$, so $e = 0$ is an equilibrium of (8.9).

We now show how to design an adaptive controller (8.7)–(8.8) when an atclf is known.

Theorem 3.1 The following two statements are equivalent:

(1) There exists a triple $(\tilde{\alpha}, V_a, \Gamma)$ such that $\tilde{\alpha}(t, e, \theta)$ globally uniformly asymptotically stabilizes (8.9) at $e = 0$ for each $\theta \in \mathbb{R}^p$ with respect to the Lyapunov function $V_a(t, e, \theta)$.
(2) There exists an *atclf* $V_a(t, e, \theta)$ for (8.1).

Moreover, if an *atclf* $V_a(t, e, \theta)$ exists, then the adaptive tracking problem for (8.1) is solvable.

Proof $(1 \Rightarrow 2)$ Obvious because 1 implies that there exists a continuous function $W : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite in e (uniformly in t) for each θ, such that

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{\alpha}\right] \leq -W(t, e, \theta) \tag{8.13}$$

Thus $V_a(t, e, \theta)$ is a clf for (8.9) for each $\theta \in \mathbb{R}^p$, and therefore it is an *atclf* for (8.1).

$(2 \Rightarrow 1)$ The proof of this part is based on Sontag's formula (Sontag, 1989). We assume that V_a is an *atclf* for (8.1), that is, a clf for (8.9). Sontag's formula applied to (8.9) gives a control law smooth on $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$:

$$\tilde{\alpha}(t, e, \theta) = \begin{cases} -\dfrac{\dfrac{\partial V_a}{\partial t} + \dfrac{\partial V_a}{\partial e}\bar{f} + \sqrt{\left(\dfrac{\partial V_a}{\partial t} + \dfrac{\partial V_a}{\partial e}\bar{f}\right)^2 + \left(\dfrac{\partial V_a}{\partial e}g\right)^4}}{\dfrac{\partial V_a}{\partial e}g}, & \dfrac{\partial V_a}{\partial e}g(t, e, \theta) \neq 0 \\ 0, & \dfrac{\partial V_a}{\partial e}g(t, e, \theta) = 0 \end{cases} \tag{8.14}$$

where

$$\bar{f} = \tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} \tag{8.15}$$

With the choice (8.14), inequality (8.13) is satisfied with the continuous function

$$W(t,e,\theta) = \sqrt{\left(\frac{\partial V_a}{\partial t}(t,e,\theta) + \frac{\partial V_a}{\partial e}\bar{f}(t,e,\theta)\right)^2 + \left(\frac{\partial V_a}{\partial e}g(t,e,\theta)\right)^4} \tag{8.16}$$

which is positive definite in e (uniformly in t) for each θ, because (8.10) implies that

$$\frac{\partial V_a}{\partial e}g(t,e,\theta) = 0 \Longrightarrow \frac{\partial V_a}{\partial t}(t,e,\theta) + \frac{\partial V_a}{\partial e}\bar{f}(t,e,\theta) < 0, \quad \forall e \neq 0\,, t \geq 0 \tag{8.17}$$

We note that the control law $\tilde{\alpha}(t,e,\theta)$, smooth away from $e = 0$, will be also continuous at $e = 0$ if and only if the atclf V_a satisfies the following property, called the *small control property* (Sontag, 1989): for each $\theta \in \mathbb{R}^p$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that, if $e \neq 0$ satisfies $|e| \leq \delta$, then there is some $\tilde{u}$ with $|\tilde{u}| \leq \varepsilon$ such that

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u}\right] < 0 \tag{8.18}$$

for all $t \geq 0$.

Assuming the existence of an atclf we now show that the adaptive tracking problem for (8.1) is solvable. Since $(2 \Rightarrow 1)$, there exists a triple $(\tilde{\alpha}, V_a, \Gamma)$ and a function W such that (8.13) is satisfied. Consider the Lyapunov function candidate

$$V(t,e,\hat{\theta}) = V_a(t,e,\hat{\theta}) + \tfrac{1}{2}(\theta - \hat{\theta})^{\mathrm{T}}\Gamma^{-1}(\theta - \hat{\theta}) \tag{8.19}$$

With the help of (8.13), the derivative of V along the solutions of (8.5), (8.7), (8.8), is

$$\begin{aligned}
\dot{V} &= \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + F_r\tilde{\theta} - \frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau(t,e,\hat{\theta}) + g\tilde{\alpha}(t,e,\hat{\theta})\right] \\
&\quad + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(t,e,\hat{\theta}) - \tilde{\theta}^{\mathrm{T}}\tau(t,e,\hat{\theta}) \\
&= \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\hat{\theta} + g\tilde{\alpha}(t,e,\hat{\theta})\right] + \frac{\partial V_a}{\partial e}F\tilde{\theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau - \tilde{\theta}^{\mathrm{T}}\tau \\
&\leq -W(t,e,\hat{\theta}) - \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau + \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau \\
&\quad + \tilde{\theta}^{\mathrm{T}}\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - \tilde{\theta}^{\mathrm{T}}\tau
\end{aligned} \tag{8.20}$$

Choosing

$$\tau(t,e,\hat{\theta}) = \left(\frac{\partial V_a}{\partial e}F(t,e,\hat{\theta})\right)^{\mathrm{T}} \tag{8.21}$$

we get

$$\dot{V} \leq -W(t, e, \hat{\theta}) \tag{8.22}$$

Thus the equilibrium $e = 0, \tilde{\theta} = 0$ of (8.5), (8.7), (8.8) is globally stable, and by LaSalle's theorem, $e(t) \to 0$ as $t \to \infty$. By Definition 2.1, the adaptive tracking problem for (8.1) is solvable.

The adaptive controller constructed in the proof of Theorem 3.1 consists of a control law $\tilde{u} = \tilde{\alpha}(t, e, \hat{\theta})$ given by (8.14), and an update law $\dot{\hat{\theta}} = \Gamma\tau(t, e, \hat{\theta})$ with (8.21). The control law $\tilde{\alpha}(t, e, \theta)$ is stabilizing for the modified system (8.9) but may not be stabilizing for the original system (8.5). However, as the proof of Theorem 3.1 shows, its certainty equivalence form $\tilde{\alpha}(t, e, \hat{\theta})$ is an adaptive globally stabilizing control law for the original system (8.5). The modified system 'anticipates' parameter estimation transients, which results in incorporating the *tuning function* τ in the control law $\tilde{\alpha}$. Indeed, the formula (8.14) for $\tilde{\alpha}$ depends on τ via

$$\frac{\partial V_a}{\partial e}\bar{f}(t, e, \theta) = \frac{\partial V_a}{\partial e}(\tilde{f} + \tilde{F}\theta) + \tau^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)^{\mathrm{T}} \tag{8.23}$$

which is obtained by combining (8.15) and (8.21). Using (8.21) to rewrite the inequality (8.13) as

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}[\tilde{f} + \tilde{F}\theta + g\tilde{\alpha}(t, e, \theta)] + \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\tau(t, e, \theta) \leq -W(t, e, \theta) \tag{8.24}$$

it is not difficult to see that the control law (8.14) containing (8.23) prevents τ from destroying the nonpositivity of the Lyapunov derivative.

Example 3.1 Consider the problem of designing an adaptive tracking controller for the system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned} \tag{8.25}$$

In light of (8.1), $f(x) = [x_2, 0]^{\mathrm{T}}$, $F(x) = [\varphi(x_1), 0]^{\mathrm{T}}$, $g(x) = [0, 1]^{\mathrm{T}}$. For any given $\mathcal{C}^2$ function $y_r(t)$, the function $\rho(t, \theta) = [\rho_1(t), \ \rho_2(t, \theta)]^{\mathrm{T}}$ is given by $\rho_1(t) = y_r(t)$ and $\rho_2(t, \theta) = \dot{y}_r(t) - \varphi(y_r)^{\mathrm{T}}\theta$, and the reference input is $\alpha_r(t, \theta) = \ddot{y}_r(t) - \frac{\partial \varphi(y_r)}{\partial y_r}\theta\dot{y}_r(t)$. Hence Assumption (A1) is satisfied.

With the signal $x_r(t) = \rho(t, \hat{\theta})$ and the tracking error $e = x - x_r$, we get the error system

$$\begin{aligned}\dot{e}_1 &= e_2 + \tilde{\varphi}^{\mathrm{T}}\theta + \varphi_r^{\mathrm{T}}\tilde{\theta} \\ \dot{e}_2 &= \tilde{u} - \frac{\partial \rho_2}{\partial \hat{\theta}}\dot{\hat{\theta}}\end{aligned} \tag{8.26}$$

where $\tilde{\varphi} = \varphi(x_1) - \varphi(x_{r1})$, $\varphi_r = \varphi(x_{r1}) = \varphi(y_r)$, $\tilde{u} = u - \alpha_r$. The modified nonadaptive error system is

$$\begin{aligned}\dot{e}_1 &= e_2 + \tilde{\varphi}^{\mathrm{T}}\theta + \varphi^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} \\ \dot{e}_2 &= \tilde{u} - \frac{\partial \rho_2}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e_1}\varphi^{\mathrm{T}}\right)^{\mathrm{T}}\end{aligned} \tag{8.27}$$

The control law

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}(t, e, \theta) \\ &= -z_1 - c_2 z_2 - (-c_1 z_1 + z_2)\left(c_1 + \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right) - \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial t}\theta - \varphi^{\mathrm{T}}\Gamma\varphi\left[1,\ c_1 + \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right] z,\end{aligned} \tag{8.28}$$

where $z_1 = e_1$, $z_2 = c_1 e_1 + e_2 + \tilde{\varphi}^T\theta$, and $c_1,\ c_2 > 0$, globally uniformly asymptotically stabilizes (8.27) at $e = 0$ with respect to $V_a = \frac{1}{2}(z_1^2 + z_2^2)$ with $W(t, e, \theta) = c_1 z_1^2 + c_2 z_2^2$. By Theorem 3.1, the adaptive tracking problem for (8.25) is solved with the control law $\tilde{u} = \tilde{\alpha}(t, e, \hat{\theta})$ and the update law

$$\dot{\hat{\theta}} = \Gamma\tau(t, e, \hat{\theta}) = \Gamma\varphi(t, e, \hat{\theta})\left[1, c_1 + \frac{\partial \tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\hat{\theta}\right] z \tag{8.29}$$

As it is always the case in adaptive control, in the proof of Theorem 3.1 we used a Lyapunov function $V(t, e, \hat{\theta})$ given by (8.19), which is quadratic in the parameter error $\theta - \hat{\theta}$. The quadratic form is suggested by the linear dependence of (8.5) on θ, and the fact that θ cannot be used for feedback. We will now show that the quadratic form of (8.19) is both necessary and sufficient for the existence of an atclf.

Definition 3.2 The *adaptive quadratic tracking* problem for (8.1) is solvable if the *adaptive tracking* problem for (8.1) is solvable and, in addition, there exist a smooth function $V_a(t, e, \theta)$ positive definite, decrescent, and proper in e (uniformly in t) for each θ, and a continuous function $W(t, e, \theta)$ positive definite in e (uniformly in t) for each θ, such that the derivative of (8.19) along the solutions of (8.5), (8.7), (8.8) is given by (8.22).

Corollary 3.1 The adaptive quadratic tracking problem for the system (8.1) is solvable if and only if there exists an *atclf* $V_a(t, e, \theta)$.

Proof The 'if' part is contained in the proof of Theorem 3.1 where the Lyapunov function $V(t, e, \hat{\theta})$ is in the form (8.19). To prove the 'only if' part, we start by assuming global adaptive quadratic stabilizability of (8.5), and first show that $\tau(t, e, \hat{\theta})$ must be given by (8.21). The derivative of V along the solutions of (8.5), (8.7), (8.8), given by (8.20), is rewritten as

$$\dot{V} = \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}[\tilde{f} + \tilde{F}\hat{\theta} + g\tilde{\alpha}(t, e, \hat{\theta})] - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \hat{\theta}}\Gamma\tau(t, e, \hat{\theta}) + \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\tau(t, e, \hat{\theta}) \\ - \hat{\theta}^{\mathrm{T}}\left(\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - \tau\right) + \theta^{\mathrm{T}}\left(\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - \tau\right) \tag{8.30}$$

This expression has to be nonpositive to satisfy (8.22). Since it is affine in θ, it can be nonpositive for all $\theta \in \mathbb{R}^p$ only if the last term is zero, that is, only if τ is defined as in (8.21). Then, it is straightforward to verify that

$$\begin{aligned} &\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\hat{\theta} + F\Gamma\left(\frac{\partial V_a}{\partial \hat{\theta}}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{\alpha}\right] \\ &\quad = \dot{V} + \left(\tilde{\theta}^{\mathrm{T}} - \frac{\partial V_a}{\partial \hat{\theta}}\Gamma\right)\left(\tau - \left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}\right) \\ &\quad \le -W(t, e, \hat{\theta}) \end{aligned} \tag{8.31}$$

for all $(t, e, \hat{\theta}) \in \mathbb{R}_+ \times \mathbb{R}^{n+p}$. By $(1 \Rightarrow 2)$ in Theorem 3.1, $V_a(t, e, \theta)$ is an atclf for (8.1).

8.4 Adaptive backstepping

With Theorem 3.1, the problem of adaptive stabilization is reduced to the problem of finding an atclf. This problem is solved recursively via backstepping.

Lemma 4.1 If the adaptive quadratic tracking problem for the system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)u \\ y &= h(x) \end{aligned} \tag{8.32}$$

is solvable with a $\mathcal{C}^1$ control law, then the adaptive quadratic tracking problem for the augmented system

$$
\begin{aligned}
\dot{x} &= f(x) + F(x)\theta + g(x)\xi \\
\dot{\xi} &= u \\
y &= h(x)
\end{aligned}
\tag{8.33}
$$

is also solvable.

Proof Since the adaptive quadratic tracking problem for the system (8.32) is solvable, by Corollary 3.1 there exists an atclf $V_a(t, e, \theta)$ for (8.32), and by Theorem 3.1 it satisfies (8.13) with a control law $\tilde{\alpha}(t, e, \theta)$. Define $\tilde{\xi} = \xi - \alpha_r(t, \hat{\theta})$ and consider the system

$$
\begin{aligned}
\dot{e} &= \tilde{f}(t, e, \hat{\theta}) + \tilde{F}(t, e, \hat{\theta})\theta + F_r(t, \hat{\theta})\tilde{\theta} - \frac{\partial \rho}{\partial \hat{\theta}}\dot{\hat{\theta}} + g(t, e, \hat{\theta})\tilde{\xi} \\
\dot{\tilde{\xi}} &= \tilde{u} - \frac{\partial \alpha_r}{\partial \hat{\theta}}\dot{\hat{\theta}}
\end{aligned}
\tag{8.34}
$$

where $\tilde{u} = u - \alpha_{r1}(t, \hat{\theta})$ and $\alpha_{r1}(t, \hat{\theta}) = \dfrac{\partial \alpha_r(t, \hat{\theta})}{\partial t}$. We will now show that

$$
V_1(t, e, \tilde{\xi}, \theta) = V_a(t, e, \theta) + \tfrac{1}{2}(\tilde{\xi} - \tilde{\alpha}(t, e, \theta))^2 \tag{8.35}
$$

is an atclf for the augmented system (8.33) by showing that it is a clf for the modified nonadaptive system

$$
\begin{aligned}
\dot{e} &= \tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_1}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + g\tilde{\xi} \\
\dot{\tilde{\xi}} &= \tilde{u} - \frac{\partial \alpha_r}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}}
\end{aligned}
\tag{8.36}
$$

We present a constructive proof which shows that the control law

$$
\begin{aligned}
\tilde{u} &= \tilde{\alpha}_1(t, e, \tilde{\xi}, \theta) \\
&= -c(\tilde{\xi} - \tilde{\alpha}) + \frac{\partial \tilde{\alpha}}{\partial t} - \frac{\partial V_a}{\partial e}g + \frac{\partial \tilde{\alpha}}{\partial e}(\tilde{f} + \tilde{F}\theta + g\tilde{\xi}) \\
&\quad + \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}, \quad c > 0,
\end{aligned}
\tag{8.37}
$$

satisfies

$$
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial(e, \tilde{\xi})}\begin{bmatrix} \tilde{f} + \tilde{F}\theta + F\Gamma\left(\dfrac{\partial V_1}{\partial \theta}\right)^{\mathrm{T}} - \dfrac{\partial \rho}{\partial \theta}\Gamma\left(\dfrac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + g\tilde{\xi} \\ \tilde{\alpha}_1(t, e, \tilde{\xi}, \theta) - \dfrac{\partial \alpha_r}{\partial \theta}\Gamma\left(\dfrac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} \end{bmatrix} \le -W - c(\tilde{\xi} - \tilde{\alpha})^2
\tag{8.38}
$$

Let us start by introducing for brevity a new error state $z = \tilde{\xi} - \tilde{\alpha}(t, e, \theta)$. With (8.35) we compute

$$
\begin{aligned}
&\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial (e, \tilde{\xi})} \begin{bmatrix} \tilde{f} + \tilde{F}\theta + g\tilde{\xi} \\ \tilde{\alpha}_1(t, e, \tilde{\xi}, \theta) \end{bmatrix} \\
&= \frac{\partial V_a}{\partial t} - z\frac{\partial \tilde{\alpha}}{\partial t} + \left(\frac{\partial V_a}{\partial e} - z\frac{\partial \tilde{\alpha}}{\partial e} \right)(\tilde{f} + \tilde{F}\theta + g\tilde{\xi}) + \frac{\partial V_1}{\partial \tilde{\xi}}\tilde{\alpha}_1 \\
&= \frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left(\tilde{f} + \tilde{F}\theta + g\tilde{\alpha} \right) + z\left[\tilde{\alpha}_1 - \frac{\partial \tilde{\alpha}}{\partial t} + \frac{\partial V_a}{\partial e}g - \frac{\partial \tilde{\alpha}}{\partial e}(\tilde{f} + \tilde{F}\theta + g\tilde{\xi}) \right]
\end{aligned}
\tag{8.39}
$$

On the other hand, in view of (8.35), we have

$$
\begin{aligned}
&\frac{\partial V_1}{\partial (e, \tilde{\xi})} \begin{bmatrix} F\Gamma\left(\frac{\partial V_1}{\partial \theta} \right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F \right)^{\mathrm{T}} \\ -\frac{\partial \alpha_r}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F \right) \end{bmatrix} \\
&= \left(\frac{\partial V_a}{\partial e} - z\frac{\partial \tilde{\alpha}}{\partial e} \right)\left[F\Gamma\left(\frac{\partial V_a}{\partial \theta} - z\frac{\partial \tilde{\alpha}}{\partial \theta} \right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F - z\frac{\partial \tilde{\alpha}}{\partial e}F \right)^{\mathrm{T}} \right] \\
&\quad - z\frac{\partial \alpha_r}{\partial \theta}\Gamma\left(\frac{\partial V_1}{\partial e}F \right)^{\mathrm{T}} \\
&= \frac{\partial V_a}{\partial e}F\Gamma\left(\frac{\partial V_a}{\partial \theta} \right)^{\mathrm{T}} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F \right)^{\mathrm{T}} \\
&\quad - z\left[\left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta} \right)\Gamma\left(\frac{\partial V_1}{\partial e}F \right)^{\mathrm{T}} + \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta} \right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F \right)^{\mathrm{T}} \right]
\end{aligned}
\tag{8.40}
$$

Adding (8.39) and (8.40), with (8.13) and (8.37) we get (8.38).This proves by Theorem 3.1 that $V_1(t, e, \tilde{\xi}, \theta)$ is an atclf for system (8.33), or, an aclf for (8.34), and by Corollary 3.1 the adaptive quadratic tracking problem for this system is solvable.

The new tuning function is determined by the new atclf V_1 and given by

$$\tau_1(t,e,\tilde{\xi},\theta) = \left(\frac{\partial V_1}{\partial(e,\tilde{\xi})}\begin{bmatrix}F\\0\end{bmatrix}\right)^{\mathrm{T}} = \left(\frac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} = \left[\left(\frac{\partial V_a}{\partial e} - (\tilde{\xi}-\tilde{\alpha})\frac{\partial\tilde{\alpha}}{\partial e}\right)F\right]^{\mathrm{T}}$$
$$= \tau(t,e,\theta) - \left(\frac{\partial\tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}}(\tilde{\xi}-\tilde{\alpha}) \tag{8.41}$$

The control law $\tilde{\alpha}_1(t,e,\tilde{\xi},\theta)$ in (8.37) is only one out of many possible control laws. Once we have shown that V_1 given by (8.35) is an atclf for (8.33), or, an aclf for (8.34), we can use, for example, the C^0 control law $\tilde{\alpha}_1$ given by Sontag's formula (8.14).

Example 4.1 (Example 3.1 continued) Let us consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\\ y &= x_1\end{aligned} \tag{8.42}$$

We treat the state x_3 as an integrator added to the (x_1, x_2)-subsystem for Example 3.1, so Lemma 4.1 is applicable. Defining $z_3 = \tilde{x}_3 - \tilde{\alpha}(t,e,\theta)$, where $\tilde{x}_3 = x_3 - \alpha_r$, by Lemma 4.1, the function $V_1(t,e,\tilde{x}_3,\theta) = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2)$ is an atclf for the system (8.42). With (8.37) and (8.41) we obtain

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}_1(t,e,\tilde{\xi},\theta)\\ &= -z_2 - cz_3 + \frac{\partial\tilde{\alpha}}{\partial t} + \frac{\partial\tilde{\alpha}}{\partial e}\begin{bmatrix}e_2 + \tilde{\varphi}^{\mathrm{T}}\theta\\ \tilde{x}_3\end{bmatrix}\\ &\quad + \left(\frac{\partial\alpha_r}{\partial\theta} + \frac{\partial\tilde{\alpha}}{\partial\theta} + \frac{\partial\tilde{\alpha}}{\partial e_2}\varphi(y_r)^{\mathrm{T}}\right)\Gamma\tau_1 + \varphi^{\mathrm{T}}\Gamma\varphi\frac{\partial\tilde{\alpha}}{\partial e_1}z_2\end{aligned} \tag{8.43}$$

$$\tau_1 := \tau_1(t,e,\tilde{x}_3,\theta) = \tau - \frac{\partial\tilde{\alpha}}{\partial e_1}\varphi z_3 \tag{8.44}$$

The actual control is $u = \tilde{u} + \alpha_{r1}$ where $\alpha_{r1} = \frac{\partial\alpha_r}{\partial t} = \ddot{y}_r(t) - \frac{\partial\varphi(y_r)^{\mathrm{T}}}{\partial y_r}\theta\ddot{y}_r(t) - \frac{\partial^2\varphi(y_r)^{\mathrm{T}}}{\partial y_r^2}\theta(\dot{y}_r(t))^2$.

A repeated application of Lemma 4.1 (generalized as in Krstić *et al*., 1995 (page 138, to the case where $\dot{\xi} = u + \phi(x,\xi)^{\mathrm{T}}\theta$) recovers our earlier result (Krstić *et al*., 1992).

Corollary 4.1 [Krstić *et al*., 1992] The adaptive quadratic tracking problem for the following system is solvable

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^{\mathrm{T}}\theta, \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= u + \varphi_n(x_1, \ldots, x_n)^{\mathrm{T}}\theta \\ y &= x_1\end{aligned} \tag{8.45}$$

8.5 Inverse optimal adaptive tracking

While in the previous sections our objective was only to achieve adaptive tracking, in this section our objective is to achieve its optimality in a certain sense.

Definition 5.1 The *inverse optimal adaptive tracking* problem for system (8.1) is solvable if there exist a positive constant β, a positive real-valued function $r(t, e, \theta)$, a real-valued function $l(t, e, \theta)$ positive definite in e for each θ, and a dynamic feedback law (8.7), (8.8) which solves the adaptive quadratic tracking problem and also minimizes the cost functional

$$J = \beta \lim_{t\to\infty} |\theta - \hat{\theta}(t)|^2_{\Gamma^{-1}} + \int_0^\infty \Big(l(t, e, \hat{\theta}) + r(t, e, \hat{\theta})\tilde{u}^2 \Big) dt \tag{8.46}$$

for any $\theta \in \mathbb{R}^p$.

This definition of optimality puts penalty on e and $\tilde{u}$ as well as on the terminal value of $|\tilde{\theta}|$. Even though $\tilde{\theta}(t)$ is not guaranteed to have a limit in the general tracking case (it is guaranteed to have a limit in the case of set-point regulation (Krstić, 1996; Li and Krstić, 1996), the existence of $\lim_{t\to\infty} |\tilde{\theta}|^2_{\Gamma^{-1}}$ is assured by the assumption that the adaptive *quadratic* tracking problem is solvable. This can be seen by noting that, since $V(t) \geq 0$ and from (8.22) $V(t)$ is nonincreasing, $V(t)$ has a limit. Since (8.22) guarantees that $V_a(t) \to 0$, it follows from (8.19) that $|\tilde{\theta}|^2_{\Gamma^{-1}}$ has a limit. The absence of an integral penalty on $\tilde{\theta}$ in (8.46) should not be surprising because adaptive feedback systems, in general, do not guarantee parameter convergence to a true value.

Theorem 5.1 Suppose there exists an atclf $V_a(t, e, \theta)$ for (8.1) and a control law $\tilde{u} = \tilde{\alpha}(t, e, \theta)$ that stabilizes the system

$$\dot{e} = \underbrace{\tilde{f} + \tilde{F}\theta + F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}}_{\bar{f}} + g\tilde{u} \tag{8.47}$$

has the form

$$\tilde{\alpha}(t, e, \theta) = -r^{-1}(t, e, \theta)\frac{\partial V_a}{\partial e}g \tag{8.48}$$

where $r(t, e, \theta) > 0$ for all t, e, θ. Then

(1) The nonadaptive control law

$$\tilde{u} = \tilde{\alpha}^*(t, e, \theta) = \beta\tilde{\alpha}(t, e, \theta), \quad \beta \geq 2 \tag{8.49}$$

minimizes the cost functional

$$J_a = \int_0^\infty \big(l(t, e, \theta) + r(t, e, \theta)\tilde{u}^2\big)dt, \quad \forall\theta \in \mathbb{R}^p \tag{8.50}$$

along the solutions of the nonadaptive system (8.47), where

$$l(t, e, \theta) = -2\beta\left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left(\bar{f} + g\tilde{\alpha}\right)\right] + \beta(\beta-2)r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 \tag{8.51}$$

(2) The inverse optimal adaptive tracking problem is solvable.

Proof (Part 1) In light of (8.13), we have

$$\begin{aligned} l(t, e, \theta) &= -2\beta\left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left(\bar{f} + g\tilde{\alpha}\right)\right] + \beta(\beta-2)r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 \\ &\geq 2\beta W(t, e, \theta) + \beta(\beta-2)r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 \end{aligned} \tag{8.52}$$

Since $\beta \geq 2$, $r(t, e, \theta) > 0$, and $W(t, e, \theta)$ is positive definite, $l(t, e, \theta)$ is also positive definite. Therefore J_a defined in (8.50) is a meaningful cost functional, which puts penalty both on e and $\tilde{u}$. Substituting $l(t, e, \theta)$ and

$$v = \tilde{u} - \tilde{\alpha}^* = \tilde{u} + \beta r^{-1}\frac{\partial V_a}{\partial e}g \tag{8.53}$$

into J_a, we get

$$\begin{aligned} J_a &= \int_0^\infty \Bigg[-2\beta\frac{\partial V_a}{\partial t} - 2\beta\frac{\partial V_a}{\partial e}\bar{f} + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 + rv^2 - 2\beta v\frac{\partial V_a}{\partial e}g \\ &\quad + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2\Bigg]dt \\ &= -2\beta\int_0^\infty \left[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left(\bar{f} + g\tilde{u}\right)\right]dt + \int_0^\infty rv^2 dt \\ &= -2\beta\int_0^\infty dV_a + \int_0^\infty rv^2 dt \\ &= 2\beta V_a(0, e(0), \hat{\theta}(0)) - 2\beta\lim_{t\to\infty} V_a(t, e(t), \hat{\theta}(t)) + \int_0^\infty rv^2 dt \end{aligned} \tag{8.54}$$

Since the control input $\tilde{u}(t)$ solves the adaptive quadratic tracking problem, $\lim_{t\to\infty} e(t) = 0$, and we have that $\lim_{t\to\infty} V_a(t, e(t), \hat{\theta}(t)) = 0$. Therefore, the minimum of (8.54) is reached only if $v = 0$, and hence the control $\tilde{u} = \tilde{\alpha}^*(t, e, \theta)$ is an optimal control.

(Part 2) Since there exists an atclf V_a for (8.1), the adaptive quadratic tracking problem is solvable. Next, we show that the dynamic control law

$$\tilde{u} = \tilde{\alpha}^*(t, e, \hat{\theta}) \tag{8.55}$$

$$\dot{\hat{\theta}} = \Gamma\tau(t, e, \hat{\theta}) = \Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} \tag{8.56}$$

minimizes the cost functional (8.46). The choice of the update law (8.56) is due to the requirement that (8.55), (8.56) solves the adaptive *quadratic* tracking problem (see the proof of Corollary 3.1). Substituting $l(t, e, \hat{\theta})$ and

$$v = \tilde{u} + \beta r^{-1}\frac{\partial V_a}{\partial e}g \tag{8.57}$$

into J, along the solutions of (8.5) and (8.56) we get

$$\begin{aligned}
J &= \beta \lim_{t\to\infty} |\tilde{\theta}|^2_{\Gamma^{-1}} + \int_0^\infty \Bigg[-2\beta\frac{\partial V_a}{\partial t} - 2\beta\frac{\partial V_a}{\partial e}\left(\tilde{f} + \tilde{F}\theta + F_r\tilde{\theta} - \frac{\partial\rho}{\partial\hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}\right) \\
&\quad - 2\beta\left(\frac{\partial V_a}{\partial\hat{\theta}} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2 \\
&\quad + rv^2 - 2\beta v\frac{\partial V_a}{\partial e}g + \beta^2 r^{-1}\left(\frac{\partial V_a}{\partial e}g\right)^2\Bigg] dt \\
&= \beta \lim_{t\to\infty} |\tilde{\theta}|^2_{\Gamma^{-1}} - 2\beta\int_0^\infty \Bigg[\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left(\tilde{f} + \tilde{F}\theta + F_r\tilde{\theta} - \frac{\partial\rho}{\partial\hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + g\tilde{u}\right) \\
&\quad + \left(\frac{\partial V_a}{\partial\hat{\theta}} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}\Bigg] dt + \int_0^\infty rv^2 dt \\
&= \beta \lim_{t\to\infty} |\tilde{\theta}|^2_{\Gamma^{-1}} - 2\beta\int_0^\infty d\left(V_a + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}\right) + \int_0^\infty rv^2 dt \\
&= 2\beta V_a(0, e(0), \hat{\theta}(0)) + \beta|\tilde{\theta}(0)|^2_{\Gamma^{-1}} - 2\beta\lim_{t\to\infty} V_a(t, e(t), \hat{\theta}(t)) + \int_0^\infty rv^2 dt
\end{aligned} \tag{8.58}$$

Again, since $\tilde{u}(t)$ solves the adaptive quadratic tracking problem, $\lim_{t\to\infty} e(t) = 0$, and we have that $\lim_{t\to\infty} V_a(t, e(t), \hat{\theta}(t)) = 0$. Therefore, the minimum of (8.58) is reached only if $v = 0$, thus the control $\tilde{u} = \tilde{\alpha}^*(t, e, \hat{\theta})$ minimizes the cost functional (8.46).

Remark 5.1 Even though not explicit in the proof of the above theorem, the atclf $V_a(t, e, \theta)$ solves the following family of HJB equations parametrized in $\beta \geq 2$:

$$\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\left[\tilde{f} + \tilde{F}\theta + \underbrace{F\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}}\right] - \frac{\beta}{2r(t,e,\theta)}\left(\frac{\partial V_a}{\partial e}g\right)^2 + \frac{l(t,e,\theta)}{2\beta} = 0 \tag{8.59}$$

The under-braced terms represent the 'non-certainty equivalence' part of this HJB equation. Their role is to take into account the time-varying effect of parameter adaptation and make the control law optimal *in the presence of an update law*.

Remark 5.2 The freedom in selecting the parameter $\beta \geq 2$ in the control law (8.49) means that the inverse optimal adaptive controller has an infinite gain margin.

Example 5.1 Consider the scalar *linear* system

$$\begin{aligned} \dot{x} &= u + \theta x \\ y &= x \end{aligned} \tag{8.60}$$

For simplicity, we focus on the regulation case, $y_r(t) \equiv 0$. Since the system is scalar, $V_a = \frac{1}{2}x^2$, $L_g V_a = x$, $L_{\bar{f}} V_a = x^2\theta$. We choose the control law based on Sontag's formula

$$u_s = -x\left(\theta + \sqrt{\theta^2 + 1}\right) = -2r^{-1}(\theta)x \tag{8.61}$$

where

$$r(\theta) = \frac{2}{\theta + \sqrt{\theta^2 + 1}} > 0, \quad \forall \theta \tag{8.62}$$

The control $\frac{u_s}{2}$ is stabilizing for the system (8.60) because

$$\dot{V}_a|_{\frac{u_s}{2}} = -\tfrac{1}{2}\left(-\theta + \sqrt{\theta^2 + 1}\right)x^2 \tag{8.63}$$

By Theorem 5.1, the control u_s is optimal with respect to the cost functional

$$J_a = \int_0^\infty \left(l(x,\theta) + ru^2\right)dt = 2\int_0^\infty \frac{x^2 + u^2}{\theta + \sqrt{\theta^2 + 1}}dt \tag{8.64}$$

with a value function $J_a^*(x) = 2x^2$. Meanwhile, the dynamic control

$$u_s = -x\left(\hat{\theta} + \sqrt{\hat{\theta}^2 + 1}\right) \tag{8.65}$$

$$\dot{\hat{\theta}} = x^2 \tag{8.66}$$

is optimal with respect to the cost functional

$$J = 2(\theta - \hat{\theta}(\infty))^2 + 2\int_0^\infty \frac{x^2 + u^2}{\hat{\theta} + \sqrt{\hat{\theta}^2 + 1}} dt \tag{8.67}$$

with a value function $J^*(x, \hat{\theta}) = [x^2 + (\theta - \hat{\theta})^2]$.

We point out that $\hat{\theta}(\infty)$ exists both due to the scalar (in parameter θ) nature of the problem and because it is a problem of regulation (Krstić, 1996). Note that, even though the penalty coefficient on x and u in (8.67) varies with $\hat{\theta}(t)$, the penalty coefficient is always positive and finite.

Remark 5.3 The control law (8.61) is, in fact, a linear-quadratic-regulator (LQR) for the system (8.60) when the parameter θ is known. The control law can be also written as $u_s = -p(\theta)x$ where $p(\theta)$ is the solution of the Riccati equation

$$p^2 - 2\theta p - 1 = 0 \tag{8.68}$$

It is of interest to compare the approach in this chapter with 'adaptive LQR schemes' for linear systems in Ioannou and Sun (1995, Section 7.4.4).

- Even though both methodologies result in the same control law (8.61) for the *scalar linear* system in Example 5.1, they employ different update laws. The *gradient* update law in Ioannou and Sun (1995, Section 7.4.4) is optimal with respect to an (instantaneous) cost on an *estimation* error; however, when its estimates $\hat{\theta}(t)$ are substituted into the control law (8.65), this control law is not (guaranteed to be) optimal for the overall system. (Even its proof of stability is a non-trivial matter!) In contrast, the update law (8.66) guarantees optimality of the control law (8.65) for the overall system with respect to the meaningful cost (8.67).
- The true difference between the approach here and the adaptive LQR scheme in Ioannou and Sun (1995, Section 7.4.4) arises for systems of higher order. Then the under-braced non-certainty equivalence terms in (8.59) start to play a significant role. The CE approach in Ioannou and Sun (1995) would be to set $\Gamma = 0$ in the HJB (Riccati – for linear systems) equation (8.59) and combine the resulting control law with a gradient or least-squares update law. The optimality of the nonadaptive controller would be lost in the presence of adaptation due to the time-varying $\hat{\theta}(t)$. In contrast, a solution to Example (5.1) with $\Gamma > 0$ would lead to optimality with respect to (8.46).

Corollary 5.1 If there exists an atclf $V_a(t, e, \theta)$ for (8.1) then the inverse optimal adaptive tracking problem is solvable.

Proof Consider the Sontag-type control law $u_s = \tilde{\alpha}(t, e, \theta)$ where $\tilde{\alpha}(t, e, \theta)$ is defined by (8.14).The control law $\frac{\tilde{u}_s}{2} = \frac{1}{2}\tilde{\alpha}(t, e, \theta)$ is an asymptotic stabilizing controller for system (8.9) because inequality (8.13) is satisfied with

$$W(t, e, \theta) = \tfrac{1}{2}\left[-\left(\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f}\right) + \sqrt{\left(\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f}\right)^2 + \left(\frac{\partial V_a}{\partial e}g\right)^4}\,\right] \tag{8.69}$$

which is positive definite in e (uniformly in t) for each θ, since $\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f} < 0$ whenever $\frac{\partial V_a}{\partial e}g = 0$, $e \neq 0$ and $t \geq 0$. Since $\frac{1}{2}\tilde{\alpha}(t, e, \theta)$ is of the form $\frac{1}{2}\tilde{\alpha}(t, e, \theta) = -r^{-1}\frac{\partial V_a}{\partial e}g$ with $r(t, e, \theta) > 0$ given by

$$r(t, e, \theta) = \begin{cases} \dfrac{2\left(\frac{\partial V_a}{\partial e}g\right)^2}{\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f} + \sqrt{\left(\frac{\partial V_a}{\partial t} + \frac{\partial V_a}{\partial e}\bar{f}\right)^2 + \left(\frac{\partial V_a}{\partial e}g\right)^4}}, & \frac{\partial V_a}{\partial e}g \neq 0 \\ \text{any positive real number}, & \frac{\partial V_a}{\partial e}g = 0 \end{cases} \tag{8.70}$$

by Theorem 5.1, the inverse optimal adaptive tracking problem is solvable. The optimal control is the formula (8.14) itself.

Corollary 5.2 The inverse optimal adaptive tracking problem for the following system is solvable

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i)^{\mathrm{T}}\theta, \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= u + \varphi_n(x_1, \ldots, x_n)^{\mathrm{T}}\theta \\ y &= x_1 \end{aligned} \tag{8.71}$$

Proof By Corollary 4.1 and Corollary 3.1, there exists an atclf for (8.71). It then follows from Corollary 5.1 that the inverse optimal adaptive tracking problem for system (8.71) is solvable.

8.6 Inverse optimality via backstepping

With Theorem 5.1, the problem of inverse optimal adaptive tracking is reduced to the problem of finding an atclf. However, the control law (8.14) based on Sontag's formula is not guaranteed to be smooth at the origin. In this section we develop controllers based on backstepping which are *smooth everywhere*, and, hence, can be employed in a recursive design.

Lemma 6.1 If the adaptive quadratic tracking problem for the system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)u \\ y &= h(x) \end{aligned} \tag{8.72}$$

is solvable with a *smooth* control law $\tilde{\alpha}(t, e, \theta)$ and (8.13) is satisfied with $W(t, e, \theta) = e^{\mathrm{T}}\Omega(t, e, \theta)e$, where $\Omega(t, e, \theta)$ is positive definite and symmetric for all t, e, θ; then the inverse optimal adaptive tracking problem for the augmented system

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi \\ \dot{\xi} &= u \\ y &= h(x) \end{aligned} \tag{8.73}$$

is also solvable with a *smooth* control law.

Proof Since the adaptive quadratic tracking problem for the system (8.72) is solvable, by Lemma 4.1 and Corollary 3.1, $V_1(t, e, \tilde{\xi}, \theta) = V_a(t, e, \theta) + \frac{1}{2}(\tilde{\xi} - \tilde{\alpha}(t, e, \theta))^2$ is an atclf for the augmented system (8.73), i.e. a clf for the modified nonadaptive error system (8.36). Adding (8.39) and (8.40), with (8.13), we get

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial (e, \tilde{\xi})} \begin{bmatrix} \tilde{f} + \tilde{F}\theta + F\Gamma\left(\dfrac{\partial V_1}{\partial \theta}\right)^{\mathrm{T}} - \dfrac{\partial \rho}{\partial \theta}\Gamma\left(\dfrac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} + g\tilde{\xi} \\ \tilde{u} - \dfrac{\partial \alpha_r}{\partial \theta}\Gamma\left(\dfrac{\partial V_1}{\partial e}F\right)^{\mathrm{T}} \end{bmatrix} \\ &\leq -W + z\left[\tilde{u} - \frac{\partial \tilde{\alpha}}{\partial t} + \frac{\partial V_a}{\partial e}g - \frac{\partial \tilde{\alpha}}{\partial e}(\tilde{f} + \tilde{F}\theta + g(\tilde{\alpha} + z))\right. \\ &\quad - \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} + \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} z \\ &\quad \left. - \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)\right] \end{aligned} \tag{8.74}$$

where $z = \tilde{\xi} - \tilde{\alpha}(t, e, \theta)$. To render $\dot{V}_1$ negative definite, one choice is (8.37) which cancels all the nonlinear terms inside the bracket in (8.74). However, the cancellation controller (8.37) is not (guaranteed to be) optimal. Therefore, we have to use other techniques in the design of our control law. One such technique we will use here is 'nonlinear damping' (Krstić *et al.*, 1995).

Since $\tilde{\alpha}$, $\frac{\partial \tilde{\alpha}}{\partial t}$, $\tilde{f}$, $\tilde{F}$, $\frac{\partial V_a}{\partial e}$, $\frac{\partial V_a}{\partial \theta}$ are smooth and vanish for $e = 0$, then we can write

$$\begin{aligned}
&-\frac{\partial \tilde{\alpha}}{\partial t} + \frac{\partial V_a}{\partial e} g - \frac{\partial \tilde{\alpha}}{\partial e}(\tilde{f} + \tilde{F}\theta + g\tilde{\alpha}) \\
&-\left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}} - \left(\frac{\partial V_a}{\partial \theta} - \frac{\partial V_a}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} \qquad (8.75) \\
&= \Psi_1(t, e, \theta)^{\mathrm{T}}\Omega(t, e, \theta)^{1/2} e
\end{aligned}$$

where $\Psi_1(t, e, \theta)$ is a vector-valued smooth function and $\Omega(t, e, \theta)^{1/2}$ is invertible for all t, e, θ. In addition, let us denote

$$-\frac{\partial \tilde{\alpha}}{\partial e} g + \left(\frac{\partial \alpha_r}{\partial \theta} + \frac{\partial \tilde{\alpha}}{\partial \theta} - \frac{\partial \tilde{\alpha}}{\partial e}\frac{\partial \rho}{\partial \theta}\right)\Gamma\left(\frac{\partial \tilde{\alpha}}{\partial e}F\right)^{\mathrm{T}} = \Psi_2(t, e, \theta) \qquad (8.76)$$

Then (8.74) is re-written as

$$\dot{V}_1 \leq -|\Omega^{1/2} e|^2 + z\tilde{u} + z\Psi_1\Omega^{1/2} e + \Psi_2 z^2 \qquad (8.77)$$

The choice

$$\tilde{u} = \tilde{\alpha}_1(t, e, \tilde{\xi}, \theta) = -\left(c + \frac{|\Psi_1|^2}{2} + \frac{\Psi_2^2}{2c}\right) z, \quad c > 0 \qquad (8.78)$$

renders

$$\dot{V}_1 \leq -\tfrac{1}{2}|\Omega^{1/2} e|^2 - \frac{c}{2} z^2 \qquad (8.79)$$

Since the control law $\tilde{u} = \tilde{\alpha}_1$ defined in (8.78) is of the form

$$\tilde{\alpha}_1(t, e, \tilde{\xi}, \theta) = -R^{-1}(t, e, \tilde{\xi}, \theta)\frac{\partial V_1}{\partial(e, \tilde{\xi})}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (8.80)$$

where

$$R^{-1}(t, e, \tilde{\xi}, \theta) = \left(c + \frac{|\Psi_1|^2}{2} + \frac{\Psi_2^2}{2c}\right) > 0, \quad \forall t, e, \tilde{\xi}, \theta, \qquad (8.81)$$

by Theorem 5.1, the dynamic feedback control (adaptive control) $\tilde{u}^* = \beta\tilde{\alpha}_1(t, e, \tilde{\xi}, \hat{\theta})$, $\beta \geq 2$, with

$$\dot{\hat{\theta}} = \Gamma\tau_1(t, e, \tilde{\xi}, \hat{\theta}) = \Gamma\left(\frac{\partial V_1}{\partial e}F(t, e, \tilde{\xi}, \hat{\theta})\right)^{\mathrm{T}} \qquad (8.82)$$

is optimal for the closed loop tracking error system (8.34) and (8.32).

Example 6.1 (Example 4.1 revisited) For the system (8.42), we designed a controller (8.43) which is not optimal due to its cancellation property. With Lemma 6.1, we can design an optimal control as follows. First we note that $\tilde{\alpha}$ given by (8.28) in Example 3.1 is of the form

$$\begin{aligned}\tilde{\alpha}(t,e,\theta) &= -\left[1+\varphi^{\mathrm{T}}\Gamma\varphi-\left(c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)c_1+\frac{\partial\phi^{\mathrm{T}}}{\partial t}\theta\right]z_1\\ &\quad -\left[c_2+\left(c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)(1+\varphi^{\mathrm{T}}\Gamma\varphi)\right]z_2\\ &:= a(t,e,\theta)z_1+b(t,e,\theta)z_2\end{aligned} \tag{8.83}$$

because $\tilde{\varphi}=\varphi(x_1)-\varphi(x_{r1})=\varphi(e_1+x_{r1})-\varphi(x_{r1})=e_1\phi(e_1)$ and $\frac{\partial\tilde{\varphi}}{\partial t}=z_1\frac{\partial\phi}{\partial t}$. Instead of (8.43) we choose the 'nonlinear damping' control suggested by Lemma 6.1:

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}_1(t,e,\tilde{\xi},\theta)\\ &= -\left\{c_3+\frac{1}{2c_1}\left[\frac{\partial\tilde{\alpha}}{\partial e_1}c_1-\frac{\partial a}{\partial t}-\frac{\partial\tilde{\alpha}}{\partial e_2}a-\left(\frac{\partial\alpha_r}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial e_2}\varphi_r^{\mathrm{T}}\right)\Gamma\varphi\right]^2\right.\\ &\quad +\frac{1}{2c_2}\left[1+\frac{\partial\tilde{\alpha}}{\partial e_1}\varphi^{\mathrm{T}}\Gamma\varphi-\frac{\partial\tilde{\alpha}}{\partial e_1}-\frac{\partial b}{\partial t}-\frac{\partial\tilde{\alpha}}{\partial e_2}b\right.\\ &\quad \left.\left.-\left(\frac{\partial\alpha_r}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial\theta}+\frac{\partial\tilde{\alpha}}{\partial e_2}\varphi_r^{\mathrm{T}}\right)\Gamma\varphi\left(c_1+\frac{\partial\tilde{\varphi}^{\mathrm{T}}}{\partial e_1}\theta\right)\right]^2+\frac{1}{2c_3}\left(\frac{\partial\tilde{\alpha}}{\partial e_2}\right)^2\right\}z_3\end{aligned} \tag{8.84}$$

The tuning function τ_1 is the same as in (8.44). The control law $\tilde{u}^*=\beta\tilde{\alpha}_1(t,e,\tilde{\xi},\hat{\theta})$, $\beta\geq 2$, with $\dot{\hat{\theta}}=\Gamma\tau_1$ is optimal.

8.7 Design for strict-feedback systems

We now consider the *parametric strict-feedback systems* (Krstić *et al.*, 1995):

$$\begin{aligned}\dot{x}_i &= x_{i+1}+\varphi_i(\bar{x}_i)^{\mathrm{T}}\theta, \quad i=1,\ldots,n-1\\ \dot{x}_n &= u+\varphi_n(x)^{\mathrm{T}}\theta\\ y &= x_1\end{aligned} \tag{8.85}$$

where we use the compact notation $\bar{x}_i=(x_1,\ldots,x_i)$, and develop a procedure for optimal adaptive tracking of a given signal $y_r(t)$. An inverse optimal design following from Corollary 5.2 (based on Sontag's formula) would be non-smooth at $e=0$. In this section we develop a design which is smooth everywhere. This design is also different from the nonoptimal design in Krstić *et al.* (1992, 1995).

For the class of systems (8.85), (A1) is satisfied for any function $y_r(t)$, and there exist functions $\rho_1(t)$, $\rho_2(t,\theta)$,…,$\rho_n(t,\theta)$, and $\alpha_r(t,\theta)$ such that

$$\begin{aligned}\dot{\rho}_i &= \rho_{i+1} + \varphi_i(\bar{\rho}_i)^{\mathrm{T}}\theta, \quad i = 1, \ldots, n-1 \\ \dot{\rho}_n &= \alpha_r(t, \theta) + \varphi_n(\rho)^{\mathrm{T}}\theta \\ y_r(t) &= \rho_1(t)\end{aligned} \tag{8.86}$$

Consider the signal $x_r(t) = \rho(t, \hat{\theta})$ which is governed by

$$\begin{aligned}\dot{x}_{ri} &= x_{r,i+1} + \varphi_{ri}(\bar{x}_{ri})^{\mathrm{T}}\hat{\theta} + \frac{\partial \rho_i}{\partial \hat{\theta}}\dot{\hat{\theta}}, \quad i = 1, \ldots, n-1 \\ \dot{x}_{rn} &= \alpha_r(t, \hat{\theta}) + \varphi_{rn}(x_r)^{\mathrm{T}}\hat{\theta} + \frac{\partial \rho_n}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ y_r &= x_{r1}\end{aligned} \tag{8.87}$$

The tracking error $e = x - x_r$ is governed by the system

$$\begin{aligned}\dot{e}_i &= e_{i+1} + \tilde{\varphi}_i(t, \bar{e}_i, \hat{\theta})^{\mathrm{T}}\theta + \varphi_{ri}(t, \hat{\theta})^{\mathrm{T}}\tilde{\theta} - \frac{\partial \rho_i}{\partial \hat{\theta}}\dot{\hat{\theta}}, \quad i = 1, \ldots, n-1 \\ \dot{e}_n &= \tilde{u} + \tilde{\varphi}_n(t, e, \hat{\theta})^{\mathrm{T}}\theta + \varphi_{rn}(t, \hat{\theta})^{\mathrm{T}}\tilde{\theta} - \frac{\partial \rho_n}{\partial \hat{\theta}}\dot{\hat{\theta}}\end{aligned} \tag{8.88}$$

where $\tilde{u} = u - \alpha_r(t, \hat{\theta})$ and $\tilde{\varphi}_i = \varphi_i(\bar{x}_i) - \varphi_{ri}(\bar{x}_{ri})$, $i =, \ldots, n$. For an atclf V_a, the modified nonadaptive error system is

$$\begin{aligned}\dot{e}_i &= e_{i+1} + \tilde{\varphi}_i^{\mathrm{T}}\theta + \varphi_i^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho_i}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}, \quad i = 1, \ldots, n-1 \\ \dot{e}_n &= \tilde{u} + \tilde{\varphi}_n(t, e, \hat{\theta})^{\mathrm{T}}\theta + \varphi_n^{\mathrm{T}}\Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}} - \frac{\partial \rho_n}{\partial \hat{\theta}}\Gamma\left(\frac{\partial V_a}{\partial e}F\right)^{\mathrm{T}}\end{aligned} \tag{8.89}$$

where $F = [\varphi_1, \ldots, \varphi_n]^{\mathrm{T}}$.

First, we search for an atclf for the system (8.85). Repeated application of Lemma 4.1 gives an atclf

$$\begin{aligned}V_a &= \tfrac{1}{2}\sum_{i=1}^{n} z_i^2 \\ z_i &= e_i - \tilde{\alpha}_{i-1}(t, \bar{e}_{i-1}, \theta)\end{aligned} \tag{8.90}$$

where $\tilde{\alpha}_i$'s are to be determined. For notational convenience we define $z_0 := 0$, $\tilde{\alpha}_0 := 0$. We then have

$$\frac{\partial V_a}{\partial \theta} = -\sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} z_j \tag{8.91}$$

$$\left(\frac{\partial V_a}{\partial e} F\right)^{\mathrm{T}} = \sum_{j=1}^{n} \frac{\partial V_a}{\partial e_j} \varphi_j = \sum_{j=1}^{n} \left(z_j - \sum_{k=j+1}^{n} \frac{\partial \tilde{\alpha}_{k-1}}{\partial e_j} z_k \right) \varphi_j = \sum_{j=1}^{n} w_j z_j \tag{8.92}$$

where

$$w_j(t, \bar{e}_j, \theta) = \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \tilde{\alpha}_{j-1}}{\partial e_k} \varphi_k \tag{8.93}$$

Therefore the modified nonadaptive error system (8.89) becomes

$$\begin{aligned} \dot{e}_i &= e_{i+1} + \tilde{\varphi}_i^{\mathrm{T}} \theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_i z_j - \sum_{j=1}^{n} \frac{\partial \rho_i}{\partial \theta} \Gamma w_j z_j, \quad i = 1, \ldots, n-1 \\ \dot{e}_n &= \tilde{u} + \tilde{\varphi}_n^{\mathrm{T}} \theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_n z_j - \sum_{j=1}^{n} \frac{\partial \rho_n}{\partial \theta} \Gamma w_j z_j \end{aligned} \tag{8.94}$$

The functions $\tilde{\alpha}_1,\ldots,\tilde{\alpha}_{n-1}$ are yet to be determined to make V_a defined in (8.90) a clf for system (8.94). To design these functions, we apply the backstepping technique as in Krstić *et al.* (1995). We perform cancellations at all the steps before step n. At the final step n, we depart from Krstić *et al.* (1995) and choose the actual control $\tilde{u}$ in a form which, according to Lemma 6.1, is inverse optimal.

Step $i = 1, \ldots, n-1$:

$$\begin{aligned} \tilde{\alpha}_i(t, \bar{e}_i, \theta) = &-z_{i-1} - c_i z_i + \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} + \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} - \tilde{w}_i^{\mathrm{T}} \theta \\ &- \sum_{k=1}^{i-1} (\sigma_{ki} + \sigma_{ik}) z_k - \sigma_{ii} z_i, \quad c_i > 0 \end{aligned} \tag{8.95}$$

$$\tilde{w}_i(t, \bar{e}_i, \theta) = \tilde{\varphi}_i - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} \tilde{\varphi}_k \tag{8.96}$$

$$\sigma_{ik} = -\left(\frac{\partial \tilde{\alpha}_{i-1}}{\partial \theta} + \frac{\partial \rho_i}{\partial \theta} - \sum_{j=2}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_j} \frac{\partial \rho_j}{\partial \theta} \right) \Gamma w_k \tag{8.97}$$

Step n: With the help of Lemma A.1 in the appendix, the derivative of V_a is

$$\begin{aligned}\dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \Bigg[z_{n-1} + \sum_{k=1}^{n-1} (\sigma_{kn} + \sigma_{nk}) z_k + \sigma_{nn} z_n + \tilde{u} \\ - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta \Bigg]\end{aligned} \tag{8.98}$$

We are now at the position to choose the actual control $\tilde{u}$. We may choose $\tilde{u}$ such that all the terms inside the bracket are cancelled and the bracketed term multiplying z_n is equal to $-c_n z_n^2$ as in Krstić *et al.* (1995), but the controller designed in that way is not guaranteed to be inverse optimal. To design a controller which is inverse optimal, according to Theorem 5.1, we should choose a control law that is of the form

$$\tilde{u} = \tilde{\alpha}_n(t, e, \theta) = -r^{-1}(t, e, \theta) \frac{\partial V_a}{\partial e} g \tag{8.99}$$

where $r(t, e, \theta) > 0$, $\forall t, e, \theta$. In light of (8.94) and (8.90), (8.99) simplifies to

$$\tilde{u} = \tilde{\alpha}_n(t, e, \theta) = -r^{-1}(t, e, \theta) z_n \tag{8.100}$$

i.e. we must choose $\tilde{\alpha}_n$ with z_n as a factor.

Since $e_{k+1} = z_{k+1} + \tilde{\alpha}_k$, $k = 1, \ldots, n-1$, and the expression in the second line in (8.98) vanishes at $e = 0$, it is easy to see that it also vanishes for $z = 0$. Therefore, there exist smooth functions ϕ_k, $k = 1, \ldots, n$, such that

$$-\frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta = \sum_{k=1}^{n} \phi_k z_k \tag{8.101}$$

Thus (8.98) becomes

$$\dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \tilde{u} + \sum_{k=1}^{n} z_n \Phi_k z_k \tag{8.102}$$

where

$$\begin{aligned}\Phi_k &= \sigma_{kn} + \sigma_{nk} + \phi_k, \quad k = 1, \ldots, n-2 \\ \Phi_{n-1} &= 1 + \sigma_{n-1,n} + \sigma_{n,n-1} + \phi_{n-1} \\ \Phi_n &= \sigma_{nn} + \phi_n\end{aligned} \tag{8.103}$$

A control law of the form (8.100) with

$$r(t, e, \theta) = \left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k} \right)^{-1} > 0, \quad c_k > 0, \quad \forall t, e, \theta \tag{8.104}$$

results in

$$\dot{V}_a = -\tfrac{1}{2}\sum_{k=1}^{n} c_k z_k^2 - \sum_{k=1}^{n} \frac{c_k}{2}\left(z_k - \frac{\Phi_k}{c_k} z_n\right)^2 \tag{8.105}$$

By Theorem 5.1, the inverse optimal adaptive tracking problem is solved through the dynamic feedback control (adaptive control) law

$$\begin{aligned}\tilde{u} &= \tilde{\alpha}_n^*(t, e, \hat{\theta}) = 2\tilde{\alpha}_n(t, e, \hat{\theta}) \\ \dot{\hat{\theta}} &= \Gamma\left(\frac{\partial V_a}{\partial e} F\right)^{\mathrm{T}} = \Gamma \sum_{j=1}^{n} w_j z_j\end{aligned} \tag{8.106}$$

8.8 Transient performance

In this brief section, we give an $\mathcal{L}_2$ bound on the error state z and control $\tilde{u}$ for the inverse optimal adaptive controller designed in Section 8.7. According to Theorem 5.1, the control law (8.106) is optimal with respect to the cost functional

$$\begin{aligned}J &= 2\lim_{t\to\infty} |\theta - \hat{\theta}(t)|^2_{\Gamma^{-1}} \\ &+ 2\int_0^\infty \left[\sum_{k=1}^{n} c_k z_k^2 + \sum_{k=1}^{n} c_k\left(z_k - \frac{\Phi_k}{c_k} z_n\right)^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k}\right)}\right] dt\end{aligned} \tag{8.107}$$

with a value function

$$J^* = 2|\theta - \hat{\theta}|^2_{\Gamma^{-1}} + 2|z|^2 \tag{8.108}$$

In particular, we have the following $\mathcal{L}_2$ performance result.

Theorem 8.1 In the adaptive system (8.88), (8.106), the following inequality holds

$$\int_0^\infty \left[\sum_{k=1}^{n} c_k z_k^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^{n} \frac{\Phi_k^2}{2c_k}\right)}\right] dt \le |\tilde{\theta}(0)|^2_{\Gamma^{-1}} + |z(0)|^2 \tag{8.109}$$

This theorem presents the first performance bound in the adaptive control literature that includes an estimate of *control effort*.

The bound (8.109) depends on $z(0)$ which is dependent on the design parameters $c_1, \ldots, c_n$, and Γ. To eliminate this dependency and allow a systematic improvement of the bound on $\|z\|_2$, we employ *trajectory initialization* as in Krstić *et al.* (1995, Section 4.3.2) to set $z(0) = 0$ and obtain:

$$\int_0^\infty \left[\sum_{k=1}^n c_k z_k^2 + \frac{\tilde{u}^2}{2\left(c_n + \sum_{k=1}^n \frac{\Phi_k^2}{2c_k}\right)} \right] dt \leq |\tilde{\theta}(0)|^2_{\Gamma^{-1}} \tag{8.110}$$

8.9 Conclusions

In this chapter we showed that the solvability of the inverse optimal adaptive control problem for a given system is implied by the solvability of the HJB (nonadaptive) equation for a *modified* system. Our results can be readily extended to the multi-input case and the case where the input vector field depends on the unknown parameter.

In constructing an inverse optimal adaptive controller for strict-feedback-systems we followed the simplest path of using an atclf designed by the tuning functions method in Krstić *et al.* (1995). Numerous other (possibly better) choices are possible, including an inverse optimal adaptive design at each step of backstepping. The relative merit of different approaches is yet to be established.

Appendix A technical lemma

Lemma A.1 The time derivative of V_a in (8.90) along the solutions of system (8.94) with (8.95)–(8.97) is given by

$$\begin{aligned} \dot{V}_a = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \Bigg[& z_{n-1} + \sum_{k=1}^{n-1} (\sigma_{kn} + \sigma_{nk}) z_k + \sigma_{nn} z_n + \tilde{u}. \\ & - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta \Bigg] \end{aligned} \tag{A.1}$$

Proof First we prove that the closed loop system after i steps is

$$
\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -c_1 & 1+\pi_{12} & \pi_{13} & \cdots & \pi_{1,i-1} & \pi_{1i} \\ -1-\pi_{12} & -c_2 & 1+\pi_{23} & \cdots & \pi_{2,i-1} & \pi_{2i} \\ -\pi_{13} & -1-\pi_{23} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\pi_{i-2,i-1} & \pi_{i-2,i} \\ -\pi_{1,i-1} & -\pi_{2,i-1} & \cdots & -1-\pi_{i-2,i-1} & -c_{i-1} & 1+\pi_{i-1,i} \\ -\pi_{1i} & -\pi_{2i} & \cdots & -\pi_{i-2,i} & -1-\pi_{i-1,i} & -c_i \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_i \end{bmatrix}
$$
$$
+ \begin{bmatrix} \pi_{1,i+1} & \pi_{1,i+2} & \cdots & \pi_{1n} \\ \pi_{2,i+1} & \pi_{2,i+2} & \cdots & \pi_{2n} \\ \vdots & & & \\ \pi_{i-1,i+1} & \pi_{i-1,i+2} & \cdots & \pi_{i-1,n} \\ 1+\pi_{i,i+1} & \pi_{i,i+2} & \cdots & \pi_{i,n} \end{bmatrix} \begin{bmatrix} z_{i+1} \\ \vdots \\ z_n \end{bmatrix} \tag{A.2}
$$

and the resulting $\dot{V}_i$ is

$$
\dot{V}_i = -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i} \pi_{kj} z_k \tag{A.3}
$$

where

$$
\pi_{ik} = \eta_{ik} + \xi_{ik} \tag{A.4}
$$

$$
\eta_{ik} = -\frac{\partial \tilde{\alpha}_{k-1}}{\partial \theta} \Gamma w_i \tag{A.5}
$$

$$
\xi_{ik} = -\left(\frac{\partial \rho_i}{\partial \theta} - \sum_{j=2}^{i-1} \frac{\partial \tilde{\alpha}_{k-1}}{\partial e_j} \frac{\partial \rho_j}{\partial \theta} \right) \Gamma w_k \tag{A.6}
$$

The proof is by induction. Step 1: Substituting $\tilde{\alpha}_1 = -c_1 z_1 - \tilde{\varphi}_1^{\mathrm{T}} \theta$ into (8.94) with $i = 1$, using (8.90) and noting $\tilde{\alpha}_0 = 0$ and $\dfrac{\partial \rho_1}{\partial \theta} = 0$, we get

$$
\dot{z}_1 = -c_1 z_1 + z_2 - \sum_{j=1}^{n} z_j \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma w_1 = -c_1 z_1 + z_2 - \pi_{12} z_3 - \cdots - \pi_{1n} z_n \tag{A.7}
$$

and

$$
\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 - \sum_{j=1}^{n} z_j \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma w_1 z_1 = -c_1 z_1^2 + z_1 z_2 - \sum_{j=2}^{n} z_j \pi_{1j} z_1 \tag{A.8}
$$

which shows that (A.2) and (A.3) are true for $i = 1$.

Assume that (A.2) and (A.3) are true for $i - 1$, that is

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_{i-1} \end{bmatrix} = \begin{bmatrix} -c_1 & 1+\pi_{12} & \pi_{13} & \cdots & \pi_{1,i-2} & \pi_{1,i-1} \\ -1-\pi_{12} & -c_2 & 1+\pi_{23} & \cdots & \pi_{2,i-2} & \pi_{2,i-1} \\ -\pi_{13} & -1-\pi_{23} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\pi_{i-2,i-2} & \pi_{i-2,i-1} \\ -\pi_{1,i-2} & -\pi_{2,i-2} & \cdots & -1-\pi_{i-3,i-2} & -c_{i-2} & 1+\pi_{i-2,i-1} \\ -\pi_{1,i-1} & -\pi_{2,i-1} & \cdots & -\pi_{i-3,i-1} & -1-\pi_{i-2,i-1} & -c_{i-1} \end{bmatrix}$$

$$\times \begin{bmatrix} z_1 \\ \vdots \\ z_{i-1} \end{bmatrix} + \begin{bmatrix} \pi_{1,i} & \pi_{1,i+1} & \cdots & \pi_{1n} \\ \pi_{2,i} & \pi_{2,i+1} & \cdots & \pi_{2n} \\ \vdots & & & \\ \pi_{i-2,i} & \pi_{i-2,i+1} & \cdots & \pi_{i-2,n} \\ 1+\pi_{i-1,i} & \pi_{i-1,i+1} & \cdots & \pi_{i-1,n} \end{bmatrix} \begin{bmatrix} z_i \\ \vdots \\ z_n \end{bmatrix} \tag{A.9}$$

and

$$\dot{V}_{i-1} = -\sum_{k=1}^{i-1} c_k z_k^2 + z_{i-1} z_i + \sum_{j=i}^{n} z_j \sum_{k=1}^{i-1} \pi_{kj} z_k \tag{A.10}$$

The z_i-subsystem is

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \tilde{\alpha}_i + \tilde{\varphi}_i^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_i z_j - \sum_{j=1}^{n} \frac{\partial \rho_i}{\partial \theta} \Gamma w_j z_j \\ &\quad - \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} \left(e_{k+1} + \tilde{\varphi}_k^{\mathrm{T}}\theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_k z_j - \sum_{j=1}^{n} \frac{\partial \rho_k}{\partial \theta} \Gamma w_j z_j \right) \\ &= z_{i+1} + \tilde{\alpha}_i - \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i{}^{\mathrm{T}}\theta + \sum_{j=1}^{n} \pi_{ij} z_j \end{aligned} \tag{A.11}$$

The derivative of $V_i = V_{i-1} + \frac{1}{2} z_i^2$ is calculated as

$$\begin{aligned}\dot{V}_i = & -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i-1} \pi_{kj} z_k + \sum_{j=i+1}^{n} z_j \pi_{ij} z_i \\ & + z_i \left[z_{i-1} + \sum_{k=1}^{i-1} \pi_{ki} z_k + \tilde{\alpha}_i - \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i^{\mathrm{T}} \theta + \sum_{k=1}^{i} \pi_{ik} z_k \right] \\ = & -\sum_{k=1}^{i-1} c_k z_k^2 + z_i z_{i+1} + \sum_{j=i+1}^{n} z_j \sum_{k=1}^{i} \pi_{kj} z_k \\ & + z_i \left[\tilde{\alpha}_i + z_{i-1} - \frac{\partial \tilde{\alpha}_{i-1}}{\partial t} - \sum_{k=1}^{i-1} \frac{\partial \tilde{\alpha}_{i-1}}{\partial e_k} e_{k+1} + \tilde{w}_i^{\mathrm{T}} \theta + \sum_{k=1}^{i-1} (\pi_{ki} + \pi_{ik}) z_k + \pi_{ii} z_i \right]\end{aligned} \tag{A12}$$

From the definitions of π_{ik}, η_{ik}, δ_{ik} and σ_{ik}, it is easy to show that $\pi_{ki} + \pi_{ik} = \sigma_{ki} + \sigma_{ik}$ and that $\pi_{ii} = \sigma_{ii}$. Therefore the choice of $\tilde{\alpha}_i$ as in (8.95) results in (A.3) and

$$\dot{z}_i = -\sum_{k=1}^{i-2} \pi_{ki} z_k - (1 + \pi_{i-1,i}) z_{i-1} - c_i z_i + (1 + \pi_{i,i+1}) z_{i+1} + \sum_{k=i+2}^{n} \pi_{ik} z_k \tag{A.13}$$

Combining (A.13) with (A.9), we get (A.2).

We now rewrite the last equation of (8.94) as

$$\begin{aligned}\dot{z}_n = & \tilde{u} + \tilde{\varphi}_n^{\mathrm{T}} \theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_n z_j - \sum_{j=1}^{n} \frac{\partial \rho_n}{\partial \theta} \Gamma w_j z_j \\ & - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} \left(e_{k+1} + \tilde{\varphi}_k^{\mathrm{T}} \theta - \sum_{j=1}^{n} \frac{\partial \tilde{\alpha}_{j-1}}{\partial \theta} \Gamma \varphi_k z_j - \sum_{j=1}^{n} \frac{\partial \rho_k}{\partial \theta} \Gamma w_j z_j \right) \\ = & \tilde{u} - \frac{\partial \tilde{\alpha}_{n-1}}{\partial t} - \sum_{k=1}^{n-1} \frac{\partial \tilde{\alpha}_{n-1}}{\partial e_k} e_{k+1} + \tilde{w}_n^{\mathrm{T}} \theta + \sum_{j=1}^{n} \pi_{nj} z_j\end{aligned} \tag{A.14}$$

where $\tilde{w}_n$ follows the same definition as in (8.96). Noting that $V_a = V_{n-1} + \frac{1}{2} z_n^2$ and $\pi_{kn} + \pi_{nk} = \sigma_{kn} + \sigma_{nk}$, and $\pi_{nn} = \sigma_{nn}$, (A.1) follows readily from (A.3) and (A.4).

References

[1] Artstein, Z. (1983) 'Stabilization with Relaxed Controls', *Nonlinear Analysis*, **TMA–7**, 1163–1173.

[2] Didinsky, G. and Başar, T. (1994) 'Minimax Adaptive Control of Uncertain Plants', *Proceedings of the 33rd IEEE Conference on Decision and Control*, 2839–2844, Lake Buena Vista, FL.

[3] Freeman, R. A. and Kokotović, P. V. (1996a) 'Inverse Optimality in Robust Stabilization', *SIAM Journal on Control and Optimization*, **34**, 1365–1391.

[4] Freeman, R. A. and Kokotović, P. V. (1996b) *Robust Nonlinear Control Design*. Birkhause, Boston.

[5] Ioannou, P. A. and Sun, J. (1995) *Robust Adaptive Control*. Prentice-Hall, New Jersey.

[6] Krstić, M. (1996) 'Invariant Manifolds and Asymptotic Properties of Adaptive Nonlinear Systems, *IEEE Transactions on Automatic Control*, **41**, 817–829.

[7] Krstić, M. and Kokotović, P. V. (1995) 'Control Lyapunov Functions for Adaptive Nonlinear Stabilization', *Systems & Control Letters*, **26**, 17–23.

[8] Krstić M., Kanellakopoulos, I. and Kokotović P. V (1992) 'Adaptive Nonlinear Control Without Overparametrization', *Systems & Control Letters*, **19**, 177–185.

[9] Krstić M., Kanellakopoulos, I. and Kokotović P. V. (1995) *Nonlinear and Adaptive Control Design*. Wiley, New York.

[10] Li, Z. H. and Krstić M. (1996) 'Geometric/Asymptotic Properties of Adaptive Nonlinear Systems with Partial Excitation, *Proceedings of the 35th IEEE Conference on Decision and Control*, 4683–4688, Kobe, Japan; also to appear in *IEEE Transac. Autom. Contr*.

[11] Marino, R. and Tomei, P. (1995) *Nonlinear Control Design: Geometric, Adaptive, and Robust*. Prentice-Hall, London.

[12] Pan, Z. and Başar, T. (1996) 'Adaptive Controller Design for Tracking and Disturbance Attenuation in Parametric-strict-feedback Nonlinear Systems', *Proceedings of the 13th IFAC Congress of Automatic Control*, **vol. F**, 323–328, San Francisco, CA.

[13] Sepulchre, R., Jankovic, M. and Kokotović P. V. (1997) *Constructive Nonlinear Control*. Springer-Verlag, New York.

[14] Sontag, E. D. (1983) 'A Lyapunov-like Characterization of Asymptotic Controllability', *SIAM Journal of Control and Optimization*, **21**, 462–471.

[15] Sontag, E. D. (1989) 'A "Universal" Construction of Artstein's Theorem on Nonlinear Stabilization, *Systems & Control Letters*, **13**, 117–123.

9

Stable adaptive systems in the presence of nonlinear parametrization

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Abstract

This chapter addresses the problem of adaptive control when the unknown parameters occur nonlinearly in a dynamic system. The traditional approach used in linearly parametrized systems employs a gradient-search principle in estimating the unknown parameters. Such an approach is not sufficient for nonlinearly parametrized (NLP) systems. Over the past two years, a new algorithm based on a min–max optimization scheme has been developed to address NLP adaptive systems. This algorithm is shown to result in globally stable closed loop systems when the states of the plant are accessible for measurement. We present the fundamental principles behind this approach and the stability properties of the resulting adaptive system in this chapter. Several examples from practical applications are presented which possess the NLP property. Simulation studies that illustrate the performance that results from the proposed algorithm and the improvement over the gradient scheme are also presented.

9.1 Introduction

The field of adaptive control has, by and large, treated the control problem in the presence of parametric uncertainties with the assumption that the unknown parameters occur linearly [1]. This assumption has been fairly central in the development of adaptive estimation and control strategies, and commonly made in both discrete-time and continuous-time plants. Whether in an adaptive observer or an adaptive controller, the assumption of linear parametrization has dictated the choice of the structure of the dynamic system. For instance, in adaptive observers, the focus has been on structures that allow measurable outputs to be expressed as linear, but unknown, combinations of accessible system variables. In direct adaptive control, a model-based controller is chosen so as to allow the desired closed loop output to be linear in the control parameters. In indirect adaptive control, estimators and controllers are often chosen so as to retain the linearity in the parameters being estimated. The design of stable adaptive systems using the classical augmented approach as in [1] or using adaptive nonlinear designs as in [2] relies heavily on linear parametrization.

The problem is an important one both in linear and nonlinear dynamic systems, albeit for different reasons. In nonlinear dynamic systems, despite the fact that the majority of results have sought to extend the ideas of feedback linearization to systems with parametric uncertainties using the certainty equivalence principle, it is only within the context of linearly parametrizable nonlinear dynamics that global results have been available. Obviously, it is a nontrivial task to find transform methods for general nonlinear systems so as to convert them into systems with linear parametrizations. In linear dynamic systems, it is quite possible to transform the problem into one where unknown parameters occur linearly. However, such a transformation also can result in a large dimension of the space of linear parameters. This has a variety of consequences. The first is due to overparametrization which requires much larger amounts of persistent excitation or results in a lower degree of robustness. The second is that it can introduce undue restrictions in the allowable parametric uncertainty due to possible unstable pole-zero cancellations [3].

Nonlinearly parameterized systems are ubiquitous in nature, and as such unavoidable in many characterizations of observed complex behaviour. Friction dynamics [4], dynamics of magnetic bearing [5], and biochemical processes [6] are some examples where more than one physical parameter occurs nonlinearly in the underlying dynamic model. Several biological models of complex systems such as Hammerstein, Uryson, and Wiener models, consist of combinations of static nonlinearities and linear dynamic systems, which invariably result in NLP systems. The recent upsurge of interest in neural networks also stems from the fact that they can parsimoniously represent complex behaviour by including nonlinear parameterization.

The fundamental property that linearly parametrized (LP) systems possess is in the cost function related to the parameter estimation. The latter can be posed as a hill-climbing problem, where the objective is to get to the bottom of the hill which is a measurable cost function of the parameter error. In LP systems, this hill is unimodal due to the linearity as a result of which, gradient rules suffice to ensure convergence to the bottom of the hill which is guaranteed to be unimodal. In NLP systems, such a property is no longer preserved and hence, the gradient algorithm is not sufficient to ensure either stability or convergence. We show how algorithms that guarantee desired stability properties can be designed for general NLP systems in this chapter.

Our proposed approach involves the introduction of a new adaptive law that utilizes a min–max optimization procedure, properties of concave functions and their covers, and a new error model. In the min–max optimization, the maximization is that of a tuning function over all possible values of the nonlinear parameter, and the minimization is over all possible sensitivity functions that can be used in the adaptive law. These two tuning functions lead to sign definiteness of an underlying nonlinear function of the states and the parameter estimates. Such a property is then utilized to establish closed loop system stability. Concave/convex parametrization can be viewed as a special case of nonlinear parametrization. That is, if θ is an unknown parameter and is known to lie in a compact set Θ, then the underlying nonlinearity is either concave with respect to all $\theta \in \Theta$ or convex with respect to all $\theta \in \Theta$. We make use of properties of concave and convex functions in the development of the adaptive algorithms. To address adaptive control for general NP systems, a concave cover is utilized.The third feature of our approach is the development of an error model for NP systems. In any adaptive system, typically, there are two kinds of errors, which include a tuning error and parameter error. The goal of the adaptive system is to drive the former to zero while the latter is required to be at least bounded. Depending on the complexity of the dynamic system and the parametrization, the relationship between the two errors is correspondingly complex. A new error model is introduced in this chapter that is applicable to stable NLP adaptive system design.

Very few results are available in the literature that address adaptive control in the presence of nonlinear parametrization [7, 8]. For these systems, the use of a gradient algorithm as in systems with linear parametrization will not only prove to be inadequate but can actually cause the system errors to increase. The approach used in [7], for example, was a gradient-like adaptive law which suffices for stabilization and concave functions but would be inadequate either for tracking or for convex functions. In [9, 10, 11, 12], adaptive estimation and control of general NLP systems, both in continuous-time and discrete-time has been addressed, stability conditions for the closed loop system have been derived, and the resulting performance in various practical applications such as friction dynamics, magnetic bearing, and chemical reactors, has been addressed. An overview of the approach developed in these papers and the major highlights are presented in this chapter.

In Section 9.2, we provide the statement of the problem, the applicability of the gradient rule, and the motivation behind the new algorithm. In Section 9.3, we present certain preliminaries that provide the framework for the proposed adaptive algorithms. These include definitions, properties of concave/convex functions, concave covers, solutions of min–max optimization problems, and sign definiteness of related nonlinear functions. In Section 9.4, the new error model is introduced, and its properties are derived. This is followed by stable adaptive system designs for both a controller and an observer. The first addresses NLP systems when states are accessible, and the second addresses the problem of estimation in NLP systems when states are not accessible. Together, they set the stage for the general adaptive control problem when only the input and output are available for measurement. In Section 9.5, the performance of the proposed adaptive controller is explored in the context of physical examples, such as the position control of a single mass system subjected to nonlinear friction dynamics, positioning in magnetic bearings, and temperature regulation in chemical reactors. In Section 9.6, concluding remarks and extensions to discrete-time systems and systems where the matching conditions are not satisfied are briefly discussed. Proofs of lemmas are presented in the appendix.

9.2 Statement of the problem

The class of plants that we shall consider in this chapter is of the form

$$\dot{X}_p = A_p(p)X_p + b_p\left[u + \varphi^T(t)\beta + \sum_{i=1}^{m} f_i(\phi_i(t), \theta_i)\right] \tag{9.1}$$

where u is a scalar control input, $X_p \in \mathbb{R}^n$ is the plant state assumed accessible for measurement, $p, \theta_i \in \mathbb{R}^m$, and $\beta \in \mathbb{R}^l$ are unknown parameters. $\phi_i : \mathbb{R}^+ \to \mathbb{R}^m$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^l$ are known and bounded functions of the state variable. A_p is nonlinear in p, and f_i is nonlinear in both ϕ_i and θ_i. Our goal is to find an input u such that the closed loop system has globally bounded solutions and so that X_p tracks as closely as possible the state X_m of a reference model specified as

$$\dot{X}_m = A_m X_m + b_m r \tag{9.2}$$

where r is a bounded reference input.

The plant in (9.1) has been studied in the literature extensively, whether or not the states of the system are accessible, but with the assumption that A_p is *linear* in p, and that f_i is *linear* in θ_i. Our focus here is when A_p and f_i are nonlinear functions of the parameters p and θ_i, respectively. This chapter is restricted to the case when all states of (9.1) are accessible. Further extensions to such systems with only input and output accessible are open problems and are currently under investigation.

9.2.1 *The structure of the adaptive controller*

In order to evaluate the behaviour of the adaptive controller that is employed when the plant is linearly parametrized (which shall be referred to as an LP-adaptive controller), we consider the simplest form of plants in (9.1), given by

$$\dot{x}_p = f(\phi, \theta) + u \tag{9.3}$$

where x_p and θ are scalars, and θ is an unknown parameter and ϕ is a bounded function of x_p. Choosing $A_m = -k < 0$, and $b_m = 1$, the structure of the plant in equation (9.3) suggests that when θ is known, a control input of the form

$$u = -f(\phi, \theta) - kx_p + r \tag{9.4}$$

leads to a closed loop system

$$\dot{x}_p \;=\; A_m x_p + b_m r$$

and hence x_p is bounded and tracks X_m asymptotically. The question is how u can be determined when θ is unknown and f is nonlinear in θ. The LP-adaptive control approach is to choose the input as

$$u = -f(\phi, \hat{\theta}) - kx_p + r \tag{9.5}$$

Defining $e_c = x_p - x_m, \tilde{\theta} = \hat{\theta} - \theta$, and $\hat{f} = f(\phi, \hat{\theta})$, we obtain that the error equation is of the form of

$$\dot{e}_c = -ke_c + f - \hat{f} \tag{9.6}$$

Suppose f is linear in θ with

$$f(\phi, \theta) = g(\phi)\theta$$

then the gradient algorithm used in linear adaptive control approaches (see Chapter 3 in [1], for example) suggests that $\tilde{\theta}$ must be adjusted as

$$\dot{\tilde{\theta}} = e_c \nabla f_{\hat{\theta}} \tag{9.7}$$

The above adaptive law is derived from stability considerations by showing that

$$V = e_c^2 + \tilde{\theta}^2 \tag{9.8}$$

is a Lyapunov function, which follows since the gradient of f with respect to $\hat{\theta}$ is $g(\phi)$ and is independent of θ.

When f is nonlinear in θ, the application of such a gradient approach is inadequate and in fact undesirable, since it can lead to an unstable behaviour. For instance, if V is chosen as in (9.8), the error equation (9.6) with the adaptive law as in (9.7) leads to a time derivative

$$\dot{V} = -ke_c^2 + e_c\left[f - \hat{f} + \tilde{\theta}\nabla f_{\hat{\theta}}\right]$$

When f is concave, the property of a concave function enables us to conclude that $\dot{V} \leq 0$ when $e_c > 0$; it also implies that

$$\dot{V} \geq 0 \qquad \text{if } e_c < 0 \quad \text{and} \quad k|e_c| < \left|f - \hat{f} + \tilde{\theta}\nabla f_{\hat{\theta}}\right|$$

This illustrates that, for a general f, it cannot be guaranteed that $\dot{V} \leq 0$ for all e_c and $\hat{\theta}$ and thus can lead to unbounded solutions. To illustrate this, if we choose $f = \phi\theta e^{-\theta^2\phi}$, $k = 1$ and $\theta = 1$, then for any ϕ which is such that

$$|\phi(t)| \ \geq \ \phi_0 > 0, \quad \forall\, t$$

we have $\nabla f_{\hat{\theta}} < 0$ and $\tilde{\theta}\nabla f_{\hat{\theta}} > 0$ for all

$$\hat{\theta} \in \left(-\infty, -\frac{1}{\sqrt{2\phi_0}}\right) \quad \text{if } \phi(t) \geq \phi_0$$

$$\hat{\theta} \in \left(\frac{1}{\sqrt{2\phi_0}}, \infty\right) \quad \text{if } \phi(t) \leq -\phi_0$$

It follows therefore that when $\phi(t) > 0$, in the region

$$\mathcal{D} = \left\{(e_c, \hat{\theta}) \mid 0 < ke_c < f - \hat{f}, \hat{\theta} \in \left(-\infty, -\frac{1}{\sqrt{2\phi_0}}\right)\right\}$$

we have

$$\dot{V} = -ke_c^2 + e_c\left[f - \hat{f} + \tilde{\theta}\nabla f_{\hat{\theta}}\right] > 0$$

since

$$ke_c < f - \hat{f} < f - \hat{f} + \tilde{\theta}\nabla f_{\hat{\theta}}$$

Hence, $\mathcal{D}$ is an open invariant region where $\dot{V} > 0$, since $\dot{e}_c > 0$ at $e_c = 0$, $\dot{e}_c < 0$ for $0 < e_c < \dfrac{f - \hat{f}}{k}$, and $\dot{\hat{\theta}} < 0\ \forall \hat{\theta} \in \left(-\infty, -\dfrac{1}{\sqrt{2\phi_0}}\right)$.This implies that the gradient algorithm will lead to $\hat{\theta}(t) \to -\infty$ if $\phi(t) > 0$ and hence in unbounded solutions. Similarly, we can show that $\hat{\theta}(t) \to \infty$ if $\phi(t) < 0$. This clearly illustrates the inadequacy of the gradient rule for nonlinear parametrizations.

The discussion above indicates that the underlying problem is to ensure that the nonlinear function of the form

$$e_c\left[f - \hat{f} + \tilde{\theta}\omega\right]$$

has to be made nonpositive by an appropriate choice of ω. This has to be ensured independent of the sign of e_c without introducing discontinuities in the controller. We illustrate how this can be accomplished in Section 9.4.

9.3 Preliminaries

The problem of NLP systems is addressed by making use of concavity/convexity of the underlying nonlinearity wherever possible and by introducing concave covers in other cases. In this section, we introduce the basic definitions and properties of concave/convex functions as well as concave covers in

Sections 9.3.1 and 9.3.2, respectively.We show that these properties also ensure the desired sign-definiteness property which will be used to design globally stable adaptive systems in Section 9.4.

9.3.1 Concave/convex functions

9.3.1.1 Definitions

Definition 3.1 A function $f(\theta)$ is said to be (i) convex on Θ if it satisfies the inequality

$$f(\lambda\theta_1 + (1-\lambda)\theta_2) \leq \lambda f(\theta_1) + (1-\lambda)f(\theta_2), \quad \forall\theta_1, \theta_2 \in \Theta \tag{9.9}$$

and (ii) concave if it satisfies the inequality

$$f(\lambda\theta_1 + (1-\lambda)\theta_2) \geq \lambda f(\theta_1) + (1-\lambda)f(\theta_2), \quad \forall\theta_1, \theta_2 \in \Theta \tag{9.10}$$

where $0 \leq \lambda \leq 1$.

Definition 3.2 A simplex Θ_S in $\mathbb{R}^m$ is a convex polyhedron having exactly $m+1$ vertices.

In order to obtain a continuous controller, we employ the following saturation function extensively in the control design:

Definition 3.3 The saturation function, sat$(\cdot)$, is defined as

$$\text{sat}(y) = \begin{cases} 1 & y \geq 1 \\ y & |y| < 1 \\ -1 & y \leq -1 \end{cases}$$

9.3.1.2 Properties of concave/convex functions

A useful property of these functions is their relation to the gradient. When $f(\theta)$ is convex and differentiable on Θ, then it can be shown that

$$f(\theta) - f(\theta_0) \geq \nabla f_{\theta_0}(\theta - \theta_0), \quad \forall\theta, \theta_0 \in \Theta \tag{9.11}$$

and when $f(\theta)$ is concave and differentiable on Θ, then

$$f(\theta) - f(\theta_0) \leq \nabla f_{\theta_0}(\theta - \theta_0), \quad \forall\theta, \theta_0 \in \Theta \tag{9.12}$$

where $\nabla f_{\theta_0} = \frac{\partial f}{\partial \theta}|_{\theta_0}$.

Lemma 3.1 Let

$$J(\omega,\theta)=\beta\Big[f(\phi,\theta)-f(\phi,\hat{\theta})+\omega(\hat{\theta}-\theta)\Big] \tag{9.13}$$

$$a_0=\min_{\omega\in\mathbb{R}}\ \max_{\theta\in\Theta}\ J(\omega,\theta) \tag{9.14}$$

$$\omega_0=arg\ \min_{\omega\in\mathbb{R}}\ \max_{\theta\in\Theta}\ J(\omega,\theta) \tag{9.15}$$

where $\hat{\theta}\in\Theta$ and β is independent of θ. Then

$$a_0=\begin{cases}\beta\Big[f_{\min}-\hat{f}+\dfrac{f_{\max}-f_{\min}}{\theta_{\max}-\theta_{\min}}(\hat{\theta}-\theta_{\min})\Big] & \text{if } \beta f \text{ is convex on } \Theta\\ 0 & \text{if } \beta f \text{ is concave on } \Theta\end{cases} \tag{9.16}$$

$$\omega_0=\begin{cases}\dfrac{f_{\max}-f_{\min}}{\theta_{\max}-\theta_{\min}} & \text{if } \beta f \text{ is convex on } \Theta\\ \nabla f_{\hat{\theta}} & \text{if } \beta f \text{ is concave on } \Theta\end{cases} \tag{9.17}$$

where $\Theta=[\theta_{\min},\theta_{\max}]$, $f_{\max}=f(\phi,\theta_{\max})$ and $f_{\min}=f(\phi,\theta_{\min})$.

Proof See appendix.

Lemma 3.2 For any x, and $\theta,\hat{\theta}\in\Theta$,

$$x\Big[f-\hat{f}-(\theta-\hat{\theta})\omega_0-a_0\,\mathrm{sat}\Big(\frac{x}{\varepsilon}\Big)\Big]\le 0,\quad \forall\ |x|>\varepsilon \tag{9.18}$$

whether f is concave or convex, if a_0 and ω_0 are chosen as in (9.16) and (9.17) respectively, with $\beta=\mathrm{sign}(x)$.

Proof See appendix.

When f is linear in θ, Lemma 3.2 is trivially satisfied, which can be shown as follows. If $f(\phi,\theta)=g(\phi)\theta$, then, $a_0=0$ if $\omega_0=g(\phi)$. With such an a_0, the left-hand side in (9.18) is identically zero. Lemma 3.2 implies that when f is nonlinear in θ, appropriate a_0 and ω_0 can be found leading to an inequality as in (9.18) rather than an equality.

Lemma 3.3 For a vector θ, let

$$a_0=\min_{\omega\in\mathbb{R}^m}\ \max_{\theta\in\Theta_S}\ J(\omega,\theta) \tag{9.19}$$

$$\omega_0=arg\ \min_{\omega\in\mathbb{R}^m}\ \max_{\theta\in\Theta_S}\ J(\omega,\theta) \tag{9.20}$$

where $J(\omega,\theta)$ is as in (9.13), $\hat{\theta}\in\Theta_S\in\Theta$ and β is a known nonzero constant. Then

$$a_0=\begin{cases}A_1 & \text{if } \beta f \text{ is convex on } \Theta_S\\ 0 & \text{if } \beta f \text{ is concave on } \Theta_S\end{cases} \tag{9.21}$$

$$\omega_0 = \begin{cases} A_2 & \text{if } \beta f \text{ is convex on } \Theta_S \\ \nabla f & \text{if } \beta f \text{ is concave on } \Theta_S \end{cases} \tag{9.22}$$

where $A = [A_1, A_2]^T = G^{-1}b$,$A_1$ is a scalar, $A_2 \in \mathbb{R}^m$

$$G = \begin{bmatrix} -1 & \beta(\hat{\theta} - \theta_{S1})^T \\ -1 & \beta(\hat{\theta} - \theta_{S2})^T \\ \vdots & \vdots \\ -1 & \beta(\hat{\theta} - \theta_{Sm+1})^T \end{bmatrix}, \quad b = \begin{bmatrix} \beta(\hat{f} - f_{S1}) \\ \beta(\hat{f} - f_{S2}) \\ \vdots \\ \beta(\hat{f} - f_{Sm+1} \end{bmatrix} \tag{9.23}$$

and $f_{Si} = f(\phi, \theta_{Si})$.

Proof See appendix.

9.3.2 Concave covers

9.3.2.1 Definitions

Definition 3.4 A point $\theta^0 \in \theta_c$ if $\theta^0 \in \Theta$ and

$$\beta \nabla f_{\theta^0}(\theta - \theta^0) \geq \beta(f - f^0), \quad \forall \theta \in \Theta \tag{9.24}$$

where $\nabla f_{\theta^0} \triangleq \left.\dfrac{\partial f}{\partial \theta}\right|_{\theta^0}$ and $f^0 = f(\phi, \theta^0)$.

Definition 3.5 $\breve{\theta}_c \triangleq \bar{\theta}_c \cap \Theta$, where $\bar{\theta}_c$ denotes the complement of θ_c.

A concave cover of a function $\beta(f - \hat{f})$ on Θ is defined as:

Definition 3.6

$$F(\theta) = \begin{cases} \beta(f - \hat{f}), & \forall \theta \in \theta_c \\ \beta(\omega^{ij}\theta + c^{ij}) & \forall \theta \in \theta^{ij} \in \breve{\theta}_c \end{cases} \tag{9.25}$$

where

$$\omega^{ij} = \frac{f^j - f^i}{\theta^j - \theta^i}, \quad c^{ij} = f^i - \hat{f} - \omega^{ij}\theta^i, \quad f^i = f(\phi, \theta^i) \tag{9.26}$$

9.3.2.2 Properties

Lemma 3.4 $F(\theta)$ as defined in (9.25) has the following properties:

(1) $F(\theta)$ is concave on Θ and $F(\theta) \geq \beta(f - \hat{f})$, $\forall \theta \in \Theta$.

(2) The solutions a_0 and ω_0 for equations (9.14) and (9.15) are given by

$$a_0 = F(\hat{\theta}) \tag{9.27}$$

$$\omega_0 = \begin{cases} \nabla f_{\hat{\theta}} & \text{if } \hat{\theta} \in \theta_c \\ \omega^{ij} & \text{if } \hat{\theta} \in \theta^{ij} \in \breve{\theta}_c \end{cases} \tag{9.28}$$

where ω^{ij} is defined as in (9.26).

Proof See appendix.

Lemma 3.5 For any x, and $\theta, \hat{\theta} \in \Theta$

$$x\left[f - \hat{f} - (\theta - \hat{\theta})\omega_0 - a_0 \operatorname{sat}\left(\frac{x}{\varepsilon}\right)\right] \leq 0 \tag{9.29}$$

by choosing a_0 and ω_0 using (9.27) and (9.28) respectively.

Proof The proof follows along the same lines as that of Lemma 3.2.

9.3.2.3 Examples of concave covers

To illustrate the nature and construction of a concave cover for a general function f, we present a number of examples in this section. We show that the nature of the cover not only depends on the function but also on the compact set Θ that the parameter θ lies in.

Example 3.1 Let

$$f(\phi, \theta) = g(\phi)\cos\theta \tag{9.30}$$

where g is a nonlinear function of the system variable ϕ. Obviously, if Θ includes any interval whose length is larger than π, f is neither concave nor convex on Θ. We show how a concave cover $F(\theta)$ can be constructed for different intervals of Θ, assuming $\beta = 1$.

(a) $\Theta = [-\pi/2, 5\pi/2]$: Since inequality (9.24) is satisfied for all θ^0 in intervals $\theta^{01} = [-\pi/2, 0]$ and $\theta^{23} = [2\pi, 5\pi/2]$, we have that θ^{01}, $\theta^{23} \subset \theta_c$. For any $\theta^0 \in \theta^{12} = [0, 2\pi]$, the inequality (9.24) is not satisfied, and hence $\breve{\theta}_c = \theta^{12}$. At first glance, one might be tempted to conclude that the interval $[-\pi/2, \pi/2] \in \theta_c$. This is not true since for any $\theta^0 \in (0, \pi/2]$, inequality (9.24) will not be satisfied $\forall\theta \in \Theta$ since Θ includes points beyond $\pi/2$. Since $\theta^1 = 0$ and $\theta^2 = 2\pi$, we have that

$$\omega^{12} = 0 \quad \text{and} \quad c^{12} = 1 - \hat{f}$$

Hence, in this example, $F(\theta) = f - \hat{f}$ over θ^{01} and θ^{23} and is a straight line with zero gradient over θ^{12}, as shown in Figure 9.1. Note that although $F(\theta)$ depends on $\hat{f}$, the determination of the sets θ_c and $\breve{\theta}_c$ in this example is independent of $\hat{f}$ since $\hat{f}$ only affects the vertical scale of F.

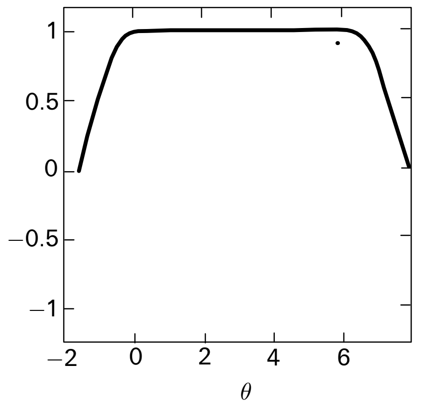

Figure 9.1 *Illustration of* $\cos(\theta)$ $(\ldots)$ *and* $F(\theta)$ (—) *for* $\Theta = [-\pi/2, 5\pi/2]$

(b) $\Theta = [-1.4\pi, 2\pi]$: Unlike case (a), there are fewer subintervals in θ_c in this case. We note that there does not exist any point $\theta^0 \in [-1.4\pi, -\pi/2]$ which satisfies (9.24). Therefore, $\theta^1 = -1.4\pi$. Next, we need to find a point θ^2 greater than $-\pi/2$ such that θ^2 satisfies (9.24). Since θ^2 lies on $f - \hat{f}$ and $\theta^{12} \in \breve{\theta}_c$, θ^2 is at the intersection of $f - \hat{f}$ and the straight line joining f^1 and f^2 whose gradient is $-\sin(\theta^2)$ at θ^2. That is, θ^2 satisfies the relation

$$\cos(-1.4\pi) - \cos(\theta^2) = (\theta^2 + 1.4\pi)\sin(\theta^2)$$

which yields $\theta^2 = -0.31365$ rad. Proceeding as in the previous example, we note that $\theta^{23} = [\theta^2, 0] \in \theta_c$ while $\theta^{34} = [0, 2\pi] \in \breve{\theta}_c$. The resulting $F(\theta)$ is illustrated in Figure 9.2. It should be noted that θ^2 is close to but not coincident with the peak at zero.

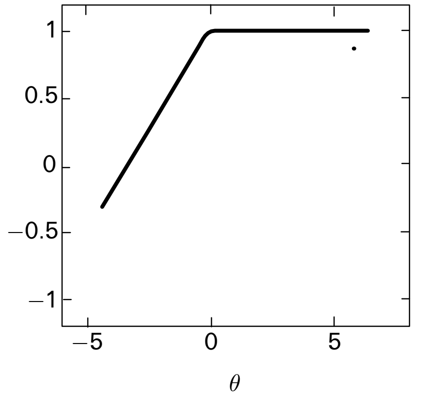

Figure 9.2 *Illustration of* $\cos(\theta)$ $(\ldots)$ *and* $F(\theta)$ (—) *for* $\Theta = [-1.4\pi, 2\pi]$

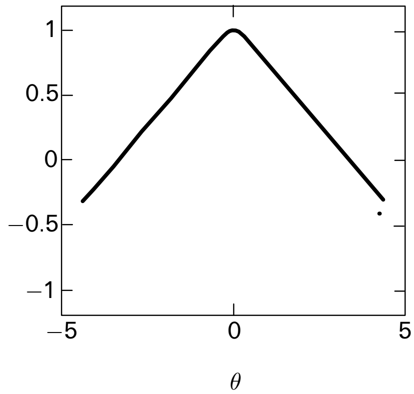

Figure 9.3 *Illustration of* $\cos(\theta)$ $(\ldots)$ *and* $F(\theta)$ (—) *for* $\Theta = [-1.4\pi, 1.4\pi]$

(c) $\Theta = [-1.4\pi, 1.4\pi]$: Case (b) implies that $\theta^{12} = [-1.4\pi, -0.31365] \in \breve{\theta}_c$. Also, $[-0.313\,65, 0] \in \theta_c$. In this case, since Θ includes only points up to 1.4π, we also have that $[0, \theta^3] \in \theta_c$ for some $\theta^3 > 0$. Proceeding as in case (b), we can show that θ^3 is the solution of the equation

$$\cos(1.4\pi) - \cos(\theta^3) = (\theta^3 - 1.4\pi)\sin(\theta^3)$$

or $\theta^3 = 0.313\,65$. We also can conclude that $\theta^{34} = [\theta^3, 1.4\pi] \in \breve{\theta}_c$. The resulting $F(\theta)$ as well as the convex cover which is computed in a similar fashion are shown in Figure 9.3.

(d) $\Theta = [-\alpha\pi, \lambda\pi]$, $n + 0.5 \leq \alpha, \lambda \leq n + 2$, $n = 0, 2, \ldots,$ even: We first define n_x as the largest even integer that is smaller than x. Then θ_c and $\breve{\theta}_c$ can be determined as

$$\theta_c = \{[-n_\alpha\pi + \theta^1, -n_\alpha\pi], [n_\lambda\pi, n_\lambda\pi + \theta^2]\}$$
$$\breve{\theta}_c = \{[-\alpha\pi, -n_\alpha\pi + \theta^1], [-n_\alpha\pi, n_\lambda\pi], [n_\lambda\pi + \theta^2, \lambda\pi]\}$$

where θ^1 and θ^2 are computed using the relations

$$\cos\alpha\pi - \cos\theta^1 = \alpha\pi\sin\theta^1 + \theta^1\sin\theta^1$$
$$\cos\lambda\pi - \cos\theta^2 = \theta^2\sin\theta^2 - \lambda\pi\sin\theta^2$$

It is worthwhile noting that when $\alpha, \lambda = n$, n even, then θ_c is an empty set and $\breve{\theta}_c = [-n\pi, n\pi]$.

We note that f in (9.30) is independent of ϕ and as a result, θ_c and $\breve{\theta}_c$ can be determined off-line which reduces the implementational burden of the proposed on-line algorithm. We next consider an example where θ_c and $\breve{\theta}_c$ are

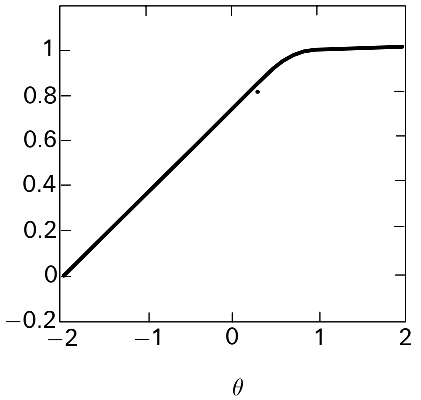

Figure 9.4 *Illustration of* $f(\phi,\theta) = \dfrac{1}{1+e^{-\phi\theta}}$ $(\ldots)$ *and* $F(\theta)$ (—) *for* $\Theta = [-2, 2]$ *and* $\phi = 5$

functions of the system variable ϕ and thus cannot be determined easily on-line.

Example 3.2 Consider f given by

$$f(\phi,\theta) = \frac{1}{1+e^{-\phi\theta}}, \qquad \theta \in [-2, 2]$$

In this case, $\theta_c = \{[\theta^1, 2]\}$ and $\breve{\theta}_c = \{[-2, \theta^1]\}$ where θ^1 is determined from the following nonlinear equation :

$$\phi(2+\theta^1)e^{-\phi\theta^1} = 1 + e^{-\phi\theta^1} \tag{9.31}$$

It is clear that the solution of equation (9.31) depends on ϕ. For a fixed ϕ, a numerical routine such as the Newton–Raphson method could be used to solve for θ^1. Figure 9.4 gives an illustration for the concave cover $F(\theta)$ for $\phi = 5$.

We note that $F(\theta)$ always exist for any bounded f. Their complexities depend on the complexity of f. We also note that the number of subintervals in θ_c (and $\breve{\theta}_c$) is proportional to the number of local maximas of f. Consequently, more computation is needed to determine all the θ^i's when f has several local maximas. In addition, if the θ^i's are independent of system variables, ϕ, then these θ^i's may be computed off-line. However, if they are dependent on ϕ, their on-line computation may prove to be too cumbersome.

In the case of $\theta \in \mathbb{R}^m$, the determination of the sets θ_c and $\breve{\theta}_c$ generally involves a search over $\theta \in \Theta$ such that

$$\theta_c = \left\{\theta^1 \mid \beta\frac{\partial f^T}{\partial\theta}\mid_{\theta_1}(\theta-\theta^1) \geq \beta(f-f^1), \forall\theta \in \Theta\right\}$$

Furthermore, given θ_c and $\breve{\theta}_c$, we need to define hyperplanes, $\omega^T\theta + c$ such that

$$F(\theta) = \beta(\omega^T\theta + c) \geq \beta(\hat{f} - f), \quad \theta \in \breve{\theta}_c$$

We believe that the solutions to θ_c, $\breve{\theta}_c$,ω and c are nontrivial due to the dimensionality of the problem and is currently a subject of on-going research.

9.4 Stable adaptive NP-systems

We now address the design of stable adaptive systems in the presence of nonlinear parametrization. We address the problem of adaptive control of systems of the form of (9.1) in Section 9.4.2 with the assumption that states are accessible, propose a controller that includes tuning functions derived from the solution of a min–max optimization problem, and show that it guarantees global boundedness. In Section 9.4.3, we address the problem when the states cannot be measured, and propose a globally stable adaptive observer. We begin our discussion by proposing a new error model for NLP-systems in Section 9.4.1.

9.4.1 A new error model for NLP-systems

All adaptive systems have, typically, two types of errors, a tracking error, e_c, that can be measured, and a parameter error,$\tilde{\theta}$, that is not accessible for measurement. The nature of the relationship between these two errors is dependent on the complexity of the underlying plant dynamics as well as on the parametrization. This relationship in turn affects the choice of the adaptive algorithms needed for parameter adjustment. It therefore is important to understand the error model that describes this relationship and develop the requisite adaptive algorithm. In this section we discuss a specific error model that seems to arise in the context of NLP-adaptive systems.

The error model for NLP-systems is of the form

$$\dot{e}_c = -e_c + \left[f - \hat{f} - a_0 \text{sat}\left(\frac{e_c}{\varepsilon}\right)\right] \tag{9.32}$$

$$\dot{\hat{\theta}} = e_\varepsilon \omega_0 \tag{9.33}$$

$$e_\varepsilon = e_c - \varepsilon \, \text{sat}\left(\frac{e_c}{\varepsilon}\right) \tag{9.34}$$

where $f = f(\phi(t), \theta)$ and $\hat{f} = f(\phi, \hat{\theta})$. In (9.32), $e_c(t) \in \mathbb{R}$ is the tracking error while $\tilde{\theta}(t) \in \mathbb{R}$ is the parameter error. $\phi(t) \in \mathbb{R}^m$ is a given measurable function, while $a_0(t)$ and $\omega_0(t)$ are tuning functions to be determined.

Lemma 4.1 If $\hat{\theta}(t) \in \Theta$ for all $t \geq t_0$, and a_0 and ω_0 are chosen as in (9.16) and (9.17) for concave–convex functions and (9.27) and (9.28) for nonconcave–convex functions, with $\beta = \text{sign}(e_c)$ in all cases, then the solutions of the error model in (9.32)–(9.34) are globally bounded, provided $\hat{\theta}(t) \in \Theta \; \forall t \geq t_0$.

Proof See Appendix.

The requirement that $\hat{\theta}(t) \in \Theta$ can be relaxed by making the following changes in the adaptive laws in equation (9.33).

$$\dot{\bar{\theta}} = e_\varepsilon \omega_0 - \gamma(\bar{\theta} - \hat{\theta}), \quad \gamma > 0$$

$$\hat{\theta} = \begin{cases} \theta & \bar{\theta} \in \Theta \\ \theta_{\max} & \bar{\theta} > \theta_{\max} \\ \theta_{\min} & \bar{\theta} < \theta_{\min} \end{cases} \tag{9.33'}$$

This results in the following corollary.

Corollary 4.1 If $\hat{\theta}(t_0) \in \Theta$ and $\hat{\theta}$ is adjusted as in (9.33′), where a_0 and ω_0 are chosen as in (9.16) and (9.17) for concave–convex functions and (9.27) and (9.28) for nonconcave–convex functions, with $\beta = \text{sign}(e_c)$ in all cases, then e_c and $\tilde{\theta}$ are always bounded.

Proof See appendix.

We note that unlike LP systems, the tracking error in (9.32) is a scalar. This raises the question as to how the error model can be applied for general adaptive system design where there is a vector of tracking errors. This is shown in the following section.

9.4.1.1 Extensions to the vector case

The following lemma gives conditions under which the stability properties of a vector of error equations can be reduced to those of a scalar equation.

Lemma 4.2 Let

$$\dot{E} = A_m E + bv \tag{9.35}$$

$$\dot{e}_c = -ke_c + v \tag{9.36}$$

If A_m is asymptotically stable, (A_m, b) controllable, and $s = -k$ is one of the eigenvalues of A_m, then $\exists h$ such that for $e_c = h^T E$:

(i) If e_c is bounded, then E is bounded; and
(ii) $\lim_{t\to\infty} e_c(t) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} E(t) = 0.$

Proof See appendix.

9.4.1.2 The scalar error model with linear parameters

Often, in NLP-systems, linear unknown parameters and multiplicative parameters are simultaneously present. This necessitates the need for the following extension of equation (9.32):

$$\dot{e}_c = -e_c + \sigma\Big[f - \hat{f} - \varphi^T\tilde{\alpha} - \tilde{\sigma}\hat{f}\Big] - u_a \tag{9.37}$$

where σ is a constant whose sign and an upper bound $|\sigma|_{\max}$are known. $\tilde{\sigma}(t) \in \mathbb{R}$, $\tilde{\theta} \in \mathbb{R}$ and $\tilde{\alpha}(t) \in \mathbb{R}^l$ are parameter errors, while $\varphi(t) \in \mathbb{R}^l$ is a measurable signal. In such a case, the following adaptive laws for adjusting $\tilde{\sigma}$, $\tilde{\theta}$ and $\tilde{\alpha}$ are needed, and the boundedness of the resulting errors is stated in Lemma 8:

$$\dot{\hat{\theta}} = \mathrm{sign}(\sigma)e_\varepsilon\omega_0 \tag{9.38}$$

$$\dot{\hat{\sigma}} = sign(\sigma)e_\varepsilon\hat{f} \tag{9.39}$$

$$\dot{\hat{\alpha}} = sign(\sigma)e_\varepsilon\varphi \tag{9.40}$$

$$u_a = a_0|\sigma|_{\max}\,\mathrm{sat}\left(\frac{e_c}{\varepsilon}\right) \tag{9.41}$$

Lemma 4.3 If $\hat{\theta}(t) \in \Theta$ for all $t \geq t_0$, and a_0 and ω_0 are chosen as in (9.16) and (9.17) for concave–convex functions and (9.27) and (9.28) for nonconcave–convex functions, using $\beta = \mathrm{sign}(\sigma e_c)$ in all cases, then the solutions of the error model in (9.37)–(9.41) are bounded.

Proof See appendix.

As in Corollary 4.1, the requirement that $\hat{\theta}(t) \in \Theta$ in Lemma 4.3 can be relaxed by making the following change in the adaptive law for $\hat{\theta}$ in equation (9.38).

$$\begin{aligned} \dot{\bar{\theta}} &= e_\varepsilon\omega_0 - \gamma(\bar{\theta} - \hat{\theta}), \quad \gamma > 0 \\ \hat{\theta} &= \begin{cases} \bar{\theta} & \bar{\theta} \in \Theta \\ \theta_{\max} & \bar{\theta} \geq \theta_{\max} \\ \theta_{\min} & \bar{\theta} \leq \theta_{\min} \end{cases} \end{aligned} \tag{9.38$'$}$$

The corresponding result is stated in the following corollary:

Corollary 4.2 If $\hat{\theta}(t_0) \in \Theta$, and a_0 and ω_0 are chosen as in (9.16) and (9.17) for concave–convex functions and (9.27) and (9.28) for nonconcave–convex functions, using $\beta = \mathrm{sign}(\sigma e_c)$ in all cases, then the solutions of the error model in (9.37), (9.38′), (9.39)–(9.41) are bounded.

Proof Proof omitted since it follows closely that of Lemma 4.3 and Corollary 4.1.

9.4.2 Adaptive controller

The dynamic system under consideration here is of the form of equation (9.1), where u is a scalar control input, $X_p \in \mathbb{R}^n$ is the plant state assumed accessible for measurement, $p, \theta_i \in \mathbb{R}$, and $\beta \in \mathbb{R}^l$ are unknown parameters. $\phi_i : \mathbb{R}^+ \rightarrow \mathbb{R}^m$, and $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^l$ are known and bounded functions of the state variable. A_p is nonlinear in p, and f_i is nonlinear in both ϕ_i and θ_i. Our goal is to find an input u such that the closed loop system has globally bounded solutions and so that X_p tracks as closely as possible the state X_m of a reference model specified in equation (9.2), where r is a bounded scalar input. We make the following assumptions regarding the plant and the model:

(A1) $X_p(t)$ is accessible for measurement.
(A2) $\theta_i \in \Theta_i$, where $\Theta_i \triangleq [\theta_{\min,i}, \theta_{\max,i}]$, and $\theta_{\min,i}$ and $\theta_{\max,i}$ are known; p is unknown and lies in a known interval $\mathcal{P} = [p_{\min}, p_{\max}] \subset \mathbb{R}$.
(A3) $\phi(t)$ and $\varphi(t)$ are known, bounded functions of the state variable X_p.
(A4) f is a known bounded function of its arguments.
(A5) All elements of $A(p)$ are known, continuous functions of the parameter p.
(A6) $b_m = b_p\alpha$ where α is an unknown scalar with a known sign and upper bound on its modulus, $|\alpha|_{\max}$.
(A7) $(A(p), b_p)$ is controllable for all values of $p \in \mathcal{P}$, with

$$A(p) + b_m g^T(p) = A_m$$

where g is a known function of p.
(A8) A_m is an asymptotically stable matrix in $\mathbb{R}^n$with

$$\det(sI - A_m) \triangleq R_m(s) = (s + k)R(s), \quad k > 0$$

Except for assumption (A1), the others are satisfied in most dynamic systems, and are made for the sake of analytical tractability. Assumption (A2) is needed due to the nonlinearity in the parametrization. Assumptions (A3)–(A5) are needed for analytical tractability. Assumptions (A6) and (A7) are matching conditions that need to be satisfied in LP-adaptive control as well. (A8) can be satisfied without loss of generality in the choice of the reference model, and is needed to obtain a scalar error model. Assumption (A1) is perhaps the most restrictive of all assumptions, and is made here to accomplish the first step in the design of stable adaptive NLP-systems.

Our objective is to construct the control input, u, so that the error, $E = X_p - X_m$, converges to zero asymptotically with the signals in the closed loop system remaining bounded. The structure of the dynamic system in equation (9.1) and assumptions (A6) and (A7) imply that when β, p, α, and θ_i are known

$$u = \alpha(g(p)^T X_p + r) - \sum_{i=1}^{m} f_i(\phi_i, \theta_i) - \varphi^T \beta \tag{9.42}$$

meets our objective since it leads to a closed loop system

$$\dot{X}_p = A_m X_p + b_m r$$

Our discussions in Section 9.2 indicate that an adaptive version of the controller in (9.42), with the actual parameters replaced by their estimates together with a gradient-rule for the adaptive law, will not suffice. We therefore propose the following adaptive controller:

$$u = \hat{\alpha}\Big(g(\hat{p})^T X_p + r\Big) - \sum_{i=1}^{m} f_i(\phi_i, \hat{\theta}_i) - \varphi^T \hat{\beta} + u_a(t) \tag{9.43}$$

$$e_\varepsilon = e_c - \varepsilon\,\mathrm{sat}\Big(\frac{e_c}{\varepsilon}\Big) \tag{9.44}$$

$$\dot{\hat{\beta}} = \mathrm{sign}(\alpha)\Gamma_\beta e_\varepsilon \varphi, \quad \Gamma_\beta > 0 \tag{9.45}$$

$$\dot{\hat{\alpha}} = -\mathrm{sign}(\alpha)\gamma_\alpha e_\varepsilon \hat{G}, \quad \gamma_\alpha > 0 \tag{9.46}$$

$$\dot{\bar{\theta}}_i = \mathrm{sign}(\alpha)\gamma_{\theta_i} e_\varepsilon \omega_i^* - \gamma_1(\bar{\theta}_i - \hat{\theta}_i), \quad \gamma_1, \gamma_{\theta_i} > 0$$

$$\hat{\theta}_i = \begin{cases} \bar{\theta}_i & \bar{\theta}_i \in \Theta_i \\ \theta_{\max,i} & \bar{\theta}_i > \theta_{\max,i} \\ \theta_{\min,i} & \bar{\theta}_i < \theta_{\min,i} \end{cases} \tag{9.47}$$

$$\dot{\bar{p}} = -\gamma_p e_\varepsilon \omega_{m+1}^* - \gamma_2(\bar{p} - \hat{p}) \qquad \gamma_2, \gamma_p > 0$$

$$\hat{p} = \begin{cases} \bar{p} & \bar{p} \in \mathcal{P} \\ p_{\max} & \bar{p} > p_{\max} \\ p_{\min} & \bar{p} < p_{\min} \end{cases} \tag{9.48}$$

$$G(x_p, p) = g(p)^T X_p + r, \hat{G} = g(\hat{p})^T X_p + r \tag{9.49}$$

$$u_a = -\mathrm{sign}(\alpha)\,\mathrm{sat}\Big(\frac{e_c}{\varepsilon}\Big)\sum_{i=1}^{m+1} a_i{}^* \tag{9.50}$$

where

$$a_i^* = \min_{\omega_i \in \mathbb{R}} \max_{\theta_i \in \Theta_i} \mathrm{sign}(\alpha e_\varepsilon)\Big[f_i - \hat{f}_i + \omega_i(\hat{\theta}_i - \theta_i)\Big], \quad i = 1, \ldots, m \tag{9.51}$$

$$a_{m+1}^* = |\alpha|_{\max} \min_{\omega_{m+1} \in \mathbb{R}} \max_{p \in \mathcal{P}} \mathrm{sign}(e_\varepsilon)\big[\hat{G} - G - \omega_{m+1}(\hat{p} - p)\big] \tag{9.52}$$

w_i are the corresponding w_i's that realize the min-max solutions in (9.51) and (9.52), and $|\alpha|_{\max}$ denotes the maximum modulus of α.The stability property of this adaptive system is given in Theorem 4.1 below.

Theorem 4.1 The closed loop adaptive system defined by the plant in (9.1), the reference model in (9.2) and (9.43)–(9.52) has globally bounded solutions if $\hat{p}(t_0) \in \mathcal{P}$ and $\hat{\theta}_i(t_0) \in \Theta_i \; \forall \; i$. In addition, $\lim_{t\to\infty} e_\varepsilon(t) = 0$.

Proof For the plant model in (9.1), the reference model in (9.2) and the control law in (9.43), we obtain the error differential equation

$$\dot{E} = A_m E + \frac{b_m}{\alpha}\left[\alpha(\hat{G} - G) + \sum_{i=1}^{m}\Big(f_i(\phi_i, \theta_i) - f_i(\phi_i, \hat{\theta}_i)\Big) - \varphi^T\tilde{\beta} + \tilde{\alpha}\hat{G} + u_a(t)\right] \tag{9.53}$$

Assumptions (A6)–(A8) and Lemma 4.2 imply that there exists a h such that $e_c = h^T E$ and

$$\dot{e_c} = -ke_c + \frac{1}{\alpha}\left[\alpha(\hat{G} - G) + \sum_{i=1}^{m}\Big(f_i(\phi_i, \theta_i) - f_i(\phi_i, \hat{\theta}_i)\Big) - \varphi^T\tilde{\beta} + \tilde{\alpha}\hat{G} + u_a(t)\right] \tag{9.54}$$

which is very similar to the error model in (9.37). Defining the tuning error, e_ε, as in (9.44), we obtain that the control law in (9.43), together with the adaptive laws in (9.45)–(9.52) lead to the following Lyapunov function:

$$V = \frac{1}{2}\left[e_\varepsilon^2 + \frac{1}{|\alpha|}\tilde{\beta}^T\Gamma_\beta^{-1}\tilde{\beta} + \frac{1}{|\alpha|}\gamma_\alpha^{-1}\tilde{\alpha}^2 + \gamma_p^{-1}\tilde{p}^2 + \frac{1}{|\alpha|}\sum_{i=1}^{m}\gamma_{\theta_i}^{-1}\tilde{\theta}_i^2 \right.$$
$$\left. +2\tilde{p}(\bar{p} - \hat{p}) + \frac{2}{|\alpha|}\sum_{i=1}^{m}\tilde{\theta}_i(\bar{\theta}_i - \hat{\theta}_i)\right] \tag{9.55}$$

This follows from Corollary 4.2 by showing that $\dot{V} \leq 0$. This leads to the global boundedness of e_ε, $\tilde{\beta}$, $\tilde{p}$, $\tilde{\alpha}$ and $\tilde{\theta}_i$ for $i = 1, \ldots, m$. Hence e_c is bounded and by Lemma 4.2, E is also bounded. As a result, ϕ_i, φ and the derivative $\dot{e}_c$ are bounded which, by Barbalat's lemma, implies that e_ε tends to zero.

Theorem 4.1 assures stable adaptive control of NLP-systems of the form in (9.3) with convergence of the errors to within a desired precision ε. The proof of boundedness follows using a key property of the proposed algorithm. This corresponds to that of the error model discussed in Section 9.4.1, which is given by Lemma 3.2. As mentioned earlier, Lemma 3.2 is trivially satisfied in adaptive control of LP-systems, where the inequality reduces to an equality for ω_0 determined with a gradient-rule and $a_0 = 0$. For NLP-systems, an inequality of the form of (9.18) needs to be satisfied. This in turn necessitates the reduction of the vector error differential equation in (9.53) to the scalar error differential equation in (9.54).

We note that the tuning functions a_i^* and ω_i^* in the adaptive controller have to be chosen differently depending on whether f is concave/convex or not, since they are dictated by the solutions to the min–max problems in (9.51) and (9.52). The concavity/convexity considerably simplifies the structure of these tuning functions and is given by Lemma 3.1. For general nonlinear parametrizations, the solutions depend on the concave cover, and can be determined using Lemma 4.3.

Extensions of the result presented here are possible to the case when only a scalar output y is possible, and the transfer function from u to y has relative degree one [10].

9.4.3 Adaptive observer

As mentioned earlier, the most restrictive assumption made to derive the stability result in Section 9.4.2 is (A1), where the states were assumed to be accessible. In order to relax assumption (A1), the structure of a suitable adaptive observer needs to be investigated. In this section, we provide an adaptive observer for estimating unknown parameters that occur nonlinearly when the states are not accessible.

The dynamic system under consideration is of the form

$$y_p = W(s,p)u, \quad W(s,p) = \frac{\sum_{i=1}^{n} b_i(p)s^{n-1}}{s^n + \sum_{i=1}^{n} a_i(p)s^{n-i}} \tag{9.56}$$

and the coefficients $a_i(p)$ and $b_i(p)$ are general nonlinear functions of an unknown scalar p. We assume that

(A2-1) p lies in a known interval $\mathcal{P} = [p_{\min}, p_{\max}]$.
(A2-2) The plant in (9.56) is stable for all $p \in \mathcal{P}$.
(A2-3) a_i and b_i are known continuous functions of p.

It is well known [1] that the output of the plant, y_p, in equation (9.56) satisfies a first order equation given by

$$\dot{y}_p = -\lambda y_p + f(p)^T \Omega, \quad \lambda > 0 \tag{9.57}$$

where

$$\dot{\omega}_1 = \Lambda\omega_1 + ku \tag{9.58}$$

$$\dot{\omega}_2 = \Lambda\omega_2 + ky_p \tag{9.59}$$

$$\Omega = \left[u, \omega_1^T, y_p, \omega_2^T\right]^T \tag{9.60}$$

$$f(p) = \left[c_0(p), c(p)^T, d_0, d(p)^T\right]^T \tag{9.61}$$

for some functions $c_0(\cdot)$, $c(\cdot)$, $d_0(\cdot)$,and $d(\cdot)$, which are linearly related to $b_i(\cdot)$ and $a_i(\cdot)$. Λ in (9.58) and (9.59) is an $(n-1)\times(n-1)$ asymptotically stable matrix and (Λ, k) is controllable.

Given the output description in equation (9.57), an obvious choice for an adaptive observer which will allow the on-line estimation of the nonlinear parameter p, and hence a_i's and b_i's in (9.56), is given by

$$\dot{\hat{y}}_p = -\lambda\hat{y}_p + \hat{f}^T\Omega - a_0\,\mathrm{sat}\left(\frac{e_1}{\varepsilon}\right) \tag{9.62}$$

where $\hat{f} = f(\hat{p})$ and e_1 is the output error $e_1 = \hat{y}_p - y_p$. It follows that the following error differential equation can be derived:

$$\dot{e}_1 = -\lambda e_1 + [\hat{f} - f]^T\Omega - a_0\,\mathrm{sat}\left(\frac{e_1}{\varepsilon}\right) \tag{9.63}$$

Equation (9.63) is of the form of the error model in (9.32) with $k_1 = \lambda$, $k_2 = 1$, $\varphi_\ell = \alpha_\ell = 0$, $m = 1$, $f_1 = -f^T\Omega$, $\phi_1 = \Omega$, and $\theta_1 = p$. Therefore, the algorithm

$$\begin{aligned}
e_\varepsilon &= e_1 - \varepsilon\,\mathrm{sat}\left(\frac{e_1}{\varepsilon}\right)\\
\dot{\bar{p}} &= -\gamma_p e_\varepsilon \omega_0 - \gamma(\bar{p} - \hat{p}), \quad \gamma_p > 0\\
\hat{p} &= \begin{cases} \bar{p} & \bar{p} \in \mathcal{P} \\ p_{\max} & \bar{p} > p_{\max} \\ p_{\min} & \bar{p} < p_{\min} \end{cases}
\end{aligned} \tag{9.64}$$

where a_0 and ω_0 are the solutions of

$$a_0 = \min_{\omega\in\mathbb{R}}\ \max_{p\in\mathcal{P}}\ \mathrm{sign}\,(e_\varepsilon)\left[\left(\hat{f} - f\right)^T\Omega - \omega(\hat{p} - p)\right]$$

$$\omega_0 = \arg\ \min_{\omega\in\mathbb{R}}\ \max_{p\in\mathcal{P}}\ \mathrm{sign}(e_\varepsilon)\left[\left(\hat{f} - f\right)^T\Omega - \omega(\hat{p} - p)\right]$$

The stability properties are summarized in Theorem 4.2 below:

Theorem 4.2 For the linear system with nonlinear parametrization given in (9.56), under the assumptions (A2-1)–(A2-3), together with the identification model in (9.62), the update law in (9.64) ensures that our parameter estimation problem has bounded errors in $\tilde{p}$ if $\hat{p}(t_0) \in \mathcal{P}$. In addition, $\lim_{t\to\infty} e_\varepsilon(t) = 0$.

Proof The proof is omitted since it follows along the same lines as that of Theorem 4.1.

We note that as in Section 9.4.2, the choices of a_0 and ω_0 are different depending on the nature of f. When f is concave or convex, these functions are simpler and easier to compute, and are given by Lemma 3.1. For general nonlinear parametrizations, these solutions depend on the concave cover and as can be seen from Lemma 3.4, are more complex to determine.

9.5 Applications

9.5.1 Application to a low-velocity friction model

Friction models have been the focus of a number of studies from the time of Leonardo Da Vinci. Several parametric models have been suggested in the literature to quantify the nonlinear relationship between the different types of frictional force and velocities. One such model, proposed in [13] is of the form

$$F = F_C \operatorname{sgn}(\dot{x}) + (F_S - F_C)\operatorname{sgn}(\dot{x}) e^{-\left(\frac{\dot{x}}{v_s}\right)^2} + F_v \dot{x} \tag{9.65}$$

where x is the angular position of the motor shaft, F is the frictional force, F_C represents the Coulomb friction, F_S stands for static friction, F_v is the viscous friction coefficient, and v_s is the Stribeck parameter. Another steady state friction model, proposed in [14] is of the form :

$$F = F_C \operatorname{sgn}(\dot{x}) + \frac{\operatorname{sgn}(\dot{x})(F_S - F_C)}{1 + (\dot{x}/v_s)^2} + F_v \dot{x} \tag{9.66}$$

Equations (9.65) and (9.66) show that while the parameters F_C, F_S and F_v appear linearly, v_s appears nonlinearly. As pointed out in [14], these parameters, including v_s, depend on a number of operating conditions such as lubricant viscosity, contact geometry, surface finish and material properties. Frictional loading, usage, and environmental conditions introduce uncertainties in these parameters, and as a result these parameters have to be estimated. This naturally motivates adaptive control in the presence of linear and nonlinear parametrization. The algorithm suggested in Section 9.4.2 in this chapter is therefore apt for the adaptive control of machines with such nonlinear friction dynamics.

In this section, we consider position control of a single mass system in the presence of frictional force F modelled as in equation (9.65). The underlying equations of motion can be written as

$$\ddot{x} = F + u \tag{9.67}$$

where u is the control torque to be determined. A similar procedure to what follows can be adopted for the model in equation (9.66) as well.

Denoting

$$\begin{aligned} &\varphi = [\operatorname{sgn}(\dot{x}), \dot{x}]^T, \quad f(\dot{x}, \theta) = \operatorname{sgn}(\dot{x}) e^{-\theta \dot{x}^2}, \quad \beta = [F_C, F_v]^T \\ &\theta = (1/v_s^2), \qquad\qquad \sigma = F_S - F_C \end{aligned} \tag{9.68}$$

it follows that the plant model is of the form

$$\ddot{x} = \sigma f(\dot{x}, \theta) + \varphi^T \beta + u \tag{9.69}$$

where $f(\dot{x}, \theta)$ is convex for all $\dot{x} > 0$ and concave for all $\dot{x} < 0$. We choose a reference model as

$$[s^2 + 2\zeta\omega_n s + \omega_n^2]x_m = \omega_n^2 r \tag{9.70}$$

where ζ and ω_n are positive values suitably chosen for the application problem. It therefore follows that a control input given by

$$u = -ke_c - D_1(s)[x] + \omega_n^2 r - \varphi^T \hat{\beta} - \hat{\sigma} f(\phi, \hat{\theta}) - a_0 \,\mathrm{sat}\left(\frac{e_c}{\varepsilon}\right) \tag{9.71}$$

where $D_1(s) = 2\zeta\omega_n s + \omega_n^2$, together with the adaptive laws

$$\dot{\hat{\beta}} = \Gamma_\beta e_\varepsilon \varphi, \qquad \Gamma_\beta > 0 \tag{9.72}$$

$$\dot{\hat{\theta}} = \gamma_\theta e_\varepsilon \omega_0\,, \qquad \gamma_\theta > 0 \tag{9.73}$$

$$\dot{\hat{\sigma}} = \gamma_\sigma e_\varepsilon \hat{f}, \qquad \gamma_\sigma > 0 \tag{9.74}$$

with a_0 and ω_0 corresponding to the min–max solutions when $\mathrm{sign}(e_\varepsilon)f$ is concave/convex, suffice to establish asymptotic tracking.

We now illustrate through numerical simulations the performance that can be obtained using such an adaptive controller. We also compare its performance with other linear adaptive and nonlinear fixed controllers. In all the simulations, the actual values of the parameters were chosen to be

$$F_C = 1\,\mathrm{N}, \quad F_S = 1.5\,\mathrm{N}, \quad F_v = 0.4\,\mathrm{Ns/m}, \quad v_s = 0.018\,\mathrm{m/s} \tag{9.75}$$

and the adaptive gains were set to

$$\Gamma_\alpha = \mathrm{diag}(1, 2), \quad \gamma_\sigma = 2, \quad \gamma_\theta = 10^8 \tag{9.76}$$

The reference model was chosen as in equation (9.70) with $\zeta = 0.707$, $\omega_n = 5\,\mathrm{rad/s}$, $r = \sin(0.2t)$.

Simulation 1 We first simulated the closed loop system with our proposed controller. That is, the control input was chosen as in equation (9.71) with $k = 1$, and adaptive laws as in equations (9.72)–(9.74) with $\varepsilon = 0.0001$. $\hat{\theta}(0)$ was set to 1370 corresponding to an initial estimate of $\hat{v}_s = 0.027$ m/s, which is 50% larger than the actual value. Figure 9.5 illustrates the tracking error, $e = x - x_m$,the control input, u, and the error in the frictional force, $e_F = F - \hat{F}$, where F is given by (9.65) and $\hat{F}$ is computed from (9.65) by replacing the true parameters with the estimated values. e, u and e_F are displayed both over [0, 6 min] and [214 min, 220 min] to illustrate the nature of the convergence. We note that the position error converges to about 5×10^{-5} rad, which is of the order of ε, and e_F to about 5×10^{-3} N. The discontinuity in u is due to the signum function in f in (9.68).

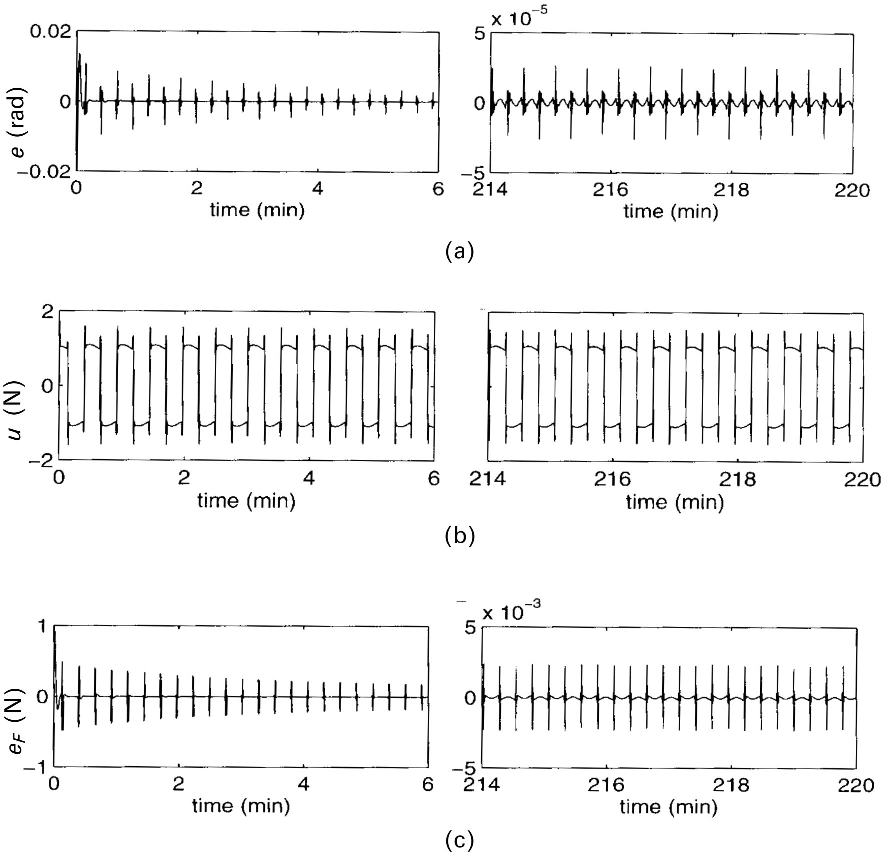

Figure 9.5 *Nonlinear adaptive control using the proposed controller. (a) e vs. time, (b) u vs. time, (c) e_F vs. time*

Simulation 2 To better evaluate our controller's performance, we simulated another adaptive controller where the Stribeck effect is entirely neglected in the friction compensation. That is

$$\hat{F} = \hat{F}_C \operatorname{sgn}(\dot{x}) + \hat{F}_v \dot{x} \tag{9.77}$$

so that the control input

$$u = -ke_c - D_1(s)[x] + \omega_n^2 r - \hat{F} \tag{9.78}$$

with estimates $\hat{F}_C$ and $\hat{F}_v$ obtained using the linear adaptive laws as in [1]. As before, the variables e, u and e_F are shown in Figure 9.6 for the first 6 minutes as well as for $T = [214\,\text{min}, 220\,\text{min}]$. As can be observed, the maximum position error does not decrease beyond 0.01 rad. It is worth noting that the control input in Figure 9.6 is similar to that in Figure 9.5 and of comparable

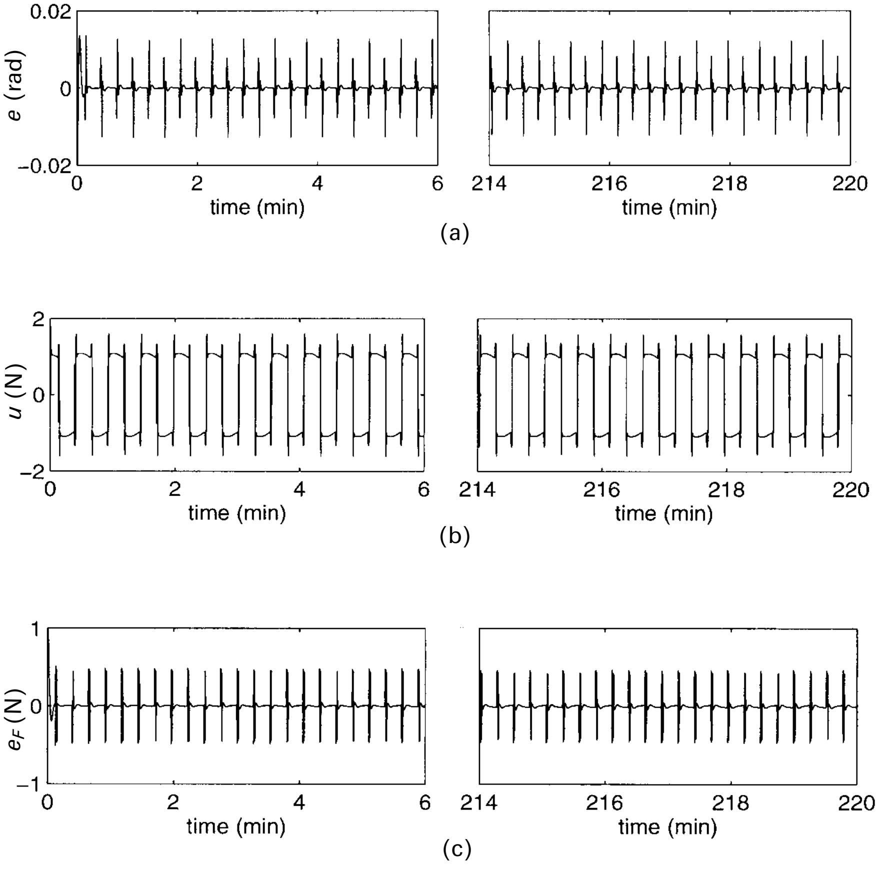

Figure 9.6 *Linear adaptive control with the Stribeck effect neglected. (a) e vs. time, (b) u vs. time, (c) e_F vs. time*

magnitude showing that our min–max algorithm does not have any discontinuities nor is it of a 'high gain' nature. Note also that the error, e_F does not decrease beyond 0.5 N.

<u>Simulation 3</u> In an attempt to avoid estimating the nonlinear parameters v_s, in [15], a friction model which is linear-in-the-parameters was proposed. The frictional force is estimated in [15] as

$$\hat{F} = \hat{F}_C \operatorname{sgn}(\dot{x}) + \hat{F}_S |\dot{x}|^{1/2} \operatorname{sgn}(\dot{x}) + \hat{F}_v \dot{x} \tag{9.79}$$

with the argument that the square-root-velocity term can be used to closely match the friction-velocity curves and linear adaptive estimation methods similar to Simulation 2 were used to derive closed loop control. The resulting

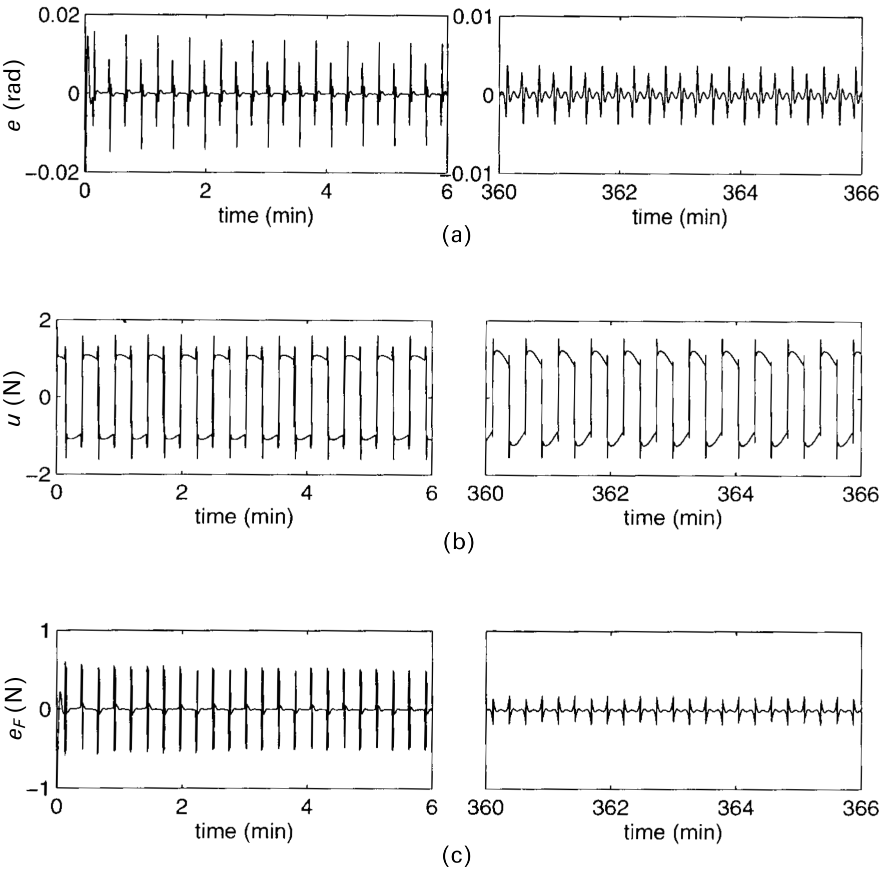

Figure 9.7 *Linear adaptive control with friction model as in (9.79). (a) e vs. time, (b) u vs. time, (c) e_F vs. time*

performance using such a friction estimate and the control input in equation (9.78) is shown in Figure 9.7 which illustrates the system variables e, u and e_F for $T = [0, 6\,\text{min}]$ and for $T = [360\,\text{min}, 366\,\text{min}]$. Though the tracking error remains bounded, its magnitude is much larger than those in Figure 9.5 obtained using our controller.

9.5.2 Stirred tank reactors (STRs)

Stirred tank reactors (STRs) are liquid medium chemical reactors of constant volume which are continuously stirred. Stirring drives the reactor medium to a uniform concentration of reactants, products and temperature. The stabilization of STRs to a fixed operating temperature proves to be difficult because a few physical parameters of the chemical reaction can dramatically alter the reaction dynamics. Defining X_1 and X_2 as the concentration of reactant and product in the inflow, respectively, r as the reaction rate, T as the temperature, h as the reaction heat released during an exothermic reaction, d as the volumetric flow into the tank, $T^* = T - T_{\text{amb}}$, it can be shown that three different energy exchanges affect the dynamics of X_1 [16]: (i) conductive heat loss with the environment at ambient temperature, T_{amb}, with a thermal heat transfer coefficient, e,(ii) temperature differences between the inflow and outflow which are T_{amb} and T respectively, (iii) a heat input, u, which acts as a control input and allows the addition of more heat into the system. This leads to a dynamic model

$$\begin{aligned}\dot{X}_1 &= -\nu_0 \exp\left(\frac{-\nu_1}{T^* + T_{\text{amb}}}\right)X_1 + d(X_{1in} - X_1)\\ \dot{X}_2 &= \nu_0 \exp\left(\frac{-\nu_1}{T^* + T_{\text{amb}}}\right)X_1 - dX_2 \\ \dot{T}^* &= -\frac{q}{\rho}T^* + \frac{1}{\rho}\left(h\nu_0 \exp\left(\frac{-\nu_1}{T^* + T_{\text{amb}}}\right)X_1 + u\right)\end{aligned} \tag{9.80}$$

The thermal input, u, is the only control input. (Volumetric feed rate, d, and inflow reactant concentration, X_{1in} are held constant.) To drive T^* from zero (T_{amb}) to the operating temperature, T^*_{oper}, we can state the problem as the tracking of the output T^*_m of a first order model, specified as

$$\dot{T}^*_m = -kT^*_m + kT^*_{\text{oper}} \tag{9.81}$$

where $k > 0$.

Driving and regulating an STR to an operating temperature is confounded by uncertainties in the reaction kinetics. Specifically, a poor knowledge of the constants, ν_0 and ν_1, in Arrhenius' law, the reaction heat, h, and thermal heat transfer coefficient, e, makes accurately predicting reaction rates nearly impossible. To overcome this problem, an adaptive controller where ν_0, ν_1, h and e are unknown may be necessary.

9.5.2.1 Adaptive control based on nonlinear parametrization

The applicability of the adaptive controller discussed in Section 9.4.2 becomes apparent with the following definitions

$$\theta_1 = \nu_0, \quad \theta_2 = \nu_1, \quad \theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \quad A_p = -q/\rho, \quad b_p = 1/\rho, \quad \beta = 0$$
$$f = \theta_1 h \exp\left(\frac{-\theta_2}{T^* + T_{\text{amb}}}\right)X_1, \quad T_e = T^* - T^*_m, \quad \phi = T^*$$

which indicates that equation (9.80) is of the form (9.1). Note that f is a convex function of θ. (It is linear in $\theta_1 = \nu_0$ and exponential in $\theta_2 = \nu_1$. If reaction heat, h, is unknown, it may be incorporated in θ_1, i.e. $\theta_1 = h\nu_0$.) Since ν_0, ν_1 and h are unknown constants within known bounds, assumption (A2) is satisfied. Assumption (A1) is satisfied since the temperatures are measurable. The system state, $T^* = \phi$, complies with (A3). Furthermore f is smooth and differentiable with known bounds and hence (A4) is satisfied. Finally, (A6)–(A8) are met due to the choice of the model as in (9.81) for $\alpha = k - \dfrac{q}{\rho}$. Since e is unknown, α is unknown and is therefore estimated. Since the plant is first order, the composite error e_c is given by $e_c = T_e$. Referring to the adaptive controller outlined in Section 9.4.2, the control input and adaptation laws are as follows:

$$u = -f(\phi, \hat{\theta}) - \hat{\alpha}T^* + \rho k T^*_{\text{oper}} - a^* \text{sat}\left(\frac{e_c}{\varepsilon}\right)$$

$$e_\varepsilon = e_c - \varepsilon sat\left(\frac{e_c}{\varepsilon}\right), \quad \varepsilon > 0$$

$$\dot{\hat{\alpha}} = -\Gamma_\alpha e_\varepsilon T^*, \quad \Gamma_\alpha > 0$$

$$\dot{\bar{\theta}} = \Gamma_\theta \left[e_\varepsilon \omega^* - (\bar{\theta} - \hat{\theta}) \right], \quad \Gamma_\theta > 0$$

$$\hat{\theta}_i = \begin{cases} \hat{\theta}_i & \text{if } \bar{\theta}_i \in \left[\theta_{i,\min}, \theta_{i,\max}\right] \\ \theta_{i,\min} & \text{if } \bar{\theta}_i \leq \theta_{i,\min} \\ \theta_{i,\max} & \text{if } \bar{\theta}_i \geq \theta_{i,\max} \end{cases}$$

Simulation results have shown that our proposed controller performs better than a linear adaptive system based on linear approximations of the nonlinear plant dynamics. Due to space constraints, however, the results are not shown in this chapter but the reader may refer to [11] for more details.

9.5.3 Magnetic bearing system

Magnetic bearings are currently used in various applications such as machine tool spindle, turbo machinery, robotic devices, and many other contact-free actuators. Such bearings have been observed to be considerably superior to mechanical bearings in many of these applications. The fact that the underlying electromagnetic fields are highly nonlinear with open loop unstable poses a challenging problem in dynamic modelling, analysis and control. As a result, controllers based on linearized dynamic models may not be suitable for applications where high rotational speed during the operation is desired. Yet another feature in magnetic bearings is the fact that the air gap, which is an underlying physical parameter, appears nonlinearly in the dynamic model. Due to thermal expansion effects, there are uncertainties associated with this parameter. The fact that dynamic models of magnetic bearings include nonlinear dynamics as well as nonlinear parametrizations suggests that an adaptive controller is needed which employs prior knowledge about these nonlinearities and uses an appropriate estimation scheme for the unknown nonlinear parameters.

To illustrate the presence of nonlinear parametrization, we focus on a specific system which employs magnetic bearings which is a magnetically levitated turbo pump [5]. The rotor is spun through an electric motor, and to actively position the rotor, a bias current i_0 is applied to both upper and lower magnets and an input u is to be determined by the control strategy. For a magnetic bearing system where rotor mass is M operating in a gravity field g, the rotor dynamics is represented by a second order differential equation of the form

$$M\ddot{z} - Mg = \frac{n^2\mu_0 A(i_0 + 0.5u)^2}{4(h_0 - z)^2} - \frac{n^2\mu_0 A(i_0 - 0.5u)^2}{4(h_0 + z)^2} \tag{9.82}$$

where n denotes the number of coils, μ_0 the air permeability, A the pole face area, i_0 the bias current (with $|u| < 2i_0$), h_0 the nominal air gap, and z the rotor position. One can rewrite equation (9.82) as

$$\ddot{z} - g = f_1(h_0, \alpha, z) + f_2(h_0, \alpha, z)u + f_3(h_0, \alpha, z)u^2, \quad |u| < 2i_0 \tag{9.83}$$

where

$$\alpha = \frac{n^2\mu_0 A}{4M}$$

$$f_1(h_0, \alpha, z) = \frac{4\alpha h_0 z i_0^2}{(h_0 - z)^2(h_0 + z)^2} = \alpha z i_0^2 \gamma_1(h_0, z)$$

$$f_2(h_0, \alpha, z) = \frac{2\alpha(h_0^2 + z^2)i_0}{(h_0 - z)^2(h_0 + z)^2} = \alpha i_0 \gamma_2(h_0, z)$$

$$f_3(h_0, \alpha, z) = \frac{\alpha h_0 z}{(h_0 - z)^2(h_0 + z)^2} = \alpha z \gamma_3(h_0, z)$$

The control objective is to track the rotor position with a stable second order model as represented by the following differential equation:

$$\ddot{z}_m + c_1\dot{z}_m + c_2 z_m = r \tag{9.84}$$

9.5.3.1 Adaptive control based on nonlinear parametrization

By examining equation (9.83), it is apparent that the parameter h_0 occurs nonlinearly while α occurs linearly. An examination of the functions f_1, f_2u, and f_3u^2 further reveals their concavity/convexity property and are

Table 9.1 *Properties of* f_1,f_2u *and* f_3u^2 *as a function of* h_i

Function	Concavity/convexity	Monotonic property	Prerequisite
$F_1 = f_1$	convex	decreasing	$0 < z < h_{\min}$
	concave	increasing	$-h_{\min} < z < 0$
$F_2 = f_2u$	convex	decreasing	$u > 0$
	concave	increasing	$u < 0$
$F_3 = f_3u^2$	convex	decreasing	$0 < z < h_{\min}$
	concave	increasing	$-h_{\min} < z < 0$

summarized in Table 9.1. Following the approach outlined in Section 9.4.2, we show that the following adaptive controller can be realized:

$$u = \frac{1}{f_2(\hat{h}_0, \hat{\alpha}, z)}\left\{-ke_\varepsilon - D_1(s) + r + u_a(t) + g - f_1(\hat{h}_0, \hat{\alpha}, z) - f_3(\hat{h}_0, \hat{\alpha}, z)u^2\right\} \tag{9.85}$$

where ε is the dead zone, c_1 and c_2 are positive constants and

$$e_\varepsilon = e_c - \varepsilon \operatorname{sat}\left(\frac{e_c}{\varepsilon}\right), \quad e_c = D(s)\left[\int (z - z_m)\, d\tau\right] \tag{9.86}$$

$$D(s) = s^2 + c_1 s + c_2, \quad D_1(s) = c_1(s) + c_2 \tag{9.87}$$

$$u_a(t) = -\operatorname{sat}\left(\frac{e_c}{\varepsilon}\right)\sum_{i=1}^{3} a_i(t) \tag{9.88}$$

The adaptive laws are determined, following the method outlined in Section 9.4.2 as the approach described in Section 9.4.2 allows us to establish adaptation laws for h_0 using h_i as follows:

$$\dot{\hat{h}}_1 = e_\varepsilon \Gamma_1 \omega_1, \quad \dot{\hat{h}}_2 = e_\varepsilon \Gamma_2 \omega_2, \quad \dot{\hat{h}}_3 = e_\varepsilon \Gamma_3 \omega_3, \quad \Gamma_i > 0 \tag{9.89}$$

and the linear parameters as

$$\dot{\hat{\alpha}}_1 = e_\varepsilon \Lambda_1 \hat{\gamma}_1 z i_0^2, \quad \dot{\hat{\alpha}}_2 = e_\varepsilon \Lambda_2 \hat{\gamma}_2 i_0 u, \quad \dot{\hat{\alpha}}_3 = e_\varepsilon \Lambda_3 \hat{\gamma}_3 z u^2, \quad \Lambda_i > 0 \tag{9.90}$$

where Λ_i are positive.

Since the functions F_i are either convex or concave, a_i and w_i are chosen as follows:

(a) F_i *is convex*

$$a_i = \begin{cases} \operatorname{sat}(e_c/\varepsilon)\alpha_{\max}\left[F_{i_{\max}} - \hat{F}_i - \dfrac{F_{i_{\max}} - F_{i_{\min}}}{h_{\max} - h_{\min}}(\hat{h} - h_{\min})\right] & \text{if } e_\varepsilon \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_i(t) = \begin{cases} -\text{sat}(e_c/\varepsilon)\dfrac{F_{i_{\max}} - F_{i_{\min}}}{h_{\max} - h_{\min}} & \text{if } e_\varepsilon \geq 0 \\ -\text{sat}(e_c/\varepsilon)\dfrac{\partial F_i}{\partial h}\Big|_{\hat{h}} & \text{otherwise} \end{cases}$$

(b) *F_i is concave*

$$a_i = \begin{cases} 0 & \text{if } e_\varepsilon \geq 0 \\ -\text{sat}(e_c/\varepsilon)\alpha_{\max}\left[\hat{F}_i - F_{i_{\min}} - \dfrac{F_{i_{\max}} - F_{i_{\min}}}{h_{\max} - h_{\min}}(\hat{h} - h_{\min})\right] & \text{otherwise} \end{cases}$$

$$\omega_i(t) = \begin{cases} \text{sat}(e_c/\varepsilon)\dfrac{\partial F_i}{\partial h}\Big|_{\hat{h}} & \text{if } e_\varepsilon \geq 0 \\ -\text{sat}(e_c/\varepsilon)\dfrac{F_{i_{\max}} - F_{i_{\min}}}{h_{\max} - h_{\min}} & \text{otherwise} \end{cases}$$

By examining equation (9.85), it is apparent that $\hat{f}_2$ cannot be arbitrarily small. This requirement is satisfied by disabling the adaptation when the magnitude of $\hat{f}_2$ reaches a certain threshold. The adaptive controller defined by equations (9.85)–(9.90) guarantees the stability of the magnetic bearing system as shown by the simulations results in [11].

9.6 Conclusions

In this chapter we have addressed the adaptive control problem when unknown parameters occur nonlinearly. We have shown that the traditional gradient algorithm fails to stabilize the system in such a case and that new solutions are needed. We present an adaptive controller that achieves global stabilization by incorporating two tuning functions which are selected by solving a min–max optimization scheme. It is shown that this can be accomplished on-line by providing closed-form solutions for the optimization problem. The forms of these tuning functions are simple when the underlying parametrization is concave or convex for all values of the unknown parameter and more complex when the parametrization is a general one.

The proposed approach is applicable to discrete-time systems as well. How stable adaptive estimation can be carried out for NLP-systems has been addressed in [12]. Unlike the manner in which the tuning functions are introduced in continuous-time systems considered in this chapter, in discrete-time systems, the tuning function a_0 is not included in the control input, but takes the form of a variable step-size ρ_t in the adaptive-law itself. It is once again shown that a min–max procedure can be used to determine ρ_t as well as the sensitivity function ω_t at each time instant. As in the solutions presented here, ω_t coincides with the gradient algorithm for half of the error space.

The class of NLP adaptive systems that was addressed in this chapter is of the form of (9.1). It can be seen that one of the striking features of this class is that it satisfies the matching conditions [17]. Our preliminary investigations [18] show that this can be relaxed as well by judicious choice of the composite scalar error in the system, thereby expanding the class of NLP adaptive systems that are globally stabilizable to include all systems that have a triangular structure [19].

Appendix Proof of lemmas

Proof of Lemma 3.1

We establish (9.14) and (9.15) by considering two cases: (a) βf is convex, and (b) βf is concave, separately.

(a) *βf is convex*. Since $\omega\beta\theta$ is linear in θ, in this case, the function $J(\omega,\theta)$ given by

$$J(\omega,\theta) = \beta\Big[f - \hat{f} + \omega(\hat{\theta} - \theta)\Big] \tag{A.1}$$

is convex in θ. Therefore, $J(\omega,\theta)$ attains its maximum at either $\theta_{\min}$ or $\theta_{\max}$ or both. The above optimization problem then becomes

$$\min_{\omega\in\mathbb{R}} \max\Big\{\beta\Big[f_{\min} - \hat{f} + \omega(\hat{\theta} - \theta_{\min})\Big],\ \beta\Big[f_{\max} - \hat{f} + \omega(\hat{\theta} - \theta_{\max})\Big]\Big\} \tag{A.2}$$

or equivalently it can be converted to a constrained linear programming problem as follows:

$$\min_{(\omega,z)\in\mathbb{R}^2} z$$

subject to

$$\begin{aligned} \beta\Big[f_{\min} - \hat{f} + \omega(\hat{\theta} - \theta_{\min})\Big] &\le z \\ \beta\Big[f_{\max} - \hat{f} + \omega(\hat{\theta} - \theta_{\max})\Big] &\le z \end{aligned} \tag{A.3}$$

By adding slack variables $\varepsilon_1 \ge 0$ and $\varepsilon_2 \ge 0$, the inequality constraints in (A.3) can be further converted into equality constraints

$$\beta\Big[f_{\min} - \hat{f} + \omega(\hat{\theta} - \theta_{\min})\Big] + \varepsilon_1 = z \tag{A.4}$$

$$\beta\Big[f_{\max} - \hat{f} + \omega(\hat{\theta} - \theta_{\max})\Big] + \varepsilon_2 = z \tag{A.5}$$

Solving for ω in equation (A.4) and substituting into equation (A.5), we have

$$z = \frac{\beta f_{\min}(\theta_{\max} - \hat{\theta})}{\theta_{\max} - \theta_{\min}} - \frac{\beta f_{\max}(\theta_{\min} - \hat{\theta})}{\theta_{\max} - \theta_{\min}} - \beta\hat{f} + \frac{\varepsilon_1(\theta_{\max} - \hat{\theta})}{\theta_{\max} - \theta_{\min}} + \frac{\varepsilon_2(\hat{\theta} - \theta_{\min})}{\theta_{\max} - \theta_{\min}} \tag{A.6}$$

The optimal solution can now be derived by considering three distinct cases: (i) $\theta_{\min} < \hat{\theta} < \theta_{\max}$, (ii) $\hat{\theta} = \theta_{\max}$ and (iii) $\hat{\theta} = \theta_{\min}$.

(i) $\theta_{\min} < \hat{\theta} < \theta_{\max}$: Since the last two terms in equation (A.6) are positive for $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$, minimum z is attained when $\varepsilon_1 = \varepsilon_2 = 0$ and is given by

$$z_{\text{opt}} = \beta\left[f_{\min} - \hat{f} + \frac{f_{\max} - f_{\min}}{\theta_{\max} - \theta_{\min}}(\hat{\theta} - \theta_{\min})\right] \tag{A.7}$$

The corresponding optimal ω can be determined by substituting equation (A.7) in equation (A.4) and is given by

$$\omega_{\text{opt}} = \frac{f_{\max} - f_{\min}}{\theta_{\max} - \theta_{\min}} \tag{A.8}$$

(ii) $\hat{\theta} = \theta_{\max}$: equation (A.6) can be simplified as

$$z = \varepsilon_2$$

and thus minimum z is obtained when $\varepsilon_2 = 0$ or $z_{\text{opt}} = 0$. The corresponding optimal ω using equation (A.4) is given by

$$\omega_{\text{opt}} = \frac{\beta(f_{\max} - f_{\min}) - \varepsilon_1}{\beta(\theta_{\max} - \theta_{\min})} \tag{A.9}$$

Thus, ω_{opt} is nonunique. However, for simplicity and continuity, if we choose $\varepsilon_1 = 0$, ω_{opt} is once again given by equation (A.8).

(iii) $\hat{\theta} = \theta_{\min}$: Here equation (A.6) reduces to

$$z = \varepsilon_1$$

Hence minimum z corresponds to $\varepsilon_1 = 0$ or $z_{\text{opt}} = 0$. Using equation (A.5), it follows that ω_{opt} is given by

$$\omega_{\text{opt}} = \frac{\beta(f_{\max} - f_{\min}) + \varepsilon_2}{\beta(\theta_{\max} - \theta_{\min})} \tag{A.10}$$

Again, ω_{opt} is nonunique but can be made equal to equation (A.8) by choosing $\varepsilon_2 = 0$.

(b) *βf is concave*. Here, we prove Lemma 3.1 by showing that (i) the absolute minimum value for $a_0 = 0$, and (ii) this value can be realized when $\omega = \omega_0$ as in (9.17).

(i) For any ω, we have that

$$\beta\Big[f-\hat{f}+\omega(\hat{\theta}-\theta)\Big]\geq 0,\quad \text{for some } \theta\in\Theta$$

since $\hat{\theta}\in\Theta$. Hence, for any ω

$$\max_{\theta\in\Theta} J(\omega,\theta)\geq 0$$

and as a result

$$\min_{\omega\in\mathbb{R}}\max_{\theta\in\Theta} J(\omega,\theta)\geq 0$$

Hence, a_0 attains the absolute minimum of zero for some θ and some ω.

(ii) We now show that when $\omega=\omega_0$, $a_0=0$, where ω_0 is given by (9.17). Since βf is concave, J is concave as well. Using the concavity property in (9.12), we have

$$\beta\Big[f-\hat{f}+\nabla f_{\hat{\theta}}(\hat{\theta}-\theta)\Big]\leq 0$$

That is,

$$\max_{\theta\in\Theta}\beta\Big[f-\hat{f}+\nabla f_{\hat{\theta}}(\hat{\theta}-\theta)\Big]=0 \tag{A.11}$$

Equation (A.11) implies that if we choose $\omega=\nabla f_{\hat{\theta}}$, then

$$\min_{\omega\in\mathbb{R}}\max_{\theta\in\Theta} J(\omega,\theta)=0$$

which proves Lemma 3.1.

Proof of Lemma 3.2

We note that

$$a_0\,\mathrm{sign}(x)\,\mathrm{sat}\Big(\frac{x}{\varepsilon}\Big)\;=\;a_0,\quad \forall|x|>\varepsilon$$

by definition of the sat$(\cdot)$ function. If a_0 and ω_0 are chosen as in (9.16) and (9.17) respectively, then

$$a_0=\max_{\theta\in\Theta}\beta\Big[f-\hat{f}+\omega_0(\hat{\theta}-\theta)\Big]$$

which implies that for any $\theta\in\Theta$,

$$\Big[\mathrm{sign}(x)\Big(f-\hat{f}-(\theta-\hat{\theta})\omega_0\Big)-a_0\,\mathrm{sign}(x)\,\mathrm{sat}\Big(\frac{x}{\varepsilon}\Big)\Big]\leq 0$$

since $\beta=\mathrm{sign}(x)$. Inequality (9.18) therefore follows.

Proof of Lemma 3.3

When βf is convex, $J(\omega,\theta)$is also convex and the min–max problem

$$\min_{\omega\in\mathbb{R}^m}\max_{\theta\in\Theta_S} J(\omega,\theta)=\min_{\omega\in\mathbb{R}^m}\max_{\theta\in\Theta_S}\beta\Big[f-\hat{f}-\omega^T(\hat{\theta}-\theta)\Big] \tag{A.12}$$

can be converted into a constrained LP problem involving the $(m+1)$ vertices of the simplex Θ_S. Defining $\max_{\theta\in\Theta_S} J(\omega,\theta) = z$, (A.12) can now be expressed as:

$$\begin{aligned} &\min_{(z,\omega)\in\mathbb{R}^{m+1}} \; z \\ &\text{subject to:} \quad g(\omega,\theta_{Si}) = J(\omega,\theta_{Si}) - z \le 0, \quad i = 1,\ldots,(m+1) \end{aligned} \tag{A.13}$$

We solve (A.13) by converting into an unconstrained problem as follows. Rewriting the constraints in matrix form as

$$H(x) = Gx - b \le 0$$

where $x = [z \quad \omega]^T$ and

$$G = \begin{bmatrix} -1 & \beta(\hat{\theta}-\theta_{S1})^T \\ -1 & \beta(\hat{\theta}-\theta_{S2})^T \\ \vdots & \vdots \\ -1 & \beta(\hat{\theta}-\theta_{Sm+1})^T \end{bmatrix}, \quad b = \begin{bmatrix} \beta(\hat{f}-f_{S1}) \\ \beta(\hat{f}-f_{S2}) \\ \vdots \\ \beta(\hat{f}-f_{Sm+1}) \end{bmatrix} \tag{A.14}$$

we have that $\nabla_x H(x) = G$ and G is full rank since θ_{Si} are distinct vertices of Θ_S and β is nonzero. Defining the Lagrangian function by

$$\phi(x,\lambda) = z + \lambda^T H(x) \tag{A.15}$$

where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{m+1}]^T$, and $\lambda_i, i = 1,\ldots,m+1$ are the Lagrange multipliers, the Kuhn Tucker theorem states that

$$\nabla_x\phi(x^*,\lambda) = \begin{bmatrix} 1 - \sum_{i=1}^{m+1}\lambda_i \\ \sum_{i=1}^{m+1}\lambda_i\beta(\hat{\theta}-\theta_{Si}) \end{bmatrix} = 0 \tag{A.16}$$

$$\lambda^T\nabla_\lambda\phi(x^*,\lambda) = \lambda^T(Gx^* - b) = 0 \tag{A.17}$$

where x^* is the optimal solution. From (A.16), we have

$$\sum_{i=1}^{m}\lambda_i(\theta_{Sm+1} - \theta_{Si}) = \theta_{Sm+1} - \hat{\theta} \tag{A.18}$$

Three cases of $\hat{\theta}$ will now be considered: (a) $\hat{\theta}$ is in the interior of Θ_S, (b) $\hat{\theta}$ is on the boundary of Θ_S, (c) $\hat{\theta}$ coincides with one of the vertices, $\theta_{Si}, i = 1,\ldots,m+1$.

Case (a) This implies that

$$\hat{\theta} = \sum_{i=1}^{m+1}\alpha_i\theta_{Si}, \;\text{ with } \sum_{i=1}^{m+1}\alpha_i = 1, \;\; 0 < \alpha_i < 1$$

Substituting into equation (A.18), we have

$$\sum_{i=1}^{m} (\lambda_i - \alpha_i)(\theta_{Sm+1} - \theta_{Si}) = 0$$

Since $(\theta_{Sm+1} - \theta_{Si}), i = 1, \ldots, m$ are m independent vectors, it follows that $\lambda_i - \alpha_i = 0$ or $\lambda_i = \alpha_i, \lambda_{m+1} = 1 - \sum_{i=1}^{m} \alpha_i$. Thus $\lambda_i > 0$ for all i. Therefore, from equation (A.17), we require $Gx^* - b = 0$ and thus the optimal solution is given by $x^* = G^{-1}b$ or $a_0 = (G^{-1}b)_{11}$where $(A)_{ij}$ refers to the ijth element of a matrix A.

Case (b) $\hat{\theta}$ is on the boundary of Θ_S. In this case, $\hat{\theta}$ is a linear combination of at most m vertices. Suppose

$$\hat{\theta} = \sum_{i=1}^{r} \alpha_i \theta_{Si}, \quad \sum_{i=1}^{r} \alpha_i = 1, \quad 0 < \alpha_i < 1, \quad 1 \leq r \leq m$$

where, for convenience, we have reordered the vertices such that $\hat{\theta}$ is a linear combination of the first r vertices. From (A.18), we have that

$$\sum_{i=1}^{r} (\lambda_i - \alpha_i)(\theta_{Sm+1} - \theta_{Si}) = 0$$

Thus, $\lambda_i = \alpha_i, i = 1, \ldots, r$ and $\lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_m = 0$. Equation (A.17) requires that

$$\sum_{i=1}^{r} \lambda_i(\beta f_{Si} - \beta\hat{f} + \beta(\hat{\theta} - \theta_{Si})^T \omega - z) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} \lambda_i \beta f_{Si} - \beta\hat{f} + \beta(\hat{\theta} - \sum_{i=1}^{r} \lambda_i \theta_{Si})^T \omega - z = 0$$

since $\sum_{i=1}^{r} \lambda_i = 1$. Since the term within the parentheses is zero, we have that

$$z = a_0 = \sum_{i=1}^{r} \lambda_i \beta f_{Si} - \beta\hat{f} \tag{A.19}$$

While equation (A.19) gives a nice closed form solution, this optimal solution is not so readily computable because we require the values of the Lagrange multipliers which in turn can only be computed by decomposing the estimate $\hat{\theta}$ in terms of the $m+1$ vertices. We avoid this by showing that the optimal solution in (A.19) coincides onces again with $(G^{-1}b)_{11}$.

Let G be partitioned as

$$G = \begin{bmatrix} -1 & \beta(\hat{\theta} - \theta_{S1})^{\mathrm{T}} \\ E_{m\times 1} & A_{m\times m} \end{bmatrix}, \quad b = \begin{bmatrix} \beta(\hat{f} - f_{S1}) \\ B_{m\times 1} \end{bmatrix}$$

Then

$$G^{-1} = \begin{bmatrix} -\lambda_1 & (-\lambda_2 \ -\lambda_3 \cdots - \lambda_r \ 0 \cdots 0) \\ ME & M \end{bmatrix}$$

where $M = \left[E\beta(\hat{\theta} - \theta_{S1})^T + A\right]^{-1}$ and hence $(G^{-1}b)_{11} = \sum_{i=1}^{r} \lambda_i \beta f_{Si} - \beta \hat{f} = a_0$.

Case (c) $\hat{\theta}$ coincides with one of the vertices, θ_{Sj}, for some $1 \le j \le m+1$: From (A.18), $\lambda_j = \alpha_j = 1, \lambda_i = \alpha_i = 0, \forall i \ne j, 1 \le i \le m+1$. In order to satisfy equation (A.17), we require that

$$\beta f_{Sj} - \beta\hat{f} + \beta(\hat{\theta} - \theta_{Sj})^T \omega - z = 0 \tag{A.20}$$

Since $\hat{\theta} = \theta_{Sj}$, (A.20) implies that the optimal solution is $z = a_0 = 0$. We can show that $a_0 = \left(G^{-1}b\right)_{11}$ in this case as well as in what follows.

Rewriting G such that the jth constraint corresponding to $\hat{\theta} = \theta_{Sj}$ is the first constraint, we have that

$$G = \begin{bmatrix} -1 & 0_{1\times m} \\ E_{m\times 1} & A_{m\times m} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ B_{m\times 1} \end{bmatrix}$$

and hence its inverse is given by

$$G^{-1} = \begin{bmatrix} -1 & 0_{1\times m} \\ A^{-1}E & A^{-1} \end{bmatrix}$$

Hence, $(G^{-1}b)_{11} = 0 = a_0$.

The proof for the case when βf is concave is the same as part (b) in the proof of Lemma 3.1.

Proof of Lemma 3.4

(1) For $\theta \in \theta_c$, $F(\theta) = \beta(f - \hat{f})$ and is concave since f is concave on θ_c. For $\theta \in \breve{\theta}_c$, $F(\theta)$ is linear in θ and thus is also concave. At θ^i, $i = 0, 1, \ldots, n+1$, $F(\theta^i) = \beta(f^i - \hat{f})$. Thus $F(\theta)$ is a continuous concave function on Θ. In addition, for $\theta \in \theta^{ij}$,

$$F(\theta) = \beta(\lambda f^j + (1-\lambda) f^i - \hat{f}), \quad \lambda = \frac{\theta - \theta^i}{\theta^j - \theta^i}$$

Since $0 \le \lambda \le 1$ and βf is not concave on each θ^{ij}, it follows that $\beta f(\theta) \le \lambda \beta f^j + (1-\lambda)\beta f^i$ for all θ in each θ^{ij} and hence

$$F(\theta) \ge \beta(f - \hat{f}), \ \forall \theta \in \breve{\theta}_c$$

(2) We first consider the expression

$$J(\omega, \theta) = \beta\left[f - \hat{f} + \omega(\hat{\theta} - \theta)\right]$$

For any β, $\omega \in \mathbb{R}$ and $\hat{\theta} \in \Theta$, there exists some $\theta \in \Theta$ such that $J \geq 0$, implying that

$$\min_{\omega \in \mathbb{R}} \max_{\theta \in \Theta} J(\omega, \theta) \geq 0. \tag{A.21}$$

(A.21) implies that the absolute minimum for the min–max problem is zero. Now, if

$$\beta(f - \hat{f}) \leq \beta(\omega\theta + c), \quad \text{for some } \omega \text{ and } c$$

then we have that

$$J(\omega, \theta) \leq \beta(\omega\hat{\theta} + c)$$

Hence, for some ω and c,

$$\max_{\theta \in \Theta} J(\omega, \theta) = \beta(\omega\hat{\theta} + c)$$

Therefore,

$$\min_{\omega \in \mathbb{R}} \max_{\theta \in \Theta} J(\omega, \theta) = \begin{array}{rc} & \min_{\omega, c \in \mathbb{R}} \ \beta(\omega\hat{\theta} + c) \\ \text{subject to} & \beta(f - \hat{f}) \leq \beta(\omega\theta + c) \end{array} \tag{A.22}$$

We have thus converted the min–max problem into a constrained linear-problem in (A.22). We can now establish (9.27) by considering the equivalent problem in (A.22) for two cases of θ: (a) $\hat{\theta} \in \theta_c$, and (b) $\hat{\theta} \in \breve{\theta}_c$.

(a) $\hat{\theta} \in \theta_c$: By the definition of θ_c, we have that

$$\beta \nabla f_{\hat{\theta}}(\theta - \hat{\theta}) \geq \beta(f - \hat{f}), \qquad \forall \theta \in \Theta \tag{A.23}$$

which satisfies the constraints in (A.22) if we choose $\omega = \nabla f_{\hat{\theta}}$ and $c = -\nabla f_{\hat{\theta}} \hat{\theta}$. For these choices of ω and c, it follows that

$$\min_{\omega, c \in \mathbb{R}} \beta(\omega\hat{\theta} + c) = 0$$

Since $\hat{\theta} \in \theta_c$, $F(\hat{\theta}) = 0$ and hence the min–max problems in (9.14) and (9.15) have solutions

$$a_0 = F(\hat{\theta}) = 0, \qquad \omega_0 = \left.\frac{\partial f}{\partial \theta}\right|_{\hat{\theta}}$$

(b) $\hat{\theta} \in \breve{\theta}_c$: Suppose

$$\hat{\theta} \in \theta^{ij} \text{ for some } i, j \tag{A.24}$$

From (A.22), we have

$$\min_{\omega \in \mathbb{R}} \max_{\theta \in \Theta} \beta\Big[f - \hat{f} + \omega(\hat{\theta} - \theta)\Big]$$

$$\equiv \begin{array}{rc} & \min_{\omega, c} \ \beta(\omega\hat{\theta} + c) \\ \text{s.t.} & \beta(f - \hat{f}) \leq \beta(\omega\theta + c), \quad \forall \theta \in \Theta \end{array} \tag{A.25}$$

$$\leq \begin{array}{rc} & \min_{\omega, c} \ \beta(\omega\hat{\theta} + c) \\ \text{s.t.} & F(\theta) \leq \beta(\omega\theta + c), \quad \forall \theta \in \Theta \end{array} \tag{A.26}$$

since $F(\theta) \geq \beta(f - \hat{f})$. However, since $F(\theta)$ is concave on Θ, we have

$$F(\theta) \leq \nabla F_{\theta^1}(\theta - \theta^1) + F(\theta^1), \quad \forall \theta, \theta^1 \in \Theta \tag{A.27}$$

Since from (9.25),

$$F(\theta) = \beta(\omega^{kl}\theta + c^{kl}), \quad \forall \theta \in \theta^{kl} \tag{A.28}$$

for any $\theta^1 \in \theta^{kl}$, it follows that $\nabla F_{\theta^1} = \beta\omega^{kl}$. Therefore, (A.27) can be rewritten as

$$F(\theta) \leq \beta(\omega^{kl}\theta + c^{kl}), \quad \forall \theta \in \Theta \tag{A.29}$$

(A.28) and (A.29) imply that

$$F(\theta) < \beta(\omega^{kl}\theta + c^{kl}), \quad \forall \theta \notin \theta^{kl} \tag{A.30}$$

since the intervals $\theta^{kl}\,\forall k, l$ are unique and equality only holds for the interval θ^{kl}. Therefore, the optimization problem in (A.26) can be further transformed as follows:

$$\begin{array}{c} \min_{\omega,c}\ \beta(\omega\hat{\theta} + c) \\ \text{s.t.} \quad F(\theta) \leq \beta(\omega\theta + c), \quad \forall \theta \in \Theta \end{array} \leq \begin{array}{c} \min_{\omega,c}\ \beta(\omega\hat{\theta} + c) \\ \text{s.t.} \quad F(\theta) = \beta(\omega^{kl}\theta + c^{kl}), \quad \theta \in \theta^{kl} \\ F(\theta) < \beta(\omega^{kl}\theta + c^{kl}), \quad \theta \notin \theta^{kl} \end{array} \tag{A.31}$$

Since the active constraints on the right-hand side of (A.31) occur only in the set θ^{kl}, the optimal solution of (A.31) is simply given by $\beta(\omega^{kl}\hat{\theta} + c^{kl})$. We note that the expansion of the constraint in (A.26) into constraints in (A.31) is not unique since the choice of k and l are arbitrary. Hence, the optimal solution of (A.26) has to be the minimum of all possible solutions of (A.26) derived from considering all the possible sets of constraints that can be derived from the single constraint in (A.26). In other words

$$\begin{array}{c} \min_{\omega,c}\ \beta(\omega\hat{\theta} + c) \\ \text{s.t.} \quad F(\theta) \leq \beta(\omega\theta + c), \quad \forall \theta \in \Theta \end{array} = \min\Big\{\beta(\omega^{kl}\hat{\theta} + c^{kl}), \quad k, l = 1, \ldots, n\Big\} \tag{A.32}$$

With i, j chosen as in (A.24), using equation (A.28), we obtain that

$$F(\hat{\theta}) = \beta(\omega^{ij}\hat{\theta} + c^{ij})$$

since $\hat{\theta} \in \theta^{ij}$.When $i \neq k, j \neq l$, since $\hat{\theta} \notin \theta^{kl}$, inequality (A.30) implies that

$$F(\hat{\theta}) < \beta(\omega^{kl}\hat{\theta} + c^{kl}), \quad \forall k \neq i,\ l \neq j.$$

That is, the minimum solution in (A.32) occurs when $k = i$, $l = j$ and therefore,

$$\begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & F(\theta) \le \beta(\omega\theta+c) \end{array} = F(\hat{\theta}) = \beta(\omega^{ij}\hat{\theta}+c^{ij}) \tag{A.33}$$

The corresponding optimal ω is given by $\omega_0 = \omega^{ij}$. We next show how the inequality in (A.26) is actually attained with equality by establishing that if the concave cover $F(\theta)$ was constructed such that it is strictly greater than $\beta(f-\hat{f})$, then the corresponding optimal solution will be larger. On the other hand, if $F(\theta)$ was constructed such that it is less than $\beta(f-\hat{f})$ for some θ, the corresponding optimal solution will be smaller. Hence if $F(\theta) \ge \beta(f-\hat{f})$, the two solutions will be equal. This is established formally below.

From (A.26) and (A.33), we have that

$$\begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & \beta(f-\hat{f}) \le \beta(\omega\theta+c) \end{array} \le \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & F(\theta) \le \beta(\omega\theta+c) \end{array}$$
$$= \beta(\omega^{ij}\hat{\theta}+c^{ij}).$$

Suppose we perturb $F(\theta)$ as $F'(\theta) = F(\theta)+\varepsilon \ge \beta(f-\hat{f})+\varepsilon,\ \varepsilon > 0$, then

$$\begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & \beta(f-\hat{f}) \le \beta(\omega\theta+c) \end{array} \le \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & F'(\theta) \le \beta(\omega\theta+c) \end{array} \tag{A.34}$$

$$= \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & F(\theta) \le \beta(\omega\theta+c)-\varepsilon \end{array} \tag{A.35}$$

Let $c' = c - \varepsilon/\beta$ in (A.35). Then the optimal solution of (A.35) is given by

$$\begin{array}{rl} \min_{\omega,c'} & \beta(\omega\hat{\theta}+c')+\varepsilon \\ \text{s.t.} & F(\theta) \le \beta(\omega\theta+c') \end{array} = \beta(\omega^{ij}\hat{\theta}+c^{ij})+\varepsilon \tag{A.36}$$

following the result in (A.33).

On the other hand, suppose that $F'(\theta) = F(\theta)-\varepsilon$. This implies that $F'(\theta) \le \beta(f-\hat{f})$ for some θ. Using the same arguments as in (A.34)–(A.36), we obtain that

$$\begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & \beta(f-\hat{f}) \le \beta(\omega\theta+c) \end{array} \ge \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & F'(\theta) \le \beta(\omega\theta+c) \end{array} = \beta(\omega^{ij}\hat{\theta}+c^{ij})-\varepsilon.$$

Hence, we have

$$\beta(\omega^{ij}\hat{\theta}+c^{ij})-\varepsilon \le \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & \beta(f-\hat{f}) \le \beta(\omega\theta+c) \end{array} \le \beta(\omega^{ij}\hat{\theta}+c^{ij})+\varepsilon \tag{A.37}$$

Since the concave cover, $F(\theta)$, was constructed as a tight bound over $\beta(f-\hat{f})$ i.e. $F(\theta) \geq \beta(f-\hat{f})$, we have that $\varepsilon = 0$ in (A.37), and hence

$$\begin{aligned} \begin{array}{rl} \min_{\omega,c} & \beta(\omega\hat{\theta}+c) \\ \text{s.t.} & \beta(f-\hat{f}) \leq \beta(\omega\theta+c) \end{array} &= \beta(\omega^{ij}\hat{\theta}+c^{ij}) \\ &= F(\hat{\theta}) \end{aligned} \tag{A.38}$$

for a corresponding optimal ω given by $\omega_0 = \omega^{ij}$. Therefore, statement (2) in Lemma 3.4 holds.

Proof of Lemma 4.1

Suppose we choose a Lyapunov candidate given by

$$V = \tfrac{1}{2}(e_\varepsilon^2 + \tilde{\theta}^2) \tag{A.39}$$

where $\tilde{\theta} = \hat{\theta} - \theta$. Then taking the derivative of V with respect to time yields

$$\dot{V} = e_\varepsilon \dot{e}_\varepsilon + \tilde{\theta}\dot{\tilde{\theta}} \tag{A.40}$$

Let $y = e_\varepsilon^2$. Since the discontinuity at $|e_c| = \varepsilon$ is of the first kind and since $e_\varepsilon = 0$ when $|e_c| \leq \varepsilon$, it follows that the derivative $\dot{V}$ exists for all e_c, and is given by

$$\dot{V} = 0 \quad \text{when } |e_c| \leq \varepsilon \tag{A.41}$$

When $|e_c| > \varepsilon$, substituting (9.32) and (9.33) into (A.40), we have

$$\dot{V} = -e_\varepsilon e_c + e_\varepsilon\left(f - \hat{f} + \tilde{\theta}\omega_0 - a_0 \,\text{sat}\left(\frac{e_c}{\varepsilon}\right)\right) \tag{A.42}$$

Equation (A.42) can be simplified, by the choice of e_ε as

$$\dot{V} \leq -e_\varepsilon^2 + e_\varepsilon\left[f - \hat{f} + \tilde{\theta}\omega_0 - a_0 \,\text{sat}\left(\frac{e_c}{\varepsilon}\right)\right] \tag{A.43}$$

If a_0 and ω_0 are chosen as in (9.16) and (9.17) for concave-convex functions and (9.27) and (9.28) for nonconcave–convex functions, from Lemmas 3.2 and 3.5, it follows that

$$e_c\left[f - \hat{f} + \tilde{\theta}\omega_0 - a_0\text{sat}\left(\frac{e_c}{\varepsilon}\right)\right] \leq 0$$

Since

$$\text{sign}(e_c) = \text{sign}(e_\varepsilon), \quad \forall\, |e_c| > \varepsilon$$

it follows that

$$e_\varepsilon\left[f - \hat{f} + \tilde{\theta}\omega_0 - a_0 \,\text{sat}\left(\frac{e_c}{\varepsilon}\right)\right] \leq 0$$

As a result, equation (A.43) reduces to

$$\dot{V} \leq -e_\varepsilon^2 \leq 0, \quad \text{for } |e_c| \geq \varepsilon \tag{A.44}$$

(A.41) and (A.44) imply that V is indeed a Lyapunov function which leads to global boundedness of e_ε and $\tilde{\theta}$. From the definition of e_ε, it follows that e_c is also bounded.

Proof of Corollary 4.1

To account for the projection terms in equation (9.3′), we modify our Lyapunov function candidate as

$$V = \tfrac{1}{2}\left(e_\varepsilon^2 + \tilde{\theta}^2 + 2\tilde{\theta}(\bar{\theta} - \hat{\theta})\right) \tag{A.45}$$

From the definition of $\hat{\theta}$ and $\bar{\theta}$, it follows that V is positive definite and radially unbounded with respect to e_ε, $\tilde{\theta}$ and $\bar{\theta}$ [20]. Equations (A.45) and (9.33′) yield a time derivative

$$\dot{V} = -e_\varepsilon^2 + e_\varepsilon\left[f - \hat{f} + \tilde{\theta}\omega^* - a_0 \,\mathrm{sat}\left(\frac{e_c}{\varepsilon}\right)\right] + m(\hat{\theta}, \bar{\theta})$$

where

$$m(\hat{\theta}, \bar{\theta}) = -\gamma(\hat{\theta} - \theta)(\bar{\theta} - \hat{\theta}) + \gamma_\theta^{-1}\dot{\hat{\theta}}(\bar{\theta} - \hat{\theta})$$

If $m \le 0$ for all $\hat{\theta}$ and $\bar{\theta}$, then we can obtain an expression for $\dot{V}$ similar to (A.43). We show this by considering the following three cases: (a) $\bar{\theta} \in \Theta$, (b) $\bar{\theta} > \theta_{\max}$, (c) $\bar{\theta} < \theta_{\min}$. In case (a), $\hat{\theta} = \bar{\theta}$ and hence $m = 0$. In both cases (b) and (c), $\dot{\hat{\theta}} = 0$, and by the choice of $\hat{\theta}$ and $\bar{\theta}$ and since $\theta \in \Theta$, it follows that $m \le 0$. As a result,

$$\dot{V} \le -e_\varepsilon^2 + e_\varepsilon\left[f - \hat{f} + \tilde{\theta}\omega_0 - a_0 \,\mathrm{sat}\left(\frac{e_c}{\varepsilon}\right)\right]$$

Also, the algorithm in equation (9.33′) ensures that $\hat{\theta}(t) \in \Theta$ for all $t \ge t_0$. Therefore, the same arguments following equation (A.43) in the proof of Lemma 4.1 can be used to conclude that

$$\dot{V} \le 0, \quad \forall e_c$$

Therefore, V is a Lyapunov function, which leads to the boundedness of $e_\varepsilon, \tilde{\theta}$ and $\bar{\theta}$, [20]. By the definition of e_ε, e_c is also bounded, thus proving Corollary 4.1.

Proof of Lemma 4.2

Let the characteristic polynomial of A_m be

$$a(s) = (s + k)R(s)$$

where $R(s)$ is a Hurwitz polynomial. By the choice of A_m, we have that

$$a(s)(sI - A_m)^{-1}b = P(s) \tag{A.46}$$

where $P(s)$ is an $n \times 1$ matrix whose elements are polynomials with degrees $\leq n-1$. Denoting $S = [1,\ s, \ldots, s^{n-1}]^T$, we have, from the controllability of (A_m, b) that

$$P(s) = MS \quad \text{where } M \text{ is nonsingular}$$

We choose h as

$$h = \left(M^T\right)^{-1} p \tag{A.47}$$

where p is an n-dimensional vector such that $p^T S = R(s)$. Therefore,

$$\begin{aligned} a(s)h^T(sI - A_m)^{-1}b &= h^T MS \\ &= R(s) \quad \text{using equation (A.47)} \end{aligned}$$

As a result, we have

$$h^T(sI - A_m)^{-1}b = \frac{1}{s+k} \tag{A.48}$$

and it follows by the choice of e_c that

$$\dot{e}_c = -ke_c + v$$

(i) Since $R(s)$ is a Hurwitz polynomial, we have that

$$\frac{s^i}{R(s)} e_c \in \mathcal{L}^\infty \quad \text{if } e_c \in \mathcal{L}^\infty$$

Since from (A.48)

$$\frac{1}{a(s)} v = \frac{1}{R(s)} e_c \tag{A.49}$$

we have that

$$\frac{s^i}{a(s)} v \in \mathcal{L}^\infty \quad \text{if } e_c \in \mathcal{L}^\infty$$

and therefore, since (A_m, b) is controllable, it follows that $E \in \mathcal{L}^\infty$. (ii) follows using the same arguments.

Proof of Lemma 4.3

Let the Lyapunov candidate be chosen as

$$V = \tfrac{1}{2}(e_\varepsilon^2 + |\sigma|\tilde{\theta}^2 + |\sigma|\tilde{\sigma}^2 + |\sigma|\tilde{\alpha}^2)$$

For $|e_c| \leq \varepsilon$, $\dot{V} = 0$, and for $|e_c| > \varepsilon$,

$$\dot{V} \leq e_\varepsilon|\sigma|\left[\text{sign}(\sigma)(f - \hat{f} + \tilde{\theta}\omega_0) - \frac{|\sigma|_{\max}}{|\sigma|} a_0 \,\text{sat}\left(\frac{e_c}{\varepsilon}\right)\right] \tag{A.50}$$

With a_0, ω_0 and β as in Lemma 4.3, it follows that

$$a_0 = \max_{\theta \in \Theta} \ \text{sign}(\sigma e_\varepsilon)\left[f - \hat{f} + \tilde{\theta}\omega_0\right]$$

In addition, for any $\hat{\theta} \in \Theta$, there exists some $\theta \in \Theta$ such that

$$\text{sign}(\sigma e_\varepsilon)\left[f - \hat{f} + \tilde{\theta}\omega_0\right] \geq 0$$

implying that $a_0 \geq 0$ and therefore

$$a_0 \frac{|\sigma_{\max}|}{|\sigma|} \geq \text{sign}(\sigma e_\varepsilon)\left[f - \hat{f} + \tilde{\theta}\omega_0\right]$$

Hence, from (A.50), we have that $\dot{V} \leq 0$ for all e_ε, thus proving the boundedness of e_c, $\tilde{\theta}$, $\tilde{\alpha}$ and $\tilde{\sigma}$.

References

[1] Narendra, K. S. and Annaswamy. A. M. (1989). *Stable Adaptive Systems*. Prentice-Hall, Inc., Englewood Cliffs, NJ.

[2] Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1994). 'Nonlinear Design of Adaptive Controllers for Linear Systems', *IEEE Transac. Autom. Contr.*, Vol. 39(4), 738–752.

[3] Morse, A. S. (1993). 'Supervisory Control of Families of Linear Set-point Controllers – Part I: Exact Matching', Technical Report 9301, Yale University, March.

[4] Armstrong-Hélouvry, B., Dupont, P. and Canudas de Wit, C. (1994). 'A Survey of Models, Analysis Tools and Compensation Methods for the Control of Machines with Friction', *Automatica*, Vol. 30(7), 1083–1138.

[5] Ting-Jen Yeh (1996). 'Modeling, Analysis and Control of Magnetically Levitated Rotating Machines', PhD thesis, MIT, Massachusetts, MA.

[6] Bošković, J. D. (1994). 'Stable Adaptive Control of a Class of Fed-batch Fermentation Processes', in *Proceedings of the Eighth Yale Workshop on Adaptive and Learning Systems*, 263–268, Center for Systems Science, Dept. of Electrical Engineering, Yale University, New Haven, CT 06520-1968, June.

[7] Fomin, V., Fradkov, A. and Yakubovich, V. (1981). *Adaptive Control of Dynamical Systems*. Eds. Nauka, Moscow.

[8] Ortega, R. (1996). 'Some Remarks on Adaptive Neuro-fuzzy Systems', *International Journal of Adaptive Control and Signal Processing*, Vol. 10, 79–83.

[9] Annaswamy, A. M., Loh, A. P. and Skantze, F. P. (1998). 'Adaptive Control of Continuous Time Systems with Convex/concave Parametrization', *Automatica*, Vol. 34, 33–49, January.

[10] Loh, A. P., Annaswamy, A. M. and Skantze, F. P. (1997). 'Adaptation in the Presence of a General Nonlinear Parametrization: An Error Model Approach, *IEEE Transac. Autom. Contr.* (submitted).

[11] Annaswamy, A. M., Mehta, N., Thanomsat, C. and Loh, A. P. (1997). 'Application of Adaptive Controllers Based on Nonlinear Parametrization to Chemical Reactors and Magnetic Bearings', in *Proceedings of the ASME Dynamic Systems and Control Division*, Dallas, TX, November .

[12] Skantze, F. P., Loh, A. P. and Annaswamy, A. M. (1997). 'Adaptive Estimation of Discrete-time Systems with Nonlinear Parametrization', *Automatica* (submitted).

[13] Armstrong-Hélouvry, B. (1991). *Control of Machines with Friction*, Kluwer Academic Publishers, Norwell, MA.

[14] Hess, D. P. and Soom, A. (1990). 'Friction at a Lubricated Line Contact Operating at Oscillating Sliding Velocities', *J. of Tribology*, Vol. 112(1), 147–152.

[15] Canudas de Wit, C., Noel, P., Aubin, A. and Brogliato, B. (1991). 'Adaptive Friction Compensation in Robot Manipulators: Low Velocities', *International Journal of Robotics Research*, Vol. 10(3), 189–199.

[16] Viel, F., Jadot, F. and Bastin, G. (1997). 'Robust Feedback Stabilization of Chemical Reactors', *IEEE Transac. Autom. Contr.*, Vol. 42, 473–481.

[17] Taylor, D. G., Kokotovic, P. V., Marino, R. and Kanellakopoulos, I. (1989). 'Adaptive Regulation of Nonlinear Systems with Unmodeled Dynamics', *IEEE Transac. Autom. Contr.*, Vol. 34, 405–412.

[18] Kojic, A., Annaswamy, A M., Loh, A.-P. and Lozano, R. (1998). 'Adaptive Control of a Class of Second Order Nonlinear Systems with Convex/concave Parameterization', in *Conference on Decision and Control*, Tampa, FL (to appear).

[19] Seto, D., Annaswamy, A. M. and Baillieul, J. (1994). 'Adaptive Control of Nonlinear Systems with a Triangular Structure', *IEEE Transac. Autom. Contr.*, Vol. 39(7), 1411–1428, July.

[20] Bakker, R. and Annaswamy, A. (1996). 'Stability and Robustness Properties of a Simple Adaptive Controller', *IEEE Transac. Autom. Contr.*, Vol. 41, 1352–1358.

10

Adaptive inverse for actuator compensation

G. Tao

Abstract

A general adaptive inverse approach is developed for control of plants with actuator imperfections caused by *nonsmooth* nonlinearities such as dead zone, backlash, hysteresis and other piecewise-linear characteristics. An adaptive inverse is employed for cancelling the effect of an actuator nonlinearity with *unknown* parameters, and a linear feedback control law is used for controlling the dynamics of a linear or *smooth* nonlinear part following the actuator nonlinearity. State feedback and output feedback control designs are presented which all lead to linearly parametrized error models suitable for developing adaptive laws to update the inverse parameters. This approach suggests that control systems with commonly used linear or nonlinear feedback controllers such as those with model reference, PID, pole placement or other dynamic compensation designs can be combined with an adaptive inverse for improving system tracking performance despite the presence of actuator imperfections.

10.1 Introduction

Adaptive control is a control methodology which provides adaptation mechanisms to adjust controllers for systems with parametric, structural and environmental uncertainties to achieve desired system performance. Payload variation or component ageing causes parametric uncertainties, component failure leads to structural uncertainties and external noises are typical environmental uncertainties. Such uncertainties often appear in control systems such as those in electrical, mechanical, chemical, aeronautical and biomedical engineering.

Adaptive control of linear systems has been extensively studied. Systematic design procedures have been developed for model reference adaptive control, pole placement control, self-tuning regulators and multivariable adaptive control. Robustness of adaptive control schemes with respect to modelling errors such as unmodelled dynamics, parameter variations and external disturbances has been a hot research topic. Recently adaptive controllers have been developed for nonlinear systems such as pure-feedback systems and feedback linearizable systems under the assumption that the nonlinearities are sufficiently *smooth*.

Nonsmooth nonlinear characteristics such as dead zones, backlash and hysteresis are common in actuators, such as mechanical connections, hydraulic servo-valves, piezoelectric translators and electric servomotors, and also appear in biomedical systems [3, 5, 7, 8, 10, 18]. They are usually poorly known and may vary with time. They often severely limit system performance, giving rise to undesirable inaccuracy or oscillations or even leading to instability. The development of adaptive control schemes for systems with actuator imperfections has been a task of major practical interest.

Recently, an adaptive inverse approach was proposed to deal with systems with nonsmooth actuator nonlinearities [10, 12]. The control scheme of [10] consists of an adaptive inverse for cancelling the effect of an unknown actuator nonlinearity such as a dead zone or a two-segment piecewise linearity and a linear state or output feedback design for a linear dynamics. In [12], the adaptive inverse approach is unified for adaptive output feedback control of linear systems with unknown actuator or/and sensor dead zone, backlash and hysteresis, based on a model reference control method for systems with stable zeros.

In this chapter, we describe how such an adaptive inverse approach can be combined with other control methods such as pole placement, PID, and how it may be applied to multivariable or nonlinear dynamics with actuator nonlinearities. In Section 10.2, we present a general parametrized actuator nonlinearity model illustrated by a dead-zone characteristic. In Section 10.3, we propose a parametrized inverse for cancelling the actuator nonlinearity, illustrated by a dead-zone inverse. In Section 10.4, we design state feedback adaptive inverse control schemes, while in Section 10.5, we develop a general output feedback adaptive inverse control scheme. In Section 10.6, we present three output feedback designs: model reference, pole placement, and PID (as illustrated by a backlash compensation example with simulation results), as examples of the general control scheme of Section 10.5. We also present feedback adaptive inverse control schemes in Section 10.7 for multivariable linear plants with actuator nonlinearities, using adaptive parameter update laws based on a coupled estimation error model or a Lyapunov design, and in Section 10.8 for smooth nonlinear dynamics with nonsmooth actuator nonlinearities, using an adaptive backstepping design.

10.2 Plants with actuator nonlinearities

Consider the plant with a nonlinearity $N(\cdot)$, with *unknown* parameters, at the input of a *known* linear part $G(D)$:

$$y(t) = G(D)[u](t), \quad u(t) = N(v(t)) \tag{10.1}$$

where $N(\cdot)$ represents an actuator uncertainty such as a dead zone, backlash, hysteresis or piecewise-linear characteristic, $v(t)$ is the applied control, $u(t)$ is not accessible for either control or measurement and $G(D)$ is a rational transfer function either in continuous time (when $D = s$ denotes either the Laplace transform variable or the time differentiation operator: $s[x](t) = \dot{x}(t)$) or in discrete time (when $D = z$ denotes either the z-transform variable or the time advance operator: $z[x](t) = x(t+1)$), for a unified presentation.

The control objective is to design an adaptive compensator to cancel the effect of the uncertain actuator nonlinearity $N(\cdot)$ so that a commonly-used control scheme for the linear part $G(D)$ can be used to ensure desired system performance. To achieve such an objective, there are two key tasks: one is the clarification of the class of actuator nonlinearities for which such compensators can be developed, and the other is the design of adaptive laws which can effectively update the compensator parameters. In this section, we fulfil the first task by presenting a parametrized nonlinearity model suitable for adaptive compensation schemes to be developed in the next sections.

Nonlinearity model. Dead-zone, backlash, hysteresis, and piecewise-linear characteristics are representatives of an actuator nonlinearity $N(\cdot)$. These nonlinear characteristics have break points so that they are nondifferentiable (nonsmooth) but they can be parametrized [10, 12, 15]. The parametrized models of these nonlinearities can be unified as

$$u(t) = N(v(t)) = N(\theta^*; v(t)) = -\theta^{*T}\omega^*(t) + a^*(t) \tag{10.2}$$

where $\theta^* \in R^{n_\theta}$ ($n_\theta \geq 1$) is an unknown parameter vector, and $\omega^*(t) \in R^{n_\theta}$ and $a^*(t) \in R$ whose components are determined by the signal motion in the nonlinear characteristic $N(\cdot)$ and therefore are also unknown.

A dead-zone example. To illustrate the nonlinearity model (10.2), let us consider a dead-zone characteristic $DZ(\cdot)$ with the input-output relationship:

$$u(t) = N(v(t)) = DZ(v(t)) = \begin{cases} m_r(v(t) - b_r) & \text{if } v(t) \geq b_r \\ 0 & \text{if } b_l < v(t) < b_r \\ m_l(v(t) - b_l) & \text{if } v(t) \leq b_l \end{cases} \tag{10.3}$$

where $m_r > 0$, $m_l > 0$, $b_r > 0$, and $b_l < 0$ are dead zone parameters.

Introducing the indicator function $\chi[X]$ of the event X:

$$\chi[X] = \begin{cases} 1 & \text{if } X \text{ is true} \\ 0 & \text{otherwise} \end{cases} \tag{10.4}$$

we define the dead-zone indicator functions

$$\chi_r(t) = \chi[u(t) > 0] \tag{10.5}$$

$$\chi_l(t) = \chi[u(t) < 0] \tag{10.6}$$

Then, introducing the dead-zone parameter vector and its regressor

$$\theta^* = (m_r, m_r b_r, m_l, m_l b_l)^T \tag{10.7}$$

$$\omega^*(t) = (-\chi_r(t)v(t), \chi_r(t), -\chi_l(t)v(t), \chi_l(t))^T \tag{10.8}$$

we obtain (10.2) with $a^*(t) = 0$ for the dead-zone characteristic (10.3).

For a parametrized nonlinearity $N(\cdot)$ in (10.2), we will develop an adaptive inverse as a compensator for cancelling $N(\cdot)$ with unknown parameters.

10.3 Parametrized inverses

The essence of the adaptive inverse approach is to employ an inverse

$$v(t) = \widehat{NI}(u_d(t)) \tag{10.9}$$

to cancel the effect of the unknown nonlinearity $N(\cdot)$, where the inverse characteristic $\widehat{NI}(\cdot)$ is parametrized by an estimate θ of θ^*, and $u_d(t)$ is a desired control signal from a feedback law. The key requirement for such an inverse is that its parameters can be updated from an adaptive law and should stay in a prespecified region needed for implementation of an inverse. In our designs, such an adaptive law is developed based on a linearly parametrized error model and the parameter boundaries are ensured by parameter projection.

Inverse model. A desirable inverse (10.9) should be parametrizable as

$$u_d(t) = -\theta^T(t)\omega(t) + a(t) \tag{10.10}$$

for some known signals $\omega(t) \in R^{n_\theta}$ and $a(t) \in R$ whose components are determined by the signal motion in the nonlinearity inverse $\widehat{NI}(\cdot)$ such that $v(t)$, $\omega(t)$ and $a(t)$ are bounded if $u_d(t)$ is. The error signal (due to an uncertain $N(\cdot)$)

$$d_n(t) = \theta^{*T}(\omega(t) - \omega^*(t)) + a^*(t) - a(t) \tag{10.11}$$

should satisfy the conditions that $d_n(t)$ is bounded, $t \geq 0$, and that $d_n(t) = 0$, $t \geq t_0$, if $\theta(t) = \theta^*$, $t \geq t_0$, and $\widehat{NI}(\cdot)$ is correctly initialized: $d_n(t_0) = 0$.

Inverse examples. As shown in [10, 12, 15], the inverses for a dead zone, backlash, hysteresis and piecewise linearity have such desired properties. Here we use the dead-zone inverse as an illustrative example. Let the estimates of $m_r b_r$, m_r, $m_l b_l$, m_l be $\widehat{m_r b_r}$, $\widehat{m_r}$, $\widehat{m_l b_l}, \widehat{m_l}$, respectively. Then the inverse for the dead-zone characteristic (10.3) is described by

$$v(t) = \widehat{NI}(u_d(t)) = \widehat{DI}(u_d(t)) = \begin{cases} \dfrac{u_d(t) + \widehat{m_r b_r}}{\widehat{m_r}} & \text{if } u_d(t) > 0 \\ 0 & \text{if } u_d(t) = 0 \\ \dfrac{u_d(t) + \widehat{m_l b_l}}{\widehat{m_l}} & \text{if } u_d(t) < 0 \end{cases} \tag{10.12}$$

For the dead zone inverse (10.12), to arrive at the desired form (10.10), we introduce the inverse indicator functions

$$\widehat{\chi_r}(t) = \chi[v(t) > 0] \tag{10.13}$$

$$\widehat{\chi_l}(t) = \chi[v(t) < 0] \tag{10.14}$$

and the inverse parameter vector and regressor

$$\theta = (\widehat{m_r}, \widehat{m_r b_r}, \widehat{m_l}, \widehat{m_l b_l})^T \tag{10.15}$$

$$\omega(t) = (-\widehat{\chi_r}(t)v(t), \widehat{\chi_r}(t), -\widehat{\chi_l}(t)v(t), \widehat{\chi_l}(t)^T \tag{10.16}$$

Then, the dead-zone inverse (12) is

$$\begin{aligned} u_d(t) &= \widehat{m_r}\widehat{\chi_r}(t)v(t) - \widehat{m_r b_r}\widehat{\chi_r}(t) + \widehat{m_l}\widehat{\chi_l}(t)v(t) - \widehat{m_l b_l}\widehat{\chi_l}(t) \\ &= -\theta^T \omega(t) \end{aligned} \tag{10.17}$$

that is, $a(t) = 0$ in (10.10). It follows from (10.7), (10.8), (10.11) and (10.16) that

$$\begin{aligned} d_n(t) &= \theta^{*T}\omega(t)\chi[u(t) = 0] \\ &= -m_r\chi[0 < v(t) < b_r](v(t) - b_r) - m_l\chi[b_l < v(t) < 0](v(t) - b_l) \end{aligned} \tag{10.18}$$

which has the desired properties that $d_n(t)$ is bounded for all $t \geq 0$ and that $d_n(t) = 0$ whenever $\theta = \theta^*$. Furthermore, $d_n(t) = 0$ whenever $v(t) \geq b_r$ or $v(t) \leq b_l$, that is, when $u(t)$ and $v(t)$ are outside the dead zone, which is the case when $\widehat{b_r} \triangleq \dfrac{\widehat{m_r b_r}}{\widehat{m_r}} \geq b_r$ and $\widehat{b_l} \triangleq \dfrac{\widehat{m_l b_l}}{\widehat{m_l}} \leq b_l$, see [12].

Control error. It is important to see that the inverse (10.10), when applied to the nonlinearity (10.2), results in the control error

$$u(t) - u_d(t) = (\theta - \theta^*)^T \omega(t) + d_n(t) \tag{10.19}$$

which is suitable for developing an adaptive inverse compensator. For the adaptive designs to be presented in the next sections, we assume that the inverse block (10.9) has the form (10.10) and $d_n(t)$ in (11) has the stated properties. We should note that the signals $a^*(t)$ in (10.2) and $a(t)$ in (10.10) are non zero if the nonlinearity $N(\cdot)$ and its inverse $NI(\cdot)$ have inner loops as in the hysteresis case [12].

10.4 State feedback designs

Consider the plant (10.1), where $G(D)$ has a *controllable* state variable realization

$$D[x](t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{10.20}$$

for some *known* constant matrices $A \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$, $n > 0$, that is, $G(D) = C(DI - A)^{-1}B$, and $r(t)$ is a bounded reference input signal.

In the presence of an actuator nonlinearity $u(t) = N(v(t))$ parametrizable as in (10.2), we propose the control scheme shown in Figure 10.1, which consists of the inverse $v(t) = \widehat{NI}(u_d(t))$ parametrizable as in (10.10) and the state feedback law

$$u_d(t) = Kx(t) + r(t) \tag{10.21}$$

where $K \in R^{1\times n}$ is a constant gain vector such that the eigenvalues of $A + BK$ are equal to some desired closed loop system poles. The choice of such a K can be made from a pole placement or linear quadratic optimal control design. A pole placement design can be used to match the closed loop transfer function $C(DI - A - BK)^{-1}B$ to a reference model if the zeros of $G(D)$ have good stability.

Error model. With the controller (10.21), the closed loop system is

$$y(t) = W_m(D)[r](t) + W_m(D)[(\theta - \theta^*)^T\omega + d_n](t) + \delta(t) \tag{10.22}$$

where $\delta(t)$ is an exponentially decaying term due to initial conditions, and

$$W_m(D) = C(DI - A - BK)^{-1}B \tag{10.23}$$

The ideal system output is that of the closed loop system *without* the actuator nonlinearity $N(\cdot)$ and its inverse $\widehat{NI}(\cdot)$, that is, with $u(t) = u_d(t)$. In this case

$$y(t) = W_m(D)[r](t) + \delta(t). \tag{10.24}$$

In view of this output, we introduce the reference output

$$y_m(t) = W_m(D)[r](t). \tag{10.25}$$

Ignoring the effect of $\delta(t)$, introducing $d(t) = W_m(D)[d_n](t)$ which is bounded because $d_n(t)$ in (10.11) is, and using (10.22) and (10.25), we have

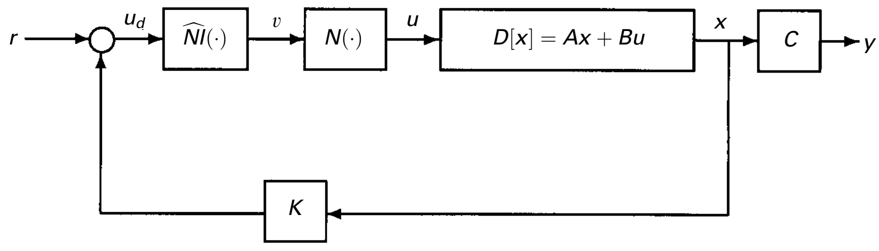

Figure 10.1 *State feedback inverse control*

$$y(t) - y_m(t) = W_m(D)[(\theta - \theta^*)^T\omega](t) + d(t)$$
$$= W_m(D)[\theta^T\omega](t) - \theta^{*T}W_m(D)[\omega](t) + d(t) \tag{10.26}$$

In view of (10.26), we define the estimation error

$$\varepsilon(t) = y(t) - y_m(t) + \xi(t) \tag{10.27}$$

where

$$\xi(t) = \theta^T(t)\zeta(t) - W_m(D)[\theta^T\omega](t) \tag{10.28}$$

$$\zeta(t) = W_m(D)[\omega](t) \tag{10.29}$$

Using (10.26)–(10.29), we arrive at the error equation

$$\varepsilon(t) = (\theta(t) - \theta^*)^T\zeta(t) + d(t) \tag{10.30}$$

which is linear in the parameter error $\theta(t) - \theta^*$, with an additional bounded error $d(t)$ due to the uncertainty of the nonlinearity $N(\cdot)$ in the plant (10.1).

A gradient projection adaptive law. The adaptive update law for $\theta(t) = (\theta_1(t), \ldots, \theta_{n_\theta}(t))^T$ can be developed using a gradient projection algorithm to minimize a cost function $J(\theta)$. The related optimization problem is to find an iterative algorithm to generate a sequence of θ which will asymptotically

$$\text{minimize } J(\theta), \text{s.t. } \theta_i \in [\theta_i^a, \theta_i^b], \quad i = 1, 2, \ldots, n_\theta \tag{10.31}$$

for some known parameter bounds $\theta_i^a < \theta_i^b$ such that $\theta_i^* \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, for $\theta^* = (\theta_i^*, \ldots, \theta_{n_\theta}^*)^T$, and that for any set of $\theta_i \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, p$, the characteristic $N(\theta; \cdot)$ renders an actuator uncertainty model in the same class to which the true model $N(\theta^*; \cdot)$ belongs, for example, for $N(\theta^*; \cdot)$ being a dead zone of the form (10.3), $N(\theta; \cdot)$ should also be a dead-zone of the form (10.3) with possibly different parameters. The constraint that $\theta_i \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, is to ensure that the nonlinearity inverse $\widehat{NI}(\cdot)$ is implementable as an inverse for the type of nonlinearity $N(\cdot)$ under consideration.

In the dead-zone case, the parameter boundaries are $m_{r1} \le m_r \le m_{r2}$, $m_{l1} \le m_l \le m_{l2}$, $0 \le b_r \le b_{r0}$, $-b_{l0} \le b_l \le 0$, for some known positive constants m_{r1}, m_{l1}, m_{r2}, m_{l2}, b_{r0}, b_{l0}. The above constraint condition, in terms of $\theta^* = (m_r, m_r b_r, m_l, m_l b_l)^T$ as in (10.7), is $\theta_i^* \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, 3, 4$, for $\theta_1^a = m_{r1}$, $\theta_2^a = 0$, $\theta_3^a = m_{l1}$, $\theta_4^a = -m_{l2} b_{l0}$, $\theta_1^b = m_{r2}$, $\theta_2^b = m_{r2} b_{r0}$, $\theta_3^b = m_{l2}$, $\theta_4^b = 0$. Then for any $\theta_i \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, 3, 4$, $DZ(\theta; \cdot)$ is a dead-zone characteristic.

For the formulated optimization problem, the gradient projection method [4, 9, 12] suggests the adaptive update law for $\theta(t)$:

$$\left.\begin{array}{l}\dot{\theta}(t)\\ \theta(t+1) - \theta(t)\end{array}\right\} = -\Gamma \frac{\partial J}{\partial \theta}(t) + f(t) \tag{10.32}$$

where $f(t)$ is a parameter projection term, and

$$\Gamma = \text{diag}\,\{\gamma_1, , \gamma_{n_\theta}\} \tag{10.33}$$

with $\gamma_i > 0$ for a continuous-time design and $0 < \gamma_i < 2$for a discrete-time design. To ensure $\theta_i \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, we introduce

$$g(t) = (g_1(t), \ldots, g_{n_\theta}(t))^T = -\Gamma \frac{\partial J}{\partial \theta}(t) \tag{10.34}$$

initialize the adaptive law (10.35) with

$$\theta_i(0) \in [\theta_i^a, \theta_i^b],\ i = 1, 2, \ldots, n_\theta \tag{10.35}$$

and set the ith component of the modification term $f(t)$ as

$$f_i(t) = \begin{cases} 0 & \text{if } \theta_i(t) \in (\theta_i^a, \theta_i^b) \\ & \text{if } \theta_i(t) = \theta_i^a, g_i(t) \ge 0, \text{or} \\ & \text{if } \theta_i(t) = \theta_i^b, g_i(t) \le 0 \\ -g_i(t) & \text{otherwise} \end{cases} \tag{10.36}$$

in a continuous-time design, or

$$f_i(t) = \begin{cases} 0 & \text{if } \theta_i(t) + g_i(t) \in [\theta_i^a, \theta_i^b] \\ \theta_i^b - \theta_i(t) - g_i(t) & \text{if } \theta_i(t) + g_i(t) > \theta_i^b \\ \theta_i^a - \theta_i(t) - g_i(t) & \text{if } \theta_i(t) + g_i(t) < \theta_i^a \end{cases} \tag{10.37}$$

in a discrete-time design, for $i = 1, 2, \ldots, n_\theta$.

For the commonly used cost function

$$J(\theta) = \frac{\varepsilon^2}{2m^2} \tag{10.38}$$

where $\varepsilon = (\theta - \theta^*)^T \zeta + d$ (see (10.30)), and

$$m(t) = \sqrt{1 + \zeta^T(t)\zeta(t)} \tag{10.39}$$

such that $\frac{\varepsilon^2}{2m^2}$ is bounded if $\theta(t)$ is bounded, it follows that

$$\frac{\partial J}{\partial \theta}(t) = \frac{\varepsilon(t)\zeta(t)}{m^2(t)} \tag{10.40}$$

Therefore, a desired adaptive update law for $\theta(t)$ is

$$\left.\begin{array}{l}\dot{\theta}(t)\\ \theta(t+1)-\theta(t)\end{array}\right\} = -\frac{\Gamma\varepsilon(t)\zeta(t)}{m^2(t)} + f(t) \tag{10.41}$$

initialized by (10.35) and projected with $f(t)$ in (10.36) or (10.37) for $g(t) = -\frac{\Gamma\varepsilon(t)\zeta(t)}{m^2(t)}$.

This adaptive law ensures that $\theta_i(t) \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, and that $\frac{\varepsilon(t)}{m(t)}$ and $\dot{\theta}(t)$ or $\theta(t+1) - \theta(t)$ are bounded by $\frac{d(t)}{m(t)}$ in a mean square sense.

A continuous-time Lyapunov design. In continuous time, with $\tilde{\theta}(t) = \theta(t) - \theta^*$, the closed loop system (10.20)–(10.21) is

$$\begin{aligned}\dot{x}(t) &= (A + BK)x(t) + Br(t) + B(\tilde{\theta}(t)\omega(t) + d_n(t))\\ y(t) &= Cx(t)\end{aligned} \tag{10.42}$$

The ideal system performance is that with $u(t) = u_d(t)$ which means $\tilde{\theta}(t)\omega(t) + d_n(t) = 0$. This motivates us to introduce the reference system

$$\begin{aligned}\dot{x}_m(t) &= (A + BK)x_m(t) + Br(t)\\ y_m(t) &= Cx_m(t)\end{aligned} \tag{10.43}$$

Defining the state error vector $\tilde{x}(t) = x(t) - x_m(t)$ and the output error $e(t) = y(t) - y_m(t)$, from (10.42) and (10.43), we obtain

$$\begin{aligned}\dot{\tilde{x}}(t) &= (A + BK)\tilde{x}(t) + B(\tilde{\theta}(t)\omega(t) + d_n(t))\\ e(t) &= C\tilde{x}(t)\end{aligned} \tag{10.44}$$

Consider the positive definite function

$$V(\tilde{x}, \tilde{\theta}) = \tfrac{1}{2}(\tilde{x}^T P\tilde{x} + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}) \tag{10.45}$$

where $P \in R^{n\times n}$ with $P = P^T > 0$ satisfying the Lyapunov equation

$$P(A + BK) + (A + BK)^T P = -2Q \tag{10.46}$$

for some $n \times n$ matrix $Q = Q^T > 0$, and $\Gamma = \mathrm{diag}\,\{\gamma_1, , \gamma_{n_\theta}\}$ with $\gamma_i > 0$,

$i = 1, 2, \ldots, n_\theta$. With (10.44) and (10.46), the time derivative of $V(\tilde{x}, \tilde{\theta})$ is

$$\begin{aligned}\dot{V}(t) &= \frac{d}{dt} V(\tilde{x}, \tilde{\theta}) \\ &= \tilde{x}^T(t) P \dot{\tilde{x}}(t) + \tilde{\theta}^T(t) \Gamma^{-1} \dot{\tilde{\theta}}(t) \\ &= -\tilde{x}^T(t) Q \tilde{x}(t) + \tilde{x}^T(t) PB(\tilde{\theta}^T(t)\omega(t) + d_n(t)) + \tilde{\theta}^T(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \quad (10.47)\end{aligned}$$

If $d_n(t)$ were absent, then the choice of $\dot{\tilde{\theta}}(t) = \dot{\theta}(t) = -\Gamma\omega(t)\tilde{x}^T(t)PB$ would make $\dot{V}(t) = -\tilde{x}^T(t)Q\tilde{x}(t) \leq 0$, as desired. In the presence of $d_n(t)$ and with the need of parameter projection, we choose the adaptive update law for $\theta(t)$ as

$$\dot{\theta}(t) = -\Gamma\omega(t)\tilde{x}^T(t)PB + f(t) \quad (10.48)$$

initialized by (10.35) and projected with $f(t)$ in (10.36) for $g(t) = -\Gamma\omega(t)\tilde{x}^T(t)PB$.

With the adaptive law (10.48), we have

$$\dot{V}(t) = -\tilde{x}^T(t)Q\tilde{x}(t) + 2\tilde{x}^T(t)PBd_n(t) + 2\tilde{\theta}^T(t)\Gamma^{-1}f(t) \quad (10.49)$$

From the parameter projection algorithm (10.36), it follows that

$$\theta_i(t) \in [\theta_i^a, \theta_i^b], \quad i = 1, 2, \ldots, n_\theta \quad (10.50)$$

$$(\theta_i(t) - \theta_i^*) f_i(t) \leq 0, \quad i = 1, 2, \ldots, n_\theta \quad (10.51)$$

that is, $\tilde{\theta}^T(t)\Gamma^{-1}f(t) \leq 0$. Since $Q = Q^T > 0$ and $d_n(t)$ is bounded, we have, from (10.49) and (10.51), the boundedness of $\tilde{x}(t)$, from (10.43), that of $x_m(t)$, and in turn, from $\tilde{x}(t) = x(t) - x_m(t)$, that of $x(t)$, from (10.21), that of $u_d(t)$, and from (10), that of $\omega(t)$, $a(t)$ and $v(t)$. Thus all closed-loop signals are bounded. Finally, from (10.48), (10.49) and (10.51), it can be verified that $\tilde{x}(t)$, $e(t) = y(t) - y_m(t)$ and $\dot{\theta}(t)$ are all bounded by $d_n(t)$ in a mean square sense.

10.5 Output feedback inverse control

When only the output $y(t)$ of the plant (10.1) is available for measurement, we need to use an output feedback control law to generate $u_d(t)$ for the inverse (10.9). As shown in Figure 10.2, such a controller structure is

$$u_d(t) = C_1(D)[w](t), \; w(t) = r(t) - C_2(D)[y](t) \quad (10.52)$$

where $r(t)$ is a reference input, and $C_1(D)$ and $C_2(D)$ are linear blocks which are designed as if the nonlinearity $N(\cdot)$ were absent, that is, when $u(t) = u_d(t)$.

Reference system. The desired reference output $y_m(t)$ to be tracked by the plant output $y(t)$ now is defined as the output of the closed loop system with the controller (10.21) applied to the plant (10.1) *without* the actuator nonlinearity $N(\cdot)$, that is, when $u(t) = u_d(t)$. With $u(t) = u_d(t)$, the closed loop output is

$$y(t) = y_m(t) \triangleq \frac{G(D)C_1(D)}{1 + G(D)C_1(D)C_2(D)}[r](t) \tag{10.53}$$

Hence $C_1(D)$ and $C_2(D)$ should be chosen such that the transfer function

$$W_m(D) = \frac{G(D)C_1(D)}{1 + G(D)C_1(D)C_2(D)} \tag{10.54}$$

has all its poles at some desired stable locations, and that the closed loop system is internally stable when $u(t) = u_d(t)$.

Error model. To develop an adaptive law for updating the parameters of the inverse $\widehat{NI}(\cdot)$, we express the closed loop system as

$$y(t) = y_m(t) + W(D)[(\theta - \theta^*)^T \omega](t) + d(t) \tag{10.55}$$

where

$$W(D) = \frac{G(D)}{1 + G(D)C_1(D)C_2(D)} \tag{10.56}$$

$$d(t) = W(D)[d_n](t) \tag{10.57}$$

To use (10.55) to derive a desirable error model, $W(D)$ in (10.56) is required to have all its poles located in a desired stable region of the complex plane. With this requirement, the error signal $d(t)$ in (10.57) is bounded because $d_n(t)$ in (10.11) is.

Similar to (10.26)–(10.28), in view of (10.55), we define the estimation error

$$\varepsilon(t) = y(t) - y_m(t) + \xi(t) \tag{10.58}$$

where

$$\xi(t) = \theta^T(t)\zeta(t) - W(D)[\theta^T \omega](t) \tag{10.59}$$

$$\zeta(t) = W(D)[\omega](t) \tag{10.60}$$

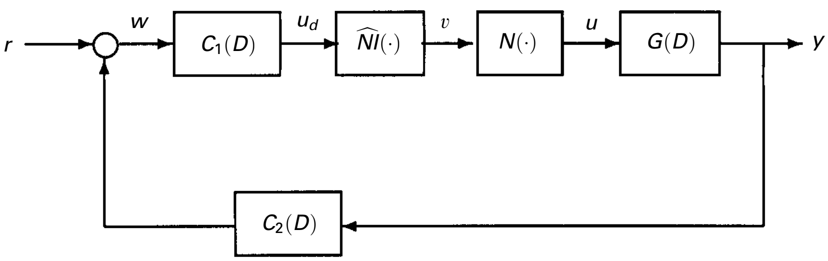

Figure 10.2 *Output feedback inverse control*

Using (10.55), (10.58)–(10.60), we arrive at the error equation

$$\varepsilon(t) = (\theta - \theta^*)^T \zeta(t) + d(t) \tag{10.61}$$

which has the same form as that in (10.30) for a state feedback design.

Adaptive law. Based on the linearly parametrized error equation (10.61) and the gradient projection optimization method described in Section 10.4, similar to that in (10.41), we choose the adaptive update law for $\theta(t)$:

$$\left.\begin{array}{l}\dot{\theta}(t) \\ \theta(t+1) - \theta(t)\end{array}\right\} = -\frac{\Gamma\varepsilon(t)\zeta(t)}{m^2(t)} + f(t) \tag{10.62}$$

which is initialized by (10.35) and projected with $f(t)$ in (10.36) or (10.37) for $g(t) = -\frac{\Gamma\varepsilon(t)\zeta(t)}{m^2(t)}$, where Γ is given in (10.33). This adaptive law also ensures that $\theta_i(t) \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, and that the normalized error $\frac{\varepsilon(t)}{m(t)}$ and parameter variation $\dot{\theta}(t)$ or $\theta(t+1) - \theta(t)$ are bounded by $\frac{d(t)}{m(t)}$ in a mean sense.

Summary. An output feedback adaptive inverse control design for the plant (10.1) with an actuator nonlinearity $N(\cdot)$ requires:

(i) $N(\cdot)$ should be parametrized by $\theta^* = (\theta_1^*, \ldots, \theta_{n_\theta}^*)^T$ as in (10.2);
(ii) the nonlinearity inverse $\widehat{NI}(\cdot)$ in (10.9), as a compensator to cancel $N(\cdot)$, should be parametrizable as (10.10) with the stated properties;
(iii) the true nonlinearity parameters $\theta_i^* \in [\theta_i^a, \theta_i^b]$ for some known constants $\theta_i^a < \theta_i^b$, $i = 1, 2, \ldots, n_\theta$, such that for any set of $\theta_i \in [\theta_i^a, \theta_i^b]$, $i = 1, 2, \ldots, n_\theta$, the characteristic $N(\theta; \cdot)$ renders a nonlinearity model of the same type as that of the true nonlinearity model $N(\theta^*; \cdot)$; and
(iv) the compensators $C_1(D)$ and $C_2(D)$ in the controller (10.52) should ensure that both $W_m(D)$ in (10.54) and $W(D)$ in (10.56) have good stability properties.

As a comparison, the conditions (i)–(iii) are also needed for a state feedback adaptive inverse control design in Section 10.4, while the state feedback control law (10.21) can always place the eigenvalues of $A + BK$ at some desired locations under the controllability condition on the system matrices (A, B) in (10.20).

10.6 Output feedback designs

Many existing control design methods for the plant (10.1) *without* the actuator nonlinearity $N(\cdot)$ may be used to construct the compensators $C_1(D)$ and $C_2(D)$ in the controller (10.52) to be incorporated in the adaptive inverse control scheme developed in Section 10.5 for a plant *with* an actuator nonlinearity. Those designs include model reference, PID and pole placement which will be given in this section. To parametrize the linear output feedback controller (10.52), we express $G(D) = k_p \dfrac{Z(D)}{P(D)}$, where k_p is a constant gain, and $Z(D)$ and $P(D)$ are monic polynomials of degrees n and m, respectively. Using the notation D, we present continuous- and discrete-time designs in a unified framework.

Model reference design [12]. A model reference control design for (10.52) is

$$u_d(t) = \phi_1^{*T}\omega_1(t) + \phi_2^{*T}\omega_2(t) + \phi_{20}^* y(t) + \phi_3^* r(t) \tag{10.63}$$

where $\omega_1(t)$, $\omega_2(t)$ are the signals from two identical filters:

$$\omega_1(t) = \frac{a(D)}{\Lambda(D)}[u_d](t),\ \omega_2(t) = \frac{a(D)}{\Lambda(D)}[y](t) \tag{10.64}$$

with $a(D) = (1, D, \ldots, D^{n-2})^T$ and $\Lambda(D)$ being a stable polynomial of degree $n-1$, and $\phi_1^* \in R^{n-1}$, $\phi_2^* \in R^{n-1}$, $\phi_{20}^* \in R$, $\phi_3^* \in R$ satisfy $\phi_3^* = k_p^{-1}$ and

$$\phi_1^{*T}a(D)P(D) + (\phi_2^{*T}a(D) + \phi_{20}^*\Lambda(D))k_p Z(D) = \Lambda(D)(P(D) - Z(D)P_m(D)) \tag{10.65}$$

where $P_m(D)$ is a stable polynomial of degree $n-m$.

In the form of the controller (10.52), we identify

$$C_1(D) = \frac{\phi_3^*}{1 - \phi_1^{*T}\dfrac{a(D)}{\Lambda(D)}}, \quad C_2(D) = k_p\left(\phi_2^{*T}\frac{a(D)}{\Lambda(D)} + \phi_{20}^*\right) \tag{10.66}$$

With this choice of $C_1(D)$ and $C_2(D)$, we have

$$W_m(D) = \frac{1}{P_m(D)}, \quad W(D) = \frac{k_p}{P_m(D)}\left(1 - \phi_1^{*T}\frac{a(D)}{\Lambda(D)}\right) \tag{10.67}$$

It shows that both $W_m(D)$ and $W(D)$ have good stability properties from that of $P_m(D)$ and $\Lambda(D)$ independent of $G(D)$. For internal stability, the zero polynomial $Z(D)$ should be stable because model reference control cancels the dynamics of $Z(D)$ to lead to the closed loop system $y(t) = W_m(D)[r](t)$ in the absence of the nonlinearity $N(\cdot)$ and its inverse $\widehat{NI}(\cdot)$, that is, when $u(t) = u_d(t)$.

Pole placement design [14]. A pole placement control design is similar:

$$u_d(t) = \phi_1^{*T}\omega_1(t) + \phi_2^{*T}\omega_2(t) + \phi_{20}^* y(t) + \phi_3^* r(t) \tag{10.68}$$

where $\omega_1(t)$ and $\omega_2(t)$ are the same as that defined in (10.64), and $\phi_1^* \in R^{n-1}, \phi_2^* \in R^{n-1}, \phi_{20}^* \in R, \phi_3^* \in R$ satisfy $\phi_3^* = k_p^{-1}$ and

$$\phi_1^{*T} a(D)P(D) + (\phi_2^{*T} a(D) + \phi_{20}^* \Lambda(D))k_p Z(D) = \Lambda(D)(P(D) - P_d(D)) \tag{10.69}$$

where $P_d(D)$ is a stable polynomial of degree n which contains the desired closed loop poles. In this design, $C_1(D)$ and $C_2(D)$ are also expressed in (66) but

$$W_m(D) = \frac{Z(D)}{P_d(D)}, \quad W(D) = \frac{k_p Z(D)}{P_d(D)} \left(1 - \phi_1^{*T} \frac{a(D)}{\Lambda(D)}\right) \tag{10.70}$$

It also shows that both $W_m(D)$ and $W(D)$ have good stability properties from that of $P_d(D)$ and $\Lambda(D)$ independent of $G(D)$. However, the reference output $y_m(t) = W_m(D)[r](t)$ now depends on the plant zero polynomial $Z(D)$ which is allowed to be unstable. To solve (10.69) for an arbitrary but stable $P_d(D)$, it is required that $Z(D)$ and $P(D)$ are co-prime.

PID design. A typical PID design (10.52) has $C_1(D) = C(D)$ and $C_2(D) = 1$:

$$u_d(t) = C(D)[r - y](t) \tag{10.71}$$

where the PID controller $C(D)$ has the form

$$C(D) = C(s) = \alpha + \frac{\beta}{s} + \gamma s \tag{10.72}$$

for a continuous-time design, and

$$C(D) = C(z) = \alpha + \frac{\beta}{z - 1} + \frac{\gamma (z - 1)}{z} \tag{10.73}$$

for a discrete-time design, where α, β and γ are design parameters.

Unlike either a model reference design or a pole placement design, which can be systematically done for a general nth-order plant $G(D)$, a PID controller only applies to certain classes of plants. In other words, given a linear plant $G(D)$, one needs to make sure that a choice of $C(D)$ is able to ensure both $W_m(D)$ in (10.54) and $W(D)$ in (10.56) with good stability properties.

An illustrative example. Many simulation results for model reference and pole placement designs were presented in [12, 14, 15], which indicate that control systems with an adaptive inverse have significantly improved tracking performance, in addition to closed-loop signal boundedness. As an additional illustrative example, we now present a detailed PI design.

An example of systems with actuator backlash is a liquid tank where the backlash is in the valve control mechanism. The liquid level $y(t)$ is described as

$$y(t) = G(s)[u](t), \; u(t) = B(v(t)) \tag{10.74}$$

where $u(t)$ is the controlled flow, $G(s) = \dfrac{k}{s}$ with k being a constant ($k = 1$ for simplicity), $B(\cdot)$ represents the actuator backlash and $v(t)$ is the control.

A simple version of the backlash characteristic $u(t) = B(v(t))$ is described by two parallel straight lines connected with inner horizontal line segments of length $2c > 0$. The upward line is active when both $v(t)$ and $u(t)$ increase:

$$u(t) = v(t) - c, \quad \dot{v}(t) > 0, \quad \dot{u}(t) > 0 \tag{10.75}$$

the downward line is active when both $v(t)$ and $u(t)$ decrease:

$$u(t) = v(t) + c, \ \dot{v}(t) < 0, \ \dot{u}(t) < 0 \tag{10.76}$$

The motion on any inner segment is characterized by $\dot{u}(t) = 0$.

As shown in [12], in the presence of backlash, a proportional controller $v = -\alpha e$ cannot reduce the error $e(t) = y(t) - r$ to zero, and a PI controller $v(t) = -(e(t) + \int_0^t e(\tau)d\tau)$ may lead to a limit cycle (e, v) trajectory.

To cancel the effects of backlash, we use an adaptive backlash inverse

$$v(t) = \widehat{BI}(u_d(t)) \tag{10.77}$$

whose characteristic is described by two straight lines and vertical jumps between the lines. With $\widehat{c}(t)$ being the estimate of c, the downward side is

$$v(t) = u_d(t) - \widehat{c}(t), \quad \dot{u}_d(t) < 0 \tag{10.78}$$

and the upward side is

$$v(t) = u_d(t) + \widehat{c}(t), \quad \dot{u}_d(t) > 0 \tag{10.79}$$

Vertical jumps of $v(t)$ occur whenever $\dot{u}_d(t)$ changes its sign.

Introducing the backlash parameter and its estimate

$$\theta^* = c, \quad \theta(t) = \widehat{c}(t) \tag{10.80}$$

and defining the backlash inverse regressor

$$\omega(t) = 2\widehat{\chi}(t) - 1, \quad \widehat{\chi}(t) = \begin{cases} 1 & \text{if } v(t) = u_d(t) + \widehat{c}(t) \\ 0 & \text{otherwise} \end{cases} \tag{10.81}$$

we have the control error expression (10.19), where $d_n(t) \in [-2c, 2c]$ and $d_n(t) = 0, \ t \geq t_0$, if $\theta = \theta^*$ and the backlash inverse is correctly initialized: $u(t_0) = u_d(t_0)$. In other words, the backlash inverse $\widehat{BI}(\cdot)$ has the desired property:

$$u_d(t_0) = B(\widehat{BI}(\theta^*; u_d(t_0))) \quad \Rightarrow \quad B(\widehat{BI}(\theta^*; u_d(t))) = u_d(t), \quad \forall t \geq t_0 \tag{10.82}$$

Combined with the backlash inverse (10.77) is the PI controller

$$u_d(t) = \left(\alpha + \frac{\beta}{s}\right)[r - y](t), \quad \alpha > 0, \quad \beta > 0 \tag{10.83}$$

which leads to the closed loop error system

$$y(t) - \frac{\alpha s + \beta}{s^2 + \alpha s + \beta}[r](t) = \frac{s}{s^2 + \alpha s + \beta}[(\theta - \theta^*)\omega + d_n](t) \tag{10.84}$$

In view of (10.53), (10.55) and (10.84), we have

$$W_m(s) = \frac{\alpha s + \beta}{s^2 + \alpha s + \beta}, \quad W(s) = \frac{s}{s^2 + \alpha s + \beta} \tag{10.85}$$

which have desired stability properties for some α and β. Then, we can update $\theta(t)$ from the continuous-time adaptive law (10.62) with $f(t)$ chosen to ensure $\widehat{c}(t) \in [c_1, c_2]$ where $0 < c_1 < c < c_2$ with c_1 and c_2 being known constants.

For the same choice of $\alpha = \beta = 1$, $r = 10$, and $y(0) = 5$ as those led to large tracking error $e(t)$ *without* a backlash inverse [12], what happens in the closed-loop system with an adaptive backlash inverse $\widehat{BI}(\cdot)$ is shown in Figure 10.3(a)–(f) for $\widehat{c}(0) = 0.5$ (other choices of $\widehat{c}(0)$ led to similar results). These results clearly show that an adaptive backlash inverse is able to adaptively cancel the effects of the unknown backlash so that the system tracking performance is significantly improved: the tracking error $e(t) = y(t) - r$ reduces to very small values, and the limit cycle, which appeared with a PI controller alone [12], is eliminated.

10.7 Designs for multivariable systems

Many control systems are multivariable. Coupling dynamics in a multivariable system often make it more difficult to control than a single-variable system. In this section, state feedback and output feedback adaptive inverse control schemes are presented for compensation of *unknown* nonsmooth nonlinearities at the input of a *known* multivariable linear plant, based on a multivariable parameter estimation algorithm for coupled error models.

Consider a multivariable plant with actuator nonlinearities, described by

$$y(t) = G(D)[u](t), \quad u(t) = N(v(t)) \tag{10.86}$$

where $G(D) = \{g_{ij}(D)\}$ is an $m \times m$ strictly proper transfer matrix, $u(t) = (u_1(t), \ldots, u_m(t))^T$ and $v(t) = (v_1(t), \ldots, v_m(t))^T$ are the output and input of the multivariable actuator nonlinearity $N(\cdot) = (N_1(\cdot), \ldots, N_m(\cdot))^T$ such that

$$u_i(t) = N_i(v_i(t)), \quad i = 1, 2, \ldots, m \tag{10.87}$$

The control problem is to design adaptive inverses for cancelling the nonlinearities $N_i(\cdot)$, to be combined with commonly used multivariable control schemes for the linear part $G(D)$ for desired tracking performance. In this problem, $v(t)$ is the control and $u(t)$ is not accessible for either control or measurement.

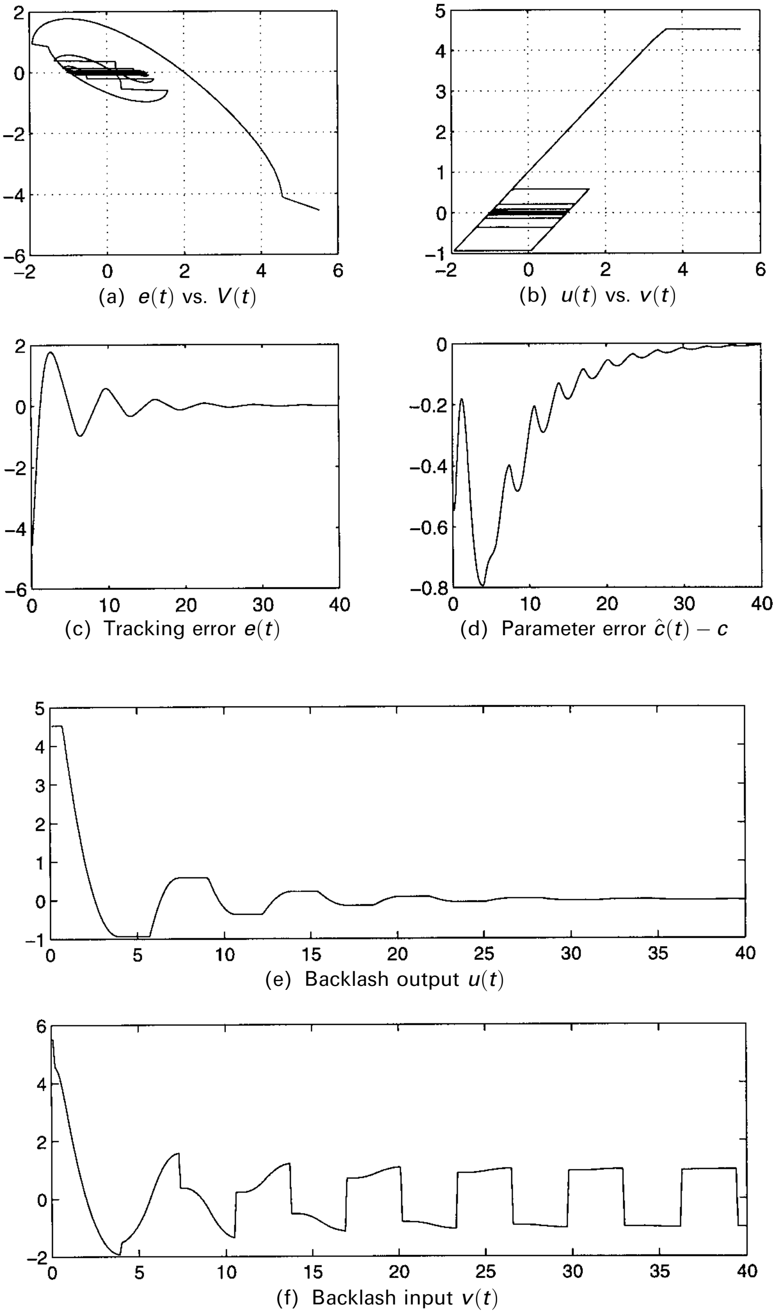

Figure 10.3 *Adaptive backlash inverse control system responses for* $\hat{c}(0) = 0.5$

Nonlinearity model and its inverse. Similar to that in Section 10.2 for $m = 1$, we assume that each $N_i(\cdot)$, which may be a dead-zone, backlash, hysteresis or other piecewise-linear characteristic, can be parametrized as

$$u_i(t) = N_i(v_i(t)) = N_i(\theta_i^*; v_i(t)) = -\theta_i^{*T}\omega_i^*(t) + a_i^*(t) \tag{10.88}$$

for some *unknown* parameter vectors $\theta_i^* \in R^{n_i}$, $n_i \geq 1$, $i = 1, \ldots, m$, and some *unknown* regressor vector signals $\omega_i^*(t) \in R^{n_i}$ and scalar signals $a_i^*(t)$.To cancel such nonlinearities, we use a multivariable nonlinearity inverse

$$v(t) = \widehat{NI}(u_d(t)) \tag{10.89}$$

where $u_d(t) = (u_{d1}(t), \ldots, u_{dm}(t))^T$ is a design vector signal to be generated for $\widehat{NI}(\cdot) = (\widehat{NI}_1(\cdot), \ldots, \widehat{NI}_m(\cdot))^T$, that is, the inverse (10.89) is equivalent to

$$v_i(t) = \widehat{NI}_i(u_{di}(t)), \quad i = 1, \ldots, m \tag{10.90}$$

We also assume that each $\widehat{NI}_i(\cdot)$ can be parametrized as

$$u_{di}(t) = -\theta_i^T(t)\omega_i(t) + a_i(t), \quad i = 1, \ldots, m \tag{10.91}$$

where $\theta_i \in R^{n_i}$ is an estimate of θ_i^*, and $\omega_i(t) \in R^{n_i}$ and $a_i(t)$ are some known signals, as in the case of an inverse for a dead zone, backlash or hysteresis.

The uncertainties in $N_i(\cdot)$ causes a control error at each input channel:

$$u_i(t) - u_{di}(t) = \tilde{\theta}_i^T(t)\omega_i(t) + d_i(t) \tag{10.92}$$

where $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$, $i = 1, 2, \ldots, m$, and the unparametrized error is

$$d_i(t) = \theta_i^{*T}(\omega_i(t) - \omega_i^*(t)) + a_i^*(t) - a_i(t) \tag{10.93}$$

which should satisfy the same condition as that for $d_n(t)$ in (10.11).

In the vector form, the control error (10.92) is

$$u(t) - u_d(t) = \tilde{\Theta}^T(t)\omega(t) + d_n(t) \tag{10.94}$$

where

$$\tilde{\Theta}^T(t) = \begin{pmatrix} \tilde{\theta}_1^T(t) & 0 & \cdots & 0 \\ 0 & \tilde{\theta}_2^T(t) & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{\theta}_m^T(t) \end{pmatrix} \tag{10.95}$$

$$\omega(t) = (\omega_1^T(t), \ldots, \omega_m^T(t))^T \tag{10.96}$$

$$d_n(t) = (d_1(t), \ldots, d_m(t))^T \tag{10.97}$$

State feedback designs. Let the plant (10.86) be in the state variable form:

$$D[x](t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{10.98}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{m\times n}$, with (A, B) being controllable. To generate $u_d(t)$ for the the inverse (10.89), we use the state feedback controller

$$u_d(t) = Kx(t) + r(t) \tag{10.99}$$

where $K \in R^{m\times n}$ is such that the eigenvalues of $A + BK$ are placed at some desired locations, and $r(t)$ is a bounded reference input signal.

Using (10.94), (10.98) and (10.99), we obtain

$$\begin{aligned} D[x](t) &= (A + BK)x(t) + Br(t) + B\tilde{\Theta}^T(t)\omega(t) + Bd_n(t) \\ y(t) &= Cx(t) \end{aligned} \tag{10.100}$$

The ideal system performance is that of (10.100) with $\tilde{\Theta}^T(t)\omega(t) + d_n(t) = 0$. Hence, we can define the reference output $y_m(t)$ for $y(t)$ as

$$y_m(t) = W_m(D)[r](t) \tag{10.101}$$

where

$$W_m(D) = C(DI - A - BK)^{-1}B \tag{10.102}$$

Then, with $d(t) = W_m(D)[d_n](t)$, the output tracking error is

$$e(t) = (e_1(t), \ldots, e_m(t))^T = y(t) - y_m(t) = W_m(D)[\tilde{\Theta}^T\omega](t) + d(t) \tag{10.103}$$

Since $W_m(D)$ is an $m \times m$ transfer matrix, the error equation (10.103) has coupled dynamics, for which the parameter estimation algorithm of [13] can be applied. To proceed, we define and the estimation errors

$$\varepsilon_i(t) = e_i(t) + \xi_{i1}(t) + \ldots + \xi_{im}(t) \tag{10.104}$$

where

$$\xi_{ij}(t) = \theta_j^T(t)\zeta_{ij}(t) - w_{ij}(D)[\theta_j^T\omega_j](t) \tag{10.105}$$

$$\zeta_{ij}(t) = w_{ij}(D)[\omega_j](t) \tag{10.106}$$

with $w_{ij}(D)$ being the (i,j)th element of $W_m(D)$, for $i, j = 1, 2, \ldots, m$.

Substituting (10.103), (10.105), (10.106) in (10.104) gives

$$\varepsilon_i(t) = \sum_{j=1}^{m}(\theta_j(t) - \theta_j^*)^T\zeta_{ij}(t) + d_i(t) \tag{10.107}$$

where d_i is the ith component of $d(t)$. Introducing

$$\theta(t) = (\theta_1^T(t), \ldots, \theta_m^T(t))^T \tag{10.108}$$

$$\theta^* = (\theta_1^{*T}, \ldots, \theta_m^{*T})^T \tag{10.109}$$

$$\zeta_i(t) = (\zeta_{i1}^T(t), \ldots, \zeta_{im}^T(t))^T \tag{10.110}$$

we express the error model (10.107) as

$$\varepsilon_i(t) = (\theta(t) - \theta^*)^T \zeta_i(t) + d_i(t), \quad i = 1, 2, \ldots, m \tag{10.111}$$

This is a set of linear error models with bounded 'disturbances' $d_i(t)$. One important feature of these equations is that dynamic coupling in the multivariable error equation (10.103) leads to a set of m estimation errors which all contain the overall parameter vectors $\theta(t)$ and θ^*.

Introducing the cost function

$$J(\theta) = \frac{\varepsilon_1^2 + \ldots + \varepsilon_m^2}{2m^2} \tag{10.112}$$

$$m^2(t) = 1 + \sum_{i=1}^{m} \zeta_i^T(t)\zeta_i(t) \tag{10.113}$$

and applying the gradient projection optimization technique of Section 10.4, we have the adaptive law for $\theta(t)$:

$$\left.\begin{array}{l} \dot{\theta}(t) \\ \theta(t+1) - \theta(t) \end{array}\right\} = -\frac{\Gamma}{m^2(t)} \sum_{i=1}^{m} \varepsilon_i(t)\zeta_i(t) + f(t) \tag{10.114}$$

where $f(t)$ is for parameter projection and Γ is an adaptation gain matrix. Both $f(t)$ and Γ can be chosen similar to that in (10.36) or (10.37) and (10.33).

To develop the multivariable counterpart of the continuous-time adaptive scheme (10.48), with the same notation as that in (10.42)–(10.46) except that $\tilde{\theta}^T(t)\omega(t)$ is replaced by $\tilde{\Theta}^T(t)\omega(t)$, we have the time derivative of $V(\tilde{x}, \tilde{\theta})$ in (10.45) as

$$\dot{V}(t) = -\tilde{x}^T(t)Q\tilde{x}(t) + \tilde{x}^T(t)PB(\tilde{\Theta}^T(t)\omega(t) + d_n(t)) + \tilde{\theta}^T(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \tag{10.115}$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$ in (10.108) and (10.109). With the *known* signals

$$\bar{x}(t) = (\bar{x}_1(t), \ldots, \bar{x}_m(t))^T = (\tilde{x}^T PB)^T \tag{10.116}$$

$$\bar{\omega}(t) = (\bar{x}_1(t)\omega_1^T(t), \ldots, \bar{x}_m(t)\omega_m^T(t))^T \tag{10.117}$$

we express (10.115) as

$$\dot{V}(t) = -\tilde{x}^T(t)Q\tilde{x}(t) + \tilde{\theta}^T(t)\bar{\omega}(t) + \tilde{x}^T(t)PBd_n(t) + \tilde{\theta}^T(t)\Gamma^{-1}\dot{\tilde{\theta}}(t) \tag{10.118}$$

which suggests the following adaptive law for $\theta(t)$:

$$\dot{\theta}(t) = -\Gamma\bar{\omega}(t) + f(t) \tag{10.119}$$

where $f(t)$ is for parameter projection and Γ is diagonal and positive.

Output feedback designs. To develop a multivariable output feedback adaptive inverse control scheme, we assume that $G(D)$ is nonsingular with the observability index v and has the form: $G(D) = Z(D)P^{-1}(D)$, for some $m \times m$ *known* polynomial matrices $Z(D)$ and $P(D)$.

The controller structure for $u_d(t)$ is

$$u_d(t) = \Phi_1^{*T}\omega_1(t) + \Phi_2^{*T}\omega_2(t) + \Phi_{20}^*y(t) + \Phi_3^*r(t) \tag{10.120}$$

where

$$\omega_1(t) = \frac{A(D)}{n(D)}[u_d](t), \quad \omega_2(t) = \frac{A(D)}{n(D)}[y](t) \tag{10.121}$$

$$A(D) = (I, DI, \ldots, D^{\nu-2}I)^T \tag{10.122}$$

$n(D)$ is a monic and stable polynomial of degree $\nu - 1$, and $\Phi_1^* \in R^{m(\nu-1)\times m}$, $\Phi_2^* \in R^{m(\nu-1)\times m}$, $\Phi_{20}^* \in R^{m\times m}$, and $\Phi_3^* \in R^{m\times m}$ satisfy

$$\Phi_1^{*T}A(D)P(D) + (\Phi_2^{*T}A(D) + \Phi_{20}^*n(D))A(D)N(D) = n(D)(P(D) - \Phi_3^*P_d(D)) \tag{10.123}$$

for a given $m \times m$ stable polynomial matrix $P_d(D)$.

A model reference design needs a stable $Z(D)$. The choice to meet (10.123) is: $P_d(D) = \xi_m(D)Z(D)$ and $\Phi_3^* = K_p^{-1}$, where $\xi_m(D)$ is a modified interactor matrix (which is a stable polynomial matrix) of $G(D)$ such that

$$\lim_{D\to\infty} \xi_m(D)G(D) = K_p \tag{10.124}$$

is finite and nonsingular [11]. For a pole placement design which does not need a stable $Z(D)$, $\Phi_3^*P_d(D)$ is chosen to have a structure similar to that of $P(D)$ so that (10.123) has a solution under the right co-primeness of $Z(D)$ and $P(D)$ [1].

With the controller (10.126) and the inverse (10.89), we have the error equation

$$e(t) = y(t) - y_m(t) = W(D)[\tilde{\Theta}^T\omega](t) + d(t) \tag{10.125}$$

where

$$y_m(t) = Z(D)P_d^{-1}(D)[r](t) \tag{10.126}$$

$$W(D) = Z(D)P_d^{-1}(D)\Phi_3^{*-1}\left(I - \Phi_1^{*T}\frac{A(D)}{n(D)}\right) \tag{10.127}$$

and $d(t) = W(D)[d_n](t)$ is bounded because $d_n(t)$ is and $W(D)$ is stable. Since the error equation (10.125) has the same form as that in (10.89), similar to (10.104)–(10.114), one can also develop an adaptive update law for $\theta_i(t)$, $i = 1, \ldots, m$.

10.8 Designs for nonlinear dynamics

The adaptive inverse approach developed in Sections 10.4–10.6 can also be applied to systems with *nonsmooth* nonlinearities at the inputs of *smooth* nonlinear dynamics [16, 17]. State feedback and output feedback adaptive inverse control schemes may be designed for some special cases of a nonlinear plant

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad u(t) = N(v(t))$$
$$y(t) = h(x(t)) \tag{10.128}$$

where $f(x) \in R^n$, $g(x) \in R^n$, and $h(x) \in R$ are smooth functions of $x \in R^n$, $N(\cdot)$ represents an *unknown* nonsmooth actuator uncertainty as a dead-zone, backlash, hysteresis or piecewise-linear characteristic, and $v(t) \in R$ is the control input, while $u(t)$ is not accessible for either control or measurement.

The main idea for the control of such plants is to use an adaptive inverse

$$v(t) = \widehat{NI}(u_d(t)) \tag{10.129}$$

to cancel the effect of the unknown nonlinearity $N(\cdot)$ so that feedback control schemes designed for (10.128) *without* $N(\cdot)$, with the help of $\widehat{NI}(\cdot)$, can be applied to (10.128) *with* $N(\cdot)$, to achieve desired system performance. To illustrate this idea, we present an adaptive inverse design [16] for a third-order *parametric-strict-feedback* nonlinear plant [6] with an actuator nonlinearity $N(\cdot)$:

$$\dot{x}_1 = x_2 + \theta_s^{*T}\varphi_1(x_1)$$
$$\dot{x}_2 = x_3 + \theta_s^{*T}\varphi_2(x_1, x_2) \tag{10.130}$$
$$\dot{x}_3 = \varphi_0(x) + \theta_s^{*T}\varphi_3(x) + \beta_0(x)u, \quad u = N(v)$$

where $x = (x_1, x_2, x_3)^T$, x_i, $i = 1, 2, 3$, are the state variables available for measurement, $\theta_s^* \in R^{n_s}$ is a vector of *unknown* constant parameters with a *known* upper bound M_s: $\|\theta_s^*\|_2 < M_s$, for the Euclidean vector norm $\|\cdot\|_2$, $\varphi_0 \in R$, $\beta_0 \in R$ and $\varphi_i \in R^p$, $i = 1, 2, 3$, are known smooth nonlinear functions, and $b_1 < |\beta_0(x)| < b_2$, for some positive constants b_1, b_2 and $\forall x \in R^3$.

To develop an adaptive inverse controller for (10.130), we assume that the nonlinearity $N(\cdot)$ is parametrized by $\theta^* \in R^{n_\theta}$ as in (10.2) and its inverse $\widehat{NI}(\cdot)$ is parametrized by $\theta \in R^{n_\theta}$, an estimate of θ^*, as in (10.10), with the stated properties. Then, the adaptive backstepping method [6] can be combined with an adaptive inverse $\widehat{NI}(\cdot)$ to control the plant (10.130), in a three-step design:

Step 1: Let the desired output be $r(t)$ to be tracked by the plant output $y(t)$, with bounded derivatives $r^{(k)}(t)$, $k = 1, 2, 3$. Defining $z_1 = x_1 - r$ and $z_2 = x_2 - \alpha_1$, where α_1 is a design signal to be determined, we have

$$\dot{z}_1 = z_2 + \alpha_1 + \theta_s^{*T}\varphi_1 - \dot{r} \tag{10.131}$$

to be stabilized by α_1 with respect to the partial Lyapunov function

$$V_1 = \tfrac{1}{2} z_1^2 + \tfrac{1}{2}(\theta_s - \theta_s^*)^T \Gamma_s^{-1} (\theta_s - \theta_s^*) \tag{10.132}$$

where $\theta_s(t) \in R^{n_s}$ is an estimate of the *unknown* parameter vector θ_s^*, and $\Gamma_s = \Gamma_s^T > 0$. The time derivative of V_1 is

$$\dot{V}_1 = z_1(z_2 + \alpha_1 + \theta_s^T \varphi_1 - \dot{r}) + (\theta_s - \theta_s^*)^T \Gamma_s^{-1} (\dot{\theta}_s - \Gamma_s z_1 \varphi_1) \tag{10.133}$$

Choosing the first stabilizing function as

$$\alpha_1 = -c_1 z_1 - \theta_s^T \varphi_1 + \dot{r} \tag{10.134}$$

with $c_1 > 0$, and defining $\tau_1 = \Gamma_s z_1 \varphi_1$, we obtain

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + (\theta_s - \theta_s^*)^T \Gamma_s^{-1} (\dot{\theta}_s - \tau_1) \tag{10.135}$$

Step 2: Introducing $z_3 = x_3 - \alpha_2$ with α_2 to be determined, we have

$$\dot{z}_2 = z_3 + \alpha_2 + \theta_s^{*T} \varphi_2 - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \theta_s^{*T} \varphi_1) - \frac{\partial \alpha_1}{\partial \theta_s} \dot{\theta}_s - \frac{\partial \alpha_1}{\partial r} \dot{r} - \frac{\partial \alpha_1}{\partial \dot{r}} \ddot{r} \tag{10.136}$$

To derive α_2, together with α_1 in (10.134), to stabilize (10.131) and (10.136) with respect to $V_2 = V_1 + \frac{1}{2} z_2^2$, we express the time derivative of V_2 as

$$\begin{aligned}\dot{V}_2 = -c_1 z_1^2 + z_2 \Bigg[& z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \theta_s} \dot{\theta}_s - \frac{\partial \alpha_1}{\partial r} \dot{r} - \frac{\partial \alpha_1}{\partial \dot{r}} \ddot{r} \\ & + \theta_s^T \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1\right) \Bigg] + (\theta_s - \theta_s^*)^T \Gamma_s^{-1} [\dot{\theta}_s - \tau_2]\end{aligned} \tag{10.137}$$

where $\tau_2 = \tau_1 + \Gamma_s z_2 \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1\right)$. Suggested by (10.137), we choose the second stabilizing function α_2 as

$$\begin{aligned}\alpha_2 = & -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \theta_s} (\tau_2 - \Gamma_s f_s(\theta_s) \theta_s) \\ & + \frac{\partial \alpha_1}{\partial r} \dot{r} + \frac{\partial \alpha_1}{\partial \dot{r}} \ddot{r} - \theta_s^T \left(\varphi_2 - \frac{\partial \alpha_1}{1} \varphi_1\right)\end{aligned} \tag{10.138}$$

where $c_2 > 0$, and

$$f_s(\theta_s) = \begin{cases} 0 & \text{if } \|\theta_s\|_2 \leq M_s \\ f_0(1 - e^{-\alpha_0(\|\theta_s\|_2 - M_s)^2} & \text{if } \|\theta_s\|_2 > M_s \end{cases} \tag{10.139}$$

with $f_0 > 0$, $\alpha_0 > 0$, and $M_s > \|\theta_s^*\|_2$. Then we can rewrite (10.137) as

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 - z_2 \frac{\partial \alpha_1}{\partial \theta_s}\left(\dot{\theta}_s - \tau_2 + \Gamma_s f_s(\theta_s)\theta_s\right) + (\theta_s - \theta_s^*)^T \Gamma_s^{-1}(\dot{\theta}_s - \tau_2) \tag{10.140}$$

Step 3: Choose the feedback control $u_d(t)$ for (10.129) as

$$u_d = \frac{1}{\beta_0}\Bigg[- z_2 - c_3 z_3 - \varphi_0 + \sum_{k=1}^{2} \frac{\partial \alpha_2}{\partial x_k} x_{k+1} + \frac{\partial \alpha_2}{\partial \theta_s}\dot{\theta}_s + \sum_{k=1}^{3} \frac{\partial \alpha_2}{\partial r^{(k-1)}} r^{(k)} - \left(\theta_s^T - z_2 \frac{\partial \alpha_1}{\partial \theta_s}\Gamma_s\right)\left(\varphi_3 - \sum_{k=1}^{2}\frac{\partial \alpha_2}{\partial x_k}\varphi_k\right) - \Gamma_s f_s(\theta_s)\theta_s z_2 \frac{\partial \alpha_1}{\partial \theta_s}\Bigg] \tag{10.141}$$

where $c_3 > 0$, and the adaptive update law for $\theta_s(t)$ is

$$\dot{\theta}_s = \Gamma_s \sum_{l=1}^{3} z_l \omega_l - \Gamma_s f_s(\theta_s)\theta_s \tag{10.142}$$

with $\omega_i(x_1, \ldots, x_i, \theta_s) = \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k$, for $i = 1, 2, 3$ and $\alpha_0 = 0$.

With this choice of u_d, we have

$$\dot{z}_3 = -z_2 - c_3 z_3 - (\theta_s - \theta_s^*)^T \omega_3 + z_2 \frac{\partial \alpha_1}{\partial \theta_s}\Gamma_s \omega_3 \tag{10.143}$$

Then, the overall system Lyapunov function

$$V = V_2 + \tfrac{1}{2} z_3^2 + \tfrac{1}{2}(\theta - \theta^*)^T \Gamma^{-1}(\theta - \theta^*) \tag{10.144}$$

with Γ being diagonal and positive, has the time derivative

$$\dot{V} = -\sum_{k=1}^{3} c_k z_k^2 - f_s(\theta_s)(\theta_s - \theta_s^*)^T \theta_s + (\theta - \theta^*)^T \Gamma^{-1}(\dot{\theta} + \Gamma z_3 \beta_0 \omega) + z_3 \beta_0 d_n \tag{10.145}$$

where $\omega(t) \in R^{n_\theta}$ is the known vector signal describing the signal motion of the inverse $\widehat{NI}(\cdot)$ in (10.10), and $d_n(t)$ is the bounded unparametrized nonlinearity error in (10.11). The expression (10.145), with the need of parameter projection for $\theta(t)$ for implementing an inverse $\widehat{NI}(\cdot)$, suggests the adaptive update law for $\theta(t)$:

$$\dot{\theta} = -\Gamma z_3 \beta_0 \omega + f \tag{10.146}$$

initialized by (10.35) and projected with $f(t)$ in (10.36) for $g(t) = -\Gamma z_3 \beta_0 \omega$.

This adaptive inverse control scheme consisting of (10.129), (10.141), (10.142) and (10.146) has some desired properties. First, the modification (10.139) ensures that $-f_s(\theta_s)(\theta_s - \theta_s^*)^T \theta_s \leq 0$, and the parameter projection (10.36) ensure that $(\theta - \theta^*)^T \Gamma^{-1} f \leq 0$ and $\theta_i(t) \in [\theta_i^a, \theta_i^b]$, $i = 1, \ldots, n_\theta$, for

$\theta(t) = (\theta_1(t), \ldots, \theta_{n_\theta}(t))^T$ and θ_i^a, θ_i^b defined in (10.31). Then, from (10.145) and (10.146), we have

$$\dot{V} = -\sum_{k=1}^{3} c_k z_k^2 - f_s(\theta_s)(\theta_s - \theta_s^*)^T \theta_s + (\theta - \theta^*)^T \Gamma^{-1} f + z_3 \beta_0 d_n \qquad (10.147)$$

Since d_n and β_0 are bounded, and $f_s(\theta_s)(\theta_s - \theta_s^*)^T \theta_s$ grows unboundedly if θ_s grows unboundedly, it follows from (10.144) and (10.147) that $\theta_s, \theta \in L^\infty$, and $z_k \in L^\infty$, $k = 1, 2, 3$. Since $z_1 = x_1 - r$ is bounded, we have $x_1 \in L^\infty$and hence $\varphi_1(x_1) \in L^\infty$. It follows from (10.134) that $\alpha_1 \in L^\infty$. By $z_2 = x_2 - \alpha_1$, we have $x_2 \in L^\infty$ and hence $\varphi_2(x_1, x_2) \in L^\infty$. It follows from (10.138) that $\alpha_2 \in L^\infty$. Similarly, $x_3 \in L^\infty$, and $u_d \in L^\infty$ in (10.141). Finally, $v(t)$ in (10.129) and $u(t)$ in (10.128) are bounded. Therefore, all closed loop system signals are bounded.

Similarly, an output feedback adaptive inverse control scheme can be developed for the nonlinear plant (10.128) in an output-feedback canonical form with actuator nonlinearities [17]. For such a control scheme, a state observer [6] is needed to obtain a state estimate for implementing a feedback control law to generate $u_d(t)$ as the input to an adaptive inverse: $v(t) = \widehat{NI}(u_d(t))$, to cancel an actuator nonlinearity: $u(t) = N(v(t))$.

10.9 Concluding remarks

Thus far, we have presented a general adaptive inverse approach for control of plants with *unknown* nonsmooth actuator nonlinearities such as dead zone, backlash, hysteresis and other piecewise-linear characteristics. This approach combines an adaptive inverse with a feedback control law. The adaptive inverse is to cancel the effect of the actuator nonlinearity, while the feedback control law can be designed as if the actuator nonlinearity were absent. For parametrizable actuator nonlinearities which have parametrizable inverses, state or output feedback adaptive inverse controllers were developed which led to linearly parametrized error models suitable for the developments of gradient projection or Lyapunov adaptive laws for updating the inverse parameters.

The adaptive inverse approach can be viewed as an algorithm-based compensation approach for cancelling unknown actuator nonlinearities caused by component imperfections. As shown in this chapter, this approach can be incorporated with existing control designs such as model reference, PID, pole placement, linear quadratic, backstepping and other dynamic compensation techniques. An adaptive inverse can be added into a control system loop without the need to change a feedback control design for a known linear or nonlinear dynamics following the actuator nonlinearity.

Improvements of system tracking performance by an adaptive inverse have been shown by simulation results. Some signal boundedness properties of adaptive laws and closed loop systems have been established. However, despite the existence of a true inverse which completely cancels the actuator nonlinearity, an analytical proof of a tracking error convergent to zero with an adaptive inverse is still not available for a general adaptive inverse control design. Moreover, adaptive inverse control designs for systems with *unknown* multivariable or more general nonlinear dynamics are still open issues under investigation.

References

[1] Elliott, H. and Wolovich, W. A. (1984) 'Parametrization Issues In Multivariable Adaptive Control', *Automatica*, Vol. 20, No. 5, 533–545.

[2] Goodwin, G. C. and Sin, K. S. (1984) *Adaptive Filtering Prediction and Control*. Prentice-Hall.

[3] Hatwell, M. S., Oderkerk, B. J., Sacher, C. A. and Inbar, G. F. (1991) 'The Development of a Model Reference Adaptive Controller to Control the Knee Joint of Paraplegics', *IEEE Trans. Autom. Cont*., Vol. 36, No. 6, 683–691.

[4] Ioannou, P. A. and Sun, J. (1995) *Robust Adaptive Control*. Prentice-Hall.

[5] Krasnoselskii, M. A. and Pokrovskii, A. V. (1983) *Systems with Hysteresis*. Springer-Verlag.

[6] Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1995) *Nonlinear and Adaptive Control Design*. John Wiley & Sons.

[7] Merritt, H. E. (1967) *Hydraulic Control Systems*. John Wiley & Sons.

[8] Physik Instrumente (1990) *Products for Micropositioning*. Catalogue 108–12, Edition E.

[9] Rao, S. S. (1984) *Optimization: Theory and Applications*. Wiley Eastern.

[10] Recker, D. (1993) *'Adaptive Control of Systems Containing Piecewise Linear Nonlinearities*', Ph.D. Thesis, University of Illinois, Urbana.

[11] Tao, G. and Ioannou, P. A. (1992) 'Stability and Robustness of Multivariable Model Reference Adaptive Control Schemes', in *Advances in Robust Control Systems Techniques and Applications*, Vol. 53, 99–123. Academic Press.

[12] Tao, G. and Kokotović, P. V. (1996) *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*. John Wiley & Sons.

[13] Tao, G. and Ling, Y. (1997) 'Parameter Estimation for Coupled Multivariable Error Models', *Proceedings of the 1997 American Control Conference*, 1934–1938, Albuquerque, NM.

[14] Tao, G. and Tian, M. (1995) 'Design of Adaptive Dead-zone Inverse for Nonminimum Phase Plants', *Proceedings of the 1995 American Control Conference*, 2059–2063, Seattle, WA.

[15] Tao, G. and Tian, M. (1995) 'Discrete-time Adaptive Control of Systems with Multi-segment Piecewise-linear Nonlinearities', *Proceedings of the 1995 American Control Conference*, 3019–3024, Seattle, WA.

[16] Tian, M. and Tao, G. (1996) 'Adaptive Control of a Class of Nonlinear Systems with Unknown Dead-zones', *Proceedings of the 13th World Congress of IFAC*, Vol. E, 209–213, San Francisco, CA.

[17] Tian, M. and Tao, G. (1997) 'Adaptive Dead-zone Compensation for Output-Feedback Canonical Systems', *International Journal of Control*, Vol. 67, No. 5, 791–812.

[18] Truxal, J. G. (1958) *Control Engineers' Handbook*. McGraw-Hill.

11

Stable multi-input multi-output adaptive fuzzy/neural control

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Abstract

In this chapter, stable direct and indirect adaptive controllers are presented which use Takagi–Sugeno fuzzy systems, conventional fuzzy systems, or a class of neural networks to provide asymptotic tracking of a reference signal vector for a class of continuous time multi-input multi-output (MIMO) square nonlinear plants with poorly understood dynamics. The direct adaptive scheme allows for the inclusion of a priori knowledge about the control input in terms of exact mathematical equations or linguistics, while the indirect adaptive controller permits the explicit use of equations to represent portions of the plant dynamics. We prove that with or without such knowledge the adaptive schemes can 'learn' how to control the plant, provide for bounded internal signals, and achieve asymptotically stable tracking of the reference inputs. We do not impose any initialization conditions on the controllers, and guarantee convergence of the tracking error to zero.

11.1 Introduction

Fuzzy systems and neural networks-based control methodologies have emerged in recent years as a promising way to approach nonlinear control problems. Fuzzy control, in particular, has had an impact in the control community because of the simple approach it provides to use heuristic control knowledge for nonlinear control problems. However, in the more complicated situations where the plant parameters are subject to perturbations, or when the dynamics of the system are too complex to be characterized reliably by an explicit mathematical model, adaptive schemes have been introduced that gather data from on-line operation and use adaptation heuristics to automatically determine the parameters of the controller. See, for example, the techniques in [1]–[7]; to date, no stability conditions have been provided for these approaches. Recently, several stable adaptive fuzzy control schemes have been introduced [8]–[12]. Moreover, closely related neural control approaches have been studied [13]–[18].

In the above techniques, emphasis is placed on control of single-input single-output (SISO) plants (except for [4], which can be readily applied to MIMO plants as it is done in [5, 6], but lacks a stability analysis). In [19], adaptive control of MIMO systems using multilayer neural networks is studied. The authors consider feedback linearizable, continuous-time systems with general relative degree, and utilize neural networks to develop an indirect adaptive scheme. These results are further studied and summarized in [20]. The scheme in [19] requires the assumptions that the tracking and neural network parameter errors are initially bounded and sufficiently small, and they provide convergence results for the tracking errors to fixed neighbourhoods of the origin.

In this chapter we present direct [21] and indirect [22] adaptive controllers for MIMO plants with poorly understood dynamics or plants subjected to parameter disturbances, which are based on the results in [8]. We use Takagi–Sugeno fuzzy systems or a class of neural networks with two hidden layers as the basis of our control schemes. We consider a general class of square MIMO systems decouplable via static nonlinear state feedback and obtain asymptotic convergence of the tracking errors to zero, and boundedness of the parameter errors, as well as state boundedness provided the zero dynamics of the plant are exponentially attractive. The stability results do not depend on any initialization conditions, and we allow for the inclusion in the control algorithm of a priori heuristic or mathematical knowledge about what the control input should be, in the direct case, or about the plant dynamics, in the indirect case. Note that while the indirect approach is a fairly simple extension of the corresponding single-input single-output case in [8], the direct adaptive case is not. The direct adaptive method turns out to require more restrictive assumptions than the indirect case, but is perhaps of more interest because, as far as we are aware, no other direct adaptive methodology with stability proof for the class of MIMO systems we consider here has been presented in the literature. The results in this chapter are nonlocal in the sense that they are global whenever the change of coordinates involved in the feedback linearization of the MIMO system is global.

The chapter is organized as follows. In Section 11.2 we introduce the MIMO direct adaptive controller and give a proof of the stability results. In Section 11.3 we outline the MIMO indirect adaptive controller, giving just a short sketch of the proof, since it is a relatively simple extension of the results in [8]. In Section 11.4 we present simulation results of the direct adaptive method applied, first, to a nonlinear differential equation that satisfies all controller assumptions, as an illustration of the method, and then to a two-link robot. The robot is an interesting practical application, and it is of special interest here because it does *not* satisfy all assumptions of the controller; however, we show how the method can be made to work in spite of this fact. In Section 11.5 we provide the concluding remarks.

11.2 Direct adaptive control

Consider the MIMO square nonlinear plant (i.e. a plant with as many inputs as outputs [23, 24]) given by

$$\begin{aligned} \dot{\mathbf{X}} &= f(\mathbf{X}) + g_1(\mathbf{X})u_1 + \ldots + g_p(\mathbf{X})u_p \\ y_1 &= h_1(\mathbf{X}) \\ &\vdots \\ y_p &= h_p(\mathbf{X}) \end{aligned} \tag{11.1}$$

where $\mathbf{X} = [x_1, \ldots, x_n]^{\mathrm{T}} \in \Re^n$ is the state vector, $\mathbf{U} := [u_1, \ldots, u_p]^{\mathrm{T}} \in \Re^p$ is the control input vector, $\mathbf{Y} := [y_1, \ldots, y_p]^{\mathrm{T}} \in \Re^p$ is the output vector, and $f, g_i, h_i, i = 1, \ldots, p$ are smooth functions. If the system is feedback linearizable [24] by static state feedback and has a well-defined vector relative degree $\mathbf{r} := [r_1, \ldots, r_p]^{\mathrm{T}}$, where the r_i's are the smallest integers such that at least one of the inputs appears in $y_i^{(r_i)}$, input–output differential equations of the system are given by

$$y_i^{(r_i)} = L_j^{r_i} h_i + \sum_{j=1}^{p} L_{gj}(L_f^{r_i-1} h_i) u_j \tag{11.2}$$

with at least one of the $L_{g_j}(L_f^{r_i-1} h_i) \neq 0$ (note that $L_f h(\mathbf{X}) : \Re^n \to \Re$ is the Lie derivative of h with respect to f, given by $L_f h(\mathbf{X}) = \frac{\partial h}{\partial X} f(\mathbf{X})$). Define, for convenience, $\alpha_i(\mathbf{X}) := L_f^{r_i} h_i$ and $\beta_{ij}(\mathbf{X}) := L_{g_j}(Lf^{r_i-1} h_i)$. In this way, we may rewrite the plant's input–output equation as

$$\underbrace{\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix}}_{\mathbf{Y}^{(r)}(t)} = \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{\mathbf{A}(\mathbf{X},t)} + \underbrace{\begin{bmatrix} \beta_{11} & \cdots & \beta_{1p} \\ \vdots & \ddots & \vdots \\ \beta_{p1} & \cdots & \beta_{pp} \end{bmatrix}}_{\mathbf{B}(\mathbf{X},t)} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}}_{\mathbf{U}(t)} \tag{11.3}$$

Consider the ideal feedback linearizing control law, $\mathbf{U}^* = [u_1^*, \ldots, u_p^*]^{\mathrm{T}}$,

$$\mathbf{U}^* = \mathbf{B}^{-1}(-\mathbf{A} + \boldsymbol{v}) \tag{11.4}$$

(note that, for convenience, we are dropping the references to the independent variables except where clarification is required), where the term $\boldsymbol{v} = [\nu_1, \ldots, \nu_p]^{\mathrm{T}}$ is an input to the linearized plant dynamics. In order for $\mathbf{U}^*$ to be well defined, we need the following assumption:

(P1) Plant Assumption
The matrix $\mathbf{B}$ as defined above is nonsingular, i.e. $\mathbf{B}^{-1}$ exists and has bounded norm for all $\mathbf{X} \in S_x, t \geq 0$, where $S_x \in \Re^n$ is some compact set of allowable state trajectories. This is equivalent to assuming

$$\sigma_p(\mathbf{B}) \geq \sigma_{\min} > 0 \tag{11.5}$$

$$\|\mathbf{B}\|_2 = \sigma_1(\mathbf{B}) \leq \sigma_{\max} < \infty \tag{11.6}$$

where $\sigma_p(\mathbf{B})$ and $\sigma_1(\mathbf{B})$ are, respectively, the smallest and largest singular values of $\mathbf{B}$.

In addition, in order to be able to guarantee state boundedness under state feedback linearization, we require:

(P2) Plant Assumption
The plant is feedback linearizable by static state feedback; it has a general vector relative degree $\mathbf{r} = [r_1, \ldots, r_p]^{\mathrm{T}}$, and its zero dynamics are exponentially attractive (please refer to [24] for a review on the concept of zero dynamics and static state feedback of square MIMO systems). We also assume the state vector $\mathbf{X}$ to be available for measurement.

Our goal is to identify the unknown control function (11.4) using fuzzy systems. Here we will use generalized Takagi–Sugeno (T–S) fuzzy systems with centre average defuzzification. To briefly present the notation, take a fuzzy system denoted by $\tilde{f}(\mathbf{X}, \mathbf{W})$ (in our context, $\mathbf{X}$ could be thought of as the state vector, and $\mathbf{W}$ as a vector of possibly exogenous signals). Then, $\tilde{f}(\mathbf{X}, \mathbf{W}) = \frac{\sum_{i=1}^{R} c_i \mu_i}{\sum_{i=1}^{R} \mu_i}$. Here, singleton fuzzification of the input vectors $\mathbf{X} = [x_1, \ldots, x_n]^{\mathrm{T}}$, $\mathbf{W} = [w_1, \ldots, w_q]^{\mathrm{T}}$ is assumed; the fuzzy system has R rules, and μ_i is the value of the membership function for the premise of the ith rule given the inputs $\mathbf{X}$, $\mathbf{W}$. It is assumed that the fuzzy system is constructed in such a way that $0 \leq \mu i \leq 1$ and $\sum_{i=1}^{R} \mu_i \neq 0$ for all $\mathbf{X} \in \Re^n$, $\mathbf{W} \in \Re^q$. The parameter c_i is the consequent of the ith rule, which in this chapter will be taken as a linear combination of Lipschitz continuous functions, $z_k(\mathbf{X}) \in \Re, k = 1, \ldots, m-1$, so that $c_i = a_{i,0} + a_{i,1} z_1(\mathbf{X}) + \cdots + a_{i,m-1} z_{m-1}(\mathbf{X}), i = 1, \ldots, R$. Define

$$z = \begin{bmatrix} 1 \\ z_1(\mathbf{X}) \\ \vdots \\ z_{m-1}(\mathbf{X}) \end{bmatrix} \in \Re^m, \quad \zeta^{\mathrm{T}} = \frac{[\mu_1(\mathbf{X},\mathbf{W}),\ldots,\mu_R(\mathbf{X},\mathbf{W})]}{\sum_{i=1}^{R}\mu_i(\mathbf{X},\mathbf{W})},$$

$$A^{\mathrm{T}} = \begin{bmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,m-1} \\ a_{2,0} & a_{2,1} & \cdots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{R,0} & a_{R,1} & \cdots & a_{R,m-1} \end{bmatrix}$$

Then, the nonlinear equation that describes the fuzzy system can be written as $\tilde{f}(\mathbf{X},\mathbf{W}) = z^{\mathrm{T}}(\mathbf{X})A\zeta(\mathbf{X},\mathbf{W})$ (notice that standard fuzzy systems may be treated as special cases of this more general representation). It was shown in [8] that the T–S model can represent a class of two-layer neural networks and many standard fuzzy systems. Note that while ζ may depend on both **X** and **W** and is bounded for any value they may take, z depends on **X** only. This allows us to impose no restrictions on **W** to guarantee boundedness of the fuzzy system.

We will represent the ith component of the ideal control (11.4), $i = 1,\ldots,p$, as

$$u_i^*(\mathbf{X},\mathbf{W},t) = z_i^{\mathrm{T}}(\mathbf{X})A_i^*\zeta_i(\mathbf{X},\mathbf{W}) + d_i(\mathbf{X},\mathbf{W}) + u_{k_i}(t) \tag{11.7}$$

where $A_i^* \in \Re^{m_i \times R_i}$ is assumed to exist, and is defined by

$$A_i^* := \arg\min_{A_i\in\Omega_i}\left[\sup_{\mathbf{X}\in S_x,\mathbf{W}\in S_w,t\geq 0}|z_i^{\mathrm{T}}A_i\zeta_i + u_{k_i} - u_i^*|\right] \tag{11.8}$$

and the representation error using optimal parameters (which arises because of the finite number of basis functions used) is bounded, i.e. $d_i(\mathbf{X},\mathbf{W}) \leq D_i(\mathbf{X},\mathbf{W})$, where D_i is a known bounding function. We define Ω_i as a compact set within which the matrix of coefficients estimates, $A_i(t)$, is allowed to lie, $S_x \subseteq \Re^n$ as the subspace through which the state trajectory may travel under closed loop control, and $S_w \subseteq \Re^q$ is the subset where the vector **W** may lie (notice that we do not restrict the sizes of S_x and S_w; however, the sup in (11.8) is assumed to exist). As a result of the proof we will be able to determine that **X** actually remains within a compact subset of S_x. Note that the ideal control law (11.4) is a function not only of the states, but also of $\boldsymbol{v}$, which may depend on variables other than the states (as will be described below). The vector **W** is provided to account for this dependence. The term u_{k_i} represents a *known* part of the ideal control input, which may be available to the designer through knowledge of the plant or expertise. If it is not available, this term may be set equal to zero with all the properties of the adaptive controller still holding. The only restriction on u_{k_i} is that it must be bounded. Note that an appropriate use of u_{k_i} may help to significantly improve the *performance* of the controller, even though in principle it has no effect on stability. Thus, the fuzzy system approximation of u_i^* is given by

$$\hat{u}_i(\mathbf{X}, \mathbf{W}, t) := z_i^{\mathrm{T}}(\mathbf{X}) A_i(t) \zeta_i(\mathbf{X}, \mathbf{W}) + u_{k_i}(t) \tag{11.9}$$

The matrix $A_i(t)$ is to be adjusted adaptively on-line in order to try to improve the approximation. We define the parameter error matrix, $\Phi_i(t) := A_i(t) - A_i^*$. Let $\hat{\mathbf{U}} := [\hat{u}_1, \ldots, \hat{u}_p]^{\mathrm{T}}$.

Our objective is to have the plant outputs track a vector of reference trajectories, $\mathbf{Y_m} = [y_{m_1}, \ldots, y_{m_p}]^{\mathrm{T}}$, on which we make the following assumption:

(R1) Reference Input Assumption
The desired reference trajectories y_{m_i} are r_i times continuously differentiable, with $y_{m_i}, \ldots, y_{m_i}^{(r_i)}$ measurable and bounded, for $i = 1, \ldots, p$.

We define the output errors $e_{o_i} := y_{m_i} - y_i$. Define also the error signals

$$e_{s_i} := [k_0, \ldots, k_{r_i-2}^i, 1][e_{o_i}, \ldots, e_{o_i}^{(r_i-2)}, e_{o_i}^{(r_i-1)}]^{\mathrm{T}}$$

and

$$\bar{e}_{s_i} := \dot{e}_{s_i} - e_{o_i}^{(r_i)} = [k_0^i, \ldots, k_{r_i-2}^i][\dot{e}_{o_i}, \ldots, e_{o_i}^{(r_i-1)}]^{\mathrm{T}}$$

The coefficients of

$$\frac{1}{\hat{L}_i(s)} := \frac{1}{s^{r_i-1} + k_{r_i-2}^i s^{r_i-2} + \ldots + k_1^i s + k_0^i},$$

$i = 1, \ldots, p$, are picked so that the transfer functions are stable. Let the ith component of the parameter $\boldsymbol{\nu}$ in (11.4) be given by $\nu_i := y_{m_i}^{(r_i)} + \eta_i e_{s_i} + \bar{e}_{s_i}$, where $\eta_i > 0$ is a constant. Consider the control law

$$\mathbf{U} = \hat{\mathbf{U}} + \mathbf{U_d} \tag{11.10}$$

where $\mathbf{U_d} := [u_{d_1}, \ldots, u_{d_p}]^{\mathrm{T}}$ is a control term required to ensure stability that will be defined later.

From (11.3) and (11.4) we can derive an expression for the plant output

$$\mathbf{Y^{(r)}} = \mathbf{A} + \mathbf{BU} = \mathbf{A} + \mathbf{B}(\mathbf{U} - \mathbf{U}^*) + \mathbf{BU}^* = \boldsymbol{\nu} + \mathbf{B}(\mathbf{U} - \mathbf{U}^*) \tag{11.11}$$

Then, using the previous definitions, the ith component of the output error dynamics is given by

$$e_{o_i}^{(r_i)} = y_{m_i}^{(r_i)} - y_i^{(r_i)} = y_{m_i}^{(r_i)} - \nu_i - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) = -\eta_i e_{s_i} - \bar{e}_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) \tag{11.12}$$

so that

$$\dot{e}_{s_i} = -\eta_i e_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) \tag{11.13}$$

Before proceeding we need to introduce another set of assumptions on the plant and formalize our assumptions about the control term $\hat{\mathbf{U}}$.

(P3) Plant Assumption

Each entry of $\mathbf{B}$, besides those on the main diagonal, is bounded by known constants, $|\beta_{ij}(\mathbf{X})| \leq \bar{\beta}_{ij}, i,j = 1, \ldots, p, i \neq j$. We require that the entries in the main diagonal satisfy $0 < \beta_{ii} \leq \beta_{ii}(\mathbf{X}) \leq \bar{\beta}_{ii} < \infty, i = 1, \ldots, p$, and their derivatives be defined and satisfy $|\dot{\beta}_{ii}(\mathbf{X})| \leq M_{ii}(\mathbf{X}), i = 1, \ldots, p$, where $\underline{\beta}_{ii}$, β_{ii} and $M_{ii}(\mathbf{X})$ are known bounds. Furthermore, the bounds have to satisfy

$$\frac{1}{\underline{\beta}_{ii}} \sum_{j=1, j\neq i}^{p} \bar{\beta}_{ij} < 1, \quad i = 1, \ldots, p. \tag{11.14}$$

(C1) Direct Adaptive Control Assumption

Bounding functions $\bar{U}_i(\mathbf{X}, \mathbf{W})$ such that $|z_i^{\mathrm{T}}(\mathbf{X})\Phi(t)\zeta_i(\mathbf{X}, \mathbf{W})| \leq \bar{U}_i(\mathbf{X}, \mathbf{W})$, $i = 1, \ldots, p, \mathbf{X} \in S_x, \mathbf{W} \in S_w$, are known and they are continuous functions. Furthermore, the fuzzy systems or neural networks that define the control term $\hat{\mathbf{U}}$, are defined so that the bounding functions of the representation errors, $D_i(\mathbf{X}, \mathbf{W}), i = 1, \ldots, p$, are continuous.

Note that in (P3) the entries of the main diagonal of $\mathbf{B}$ are all assumed positive. This is only to simplify the analysis; the diagonal entries may have any sign, as long as they are bounded away from zero, and the stability analysis requires only slight modifications to accommodate such a case. In (C1), knowledge of the bounding function $\bar{U}_i(\mathbf{X}, \mathbf{W})$ is reasonable, since both $z_i(\mathbf{X})$ and $\zeta_i(\mathbf{X}, \mathbf{W})$ are known: a projection algorithm may be employed to guarantee that $A_i(t)$ stays within the compact set Ω_i of allowable parameters. Then, an upper estimate of $\|\Phi_i(t)\|$ can be computed, and $\bar{U}_i$ can be defined.

Consider the function

$$V_i = \frac{1}{2\beta_{ii}} e_{s_i}^2 + \tfrac{1}{2}\mathbf{tr}(\Phi_i^{\mathrm{T}} Q_{u_i} \Phi_i) \tag{11.15}$$

with $Q_{u_i} \in \Re^{m_i \times m_i}$ positive definite and diagonal. This function quantifies both the tracking error for the ith plant output and the approximation error for the parameter estimates of the ith term of (11.4).

Taking the derivative of (11.15) yields

$$\dot{V}_i = \frac{1}{\beta_{ii}} e_{s_i}\dot{e}_{s_i} + \mathbf{tr}(\Phi_i^{\mathrm{T}} Q_{u_i}\dot{\Phi}_i) - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2 \tag{11.16}$$

$$= \frac{1}{\beta_{ii}} e_{s_i}\left(-\eta_i e_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*)\right) + \mathbf{tr}(\Phi_i^{\mathrm{T}} Q_{u_i}\dot{\Phi}_i) - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2 \tag{11.17}$$

Define the adaptation law for the T–S fuzzy system or neural network as

$$\dot{A}_i := Q_{u_i}^{-1} z_i \zeta_i^{\mathrm{T}} e_{s_i} \tag{11.18}$$

so that, applying the properties of the trace operator and the fact that $\dot{\Phi}_i = \dot{A}_i$, we obtain $\mathbf{tr}(\Phi_i^{\mathrm{T}} Q_{u_i}\dot{\Phi}_i) = z_i^{\mathrm{T}}\Phi_i\zeta_i\bar{e}_{s_i}$. Noting that $u_i - u_i^* = u_{d_i} + z_i^{\mathrm{T}}\Phi_i\zeta_i - d_i$, we get

$$\dot{V}_i = -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2 + \frac{e_{s_i}}{\beta_{ii}}\left[-\sum_{j=1}^{p}\beta_{ij}u_{d_j} + \sum_{j=1}^{p}\beta_{ij}d_j - \sum_{j=1,j\neq i}^{p}\beta_{ij}z_j^{\mathrm{T}}\Phi_j\zeta_j\right] - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2 \tag{11.19}$$

$$\dot{V} \leq -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2 - e_{s_i}u_{d_i} + |e_{s_i}|\left[\sum_{j=1}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}D_j + \sum_{j=1,j\neq i}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}(|u_{d_j}| + |z_j^{\mathrm{T}}\Phi_j\zeta_j|)\right] + \frac{|\dot{\beta}_{ii}|}{2\beta_{ii}^2} e_{s_i}^2 \tag{11.20}$$

Define

$$\sigma_i := \sum_{j=1}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}D_j + \sum_{j=1,j\neq i}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}\bar{U}j$$

$$\rho i := e_{s_i}\left(\frac{M_{ii}(\mathbf{X})}{2\underline{\beta}_{ii}^2}\right).$$

Given these definitions, let

$$u_{d_i} := \mathrm{sgn}(e_{s_i})\left(\sigma_i + \sum_{j=1,j\neq 1}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}U_{\max}\right) + \rho_i \tag{11.21}$$

where we need to derive an expression for $U_{\max}$ such that $|u_{d_i}| \leq U_{\max}$, $i = 1, \ldots, p$. From (11.21) we have

$$|u_{d_i}| = |\sigma_i| + |\rho_i| + U_{\max}\sum_{j=1,j\neq i}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}} \leq U_{\max} \tag{11.22}$$

It follows that if we choose

$$U_{\max}(t) \geq \max_{i=1,\ldots,p}\left[\frac{|\sigma_i| + |\rho_i|}{1 - \sum_{j=i,j\neq 1}^{p}\frac{\bar{\beta}_{ij}}{\underline{\beta}_{ii}}}\right] \tag{11.23}$$

and if (P3) holds, then in fact $|u_{d_i}| \leq U_{\max}, i = 1, \ldots, p$. Using (11.21) we can now establish

$$\dot{V}_i \leq -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2 \tag{11.24}$$

We are now ready to present our main result and give its proof.

Theorem 2.1 Stability and tracking results using MIMO direct adaptive control

If the reference input assumption (R1) holds, the plant assumptions (P1), (P2) and (P3) hold, and the control law is defined by (11.10) with the control assumption (C1) and the adaptive laws (11.18) are used,

Then the following holds:

(1) The plant states, as well as its outputs and their derivatives, $y_i, \ldots, y_i^{(r_i-1)}$, $i = 1, \ldots, p$, are bounded.
(2) The control signals are bounded, i.e. $\|\mathbf{U}\| \in \mathcal{L}_\infty (\mathcal{L}_\infty = \{\phi(t) : \sup_t |\phi(t)| < \infty\})$.
(3) The magnitudes of the output errors, e_{o_i}, decrease at least asymptotically to zero, i.e. $\lim_{t\to\infty} e_{o_i} = 0, i = 1, \ldots, p$.

Proof To show part 1, consider the Lyapunov candidate

$$V := \sum_{i=1}^{p} V_i \tag{11.25}$$

The above analysis guarantees

$$\dot{V} \leq -\sum_{i=1}^{p} \frac{\eta_i}{\beta_{ii}} e_{s_i}^2 \tag{11.26}$$

so V is a positive definite function with negative semidefinite derivative. This implies that $V_i \in \mathcal{L}_\infty$; therefore, $e_{s_i}, \bar{e}_{s_i} \in \mathcal{L}_\infty$ and $\|\Phi_i\| \in \mathcal{L}_\infty$ for $i = 1, \ldots, p$ (notice that this analysis alone does not guarantee $A_i \in \Omega_i$ for all time; rather, a projection algorithm should be used to achieve this). From the definition of e_{s_i} we have $e_{o_i}^{(j)} = \hat{G}_i^j(s) e_{s_i}$, where $\hat{G}_i^j(s) := s^j / \hat{L}_i(s), j = 0, \ldots, r_i - 1$. Since, by definition, $\hat{G}_i^j$ is stable, $e_{o_i}^{(j)} \in \mathcal{L}_\infty, j = 0, \ldots, r_i - 1$, and since by assumption (R1) the signals $y_{m_i}^{(r_i)}$ are bounded, we conclude that $y_i, \ldots, y_i^{(r_i-1)} \in \mathcal{L}_\infty$, $i = 1, \ldots, p$.

With the outputs bounded, and together with assumption (P2) we have that the states $x_1, \ldots, x_n$ are bounded [23], which implies that the state trajectories are limited to a bounded subset of S_x. Let $\bar{S}_x$ be the compact ball of minimum radius that contains the bounded subset of state trajectories. Since ζ_i is continuous and z_i is Lipschitz continuous by definition in S_x, then they are uniformly continuous, and therefore bounded, on $\bar{S}_x$, and given that u_{k_i} is bounded, we have $\hat{u}_i \in \mathcal{L}_\infty, i = 1, \ldots, p$. Since $\bar{U}_i$ is defined as a continuous function, and D_i is assumed continuous for all $\mathbf{X} \in S_x, \mathbf{W} \in S_w$, both are bounded on $\bar{S}_x$, so $\sigma_i \in \mathcal{L}_\infty$, and $\rho_i \in \mathcal{L}_\infty$ because $e_{s_i} \in \mathcal{L}_\infty$ from part 1. This implies $U_{\max} \in \mathcal{L}_\infty$, so $u_{d_i} \in \mathcal{L}_\infty, i = 1, \ldots, p$ by construction. Hence, $\|\mathbf{U}\| \in \mathcal{L}_\infty$. To prove part 3 we notice that, from (11.26)

$$\int_0^\infty \sum_{i=1}^{p} \frac{\eta_i}{\beta_{ii}} e_{s_i}^2 dt \leq -\int_0^\infty \dot{V} dt \tag{11.27}$$

$$= V(0) - V(\infty) < \infty \tag{11.28}$$

This establishes that $e_{s_i} \in \mathcal{L}_2, i = 1, \ldots, p \; (\mathcal{L}_2 = \{\phi(t) : \int_0^\infty \phi^2(t)\, dt < \infty\})$. Having determined that $e_{o_i}^j \in \mathcal{L}_\infty, j = 1, \ldots, r_i - 1$, it follows that $\dot{e}_{s_i} \in \mathcal{L}_\infty$, so e_{s_i} is uniformly continuous. Since $e_{s_i} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}_{s_i} \in \mathcal{L}_\infty$, by Barbalat's lemma we have asymptotic stability of e_{s_i} (i.e. $\lim_{t\to\infty} e_{s_i} = 0$), which implies asymptotic stability of e_{o_i} (i.e. $\lim_{t\to\infty} e_{o_i} = 0$), for $i = 1, \ldots, p$. Notice that although assumption (P1) is not used explicitly in the proof, it is still necessary in order to guarantee the *existence* of $\mathbf{U}^*$, without which the argument is not sound.

Remark 2.1 Note that, although in principle the choice of the vector $\mathbf{W}$ is arbitrary, a typical choice may be an error vector, i.e. $\mathbf{W} = \mathbf{Y} - \mathbf{Y_m}$, or some other function of the plant outputs and the reference model outputs. In this way, as a result of the proof, we also obtain that $\mathbf{W}$ remains within a bounded subset of S_w.

11.3 Indirect adaptive control

Here we consider, again, the class of plants defined in (11.1). If assumptions (P1) and (P2) of Section 11.2 are satisfied, then we may rewrite the input–output form of the plant as

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1 + \alpha_{1_k} \\ \vdots \\ \alpha_p + \alpha_{p_k} \end{bmatrix}}_{\mathbf{A}(\mathbf{X},t)} + \underbrace{\begin{bmatrix} \beta_{11} + \beta_{11_k} & \cdots & \beta_{1_p} + \beta_{1p_k} \\ \vdots & \ddots & \vdots \\ \beta_{p1} + \beta_{p1_k} & \cdots & \beta_{pp} + \beta_{pp_k} \end{bmatrix}}_{\mathbf{B}(\mathbf{X},t)} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}}_{\mathbf{U}(t)} \tag{11.29}$$

where $\alpha_{i_k}(t)$ and $\beta_{ij_k}(t)$ are known components of the plant's dynamics (that may depend on the state) or exogenous time dependent signals, with the only constraint that they have to be bounded for all $t \geq 0$. Throughout the following analysis they may be set equal to zero for all t; however, as in the direct adaptive case, a good choice of these known functions may help improve the *performance* of the controller. The functions $\alpha_i(\mathbf{X})$ and $\beta_{ij}(\mathbf{X})$ represent unknown nonlinear dynamics of the plant.

Again consider the ideal feedback linearizing control law (11.4), where the term $\boldsymbol{v}$ will be redefined below. Our goal is to identify the unknown functions α_i and β_{ij} using fuzzy systems (or neural networks) in order to indirectly approximate the ideal control law $\mathbf{U}^*$. Let the fuzzy system be a Takagi–Sugeno (T–S) form with centre average defuzzification as in Section 11.2. We rewrite

$$\alpha_i(\mathbf{X}) = z_{\alpha_i}^{\mathrm{T}} A_{\alpha_i}^* \zeta_{\alpha_i} + d_{\alpha_i}(\mathbf{X}) \tag{11.30}$$

$$\beta_{ij}(\mathbf{X}) = z_{\beta_{ij}}^{\mathrm{T}} A_{\beta_{ij}}^* \zeta_{\beta_{ij}} + d_{\beta_{ij}}(\mathbf{X}) \tag{11.31}$$

where $A_{\alpha_i}^* \in \Re^{m_{\alpha_i} \times p_{\alpha_i}}$ and $A_{\beta_{ij}}^* \in \Re^{m_{\beta_{ij}} \times p_{\beta_{ij}}}$ are defined by

$$A_{\alpha_i}^* := \arg \min_{A_{\alpha_i} \in \Omega_{\alpha_i}} \left[\sup_{\mathbf{X} \in S_x} |z_{\alpha_i}^{\mathrm{T}} A_{\alpha_i} \zeta_{\alpha_i} - \alpha_i| \right] \tag{11.32}$$

$$A_{\beta_{ij}}^* := \arg \min_{A_{\beta_{ij}} \in \Omega_{\beta_{ij}}} \left[\sup_{\mathbf{X} \in S_x} |z_{\beta_{ij}}^{\mathrm{T}} A_{\beta_{ij}} \zeta_{\beta_{ij}} - \beta_{ij}| \right] \tag{11.33}$$

Note that we are assuming the ability to specify fuzzy systems in such a way that the representation errors using optimal parameters (which arise because of the finite number of basis functions used) are bounded, i.e. $d_{\alpha_i}(\mathbf{X}) \leq D_{\alpha_i}(\mathbf{X})$, $d_{\beta_{ij}}(\mathbf{X}) \leq D_{\beta_{ij}}(\mathbf{X})$, where $D_{\alpha_i}(\mathbf{X})$ and $D_{\beta_{ij}}(\mathbf{X})$ are known bounding functions. We require the representation errors $D_{\beta_{ij}}(\mathbf{X})$ to be small (later we will provide an explicit condition as to how small they have to be), which means that the matrix $\mathbf{B}$ can, ideally, be well approximated by our fuzzy systems (or neural networks) using optimal parameters. Our adaptive controller's stability will not, however, depend on its ability to identify these optimal parameters.

The compact set $S_x \subseteq \Re^n$ is defined as before, and $\Omega_{\alpha_i}, \Omega_{\beta_{ij}}$ are compact sets within which the parameter matrices estimates, $A_{\alpha_i}(t)$ and $A_{\beta_{ij}}(t)$, are allowed to lie. Thus, the fuzzy system approximations of $\alpha_i(\mathbf{X})$ and $\beta_{ij}(\mathbf{X})$ are given by

$$\hat{\alpha}_i(\mathbf{X}) := z_{\alpha_i}^{\mathrm{T}} A_{\alpha_i} \zeta_{\alpha_i} \tag{11.34}$$

$$\hat{\beta}_{ij}(\mathbf{X}) := z_{\beta_{ij}}^{\mathrm{T}} A_{\beta_{ij}} \zeta_{\beta_{ij}} \tag{11.35}$$

The matrices $A_{\alpha_i}(t)$ and $A_{\beta_{ij}}(t)$ are to be adjusted adaptively on line in order to try to improve the approximation.

We define the output errors e_{o_i} and error signals e_{s_i} and $\bar{e}_{s_i}$ as in Section 11.2. Consider the control law $\mathbf{U} := \mathbf{U_{ce}}$, where $\mathbf{U_{ce}} = [u_{ce_1}, \ldots, u_{ce_p}]^{\mathrm{T}}$ is a *certainty equivalence* control term. Define the matrix $\hat{\mathbf{B}} := [\hat{\beta}_{ij}(\mathbf{X}) + \beta_{ij_k}(t)]$, $i,j = 1, \ldots, p$. $\hat{\mathbf{B}}$ is an approximation of the ideal and unknown matrix $\mathbf{B}$. Furthermore, let $[b_{ij}] := \hat{\mathbf{B}}^{-1}$, $i,j = 1, \ldots, p$ be a matrix of the elements of the inverse. We need to ensure that $\hat{\mathbf{B}}^{-1}$ exists for all $\mathbf{X} \in S_x$ and $t \geq 0$. If the sets $\Omega_{\beta_{ij}}$ are constructed such that

$$\sigma_p(\hat{\mathbf{B}}) \geq \sigma_{\min} \tag{11.36}$$

$$\sigma_1(\hat{\mathbf{B}}) \leq \sigma_{\max} \tag{11.37}$$

for all $\mathbf{X} \in S_x$, then, as long as the matrices $A_{\beta_{ij}}$ remain within $\Omega_{\beta_{ij}}$, respectively, we can guarantee that $\hat{\mathbf{B}}^{-1}$ exists (this can be achieved using a projection algorithm). Note that if we knew the matrix $\mathbf{B}$ to be, for instance, strictly diagonally dominant (as required by the Levy–Desplanques theorem (see [25] for an explanation of this and other invertibility results)) with known lower bounds for the main diagonal entries, we could relax the conditions on the sets $\Omega_{\beta_{ij}}$ by applying instead a projection algorithm that kept $\hat{\mathbf{B}}$ in a strictly diagonally dominant form similar to $\mathbf{B}$ to ensure it is invertible. In order to cancel the unknown parameter errors we use the following adaptive laws:

$$\begin{aligned} \dot{A}_{\alpha_i} &= -Q_{\alpha_i}^{-1} z_{\alpha_i} \zeta_{\alpha_i} e_{s_i} \\ \dot{A}_{\beta_{ij}} &= -Q_{\beta_{ij}}^{-1} z_{\beta_{ij}} \zeta_{\beta_{ij}} e_{s_i} u_{ce_j} \end{aligned} \tag{11.38}$$

with $Q_{\alpha_i} \in \Re^{m_{\alpha_i} \times m_{\alpha_i}}, Q_{\beta_{ij}} \in \Re^{m_{\beta_{ij}} \times m_{\beta_{ij}}}$ positive definite and diagonal.

Now we can write an expression for $\mathbf{U_{ce}}$

$$\mathbf{U_{ce}} := \hat{\mathbf{B}}^{-1}(-\hat{\mathbf{A}} + \boldsymbol{\nu}) \tag{11.39}$$

where $\hat{\mathbf{A}} := [\hat{\alpha}_1 + \alpha_{1_k}, \ldots, \hat{\alpha}_p + \alpha_{p_k}]^{\mathrm{T}}$. Here $\boldsymbol{\nu}(t) = [\nu_1(t), \ldots, \nu_p(t)]^{\mathrm{T}}$ is chosen to provide stable tracking and to allow for robustness to parameter uncertainty. Namely, let

$$\nu_i(t) := y_{m_i}^{(r_i)} + \eta_i e_{s_i} + \bar{e}_{s_i} + D_{\alpha_i}(\mathbf{X}) \operatorname{sgn}(e_{s_i}) + U_{\max}(\mathbf{X}) \operatorname{sgn}(e_{s_i}) \sum_{j=1}^{p} D_{\beta_{ij}}(\mathbf{X}) \tag{11.40}$$

where $\eta_i > 0$ is a design parameter, and $U_{\max}(\mathbf{X})$ is a function chosen so that $|u_{ce_i}| \leq U_{\max}(\mathbf{X}), i = 1, \ldots, p$. It can be shown that the choice

$$U_{\max}(\mathbf{X}) \geq \max_{i=1,\ldots,p} \left[\frac{a_i(\mathbf{X})}{1 - c_i(\mathbf{X})} \right] \tag{11.41}$$

satisfies the requirement, where $a_i(\mathbf{X}) := \sum_{j=1}^{p} |b_{ij}|[|\hat{\alpha}_j| + |\alpha_{j_k}| + |y_{m_j}^{(r_j)}| + \eta_j |e_{s_j}| + |\bar{e}_{s_j}| + D_{\alpha j}]$ and $c_i(\mathbf{X}) := \sum_{j=1}^{p} |b_{ij}| \sum_{l=1}^{p} D_{\beta_{jl}}$. It should be noted that for many classes of plants, each β_{ij} is a smooth function easily represented by a fuzzy system. For example, if each β_{ij} may be expressed as a constant, then $D_{\beta_{ij}} = 0$ for all i, j since a fuzzy system may exactly represent a constant on a compact set. This would remove the need for the $U_{\max}(\mathbf{X})$ term to be included in (11.40).

At this point we need to formalize our general assumption about the controller:

(C2) Indirect Adaptive Control Assumption
The fuzzy systems, or neural networks, that define the approximations (11.34) and (11.35), are defined so that $D_{\alpha_i}(\mathbf{X}) \in \mathcal{L}_\infty, D_{\beta_{ij}}(\mathbf{X}) \in \mathcal{L}_\infty$, for all $\mathbf{X} \in S_x \subseteq \Re^n, i,j = 1, \ldots, p$. Furthermore, the ideal representation errors $D_{\beta_{ij}}(\mathbf{X})$ are small enough, so that we have $0 \le c_i(\mathbf{X}) < 1$, for all $\mathbf{X} \in S_x, i = 1, \ldots, p$.

Notice that the maximum sizes of $D_{\beta_{ij}}$ that satisfy (C2) can be found, since from (11.36) we have $|b_{ij}(\mathbf{X})| \le \|\hat{\mathbf{B}}^{-1}\|_2 \le \frac{1}{\alpha_{\min}}$, and as long as assumption (C2) is satisfied, we ensure that $|u_{ce_i}| \le U_{\max}(\mathbf{X}), i = 1, \ldots, p$ as desired.

We have completely specified the signals that compose the control vector $\mathbf{U}$, and now we state our main result; its proof is omitted, but follows ideas used in Section 11.2 and [8].

Theorem 3.1 Stability and tracking results using MIMO indirect adaptive control

If the reference input assumption (R1) holds, the plant assumptions (P1) and (P2) hold, and the control law is defined by (11.39) with the control assumption (C2),

Then the following holds:

(1) The plant states, as well as its outputs and their derivatives, $y_i, \ldots, y_i{}^{(r_i-1)} i, i = 1, \ldots, p$, are bounded.
(2) The control signals are bounded, i.e. $u_{ce_i} \in \mathcal{L}_\infty, i = 1, \ldots, p$.
(3) The magnitudes of the output errors, e_{o_i}, decrease at least asymptotically to zero, i.e. $\lim_{t\to\infty} e_{o_i} = 0, i = 1, \ldots, p$.

11.4 Applications

11.4.1 Illustrative example

Consider the nonlinear differential equation given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2^2 + x_3 \\ x_1 + 2x_2 + 3x_3x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3u_1 + u_2 \\ u_1 + 2(2 + 0.5\sin(x_1))u_2 \end{bmatrix} \tag{11.42}$$

Notice that these are coupled nonlinear dynamics. The $\mathbf{B}$ matrix is not constant, but contains a bounded function of the states. We are interested in the outputs $y_1 = x_1$ and $y_2 = x_3$. It is easily verified that this system has a vector relative degree of $[2, 1]^{\mathrm{T}}$. We define the error equations as $e_{s_1} = e_{o_1} + \dot{e}_{o_1}$ and $e_{s_2} = e_{o_2}$, with the output errors defined appropriately. We want the outputs of the system to track the reference vector $[Y_{m_1}(s), Y_{m_2}(s)]^{\mathrm{T}} = \left[\frac{R_1(s)}{(s+1)^2}, \frac{R_2(s)}{s+1}\right]^{\mathrm{T}}$, where $R_1(s) = \mathcal{L}\{r_1(t)\}$ and $R_2(s) = \mathcal{L}\{r_2(t)\}$ ($\mathcal{L}\{\cdot\}$ is the Laplace transform operator). Thus, $\dot{e}_{o_1}$ is computed from $\dot{y}_{m_1}$ and x_2.

Table 11.1 *Rule base*

	$\mathbf{F}_1^1$	$\mathbf{F}_2^1$	$\mathbf{F}_3^1$	$\mathbf{F}_4^1$	$\mathbf{F}_5^1$	$\mathbf{F}_6^1$	$\mathbf{F}_7^1$	$\mathbf{F}_8^1$	$\mathbf{F}_9^1$
$\mathbf{F}_1^2$	c_1^k	c_1^k	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k
$\mathbf{F}_2^2$	c_1^k	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k
$\mathbf{F}_3^2$	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k
$\mathbf{F}_4^2$	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k
$\mathbf{F}_5^2$	c_1^k	c_2^k	$\boxed{c_3^k}$	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k
$\mathbf{F}_6^2$	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k
$\mathbf{F}_7^2$	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k
$\mathbf{F}_8^2$	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k	c_9^k
$\mathbf{F}_9^2$	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k	c_9^k	c_9^k

We use two T–S fuzzy systems to produce $\hat{u}_1$ and $\hat{u}_2$, and we set the 'known' controller terms $u_{k_i} = 0, i = 1, 2$. Both fuzzy systems have e_{o_1} and e_{o_2} as their inputs (so here $\mathbf{W} = [e_{o_1}, e_{o_2}]^{\mathrm{T}}$), and we let $z_k^{\mathrm{T}} = [1, x_1, x_2, x_3], k = 1, 2$. Both fuzzy systems have nine triangular membership functions for each of the two input universes of discourse, uniformly distributed over the interval $[-1, 1]$ with 50% overlap (we use scaling gains to normalize the inputs to this interval). We saturate the outermost membership functions, and the output is computed using centre average defuzzification. Both systems' coefficient matrices, A_1 and A_2, are initialized with zeroes, and they utilize the rule base shown in Table 11.1. The labels F_i^j denote the ith fuzzy set for the jth input, where $i = 1$ corresponds to the leftmost, and $i = 9$ to the rightmost fuzzy set. Each entry of the table corresponds to one output function $c_i^k, i = 1, \ldots, 9$, where $c_i^k = z_k^{\mathrm{T}} \underline{a}_i^k$, and $\underline{a}_i^k$ is the ith column of $A_k, k = 1, 2$. As an example, consider the rule for c_3^k that is inside a box in the table

$$\textbf{If } e_{o_1} \text{ is } F_3^1 \textbf{ and } e_{o_2} \text{ is } F_5^2 \textbf{ then } c_3^k = z_k^{\mathrm{T}} \underline{a}_3^k$$

where we evaluate the **and** operation using minimum. Note that $\mu_i, i = 1, \ldots, 9$, is the result of evaluating the *premise* of the ith rule.

From the plant's equation we choose the bounds $\bar{\beta}_{11} = 3.3, \underline{\beta}_{11} = 2.7, \bar{\beta}_{22} = 5.3, \underline{\beta}_{22} = 2.7, \underline{\beta}_{21} = 1.3, \underline{\beta}_{12} = 1.3, M_{11} = 0.0$, and $M_{22} = x_2$ (these two bounds are obtained by differentiating the diagonal entries of the matrix $\mathbf{B}$). Also, the fuzzy system approximation error bounds are chosen as $D_1 = 0.1, D_2 = 0.1$ (note that this choice is not readily apparent from the definitions of the fuzzy systems; rather, the bounds are found through a trial and error procedure). Since we know the vectors z_k^{T} and $\zeta_k^{\mathrm{T}}, k = 1, 2$, an easy way to compute the bounding functions $\bar{U}_1$ and $\bar{U}_2$ is to use fuzzy systems that have the same structure as the ones used for $\hat{u}_1$ and $\hat{u}_2$. Initially we chose the entries in their coefficient matrices to be large, so that they would bound the values taken by the $\hat{u}_1$ and $\hat{u}_2$ fuzzy systems. We found, however, that this created high amplitude and high frequency oscillations of the control signals (due to the term $\mathbf{U_d}$), which are undesirable. After some tuning we determined that setting the $\bar{U}_1$ and $\bar{U}_2$ fuzzy system's coefficients to 0.1 gave us stable behaviour, good tracking and much smoother control signals. Finally, the adaptation gains were chosen as $Q_{u_1} = Q_{u_2} = 2.4I$, where I is a 2×2 identity matrix.

In Figure 11.1 we observe the results for direct adaptive control on this system. We used a fourth order Runge–Kutta numerical approximation to the differential equation solution, with a step size of 0.001. The reference inputs $r_1(t)$ and $r_2(t)$ are chosen as square waves, and the corresponding reference model outputs are plotted in Figure 11.1(a) with dashed lines, but are hard to see since x_1 and x_3 track them closely. In Figure 11.1(b) we observe the applied control inputs.

11.4.2 Application to a two-link robot arm

Next, we consider direct adaptive control of a two-degree of freedom robot arm. This system does not satisfy assumption (P3) because, as we will see, the matrix multiplying the input vector **U** contains functions of the states (similar to the example in the previous section). However, in some regions the bounds for the entries do not satisfy the diagonal dominance condition. Nevertheless, our simulation results show that the method seems to be able to provide stable tracking with adequate performance; furthermore, the controller is able to compensate for an 'unknown' change in system parameters, which represents the situation where the robot picks up an object after some time of nominal operation (i.e. when the robot is not holding any object).

The robot arm consists of two links, the first one mounted on a rigid base by means of a frictionless hinge, and the second mounted at the end of link one by means of a frictionless ball bearing. The inputs to the system are the torques τ_1 and τ_2 applied at the joints. The outputs are the joint angles θ_1 and θ_2. A mathematical model of this system can be derived using Lagrangian equations, and is given by

$$\underbrace{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_2 & -h\dot{\theta}_1 - h\dot{\theta}_2 \\ h\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{11.43}$$

where

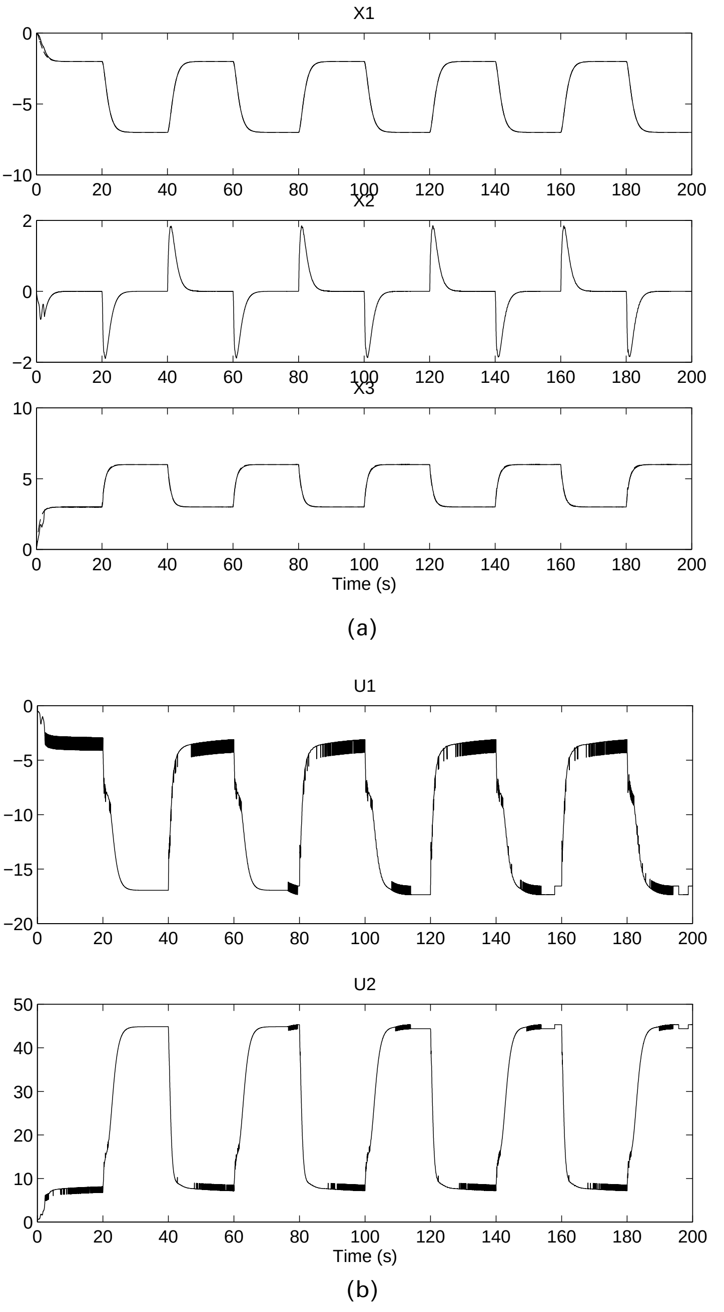

Figure 11.1 *(a) System states (solid lines) and reference model outputs (dashed lines). (b) Control inputs*

$$H_{11} = I_1 + I_2 + m_1 l_{c_1}^2 m_2[l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2}\cos(\theta_2)] + m_3(l_1^2 + l_2^2 + 2l_1 l_2 \cos(\theta_2))]$$

$$H_{22} = I_2 + m_2 l_{c_2}^2 + m_3 l_2^2$$

$$H_{12} = H_{21} = I_2 + m_2[l_{c_2}^2 + l_1 l_{c_2}\cos(\theta_2)] + m_3[l_2^2 + l_1 l_2 \cos(\theta_2)]$$

$$h = m_2 l_1 l_{c_2} \sin(\theta_2)$$

$$g_1 = m_1 l_{c_1} g \cos(\theta_1) + m_2 g[l_{c_2}\cos(\theta_1 + \theta_2) + l_1 \cos(\theta_1)]$$

$$g_2 = m_2 l_{c_2} g \cos(\theta_1 + \theta_2)$$

The matrix **H** can be shown to be positive definite, and therefore always invertible. In our simulation we use the following parameter values: $m_1 = 1.0\,\text{kg}$, mass of link one; $m_2 = 1.0\,\text{kg}$, mass of link two; $l_1 = 1.0\,\text{m}$, length of link one; $l_2 = 1.0\,\text{m}$, length of link two; $l_{c_1} = 0.5\,\text{m}$, distance from the joint of link one to its centre of gravity; $l_{c_2} = 0.5\,\text{m}$, distance from the joint of link two to its centre of gravity; $I_1 = 0.2\,\text{kg}\,\text{m}^2$, lengthwise centroidal inertia of link one; and $I_2 = 0.2\,\text{kg}\,\text{m}^2$, lengthwise centroidal inertia of link two. The mass m_3, initially set equal to zero, represents the mass of an object at the end of the link. After 100 seconds of operation, the robot 'picks up' an object of mass $m_3 = 3.0\,\text{kg}$.

We can rewrite the system dynamics in input output form

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \frac{1}{H_{11}H_{22} - H_{12}H_{21}} \left[\begin{bmatrix} H_{22} h \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) + H_{12} h \dot{\theta}_1^2 - H_{22} g_1 + H_{12} g_2 \\ -H_{21} h \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) - H_{11} h \dot{\theta}_1^2 + H_{21} g_1 - H_{11} g_2 \end{bmatrix} + \begin{bmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \right]$$

We observe that the input vector is multiplied by $\mathbf{H}^{-1}$, which contains the function $\cos(g_2)$. For some values of θ_2, $\mathbf{H}^{-1}$ is not diagonally dominant, and thus does not satisfy assumption (P3) (note, e.g., that for some θ_2 values, $H_{22} < |H_{12}|$). However, we found that not only was it possible to make the direct adaptive method work, but also that is offered relatively good performance and the ability to handle system parameter changes.

We would like the outputs θ_1 and θ_2 to track desired reference angles, which are obtained from the reference model vector $[Y_{m_1}(s), Y_{m_2}(s)]^{\mathrm{T}} = \left[\frac{0.75^2 R_1(s)}{(s+0.75)^2}, \frac{1.5^2 R_2(s)}{(s+1.5)^2}\right]^{\mathrm{T}}$, where $R_1(s) = \mathcal{L}\{r_1(t)\}$ and $R_2(s) = \mathcal{L}\{r_2(t)\}$. Clearly, the system has a vector relative degree $[2,2]^{\mathrm{T}}$, so, letting $e_{o_1} = y_{m_1} - \theta_1$ and $e_{o_2} = y_{m_2} - \theta_2$, we define the error equations $e_{s_1} = e_{o_1} + \dot{e}_{o_1}$ and $e_{s_2} = e_{o_2} + \dot{e}_{o_2}$. The error derivatives are available, since the reference inputs are twice differentiable, and the angle derivatives $\dot{\theta}_1$ and $\dot{\theta}_2$ are plant states.

When designing our fuzzy systems we assumed that there is no strong crosscoupling between the inputs and outputs, which greatly simplifies the design: we chose e_{o_1} and $\dot{e}_{o_1}$ as inputs to the fuzzy system for τ_1 (so for this fuzzy system $\mathbf{W} = [e_{o_1}, \dot{e}_{o_1}]^{\mathrm{T}}$), and e_{o_2} and $\dot{e}_{o_2}$ as inputs for τ_2. Both fuzzy systems have $z_i^{\mathrm{T}} = [1, \dot{\theta}_1, \ddot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_2]$, and are otherwise structurally identical to the systems defined in Section 11.4.1. Here, also, we initialized the coefficient matrices of both systems with zeroes. Since, as mentioned before, the system does not satisfy assumption (P3) for all θ_2, the way to choose the bounds for the $\mathbf{H}^{-1}$ matrix entries is not clear. In view of this we took a pragmatic approach, where we first picked bounds that were as close as possible to the real bounds, and yet satisfied assumption (P3): letting $\mathbf{B} = \mathbf{H}^{-1}$, substitution of the numerical values of the parameters shows that, taking into account both values of $m_3, 1.1 < \beta_{11} < 1.2, 2.3 < \beta_{12}, \beta_{21} < 2.5$, and $0.7 < \beta_{22} < 7.3$. Thus, we picked $\bar{\beta}_{11} = 1.3, \underline{\beta}_{11} = 1.1, \bar{\beta}_{22} = 2.3, \underline{\beta}_{22} = 2.5, \underline{\beta}_{21} = 2.3$, and $\underline{\beta}_{12} = 2.3$. This choice resulted in somewhat acceptable behaviour, but with highly oscillatory control signals. Therefore, we decided to tune these bounds, even though the theoretical assumptions were violated. We found that reducing the size of the bounds while meeting the diagonal dominance condition yielded satisfactory results: the magnitude of the control signals' oscillations was drastically reduced, and at the same time we obtained adequate tracking and apparent robustness to the plant parameter change we investigated. We finally chose the bounds as $\bar{\beta}_{11} = \underline{\beta}_{11} = 1.2, \bar{\beta}_{22} = \underline{\beta}_{22} = 1.2$, and $\underline{\beta}_{21} = \underline{\beta}_{12} = 0.3$. Differentation of the diagonal entries of **B** yields $\bar{M}_{11} = 0.0$, and $M_{22} = 3.1\dot{\theta}_2$. We picked the fuzzy system approximation error bounds as $D_1 = 0.1, D_2 = 0.1$ (again this is the result of a tuning process). The bounding functions $\bar{U}_1$ and $\bar{U}_2$ were picked in a way similar to Section 11.4.1, where the matrix coefficients are first chosen large, and then decreased until adequate performance is achieved; setting the coefficients to 0.1 gave us the best results. Finally, the adaptation gains were set to $Q_{u_1} = Q_{u_2} = 4.71I$, where I is a 2×2 identity matrix.

We observe the control results on Figure 11.2. We let $r_1(t)$ and $r_2(t)$ be square waves. Initially the controller has some difficulties, and tracking is not perfect: at this point, the T–S coefficient matrices are moving away from zero and possibly adapting towards values that allow for better tracking. After the first period of the square wave reference inputs we note that tracking improves significantly. At time $t = 100$ seconds, when the system dynamics change as the robot 'picks up' an object, we find the outputs exhibit virtually no transient overshoot, and tracking continues to be adequate. However, at this point we observe a high peak in τ_2, as the controller tries to compensate for the increase in the load of link two. The peaks recur at the transition points, where the references step up or down, but they tend to decrease in magnitude (we let the simulation run for 12 000 seconds, and found this pattern to hold).

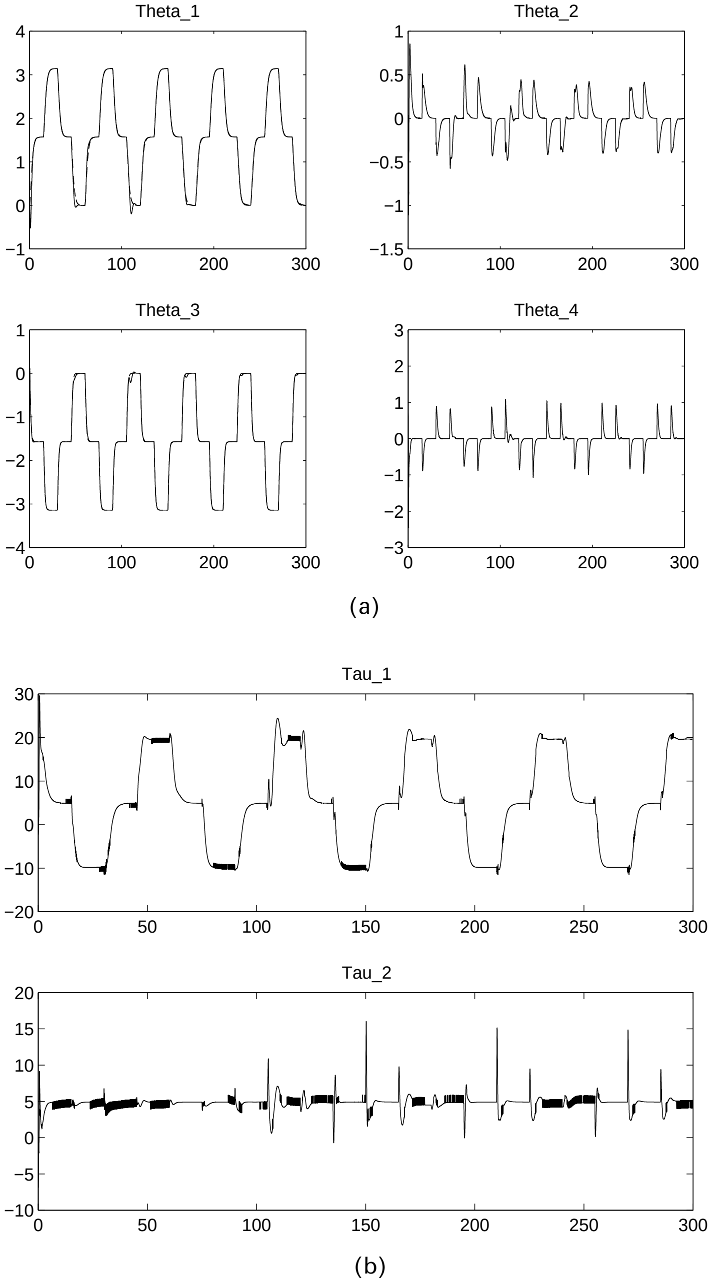

Figure 11.2 *(a) System states (solid lines) and reference model outputs (dashed lines). (b) Control inputs*

11.5 Conclusions

In this chapter we have developed direct and indirect adaptive MIMO control schemes which use Takagi–Sugeno fuzzy systems or a class of neural networks. We have proven stability of the methods and shown that they guarantee asymptotic convergence of the tracking errors to zero, as well as boundedness of all the signals and parameter errors, regardless of any initialization constraints. Both methods allow for the inclusion of previous knowledge or expertise in form of linguistics regarding what the control input should be, in the direct case, or what the plant dynamics are, in the indirect case. We show that with or without such knowledge the stability and tracking properties of the controllers hold, and present two simulations for direct adaptive control that illustrate the method.

References

[1] Procyk, T. and Mamdani, E. (1979). 'A Linguistic Self-organizing Process Controller' *Automatica*, **15**(1), 15–30.
[2] Driankov, D., Hellendoorn, H. and Reinfrank, M. M. (1993). *An Introduction to Fuzzy Control*. Springer-Verlag, Berlin Heidelberg.
[3] Layne, J. R., Passino K. M. and Yurkovich, S. (1993). 'Fuzzy Learning Control for Anti-skid Braking Systems', *IEEE Trans. Control Systems Tech.*, **1**(2) 122–129, June.
[4] Layne, J. R., and Passino, K. M. (1993). 'Fuzzy Model Reference Learning Control for Cargo Ship Steering', *IEEE Control Systems Magazine*, **13**(6), 23–34, Dec.
[5] Kwong, W. A., Passino, K. M., Lauknonen, E. G. and Yurkovich, S. (1995). 'Expert Supervision of Fuzzy Learning Systems for Fault Tolerant Aircraft Control', *Proc. of the IEEE, Special Issue on Fuzzy Logic in Engineering Applications*, **83**(3) 466–483, March.
[6] Moudgal, V. G., Kwong, W. A., Passino K. M. and Yurkovich, S. (1995). 'Fuzzy Learning Control for a Flexible-link Robot', *IEEE Transactions on Fuzzy Systems*, **3**(2), 199–210, May.
[7] Kwong, W. A. and Passino, K. M. (1996). 'Dynamically Focused Fuzzy Learning Control', *IEEE Trans. on Systems, Man, and Cybernetics*, **26**(1) 53–74, Feb.
[8] Spooner, J. T. and Passino, K. M. (1996). 'Stable Adaptive Control using Fuzzy Systems and Neural Networks', *IEEE Transactions in Fuzzy Systems*, **4**(3), 339–359, August.
[9] Wang, Li-Xin. (1994). *Adaptive Fuzzy Systems and Control: Design and Stability Analysis*. Prentice-Hall, Englewood Cliffs, NJ.
[10] Wang, Li-Xin. (1992). 'Stable Adaptive Fuzzy Control of Nonlinear Systems', in *Proc. of 31st Conf. Decision Contr.*, 2511–2516, Tucson, Arizona.
[11] Su, Chun-Yi. and Stepanenko, Y. (1994). 'Adaptive Control of a Class of Nonlinear Systems with Fuzzy Logic', *IEEE Trans. Fuzzy Systems*, **2**(4), 285–294, November.

[12] Johansen, T. A. (1994). 'Fuzzy Model Based Control: Stability, Robustness, and Performance Issues', *IEEE Trans. Fuzzy Systems*, **2**(3), 221–234, August.

[13] Nerendra, K. S. and Parthasarathy, K. (1990). 'Identification and Control of Dynamical Systems using Neural Networks', *IEEE Trans. Neural Networks*, **1**(1), 4–27.

[14] Polycarpou, M. M. and Ioannou, P. A. (1991). 'Identification and Control of Nonlinear Systems Using Neural Network Models: Design and Stability Analysis. Electrical Engineering – Systems Report 91-09-01, University of Southern California, September.

[15] Sanner, R. M. and Jean-Jacques E. Slotine (1992). 'Gaussian Networks for Direct Adaptive Control', *IEEE Trans. Neural Networks*, **3**(6), 837–863.

[16] Yeşildirek, A. and Lewis, F. L. (1994). 'A Neural Network Controller for Feedback Linearization', in *Proc. of 33rd Conf. Decision Contr.*, 2494–2499, Lake Buena Vista, FL, December.

[17] Chen, F.-C. and Khalil, H. K. (1992). Adaptive Control of Nonlinear Systems Using Neural Networks' *Int. J. Control*, **55**(6), 1299–1317.

[18] Rovithakis, G. A. and Christodoulou, M. A. (1994). 'Adaptive Control of Unknown Plants Using Dynamical Neural Networks', *IEEE Trans. Syst. Man, Cybern.*, **24**(3), 400–412, March.

[19] Liu, Chen-Chung and Chen, Fu-Chuang (1993). 'Adaptive Control of Non-linear Continuous-time Systems using Neural Networks – General Relative Degree and MIMO Cases', *International Journal of Control*, **58**(2), 317–355.

[20] Chen, Fu-Chuang and Khalil, H. K. (1995). 'Adaptive Control of a Class of Nonlinear Systems using Neural Networks', in *34th IEEE Conference on Decision and Control Proceedings*, New Orleans, LA, 2427–2432.

[21] Ordonez, R. E. and Passino, K. M. (1997). 'Stable Multi-input Multi-output Direct Adoptive Fuzzy Control', in *Proceedings of the American Control Conference*, 1271–1272, Albuquerque, NM, June.

[22] Ordonez, R. E. Spooner, J. T. and Passino, K. M. (1996). 'Stable Multi-input Multi-output Adaptive Fuzzy Control', In *IEEE Conference on Decision and Control*, 610–615, Kobe, Japan, September.

[23] Shankar Sastry, S. and Bodson, M. (1989). *Adaptive Control: Stability, Convergence, and Robustness*, Prentice-Hall, Englewood Cliffs, New Jersey.

[24] Shankar Sastry, S. and Isidori, A. (1989). 'Adaptive Control of Linearizable Systems', *IEEE Transac. Autom. Contr.*, **34**(11), 1123–1131, November.

[25] Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press, Cambridge (Cambridgeshire); New York.

12

Adaptive robust control scheme with an application to PM synchronous motors

J.-X. Xu, Q.-W. Jia and T.-H. Lee

Abstract

This chapter presents a new adaptive robust control scheme for a class of nonlinear uncertain dynamical systems. To reduce the robust control gain and widen the application scope of adaptive techniques, the system uncertainties are classified into two different categories: the structured and non-structured uncertainties. The structured uncertainty can be separated and expressed as the product of known functions of states and a set of unknown constants. The upper bounding functions of the non structured uncertainties to be addressed in this chapter is only partially known with unknown parameters. Moreover, the bounding function is convex to the set of unknown parameters, i.e. the bounding function is no longer linear in parameters. The structured uncertainty is estimated with adaptation and compensated. Meanwhile, the adaptive robust method is applied to deal with the non structured uncertainty by estimating unknown parameters in the upper bounding function. The μ-modification scheme [1] is used to cease parameter adaptation in accordance with the adaptive robust control law. The backstepping method [2] is also adopted in this chapter to deal with a system not in the *parametric–pure feedback* form, which is usually necessary for the application of backstepping control scheme. The new control scheme guarantees the uniform boundedness of the system and at the same time, the tracking error enters an arbitrarily designated zone in a finite time. The effectiveness of the proposed method is demonstrated by the application to PM synchronous motors.

12.1 Introduction

Numerous adaptive robust control algorithms for systems containing uncertainties have been developed [1]–[11]. In [3] variable structure control with an adaptive law is developed for an uncertain input–output linearizable nonlinear system, where linearity-in-parameter condition for uncertainties is assumed. The unknown gain of the upper bounding function is estimated and updated by adaptation law so that the sliding condition can be met and the error state reaches the sliding surface and stays on it. To deal with a class of nonlinear systems with partially known uncertainties, in [4] an adaptive law using a dead zone and a hysteresis function is proposed to guarantee both the uniform boundedness of all the closed loop signals and uniform ultimate boundedness of the system states. In both control schemes, it is assumed that the system uncertainties are bounded by a bounding function which is a product of a set of known functions and unknown positive constants. The objective of adaptation is to estimate these unknown constants.

In [1], a new adaptive robust control scheme is developed for a class of nonlinear uncertain systems with both parameter uncertainties and exogenous disturbances. Including the categories of system uncertainties in [3] and [4] as its subsets, it is assumed that the disturbances are bounded by a known upper bounding function. Furthermore, the input distribution matrix is assumed to be constant but unknown.

In this chapter we proposed a continuous adaptive robust control scheme which is the extension of [1] in the sense that more general classes of nonlinear uncertain dynamical systems are under consideration. The unknown input distribution matrix of the system input can be state dependent here instead of being a constant matrix in [1]. To reduce the robust control gain and widen the application scope of adaptive techniques, the system uncertainties are supposed to be composed of two different categories: the first can be separated and expressed as the product of known function of states and a set of unknown constants, and the other category is not separable but with partially known bounding functions. It is further assumed that the bounding function is convex to the set of unknown parameters, i.e. the bounding function is no longer linear in parameters. The first category of uncertainties is dealt with by means of the well-used adaptive control method. Meanwhile an adaptive robust method is applied to deal with the second category of uncertainties, where the unknown parameters in the upper bounding function are estimated with adaptation. It should also be noted that the backstepping method [2] is adopted in this chapter to deal with a system not in the *parametric–pure feedback* form, which is usually necessary for the application of a backstepping control scheme.

The proposed method is further applied to a permanent magnet synchronous (PMS) motor, which is a typical nonlinear control system. The dynamics of the PM synchronous motor can be presented by a dynamic electrical subsystem and a mechanical subsystem, which are nonlinear differential equations. Strictly speaking, most control methods for permanent magnet synchronous motors are only locally stable because the d-axis current is assumed to be zero and the design procedure is based on the reduced model. In this chapter, instead of only zeroing d-axis current, the extra d-axis control input voltage is used to deal with the nonlinear coupling part of the dynamics as well.

This chapter is organized as follows. Section 12.2 describes the class of nonlinear uncertain systems to be controlled. Section 12.3 gives the design procedure of the adaptive robust control and the stability analysis. Section 12.4 describes the application of the proposed control method to the PM synchronous motors.

12.2 Problem formulation

Consider a class of uncertain dynamical system described by

$$\dot{\mathbf{x}}_0 = \mathbf{f}_0(t) + B_0(\mathbf{p})[\mathbf{g}_0(\mathbf{p}, t) + \mathbf{x}_1 + \triangle\mathbf{g}_0(\mathbf{x}, \mathbf{p}, \omega, t)\mathbf{x}_2] \tag{12.1}$$

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}, t) + B_1(\mathbf{p})\{[I + E_1(\mathbf{x}, \mathbf{p}, t)]\mathbf{u}_1(t) + \mathbf{g}_1(\mathbf{x}, \mathbf{p}, t) + \triangle\mathbf{g}_1(\mathbf{x}, \mathbf{p}, \omega, t)\} \tag{12.2}$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}, t) + B_2(\mathbf{p})\{[I + E_2(\mathbf{x}, \mathbf{p}, t)]\mathbf{u}_2(t) + \mathbf{g}_2(\mathbf{x}, \mathbf{p}, t) + \triangle\mathbf{g}_2(\mathbf{x}, \mathbf{p}, \omega, t)\} \tag{12.3}$$

where $\mathbf{x}_i = [x_{i1}, x_{i2}, \ldots, x_{in_i}]^\top \in \mathcal{R}^{n_i}$, $i = 0, 1, 2$, are the measurable state vectors of the system, where $n_0 = n_1$ and $n_0 + n_1 + n_2 = n$; $\mathbf{x} \in \mathcal{R}^n$ is defined as $\mathbf{x} = [\mathbf{x}_0^\top, \mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top$; $\mathbf{u}_i = [u_{i1}, u_{i2}, \ldots, u_{in_i}]^\top \in \mathcal{R}^{n_i}$, $i = 1, 2$, are the control inputs of the system; $\mathbf{p} \in \mathcal{P}$ is an unknown system parameter vector and $\mathcal{P}$ is the set of admissible system parameters; $\mathbf{f}_i \in \mathcal{R}^{n_i}$, $i = 0, 1, 2$, are nonlinear function vectors; $\mathbf{g}_i \in \mathcal{R}^{n_i}$, $i = 0, 1, 2$, and $\triangle\mathbf{g}_0 \in \mathcal{R}^{n_0 \times n_2}$, $\triangle\mathbf{g}_i \in \mathcal{R}^{n_i}$, $i = 1, 2$, are nonlinear uncertain function vectors of the state $\mathbf{x}$, unknown parameter $\mathbf{p}$, time t as well as a set of random variables ω. Here we make the following assumptions:

(A1) $\mathbf{f}_0(t)$, $\mathbf{f}_1(\mathbf{x}, t)$ and $\mathbf{f}_2(\mathbf{x}, t)$ are known nonlinear function vectors. The matrices $B_i(\mathbf{p})$, $i = 0, 1, 2$, are unknown but positive definite.

(A2) For $E_i \in \mathcal{R}^{n_i \times n_i}$, $i = 1, 2$

$$\forall t \in [0, \infty) \quad \forall \mathbf{x} \in \mathcal{D} \quad \forall \mathbf{p} \in \mathcal{P}$$
$$r_{max_i} \geq \lambda(\tfrac{1}{2}E_i + \tfrac{1}{2}E_i^\top) \geq r_{min_i} > -1 \tag{12.4}$$

where $\lambda(\cdot)$ indicates the eigenvalues of '$\cdot$'.

(A3) The structured uncertainty $\mathbf{g}_i \in \mathcal{R}^{n_i}$, $i = 0, 1, 2$, are nonlinear function vectors which can be expressed as

$$\mathbf{g}_0(\mathbf{p}, t) = \Theta_0(\mathbf{p})\xi_0(t)$$
$$\mathbf{g}_1(\mathbf{x}, \mathbf{p}, t) = \Theta_1'(\mathbf{p})\xi_1'(\mathbf{x}, t)$$
$$\mathbf{g}_2(\mathbf{x}, \mathbf{p}, t) = \Theta_2(\mathbf{p})\xi_2(\mathbf{x}, t)$$
$$\Theta_i = \text{diag}\,(\Theta_{i1}^\top, \dots, \Theta_{in_i}^\top)$$
$$\xi_i = [\xi_{i1}, \xi_{i2}, \dots, \xi_{in_i}]^\top, \quad i = 0, 1, 2 \tag{12.5}$$

where Θ_0, Θ_1' and Θ_2 are unknown parameter matrices and ξ_0, ξ_1' and ξ_2 are known function vectors. The nonstructured uncertainty $\triangle\mathbf{g}_i(\mathbf{x}, \mathbf{p}, \omega, t)$, $i = 0, 1, 2$, are bounded such that

$$\forall t \in \mathcal{R}^+ \quad \forall \mathbf{x} \in \mathcal{D} \quad \forall p \in \mathcal{P}$$
$$||\triangle\mathbf{g}_i(\mathbf{x}, \mathbf{p}, \omega, t)|| \le \rho_{d_i}(\mathbf{x}, \mathbf{q}_i, t) \tag{12.6}$$

where $||\cdot||$ represents the Euclidean norm for vectors and the spectral norm for matrices; $\mathcal{D}$ is a compact subset of $\mathcal{R}^n$ in which the solution of (12.1)–(12.3) uniquely exists with respect to the given desired state trajectory $\mathbf{x}_d(t)$. $\rho_{d_i}(\mathbf{x}, \mathbf{q}_i, t)$, $i = 0, 1, 2$, are upper bounding functions with unknown parameter vectors $\mathbf{q}_i \in \mathcal{P}$. Here $\rho_{d_i}(\mathbf{x}, \mathbf{q}_i, t)$ is differentiable and convex to $\mathbf{q}_i$, that is

$$\rho_{d_i}(\mathbf{x}, \mathbf{q}_{i2}, t) - \rho_d(\mathbf{x}, \mathbf{q}_{i1}, t) \le (\mathbf{q}_{i2} - \mathbf{q}_{i1})^\top \frac{\partial \rho_d}{\partial \mathbf{q}_i}\bigg|_{\mathbf{q}_{i1}} \tag{12.7}$$

The control objective is to find suitable control inputs $\mathbf{u}_1$ and $\mathbf{u}_2$ for the state $\mathbf{x}_0$ to track the desired trajectory $\mathbf{x}_d(t) \in \mathcal{R}^{n_0}$, where $\mathbf{x}_d$ is continuously differentiable.

Remark 2.1 The sub system (12.1) has $\mathbf{x}_1$ as its input. However, it is not in the *parametric–pure feedback* form due to the existence of the nonlinear uncertain term $\triangle\mathbf{g}_0(\mathbf{x}, \mathbf{p}, \omega, t)\mathbf{x}_2$. Thus the well-used backstepping design needs to be revised to deal with the dynamical system (12.1)–(12.3).

Remark 2.2 It should be noted that $\mathbf{g}_i(\mathbf{x}, \mathbf{p}, t)$ can be absorbed into $\triangle\mathbf{g}_i(\mathbf{x}, \mathbf{p}, \omega, t)$. However, it is obviously more conservative. This can be clearly shown through the following example. Assume that the structured uncertainty is $\mathbf{g} = \Theta_1\xi_1 + \Theta_2\xi_2$ with actual values $\Theta_1 = a$, $\Theta_2 = -a$ and a is an unknown constant. Assume that the nonlinear function $\xi_2 = \xi_1 + \triangle\xi$, where $||\triangle\xi|| \ll ||\xi_1||$. Then $||\mathbf{g}|| \le ||\Theta_1|| \cdot ||\xi_1|| + ||\Theta_2|| \cdot ||\xi_2|| = \gamma^\top|\xi|_1$, where $\gamma = [||\Theta_1||, ||\Theta_2||]^\top = [||a||, ||a||]^\top$ and $|\xi|_1 = [||\xi_1||, ||\xi_2||]^\top$. The upperbound parameter to be estimated is $\gamma = [||a||, ||a||]^\top$. This implies that the actual uncertainty $\mathbf{g} = a\triangle\xi$ has been amplified to its normed product $||a|| \cdot ||\xi_1|| + ||a|| \cdot ||\xi_1 + \triangle\xi||$, which is obviously much larger than $||a|| \cdot ||\triangle\xi||$ even if the estimates converge to the true values. On the contrary, if the uncertainty is expressed by (12.5), the unknown parameters to be estimated is $[a, -a]^\top$. This means that, when the estimated parameters are near the true values, the estimated uncertainty of $\mathbf{g}$ will be able to approach the actual uncertainty $a\triangle\xi$.

12.3 Adaptive robust control with μ-modification

The adaptive robust technique is combined with backstepping method in this section to develop a controller which guarantees the global boundedness of the system. The design procedures are presented in detail as follows.

Define the measured state tracking error vector as

$$\mathbf{e}_0 = \mathbf{x}_0 - \mathbf{x}_d \tag{12.8}$$

and the parameter matrices as

$$\Phi_i \triangleq B_i^{-1}, \quad i = 0, 1, 2 \tag{12.9}$$

We further denote $\mathbf{z}_0 = \mathbf{e}_0$, $\mathbf{z}_1 = \mathbf{x}_1 - \mathbf{x}_1^{\text{ref}}$, $\mathbf{z}_2 = \mathbf{x}_2$, $\mathbf{z} = [\mathbf{z}_0^\top, \mathbf{z}_1^\top, \mathbf{z}_2^\top]^\top$, and the auxiliary control $\mathbf{x}_1^{\text{ref}}$ is defined as

$$\mathbf{x}_1^{\text{ref}} = -K_0\mathbf{z_0} - \hat{\Theta}_0\xi_0 - \hat{\Phi}_0(\mathbf{f}_0 - \dot{\mathbf{x}}_{d_0}) \tag{12.10}$$

where K_0 is a gain matrix. $\hat{\Theta}_0$ and $\hat{\Phi}_0$ are the estimates of Θ_0 and Φ_0 respectively. The first order derivative of $\mathbf{x}_1^{\text{ref}}$ is derived as follows:

$$\begin{aligned}\dot{\mathbf{x}}_1^{\text{ref}} &= -K_0\dot{\mathbf{z}}_0 - \dot{\hat{\Theta}}_0\xi_0 - \hat{\Theta}_0\dot{\xi}_0 - \dot{\hat{\Phi}}_0(\mathbf{f}_0 - \dot{\mathbf{x}}_{d_0}) - \hat{\Phi}_0(\dot{\mathbf{f}}_0 - \ddot{\mathbf{x}}_{d_0}) \\ &= -\mathbf{f}_1'' - \Theta_0'\xi_0' \end{aligned} \tag{12.11}$$

where

$$\begin{aligned}\mathbf{f}_1'' &= K_0\mathbf{f}_0' + \dot{\hat{\Theta}}_0\xi_0 + \hat{\Theta}_0\dot{\xi}_0 + \dot{\hat{\Phi}}_0(\mathbf{f}_0 - \dot{\mathbf{x}}_d) + \hat{\Phi}_0(\dot{\mathbf{f}}_0 - \ddot{\mathbf{x}}_d) \\ \mathbf{f}_0'' &= \mathbf{f}_0 - \dot{\mathbf{x}}_d \\ \Theta_0' &= [K_0B_0\Theta_0, K_0B_0] \\ \xi_0' &= [\xi_0, \mathbf{z}_1 + \mathbf{x}_1^{\text{ref}}]^\top \end{aligned} \tag{12.12}$$

Then the plant (12.1)–(12.3) can be rearranged as follows:

$$\dot{\mathbf{z}}_0 = \mathbf{f}_0'(t) + B_0(\mathbf{p})[\mathbf{g}_0(\mathbf{p}, t) + \triangle\mathbf{g}_0(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, \omega, t)\mathbf{z}_2 + \mathbf{z}_1 + \mathbf{x}_1^{\text{ref}}] \tag{12.13}$$

$$\begin{aligned}\dot{\mathbf{z}}_1 = {} & \mathbf{f}_1'(\mathbf{z}, \mathbf{x}_1^{\text{ref}}, t) + B_1(\mathbf{p})\{[I + E_1(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, t)]\mathbf{u}_1(t) + \mathbf{g}_1'(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, t) \\ & + \triangle\mathbf{g}_1(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, \omega, t)\}\end{aligned} \tag{12.14}$$

$$\begin{aligned}\dot{\mathbf{z}}_2 = {} & \mathbf{f}_2(\mathbf{z}, \mathbf{x}_1^{\text{ref}}, t) + B_2(\mathbf{p})\{[I + E_2(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, t)]\mathbf{u}_2(t) + \mathbf{g}_2(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, t) \\ & + \triangle\mathbf{g}_2(\mathbf{z}, \mathbf{p}, \mathbf{x}_1^{\text{ref}}, \omega, t)\}\end{aligned} \tag{12.15}$$

where

$$\begin{aligned}
\mathbf{f}'_1 &= \mathbf{f}_1 + \mathbf{f}''_1 \\
\mathbf{g}'_1 &= \Theta_1 \xi_1 \\
\Theta_1 &= [\Theta'_1, B_1^{-1}\Theta'_0] \\
\xi_1 &= [\xi'_1, \xi'_0]^\top
\end{aligned} \tag{12.16}$$

The adaptive robust control law. Define the parameter error matrices as

$$\tilde{\mathbf{q}}_i = \mathbf{q}_i - \hat{\mathbf{q}}_i, \tag{12.17}$$

$$\tilde{\Theta}_i = \Theta_i - \hat{\Theta}_i, \tag{12.18}$$

$$\tilde{\Phi}_i = \Phi_i - \hat{\Phi}_i, \quad i = 0, 1, 2 \tag{12.19}$$

where $\hat{\mathbf{q}}_i, \hat{\Theta}_i, \hat{\Phi}_i$ are the estimates of $\mathbf{q}_i, \Theta_i, \Phi_i$, $i = 0, 1, 2$, respectively.

The control law $\mathbf{u}_i$, $i = 1, 2$, are chosen to be

$$\mathbf{u}_i = \mathbf{u}_{c_i} + \mathbf{u}_{v_i} \tag{12.20}$$

$$\begin{aligned}
\mathbf{u}_{c_i} &= -K_i \mathbf{z}_i + (i-2)\mathbf{z}_0 - \hat{\Theta}_i \xi_i - \hat{\Phi}_i \mathbf{f}'_i - \mathbf{v}_{d_i} - (i-1)\mathbf{v}_{d_0} \\
\mathbf{u}_{v_i} &= -\frac{r^2_{\max_i} ||\mathbf{u}_{c_i}||^2 \mathbf{z}_i}{(1 + r_{\min_i})(r_{\max_i} ||\mathbf{z}_i^\top \mathbf{u}_{c_i}|| + \varepsilon_{v_i}} \\
\mathbf{v}_{d_i} &= \frac{\hat{\rho}^2_{d_i}}{\hat{\rho}_{d_i} ||\mathbf{z}_i|| + \varepsilon_{d_i}} \mathbf{z}_i \\
\mathbf{v}_{d_0} &= \frac{\hat{\rho}^2_{d_0} ||\mathbf{z}_0||^2}{\hat{\rho}_{d_0} ||\mathbf{z}_0|| ||\mathbf{z}_2|| + \varepsilon_{d_0}} \mathbf{z}_2
\end{aligned} \tag{12.21}$$

where $K_i \in \mathcal{R}^{n_i \times n_i}$, $i = 1, 2$, is a gain matrix; ε_{v_i}, $i = 1, 2$, and ε_{d_i}, $i = 0, 1, 2$, are positive constants; $\mathbf{f}'_2 \triangleq \mathbf{f}_2$; $\hat{\rho}_{d_i} \triangleq \rho_{d_i}(\mathbf{z}, \hat{\mathbf{q}}_i, \mathbf{x}_1^{\text{ref}}, t)$; The corresponding adaptive laws are defined as

$$\begin{aligned}
\dot{\hat{\Theta}}_i &= \Gamma_{i1}(\mathbf{z}_i \xi_i^\top - \mu_{i1} \hat{\Theta}_i) \\
\dot{\hat{\Phi}}_i &= \Gamma_{i2}[\mathbf{z}_i \mathbf{f}_i'^\top - \mu_{i2} \hat{\Phi}] \\
\dot{\hat{\mathbf{q}}}_i &= \Gamma_{i3}(||\mathbf{z}_i|| \frac{\partial \rho_{d_i}}{\partial \mathbf{q}_i}\Big|_{\hat{\mathbf{q}}_i} - \mu_{i3} \hat{\mathbf{q}}_i), \quad i = 0, 1, 2
\end{aligned} \tag{12.22}$$

where Γ_{ij}, $j = 1, 2, 3$ are positive definite matrices chosen to be

$$\Gamma_{ij} = \text{diag}\,(\gamma^1_{ij}, \gamma^2_{ij}, \ldots, \gamma^{n_i}_{ij}) \tag{12.23}$$

μ_{ij}, $j = 1, 2, 3$, which constitute the μ-modification scheme, are defined as

$$\mu_{ij} = \begin{cases} k_{ij}(\varepsilon_0 - ||\mathbf{z}||) & \mathbf{z} \in E_0 \\ 0 & \text{elsewhere} \end{cases} \tag{12.24}$$

where k_{ij}, $i = 0, 1, 2$, $j = 1, 2, 3$, are positive constants.

$$E_0 \triangleq \{\mathbf{e} : ||\mathbf{e}|| < \varepsilon_0\} \tag{12.25}$$

where ε_0 is a positive constant specifying the desired tracking error bound.

Convergence analysis. For the above adaptive robust controller, we have the following theorem.

Theorem 3.1 By properly choosing the control gain matrix, the proposed adaptive robust control law (12.20)–(12.24) ensures that the system trajectory enters the set E_0 in a finite time. Moreover, the tracking errors as well as the parameter estimation errors are bounded by the set

$$D = \Bigg\{\mathbf{z}, \tilde{\Theta}_i, \tilde{\Phi}_i, \tilde{\mathbf{q}}_i, \quad i = 0, 1, 2 : \quad \mathbf{z}^\top\mathbf{z} + \sum_{i=0}^{2}(\text{trace}\,\{\tilde{\Theta}_i^\top\tilde{\Theta}_i\} + \text{trace}\,\{\tilde{\Phi}_i^\top\tilde{\Phi}_i\} + \tilde{\mathbf{q}}_i^\top\tilde{\mathbf{q}}_i)$$
$$< \tfrac{1}{2}k'\left[\sum_{i=0}^{2}(k_{i1}\varepsilon_0\,\text{trace}\,\{\Theta_i^\top\Theta_i\} + k_{i2}\varepsilon_0\,\text{trace}\,\{\Phi_i^\top\Phi_i\} + k_{i3}\varepsilon_0\mathbf{q}_i^\top\mathbf{q}_i) + 2\varepsilon\right]\Bigg\} \tag{12.26}$$

where k' is defined to be

$$k' = \max\,\{k'_{ij}, \quad i = 0, 1, 2; \quad j = 1, 2, 3\}$$
$$k'_{ij} = \frac{1}{k''_{ij}\,\min\,\{\lambda_{\min}(B_i^{-1}), \lambda_{\min}(\Gamma_{ij}^{-1})\}}$$
$$k''_{ij} = \frac{2\,\min\{\lambda_{\min}(K), k_{ij}\delta\}}{\max\,\{\lambda_{\max}(B_i^{-1}), \lambda_{\max}(\Gamma_{ij}^{-1})\}}$$

$\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ indicate the maximum and minimum eigenvalues of the matrix A respectively, and ε and δ are positive values to be defined later.

Proof The following positive definite function V is selected

$$V = V_1 + V_2 + V_2 \tag{12.27}$$

where

$$V_1 = \tfrac{1}{2}\mathbf{z}_0^\top B_0^{-1}\mathbf{z}_0 + \tfrac{1}{2}\text{trace}\,\{\tilde{\Theta}_0^\top\Gamma_{01}^{-1}\tilde{\Theta}_0\} + \tfrac{1}{2}\text{trace}\,\{\tilde{\Phi}_0^\top\Gamma_{02}^{-1}\tilde{\Phi}_0\} + \tfrac{1}{2}\tilde{\mathbf{q}}_0^\top\Gamma_{03}^{-1}\tilde{\mathbf{q}}_0, \tag{12.28}$$

$$V_2 = \tfrac{1}{2}\mathbf{z}_1^\top B_1^{-1}\mathbf{z}_1 + \tfrac{1}{2}\text{trace}\,\{\tilde{\Theta}_1^\top\Gamma_{11}^{-1}\tilde{\Theta}_1\} + \tfrac{1}{2}\text{trace}\,\{\tilde{\Phi}_1^\top\Gamma_{12}^{-1}\tilde{\Phi}_1\} + \tfrac{1}{2}\tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\tilde{\mathbf{q}}_1, \tag{12.29}$$

$$V_3 = \tfrac{1}{2}\mathbf{z}_2^\top B_2^{-1}\mathbf{z}_2 + \tfrac{1}{2}\text{trace}\,\{\tilde{\Theta}_2^\top\Gamma_{21}^{-1}\tilde{\Theta}_2\} + \tfrac{1}{2}\text{trace}\,\{\tilde{\Phi}_2^\top\Gamma_{22}^{-1}\tilde{\Phi}_2\} + \tfrac{1}{2}\tilde{\mathbf{q}}_2^\top\Gamma_{23}^{-1}\tilde{\mathbf{q}}_2 \tag{12.30}$$

Take the derivatives of V_1, V_2 and V_3 along the trajectory of the dynamic system (12.13)–(12.15), we have (See Appendices A–C)

$$\dot{V}_1 = -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top \mathbf{z}_1 + \hat{\rho}_{d_0}||\mathbf{z}_0|| \cdot ||\mathbf{z}_2|| - \tfrac{1}{2}\mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top \tilde{\Theta}_0\} - \tfrac{1}{2}\mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top \tilde{\Phi}_0\} + \tfrac{1}{2}\mu_{01}\text{trace}\,\{\Theta_0^\top \Theta_0\} + \tfrac{1}{2}\mu_{02}\text{trace}\,\{\Phi_0^\top \Phi_0\} \tag{12.31}$$

$$\dot{V}_2 \le -\mathbf{z}_1^\top K_1\mathbf{z}_1 - \mathbf{z}_1^\top \mathbf{z}_0 - \tfrac{1}{2}\mu_{11}\text{trace}\,\{\tilde{\Theta}_1'^\top \tilde{\Theta}_1'\} - \tfrac{1}{2}\mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top \tilde{\Phi}_1\} - \tfrac{1}{2}\mu_{13}\tilde{\mathbf{q}}_1^\top \tilde{\mathbf{q}}_1 + \tfrac{1}{2}\mu_{11}\text{trace}\,\{\Theta_1'^\top \Theta_1'\} + \tfrac{1}{2}\mu_{12}\text{trace}\,\{\Phi_1^\top \Phi_1\} + \tfrac{1}{2}\mu_{13}\mathbf{q}_1^\top \mathbf{q}_1 + \varepsilon_{v_1} + \varepsilon_{d_1} \tag{12.32}$$

$$\dot{V}_3 \le -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} - \tfrac{1}{2}\mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \tilde{\Theta}_2\} - \tfrac{1}{2}\mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \tilde{\Phi}_2\} - \tfrac{1}{2}\mu_{23}\tilde{\mathbf{q}}_2^\top \tilde{\mathbf{q}}_2 + \tfrac{1}{2}\mu_{21}\text{trace}\,\{\Theta_2^\top \Theta_2\} + \tfrac{1}{2}\mu_{22}\text{trace}\,\{\Phi_2^\top \Phi_2\} + \tfrac{1}{2}\mu_{23}\mathbf{q}_2^\top \mathbf{q}_2 + \varepsilon_{v_2} + \varepsilon_{d_2} \tag{12.33}$$

Then we can easily get that

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\
&= -\mathbf{z}^\top K\mathbf{z} + \hat{\rho}_{d_0}||\mathbf{z}_0|| \cdot ||\mathbf{z}_2|| - \mathbf{z}_2^\top \mathbf{v}_{d_0} + \frac{1}{2}\sum_{i=0}^{2}(-\mu_{i1}\text{trace}\,\{\tilde{\Theta}_i^\top \tilde{\Theta}_i\} - \mu_{i2}\text{trace}\,\{\tilde{\Phi}_i^\top \tilde{\Phi}_i\} \\
&\quad - \mu_{i3}\tilde{\mathbf{q}}_i^\top \tilde{\mathbf{q}}_i + \mu_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\} + \mu_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + \mu_{i3}\mathbf{q}_i^\top \mathbf{q}_i) + \sum_{i=1}^{2}(\varepsilon_{v_i} + \varepsilon_{d_i}) \\
&= -\mathbf{z}^\top K\mathbf{z} + \hat{\rho}_{d_0}||\mathbf{z}_0|| \cdot ||\mathbf{z}_2|| - \frac{\hat{\rho}_{d_0}^2||\mathbf{z}_0||^2||\mathbf{z}_2||^2}{\hat{\rho}_{d_0}||\mathbf{z}_0||||\mathbf{z}_2|| + \varepsilon_{d_0}} + \frac{1}{2}\sum_{i=0}^{2}(-\mu_{i1}\text{trace}\,\{\tilde{\Theta}_i^\top \tilde{\Theta}_i\} \\
&\quad - \mu_{i2}\text{trace}\,\{\tilde{\Phi}_i^\top \tilde{\Phi}_i\} - \mu_{i3}\tilde{\mathbf{q}}_i^\top \tilde{\mathbf{q}}_i + \mu_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\}\mu_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + \mu_{i3}\mathbf{q}_i^\top \mathbf{q}_i) \\
&\quad + \sum_{i=1}^{2}(\varepsilon_{v_i} + \varepsilon_{d_i}) \\
&= -\mathbf{z}^\top K\mathbf{z} + \frac{1}{2}\sum_{i=0}^{2}(-\mu_{i1}\text{trace}\,\{\tilde{\Theta}_i^\top \tilde{\Theta}_i\} - \mu_{i2}\text{trace}\,\{\tilde{\Phi}_i^\top \tilde{\Phi}_i\} - \mu_{i3}\tilde{\mathbf{q}}_i^\top \tilde{\mathbf{q}}_i \\
&\quad + \mu_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\} + \mu_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + \mu_{i3}\mathbf{q}_i^\top \mathbf{q}_i) + \varepsilon
\end{aligned} \tag{12.34}$$

where $K = \text{diag}\,[K_0, K_1, K_2]$, and $\varepsilon = \sum_{i=0}^{2}\varepsilon_{d_i} + \sum_{i=1}^{2}\varepsilon_{v_i}$.

By choosing K such that

$$\lambda_{min}(K) \ge \frac{\varepsilon + c_1}{\varepsilon_0^2} \tag{12.35}$$

where c_1 is an arbitrary positive constant, then from (12.24) we have

$$\dot{V} \le -\mathbf{z}^\top K\mathbf{z} + \varepsilon \le -c_1\,, \quad \forall\, \mathbf{z} \in \mathcal{R}^n - E_0 \tag{12.36}$$

Note that, in terms of the adaptive robust control law (12.20)–(12.24), $\dot{V}$ is a continuous function. We can show that there exists a constant $0 < \varepsilon_0' < \varepsilon_0$ such that (see Appendix E)

$$\forall \mathbf{z} \in \mathcal{R}^n - E_0', \quad \dot{V} < 0 \tag{12.37}$$

where $E_0' \triangleq \{\mathbf{z}: \ ||\mathbf{z}|| < \varepsilon_0'\}$ is a subset of E_0. Noting the relation $E_0' \subset E_0$, (12.37) implies that the system will enter the set E_0 in a finite time.

When $\mathbf{z} \in E_0'$, it is obvious that

$$k_{ij}(\varepsilon_0 - \varepsilon_0') \leq \mu_{ij} \leq k_{ij}\varepsilon_0, \quad i = 0,1,2; \quad j = 1,2,3 \tag{12.38}$$

Define $\delta = \varepsilon_0 - \varepsilon_0' > 0$, then from (12.24), (12.34) and (12.38) we obtain

$$\begin{aligned}\dot{V} \leq & -\mathbf{z}^\top K\mathbf{z} + \frac{1}{2}\sum_{i=0}^{2}(-k_{i1}\delta \text{ trace }\{\tilde{\Theta}_i^\top \tilde{\Theta}_i\} - k_{i2}\delta \text{ trace }\{\tilde{\Phi}_i^\top \tilde{\Phi}_i\} - k_{i3}\delta \tilde{\mathbf{q}}_i^\top \tilde{\mathbf{q}}_i \\ & + k_{i1}\varepsilon_0 \text{ trace }\{\Theta_i^\top \Theta_i\} + k_{i2}\varepsilon_0 \text{ trace }\{\Phi_i^\top \Phi_i\} + k_{i3}\varepsilon_0 \mathbf{q}_i^\top \mathbf{q}_i) + \varepsilon \\ \leq & -k''V + \frac{1}{2}\sum_{i=0}^{2}(k_{i1}\varepsilon_0 \text{trace }\{\Theta_i^\top \Theta_i\} + k_{i2}\varepsilon_0 \text{trace }\{\Phi_i^\top \Phi_i\} + k_{i3}\varepsilon_0 \mathbf{q}_i^\top \mathbf{q}_i) + \varepsilon\end{aligned} \tag{12.39}$$

where

$$k'' = \min\{k_{ij}'', \quad i = 0,1,2; \quad j = 1,2,3\}$$

$$k_{ij}'' = \frac{2\min\{\lambda_{min}(K), k_{ij}\delta\}}{\max\{\lambda_{\max}(B_i^{-1}), \lambda_{\max}(\Gamma_{ij}^{-1})\}}$$

By solving (12.39) we can establish that

$$\begin{aligned}V(t) \leq e^{-k''t}V(t=0) + \frac{1}{2k''}\Bigg[&\sum_{i=0}^{2}(k_{i1}\varepsilon_0 \text{ trace }\{\Theta_i^\top \Theta_i\} + k_{i2}\varepsilon_0 \text{ trace }\{\Phi_i^\top \Phi_i\} \\ & + k_{i3}\varepsilon_0 \mathbf{q}_i^\top \mathbf{q}_i) + 2\varepsilon\Bigg]\end{aligned}$$

which implies that $\mathbf{z}$, $\tilde{\Theta}$, $\tilde{\Phi}$ and $\mathbf{q}$ converge exponentially to the residual set

$$\begin{aligned}D = \Bigg\{&\mathbf{z}, \tilde{\Theta}_i, \tilde{\Phi}_i, \tilde{\mathbf{q}}_i, \quad i = 0,1,2: \quad \mathbf{z}^\top \mathbf{z} + \sum_{i=0}^{2}(\text{trace }\{\tilde{\Theta}_i^\top \tilde{\Theta}_i\} + \text{trace }\{\tilde{\Phi}_i^\top \tilde{\Phi}_i\} + \tilde{\mathbf{q}}_i^\top \tilde{\mathbf{q}}_i) \\ & < \frac{k'}{2}\Bigg[\sum_{i=0}^{2}(k_{i1}\varepsilon_0 \text{ trace }\{\Theta_i^\top \Theta_i\} + k_{i2}\varepsilon_0 \text{ trace }\{\Phi_i^\top \Phi_i\} + k_{i3}\varepsilon_0 \mathbf{q}_i^\top \mathbf{q}_i) + 2\varepsilon\Bigg]\Bigg\}\end{aligned}$$

where

$$k' = \max\{k'_{ij}, \quad i = 0, 1, 2; \quad j = 1, 2, 3\}$$

$$k'_{ij} = \frac{1}{k''_{ij} \min\{\lambda_{\min}(B_i^{-1}), \lambda_{\min}(\Gamma_{ij}^{-1})\}}$$

12.4 Application to PM synchronous motors

Model of permanent magnet synchronous motor. A permanent magnet synchronous motor (PMSM) is described by the following subsystems: (1) a dynamic mechanical subsystem, which for the purposes of this discussion includes a single-link robot manipulator and the motor rotor; (2) a dynamic electrical subsystem which includes all of the motor's relevant electrical effects.

$$\frac{d\theta}{dt} = \omega \tag{12.40}$$

$$\frac{d\omega}{dt} = \frac{1}{J}\left\{\left[(L_d - L_q)I_d + \phi_f\right]I_q - T\sin\theta\right\} \tag{12.41}$$

$$\frac{dI_d}{dt} = \frac{1}{L_d}(u_d + \omega L_q I_q - RI_d) \tag{12.42}$$

$$\frac{dI_q}{dt} = \frac{1}{L_q}(u_q - \omega L_d I_d - RI_q - \omega\phi_f) \tag{12.43}$$

where (12.40) and (12.41) present the dynamics of mechanical subsystem, and (12.42) and (12.43) are the dynamic electrical subsystem. In those equations, u_q and u_d are the input control voltages; I_d and I_q are the motor armature current; R is the stator resistance; L_d and L_q are the self-inductances; J is the inertia angular momentum; and ϕ_f is the flux due to permanent magnet. For the above electromechanical model, we assume that the true states (i.e., θ, ω, I_d and I_q) are all measurable. This model is obtained by using circuits theory principles and a particular dq reference frame. The control objective is to develop a link position tracking controller for the electromechanical dynamics of (12.40)–(12.43) despite parametric uncertainty. In this chapter we assume that all the motor parameters are unknown.

Remark 4.1 In most existing control schemes for PM synchronous motors, the controllers are designed based on the following reduced model

$$\frac{d\theta}{dt} = \omega \tag{12.44}$$

$$\frac{d\omega}{dt} = \frac{1}{J}\left(\phi_f I_q - T\sin\theta\right) \tag{12.45}$$

$$\frac{dI_q}{dt} = \frac{1}{L_q}(u_q - RI_q - \omega\phi_f) \tag{12.46}$$

In this reduced model, component u_d is adjusted to regulate current state I_d and it is supposed that I_d exactly equals to zero. Based on this reduced model, the position or velocity control is only directly related to the command voltage u_q. Obviously, based on the reduced model, the control design will result in only a locally stable controller.

Control Design. For a given desired tracking state $\theta_d(t)$, define a quantity z_0 to be

$$z_0 = ce + \dot{e}, \quad e = \theta - \theta_d, \quad c > 0 \tag{12.47}$$

where $\theta_d(t)$ is at least twice continuously differentiable. Differentiating (12.47), multiplying by J and substituting the mechanical subsystem dynamics of (12.41), yields

$$J\dot{z}_0 = Jc\dot{e} + \left[(L_d - L_q)I_d + \phi_f\right]I_q - T\sin\theta - J\dot{\omega}_d \tag{12.48}$$

Dividing (12.48) by ϕ_f and rearranging terms yield

$$J'\dot{z}_0 = \alpha_1^\top\varphi_1 + I_q + L'I_dI_q \tag{12.49}$$

where α_1 and φ_1 are defined as

$$\alpha_1^\top = [J',\ -T'], \quad \varphi_1^\top = [c\dot{e} - \dot{\omega}_d,\ \sin\theta] \tag{12.50}$$

J', T' and L' are defined as

$$J' = \frac{J}{\phi_f}, \quad T' = \frac{T}{\phi_f}, \quad L' = \frac{L_d - L_q}{\phi_f}$$

Denote $z_1 = I_q - I_q^{\mathrm{ref}}$, $z_2 = I_d$, and the *auxiliary reference current* I_q^{ref} is defined as

$$I_q^{\mathrm{ref}} = -\hat{\alpha}_1^\top\varphi_1 - k_0z_0 \tag{12.51}$$

where k_0 is a positive constant gain. Then the PMSM model (12.40)–(12.43) can be transformed into

$$\dot{z}_0 = J'^{-1}(\alpha_1^\top\varphi_1 + z_1 + I_q^{\mathrm{ref}} + L'I_qz_2) \tag{12.52}$$

$$\dot{z}_1 = L_q^{-1}(u_q + \alpha_2^\top\varphi_2 + \alpha_3^\top\varphi_3) \tag{12.53}$$

$$\dot{z}_2 = L_d^{-1}(\alpha_4^\top\varphi_4 + u_d) \tag{12.54}$$

where the unknown constant parameter vectors α_2, α_3 and α_4 and the known regression vectors φ_2, φ_3 and φ_4 are defined as (see Appendix E for the derivation of α_2, α_3, φ_2 and φ_3)

$$
\begin{aligned}
\alpha_2^\top &= \left[L_q \frac{\phi_f}{J},\ L_q \frac{L'}{J'},\ -L_q \frac{T}{J}\right] \\
\varphi_2^\top &= [(k_1 + c\hat{\alpha}_{11})I_q,\ (k_1 + c\hat{\alpha}_{11})I_d I_q,\ (k_1 + c\hat{\alpha}_{11})\sin\theta] \\
\alpha_3^\top &= [-L_d,\ -R,\ -\phi_f,\ L_q] \\
\varphi_3^\top &= [\omega I_d,\ I_d,\ \omega,\ k_1(c\dot{e} - \dot{\omega}_d) + z_0\varphi_1^\top \Gamma_1^{-1}\varphi_1 + A_\omega] \\
\alpha_4^\top &= [-R,\ L_q] \\
\varphi_4^\top &= [I_d,\ \omega I_q]
\end{aligned}
$$

where $A_\omega = -\hat{\alpha}_{11}(c\dot{\omega}_d + \ddot{\omega}_d) + \hat{\alpha}_{12}\omega\cos\theta$. Comparing (12.52)–(12.54) with (12.13)–(12.15), it is obvious that PMSM dynamics is a particular case of the general nonlinear uncertain dynamic system (12.13)–(12.15). Therefore all the previous design and analysis can be applied directly.

Note that in practical applications, the parameters J and T are constants but may vary in a wide range due to the variation of payload. On the other hand, the unknown motor parameters have fewer deviations from its rated values (nominal values) in comparison with that of load. Therefore, it would be more appropriate for us to deal with the unknown parameters J and T by using adaptive techniques and treat the bounded motor parameters by using robust methods. In this way, referring to (12.20)–(12.24) the control inputs with the corresponding adaptive laws are given as

$$
\begin{aligned}
u_q &= -k_1 z_1 - z_0 - \hat{\alpha}_2^\top \varphi_2 - v_{d_1} \\
u_d &= -k_2 z_2 - v_{d_0} - v_{d_2} \\
\dot{\hat{\alpha}}_1 &= \varphi_1 z_0 - \mu_1 \hat{\alpha}_1 \\
\dot{\hat{\alpha}}_2 &= \varphi_2 z_1 - \mu_2 \hat{\alpha}_2 \\
v_{d_0} &= \frac{L_{\max}^2 ||I_q||^2 ||z_0||^2}{L_{\max}||I_q z_0 z_2|| + \varepsilon_{d_0}} z_2 \\
v_{d_1} &= \frac{\alpha_{\max_3}^2 ||\varphi_3||^2}{\alpha_{\max_3}||\varphi_3|| ||z_1|| + \varepsilon_{d_1}} z_1 \\
v_{d_2} &= \frac{\alpha_{\max_4}^2 ||\varphi_4||^2}{\alpha_{\max_4}||\varphi_4|| ||z_2|| + \varepsilon_{d_2}} z_2
\end{aligned}
\tag{12.55}
$$

where $||L'|| \leq L_{max}$, $||\alpha_3|| \leq \alpha_{max_3}$ and $||\alpha_4|| \leq \alpha_{max_4}$. μ_i, $i = 1, 2$, are chosen to be

$$\mu_i = \begin{cases} \varepsilon_0 - ||\mathbf{z}|| & ||\mathbf{z}|| \leq \varepsilon_0 \\ 0 & \text{elsewhere} \end{cases} \tag{12.56}$$

where $\mathbf{z} = [z_0, z_1, z_2]]^\top$.

Simulation Study. The permanent magnet synchronous motor with the following parameters is used to demonstrate the control performance

$$L_d = 25.0 \times 10^{-3}\ \text{H}, \quad L_q = 30.0 \times 10^{-3}\ \text{H}, \quad J = 1.625 \times 10^{-3}\ \text{Kg} \cdot \text{m}^2,$$

$$\phi_f = 0.90\ \text{N} \cdot \text{m/A}, \quad R = 5.0\ \Omega, \quad T = 2.2816\ \text{Kg} \cdot \text{A} \cdot \text{m/N} \cdot \text{s}^2$$

$$\alpha_1^\top = [0.0018, -2.535]$$

The following control parameters

$$c = 5.0, \quad \varepsilon_0 = 0.1, \quad \varepsilon_{d_0} = \varepsilon_{d_1} = \varepsilon_{d_2} = 0.5, \quad k_1 = k_2 = k_3 = 8.0$$

are used for the proposed control scheme. The desired trajectory is chosen as

$$\theta_d = \frac{\pi}{2}\left(1 - e^{-0.1t^3}\right)\sin\left(\frac{\pi}{5}t\right)$$

The initial states are $\theta(0) = 0.1$ rad, $\omega(0) = 0.1$ rad/sec, $I_d(0) = 0.05\ A$, $I_q(0) = 0.01\ A$. The initial values of all the parameter estimates are set zero.

The tracking error is shown in Figure 12.1. We can see that the proposed control method yields very good tracking performance. Meanwhile, the control inputs u_d and u_q are very smooth as shown in Figure 12.2. Figure 12.3 (a) shows the corresponding armature current I_q compared with *auxiliary reference current* I_q^{ref} while I_d is shown in Figure 12.3 (b). It is easy to see that the current I_q asymptotically converges to I_q^{ref}.

12.5 Conclusion

In this chapter, an adaptive robust control scheme is developed for a class of uncertain systems with both unknown parameters and system disturbances. The uncertainties are assumed to be composed of two categories: the structured category and the nonstructured category with partially known bounding functions. The structured uncertainty is estimated with the adaptive method. Meanwhile, the adaptive robust method is applied to deal with the nonstructured uncertainty, where the unknown parameters in the upper bounding function are estimated with adaptation. It is shown that the control scheme developed here can guarantee the uniform boundedness of the system and assure that the tracking error enters the arbitrarily designated zone in a finite time. The effectiveness of the control scheme is verified by theoretical analysis, as well as applications to a permanent magnet synchronous motor.

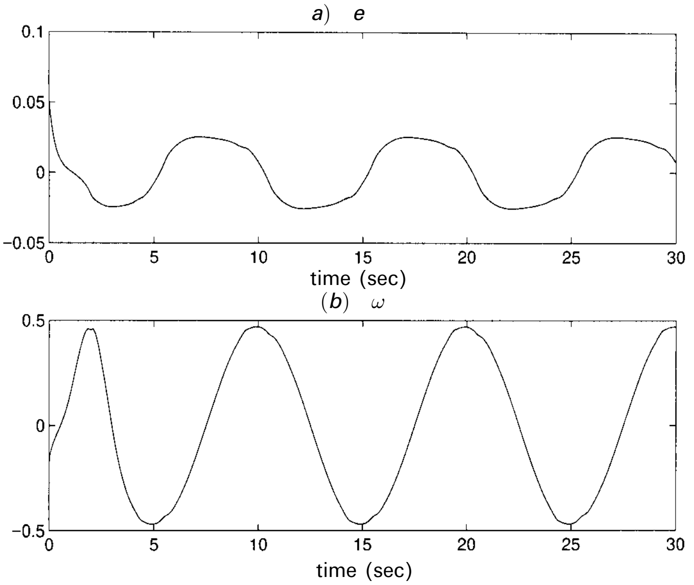

Figure 12.1 *(a) Position tracking error e. (b) Velocity tracking error ω*

Appendix A The derivative of V_1

$$
\begin{aligned}
\dot{V}_1 &= \mathbf{z}_0^\top B_0^{-1}\dot{\mathbf{z}}_0 - \text{trace}\,\{\tilde{\Theta}^\top \Gamma_{01}^{-1}\dot{\hat{\Theta}}\} - \text{trace}\,\{\tilde{\Phi}_0^\top \Gamma_{02}^{-1}\dot{\hat{\Phi}}_0\} - \tilde{\mathbf{q}}_0^\top \Gamma_{03}^{-1}\dot{\hat{\mathbf{q}}}_0 \\
&= \mathbf{z}_0^\top B_0^{-1}\{\mathbf{f}_0' + B_0[\Theta_0\xi_0 + \triangle\mathbf{g}_0\mathbf{z}_2 + \mathbf{z}_1 + \mathbf{x}_1^{\text{ref}}]\} - \text{trace}\,\{\tilde{\Theta}_0^\top \Gamma_{01}^{-1}\dot{\hat{\Theta}}_0\} \\
&\quad - \text{trace}\,\{\tilde{\Phi}_0^\top \Gamma_{02}^{-1}\dot{\hat{\Phi}}_0\} - \tilde{\mathbf{q}}_0^\top \Gamma_{03}^{-1}\dot{\hat{\mathbf{q}}}_0 \\
&= \mathbf{z}_0^\top[\Theta_0\xi_0 + \mathbf{z}_1 + \mathbf{x}_1^{ref} + \Phi_0\mathbf{f}_0' + \triangle\mathbf{g}_0\mathbf{z}_2] - \text{trace}\,\{\tilde{\Theta}_0^\top(\mathbf{z}_0\xi_0^\top - \mu_{01}\hat{\Theta}_0)\} \\
&\quad - \text{trace}\,\{\tilde{\Phi}_0^\top(\mathbf{z}_0\mathbf{f}_0'^\top - \mu_{02}\hat{\Phi}_0)\} - \tilde{\mathbf{q}}_0^\top \Gamma_{03}^{-1}\dot{\hat{\mathbf{q}}}_0 \\
&= -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top[\tilde{\Theta}_0\xi_0 + \mathbf{z}_1 + \tilde{\Phi}_0\mathbf{f}_0' + \triangle\mathbf{g}_0\mathbf{z}_2] - \mathbf{z}_0^\top\tilde{\Theta}_0\xi_0 - \mathbf{z}_0^\top\tilde{\Phi}_0\mathbf{f}_0' \\
&\quad - ||\mathbf{z}_0||(\mathbf{q}_0 - \hat{\mathbf{q}}_0)^\top \frac{\partial\rho_{d_0}}{\partial\mathbf{q}_0}\bigg|_{\hat{\mathbf{q}}_0} + \mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top\hat{\Theta}_0\} + \mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top\hat{\Phi}_0\} + \mu_{03}\tilde{\mathbf{q}}_0^\top\hat{\mathbf{q}}_0 \\
&\le -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top\mathbf{z}_1 + \rho_{d_0}||\mathbf{z}_0||\cdot||\mathbf{z}_2|| - ||\mathbf{z}_0||(\rho_{d_0} - \hat{\rho}_{d_0}) \\
&\quad + \mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top\hat{\Theta}_0\} + \mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top\hat{\Phi}_0\} + \mu_{03}\tilde{\mathbf{q}}_0^\top\hat{\mathbf{q}}_0 \\
&= -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top\mathbf{z}_1 + \hat{\rho}_{d_0}||\mathbf{z}_0||\cdot||\mathbf{z}_2|| + \mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top(\Theta_0 - \tilde{\Theta}_0)\} \\
&\quad + \mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top(\Phi_0 - \tilde{\Phi}_0)\} + \mu_{03}\tilde{\mathbf{q}}_0^\top(\mathbf{q}_0 - \tilde{\mathbf{q}}_0)
\end{aligned}
$$

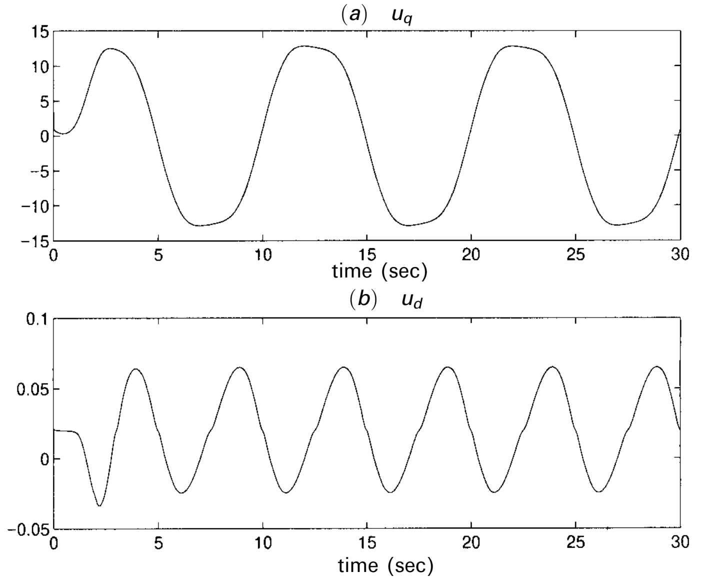

Figure 12.2 *(a) Evolution of input voltage u_q. (b) Evolution of input voltage u_d*

$$= -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top\mathbf{z}_1 + \hat{\rho}_{d_0}||\mathbf{z}_0||\cdot||\mathbf{z}_2|| - \mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top\tilde{\Theta}_0\} + \mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top\Theta_0\}$$
$$- \mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top\tilde{\Phi}_0\} + \mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top\Phi_0\} - \mu_{03}\tilde{\mathbf{q}}_0^\top\tilde{\mathbf{q}}_0 + \mu_{03}\tilde{\mathbf{q}}_0^\top\mathbf{q}_0$$
$$= -\mathbf{z}_0^\top K_0\mathbf{z}_0 + \mathbf{z}_0^\top\mathbf{z}_1 + \hat{\rho}_{d_0}||\mathbf{z}_0||\cdot||\mathbf{z}_2|| - \tfrac{1}{2}\mu_{01}\text{trace}\,\{\tilde{\Theta}_0^\top\tilde{\Theta}_0\} - \tfrac{1}{2}\mu_{02}\text{trace}\,\{\tilde{\Phi}_0^\top\tilde{\Phi}_0\}$$
$$- \tfrac{1}{2}\mu_{03}\tilde{\mathbf{q}}_0^\top\tilde{\mathbf{q}}_0\} + \tfrac{1}{2}\mu_{01}\text{trace}\,\{\Theta_0^\top\Theta_0\} + \tfrac{1}{2}\mu_{02}\text{trace}\,\{\Phi_0^\top\Phi_0\} + \tfrac{1}{2}\mu_{03}\mathbf{q}_0^\top\mathbf{q}_0\} \qquad (A.1)$$

In above derivation the following property of trace is used:

$$\text{trace}\,\{Q^\top\mathbf{v}\mathbf{w}^\top\} = \mathbf{v}^\top Q\mathbf{w} \qquad (A.2)$$

where $\mathbf{v} \in \mathcal{R}^{n\times 1}$, $\mathbf{w} \in \mathcal{R}^{l\times 1}$, and $Q \in \mathcal{R}^{n\times l}$.

Appendix B The derivative of V_2

$$\dot{V}_2 = \mathbf{z}_1^\top B_1^{-1}\dot{\mathbf{z}}_1 - \text{trace}\,\{\tilde{\Theta}_1^\top\Gamma_{11}^{-1}\dot{\hat{\Theta}}_1\} - \text{trace}\,\{\tilde{\Phi}_1^\top\Gamma_{12}^{-1}\dot{\hat{\Phi}}_1\} - \tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\dot{\hat{\mathbf{q}}}_1$$
$$= \mathbf{z}_1^\top B_1^{-1}\{\mathbf{f}_1' + B_1[(I + E_1)\mathbf{u}_1 + \Theta_1'\xi_1' + \triangle\mathbf{g}_1]\} - \text{trace}\,\{\tilde{\Theta}_1^\top\Gamma_{11}^{-1}\dot{\hat{\Theta}}_1\}$$
$$- \text{trace}\,\{\tilde{\Phi}_1^\top\Gamma_{12}^{-1}\dot{\hat{\Phi}}_1\} - \tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\dot{\hat{\mathbf{q}}}_1$$

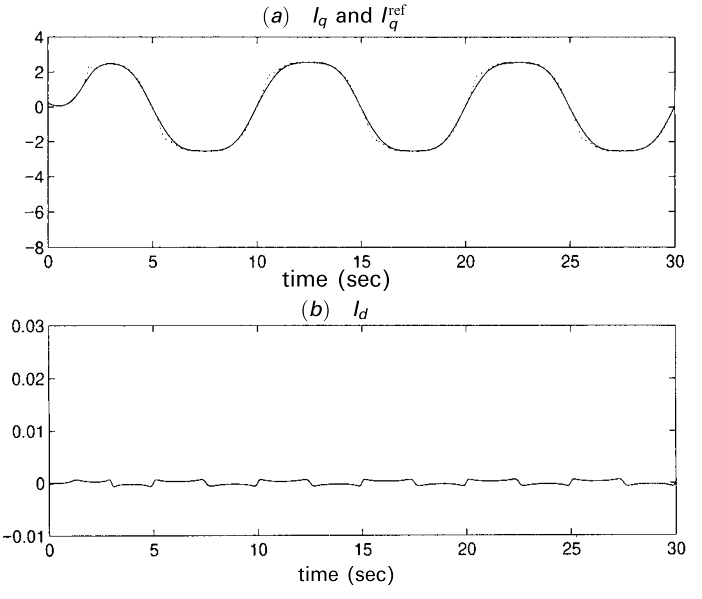

Figure 12.3 *(a) Evolution of current I_q (solid line) and I_q^{ref} (dot line). (b) Evolution of current I_d*

$$
\begin{aligned}
&= \mathbf{z}_1^\top[\Theta_1'\xi_1' + (I+E_1)(\mathbf{u}_{c_1}+\mathbf{u}_{v_1}) + \triangle\mathbf{g}_1 + \Phi_1\mathbf{f}_1'] \\
&\quad - \text{trace}\,\{\tilde{\Theta}_1'^\top(\mathbf{z}_1\xi_1'^\top - \mu_{11}\hat{\Theta}_1')\} - \text{trace}\,\{\tilde{\Phi}_1^\top(\mathbf{z}_1\mathbf{f}_1'^\top - \mu_{12}\hat{\Phi}_1)\} - \tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\dot{\hat{\mathbf{q}}}_1 \\
&= \mathbf{z}_1^\top[-K_1\mathbf{z}_1 - \mathbf{z}_0 + \tilde{\Theta}_1'\xi_1' + \tilde{\Phi}_1\mathbf{f}_1' + (I+E_1)\mathbf{u}_{v_1} + E_1\mathbf{u}_{c_1} - \mathbf{v}_{d_1} + \triangle\mathbf{g}_1] \\
&\quad - \mathbf{z}_1^\top\tilde{\Theta}_1'\xi_1' - \mathbf{z}_1^\top\tilde{\Phi}_1'\mathbf{f}_1' + \mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top\hat{\Theta}_1\} + \mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top\hat{\Phi}_1\} - \tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\dot{\hat{\mathbf{q}}}_1 \\
&\le -\mathbf{z}_1^\top K_1\mathbf{z}_1 - \mathbf{z}_1^\top\mathbf{z}_0 + \rho_{d_1}||\mathbf{z}_1|| - \mathbf{z}_1^\top\mathbf{v}_{d_1} + r_{max_1}||\mathbf{z}_1^\top\mathbf{u}_{c_1}|| \\
&\quad - \mathbf{z}_1^\top(I+E_1)\frac{r_{max_1}^2||\mathbf{u}_{c_1}||^2\mathbf{z}_1}{(1+r_{min_1})(r_{max_1}||\mathbf{z}_1^\top\mathbf{u}_{c_1}||+\varepsilon_{v_1})} + \mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top\hat{\Theta}_1\} \\
&\quad + \mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top\hat{\Phi}_1\} - \tilde{\mathbf{q}}_1^\top\Gamma_{13}^{-1}\dot{\hat{\mathbf{q}}}_1
\end{aligned} \tag{A.3}
$$

Using the fact that

$$
\begin{aligned}
\mathbf{z}_i^\top(I+E_i)\mathbf{z}_i &= \mathbf{z}_i^\top(I+\frac{E_i+E_i^\top}{2})\mathbf{z}_i \\
\frac{I+\frac{1}{2}E_i+\frac{1}{2}E_i^\top}{1+r_{min_i}} &\ge 1\,, \quad i=1,2
\end{aligned} \tag{A.4}
$$

it follows that

$$
\begin{aligned}
\dot{V}_2 \le & -\mathbf{z}_1^\top K_1 \mathbf{z}_1 - \mathbf{z}_1^\top \mathbf{z}_0 - \frac{\hat{\rho}_{d_1}^2 ||\mathbf{z}_1||^2}{\hat{\rho}_{d_1}||\mathbf{z}_1|| + \varepsilon_{d_1}} + \rho_{d_1}||\mathbf{z}_1|| - ||\mathbf{z}_1||(\mathbf{q}_1 - \hat{\mathbf{q}}_1)^\top \frac{\partial \rho_{d_1}}{\partial \mathbf{q}_i}\bigg|_{\hat{\mathbf{q}}_1} \\
& + \mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top \hat{\Theta}_1\} + \mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top \hat{\Phi}_1\} + \mu_{13}\tilde{\mathbf{q}}_1^\top \hat{\mathbf{q}}_1 + \varepsilon_{v_1} \\
\le & -\mathbf{z}_1^\top K_1 \mathbf{z}_1 - \mathbf{z}_1^\top \mathbf{z}_0 - \frac{\hat{\rho}_{d_1}^2 ||\mathbf{z}_1||^2}{\hat{\rho}_{d_1}||\mathbf{z}_1|| + \varepsilon_{d_1}} + \rho_{d_1}||\mathbf{z}_1|| - ||\mathbf{z}_1||(\rho_{d_1} - \hat{\rho}_{d_1}) \\
& + \mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top \hat{\Theta}_1\} + \mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top \hat{\Phi}_1\} + \mu_{13}\tilde{\mathbf{q}}_1^\top \hat{\mathbf{q}}_1 + \varepsilon_{v_1} \\
\le & -\mathbf{z}_1^\top K_1 \mathbf{z}_1 - \mathbf{z}_1^\top \mathbf{z}_0 + \mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top \hat{\Theta}_1\} + \mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top \hat{\Phi}_1\} \\
& + \mu_{13}\tilde{\mathbf{q}}_1^\top \hat{\mathbf{q}}_1 + \varepsilon_{v_1} + \varepsilon_{d_1} \\
\le & -\mathbf{z}_1^\top K_1 \mathbf{z}_1 - \mathbf{z}_1^\top \mathbf{z}_0 - \tfrac{1}{2}\mu_{11}\text{trace}\,\{\tilde{\Theta}_1^\top \tilde{\Theta}_1\} - \tfrac{1}{2}\mu_{12}\text{trace}\,\{\tilde{\Phi}_1^\top \tilde{\Phi}_1\} - \tfrac{1}{2}\mu_{13}\tilde{\mathbf{q}}_1^\top \tilde{\mathbf{q}}_1 \\
& + \tfrac{1}{2}\mu_{11}\text{trace}\,\{\Theta_1^{'\top} \Theta_1'\} + \tfrac{1}{2}\mu_{12}\text{trace}\,\{\Phi_1^\top \Phi_1\} + \tfrac{1}{2}\mu_{13}\mathbf{q}_1^\top \mathbf{q}_1 + \varepsilon_{v_1} + \varepsilon_{d_1} \qquad \text{(A.5)}
\end{aligned}
$$

Appendix C The derivative of V_3

$$
\begin{aligned}
\dot{V}_3 = & \mathbf{z}_2^\top B_2^{-1}\dot{\mathbf{z}}_2 - \text{trace}\,\{\tilde{\Theta}_2^\top \Gamma_{21}^{-1}\dot{\hat{\Theta}}_2\} - \text{trace}\,\{\tilde{\Phi}_2^\top \Gamma_{22}^{-1}\dot{\hat{\Phi}}_2\} - \tilde{\mathbf{q}}_2^\top \Gamma_{23}^{-1}\dot{\hat{\mathbf{q}}}_2 \\
= & \mathbf{z}_2^\top B_2^{-1}\{\mathbf{f}_2 + B_2[(I + E_2)\mathbf{u}_2 + \Theta_2\xi_2 + \triangle\mathbf{g}_2]\} - \text{trace}\,\{\tilde{\Theta}_2^\top \Gamma_{21}^{-1}\dot{\hat{\Theta}}_2\} \\
& - \text{trace}\,\{\tilde{\Phi}_2^\top \Gamma_{22}^{-1}\dot{\hat{\Phi}}_2\} - \tilde{\mathbf{q}}_2^\top \Gamma_{23}^{-1}\dot{\hat{\mathbf{q}}}_2 \\
= & \mathbf{z}_2^\top [\Theta_2\xi_2 + (I + E_2)(\mathbf{u}_{c_2} + \mathbf{u}_{v_2}) + \triangle\mathbf{g}_2 + \Phi_2\mathbf{f}_2] \\
& - \text{trace}\,\{\tilde{\Theta}_2^\top (\mathbf{z}_2\xi^\top - \mu_{21}\hat{\Theta}_2)\} - \text{trace}\,\{\tilde{\Phi}_2^\top (\mathbf{z}_2\mathbf{f}_2^\top - \mu_{22}\hat{\Phi}_2)\} - \tilde{\mathbf{q}}_2^\top \Gamma_{23}^{-1}\dot{\hat{\mathbf{q}}}_2\triangle\mathbf{g} \\
= & \mathbf{z}_2^\top [-K_2\mathbf{z}_2 + \tilde{\Theta}_2\xi_2 + \tilde{\Phi}_2\mathbf{f}_2 + (I + E_2)\mathbf{u}_{v_2} + E_2\mathbf{u}_{c_2} - \mathbf{v}_{d_0} - \mathbf{v}_{d_2} + \triangle\mathbf{g}_2] \\
& - \mathbf{z}_2^\top \tilde{\Theta}_2\xi_2 - \mathbf{z}_2^\top \tilde{\Phi}_2\mathbf{f}_2 + \mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \hat{\Theta}_2\} + \mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \hat{\Phi}_2\} - \tilde{\mathbf{q}}_2^\top \Gamma_{23}^{-1}\dot{\hat{\mathbf{q}}}_2 \\
\le & -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} - \mathbf{z}_2^\top \mathbf{v}_{d_2} + \rho_{d_2}||\mathbf{z}_2|| + r_{max_2}||\mathbf{z}_2^\top \mathbf{u}_{c_2}|| \\
& - \mathbf{z}_2^\top (I + E_2)\frac{r_{max_2}^2 ||\mathbf{u}_{c_2}||^2 \mathbf{z}_2}{(1 + r_{min_2})(r_{max_2}||\mathbf{z}_2^\top \mathbf{u}_{c_2}|| + \varepsilon_{v_2})} + \mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \hat{\Theta}_2\} \\
& + \mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \hat{\Phi}_2\} - \tilde{\mathbf{q}}_2^\top \Gamma_{23}^{-1}\dot{\hat{\mathbf{q}}}_2 \qquad \text{(A.6)}
\end{aligned}
$$

Using the fact of (A.6) it follows that

$$\dot{V}_3 \le -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} - \frac{\hat{\rho}_{d_2}^2||\mathbf{z}_2||^2}{\hat{\rho}_{d_2}||\mathbf{z}_2|| + \varepsilon_{d_2}} + \rho_{d_2}||\mathbf{z}_2|| - ||\mathbf{z}_2||(\mathbf{q}_2 - \hat{\mathbf{q}}_2)^\top \frac{\partial \rho_{d_2}}{\partial \mathbf{q}_i}\bigg|_{\hat{\mathbf{q}}_2}$$
$$+ \mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \hat{\Theta}_2\} + \mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \hat{\Phi}_2\} + \mu_{23}\tilde{\mathbf{q}}_2^\top \hat{\mathbf{q}}_2 + \varepsilon_{v_2}$$
$$\le -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} - \frac{\hat{\rho}_{d_2}^2||\mathbf{z}_2||^2}{\hat{\rho}_{d_2}||\mathbf{z}_2|| + \varepsilon_{d_2}} + \rho_{d_2}||\mathbf{z}_2|| - ||\mathbf{z}_2||(\rho_{d_2} - \hat{\rho}_{d_2})$$
$$+ \mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \hat{\Theta}_2\} + \mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \hat{\Phi}_2\} + \mu_{23}\tilde{\mathbf{q}}_2^\top \hat{\mathbf{q}}_2 + \varepsilon_{v_2}$$
$$\le -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} + \mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \hat{\Theta}_2\} + \mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \hat{\Phi}_2\} + \mu_{23}\tilde{\mathbf{q}}_2^\top \hat{\mathbf{q}}_2 + \varepsilon_{v_2} + \varepsilon_{d_2}$$
$$\le -\mathbf{z}_2^\top K_2\mathbf{z}_2 - \mathbf{z}_2^\top \mathbf{v}_{d_0} - \tfrac{1}{2}\mu_{21}\text{trace}\,\{\tilde{\Theta}_2^\top \tilde{\Theta}_2\} - \tfrac{1}{2}\mu_{22}\text{trace}\,\{\tilde{\Phi}_2^\top \tilde{\Phi}_2\} - \tfrac{1}{2}\mu_{23}\tilde{\mathbf{q}}_2^\top \tilde{\mathbf{q}}_2$$
$$+ \tfrac{1}{2}\mu_{21}\text{trace}\,\{\Theta_2^\top \Theta_2\} + \tfrac{1}{2}\mu_{22}\text{trace}\,\{\Phi_2^\top \Phi_2\} + \tfrac{1}{2}\mu_{23}\mathbf{q}_2^\top \mathbf{q}_2 + \varepsilon_{v_2} + \varepsilon_{d_2} \qquad \text{(A.7)}$$

Appendix D

From (12.34), it can be seen that $\dot{V} < 0$ if

$$\mathbf{z}^\top K\mathbf{z} > \frac{1}{2}\sum_{i=0}^{2}(\mu_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\} + \mu_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + \mu_{i2}\mathbf{q}_i^\top \mathbf{q}_i + 2\varepsilon) \qquad \text{(A.8)}$$

then ε_0' can be easily determined by solving the following equation

$$\frac{\varepsilon + c_1}{\varepsilon_0^2}||\mathbf{z}||^2 = \frac{1}{2}\sum_{i=0}^{2}(\mu_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\} + \mu_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + \mu_{i2}\mathbf{q}_i^\top \mathbf{q}_i + 2\varepsilon)$$

Substituting μ_{ij} in terms of (12.24) and letting $||\mathbf{z}|| = \varepsilon_0'$ yields

$$\frac{(\varepsilon + c_1)\varepsilon_0'^2}{\varepsilon_0^2} = \frac{1}{2}\sum_{i=0}^{2}(k_{i1}(\varepsilon_0 - \varepsilon_0')\text{trace}\,\{\Theta_i^\top \Theta_i\} + \tfrac{1}{2}k_{i2}(\varepsilon_0 - \varepsilon_0')\text{trace}\,\{\Phi_i^\top \Phi_i\}$$
$$+ \tfrac{1}{2}k_{i3}(\varepsilon_0 - \varepsilon_0')\mathbf{q}_i^\top \mathbf{q}_i + 2\varepsilon) \qquad \text{(A.9)}$$

Denote

$$a = \frac{\varepsilon + c_1}{\varepsilon_0^2}$$
$$b = \frac{1}{2}\sum_{i=0}^{2}(k_{i1}\text{trace}\,\{\Theta_i^\top \Theta_i\} + k_{i2}\text{trace}\,\{\Phi_i^\top \Phi_i\} + k_{i3}\mathbf{q}_i^\top \mathbf{q}_i)$$
$$c = \frac{1}{2}\sum_{i=0}^{2}(k_{i1}\varepsilon_0\text{trace}\,\{\Theta_i^\top \Theta_i\} + k_{i2}\varepsilon_0\text{trace}\,\{\Phi_i^\top \Phi_i\} + k_{i3}\varepsilon_0\mathbf{q}_i^\top \mathbf{q}_i + 2\varepsilon)$$

then equation (A.9) can be transformed to the following

$$a{\varepsilon'_0}^2 + b\varepsilon'_0 - c = 0$$

The solutions of above equation are

$$\varepsilon'_0 = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a} \tag{A.10}$$

It is obvious that the solutions $\in \mathcal{R}$. Notice that ε'_0 is positive, hence the desired solution is

$$\varepsilon'_0 = \frac{-b + \sqrt{b^2 + 4ac}}{2a} > 0 \tag{A.11}$$

Appendix E Definitions of α_2, α_3, φ_2 and φ_3

Differentiating z_1, yields

$$\dot{z}_1 = \dot{I}_q - \dot{I}_q^{\text{ref}} \tag{A.12}$$

From (12.51), we have

$$\begin{aligned} \dot{I}_q^{\text{ref}} &= -(z_0\Gamma_1\varphi_1)^\top\varphi_1 - \hat{\alpha}_1^\top\dot{\varphi}_1 - k_1\dot{z}_0 \\ &= -\sigma_1\varphi_1^\top\Gamma_1\varphi_1 - \hat{\alpha}_1^\top\dot{\varphi}_1 - \frac{k_1}{J'}(\alpha_1^\top\varphi_1 + I_q + L'I_dI_q) \end{aligned} \tag{A.13}$$

For $\hat{\alpha}_1^\top\dot{\varphi}_1$, it follows

$$\begin{aligned} \hat{\alpha}_1^\top\dot{\varphi}_1 &= \hat{\alpha}_{11}[c(\dot{\omega} - \dot{\omega}_d) - \ddot{\omega}_d] + \hat{\alpha}_{12}\omega\cos\theta \\ &= \frac{c\hat{\alpha}_{11}}{J}\left[(L_d - L_q)I_dI_q + \phi_f I_q - T\sin\theta\right] - \hat{\alpha}_{11}(c\dot{\omega}_d + \ddot{\omega}_d) + \hat{\alpha}_{12}\omega\cos\theta \\ &= \alpha_0^\top\varphi_0 + A_\omega \end{aligned} \tag{A.14}$$

where α_0, φ_0 and A_ω are defined by

$$\alpha_0^\top = \left[\frac{L_d - L_q}{J},\ \frac{\phi_f}{J},\ -\frac{T}{J}\right] \tag{A.15}$$

$$\varphi_0^\top = c\hat{\alpha}_{11}[I_dI_q,\ I_q,\ \sin\theta] \tag{A.16}$$

$$A_\omega = -\hat{\alpha}_{11}(c\dot{\omega}_d + \ddot{\omega}_d) + \hat{\alpha}_{12}\omega\cos\theta \tag{A.17}$$

Multiplying (A.12) by L_q and substituting (12.43), we obtain

$$
\begin{aligned}
L_q\dot{z}_1 &= u_q - \omega L_d I_d - RI_d - \omega\phi_f \\
&\quad + L_q\left[\sigma_1\varphi_1^\top\Gamma_1^{-1}\varphi_1 + \alpha_0^\top\varphi_0 + A_\omega + \frac{k_1}{J'}(\alpha_1^\top\varphi_1 + I_q + L'I_dI_q)\right] \\
&= u_q + \alpha_2^\top\varphi_2 + \alpha_3^\top\varphi_3 \qquad \text{(A.18)}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_2^\top &= \left[L_q\frac{\phi_f}{J},\ L_q\frac{L}{J},\ -L_q\frac{T}{J}\right] \\
\varphi_2^\top &= [(k_1 + c\hat{\alpha}_{11})I_q,\ (k_1 + c\hat{\alpha}_{11})I_dI_q,\ (k_1 + c\hat{\alpha}_{11})\sin\theta] \\
\alpha_3^\top &= [-L_d,\ -R,\ -\phi_f,\ L_q] \\
\varphi_3^\top &= [\omega I_d,\ I_d,\ \omega,\ k_1(c\dot{e} - \dot{\omega}_d) + \sigma_1\varphi_1^\top\Gamma_1^{-1}\varphi_1 + A_\omega]
\end{aligned}
$$

References

[1] Jian-Xin Xu, Tong-Heng Lee and Qing-Wei Jia, (1997) 'An adaptive Robust Control Scheme for a Class of Nonlinear Uncertain Systems', *International Journal of Systems Science*, Vol. 28, No. 4, 429–434.

[2] Kokotovic, P. V., (1992) 'The Joy of Feedback: Nonlinear and Adaptive', *IEEE Control systems*, Vol. 12, No.6, 7–17.

[3] Teh-Lu Liao, Li-Chen Fu and Chen-Fa Hsu, (1990) 'Adaptive Robust Tracking of Nonlinear Systems and With an Application to a Robotic Manipulator', *Systems & Control Letters*, Vol. 15, 339–348.

[4] Brogliato, B. and Trofino Neto, A. (1995) 'Practical Stabilization of a Class of Nonlinear Systems with Partially Known Uncertainties', *Automatica*, Vol. 31, No. 1, 145–150.

[5] Ioannou, P. A. and Jing Sun, (1996) *Robust Adaptive Control*. Prentice-Hall, Inc.

[6] Ioannou, P. A. and Kokotović, P. V. (1983) *Adaptive Systems with Reduced Models*. Springer-Verlag, Berlin Heidelberg.

[7] Peterson, B. B. and Narendra, K. S. (1982) 'Bounded Error Adaptive Control', *IEEE Transac. Autom. Contr.*, Vol. 27, No. 6, 1161–1168.

[8] Taylor, D. G. Kokotović, P. V. Marino, R. and Kanellakopoulos, I. (1989) 'Adaptive Regulation of Nonlinear Systems with Unmodeled Dynamics', *IEEE Trans Autom. Control,* Vol. 34, No. 1, 405–412.

[9] Sastry, S. S. and Isidori, A. (1989) 'Adaptive Control of Linearizable Systems', *IEEE Transac. Autom. Contr.*, Vol. 34, No. 11, 1123–1131.

[10] Narendra, K. S. and Annaswamy, A. M. (1989) *Stable Adaptive Systems*. Prentice-Hall, Englewood Cliffs, NJ.

[11] Kokotović, P. V. (1991) *Foundations of Adaptive Control*. Edited by Thomam, M. and Wyner, A. Springer-Verlag, CCES.

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