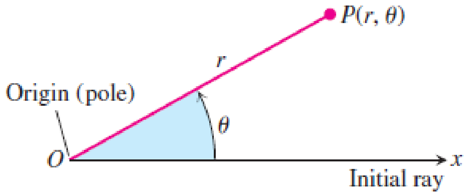
Polar Coordinates and Graphs

Polar Coordinate system



Each point P can be assigned polar Coordinates (r, θ) where:

1) r is the distance from the pole (origin) 0 to the point P. r is positive if measured from the pole along the terminal side of θ and negative if measured along the terminal side extended through the pole.

2) θ is the angle from the Initial ray to (op). The angle θ is positive if the rotation is counterclockwise and negative if the rotation is clockwise.

Review in trigonometric functions:

$$\left.\begin{aligned}\sin(-\theta) &= -\sin\theta \\ \csc(-\theta) &= -\csc\theta \\ \tan(-\theta) &= -\tan\theta \\ \cot(-\theta) &= -\cot\theta\end{aligned}\right\} \text{ odd functions}$$

$$\left.\begin{aligned}\cos(-\theta) &= \cos\theta \\ \sec(-\theta) &= \sec\theta\end{aligned}\right\} \text{ even functions}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \quad \text{if} \quad x=y \quad \Rightarrow \sin(2x) = 2\sin x\cos x$$
$$\cos(x\mp y) = \cos x\cos y \pm \sin x\sin y \quad \text{if} \quad x=y \quad \Rightarrow \cos(2x) = \cos^2 x - \sin^2 x$$
$$\tan(x\mp y) = \frac{\tan x \mp \tan y}{1\pm\tan x\tan y} \quad \text{if} \quad x=y \quad \Rightarrow \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Converting from polar to rectangular form and vice versa

We have the following relationship between rectangular Coordinates (Cartesian) (x, y) and polar Coordinates (r,θ):

$$x^2 + y^2 = r^2$$

$$\cos\theta = \frac{x}{r} \qquad \text{or} \qquad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad \text{or} \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad \text{or} \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$dA = \left.\begin{matrix} dydx \\ dxdy \end{matrix}\right\} \quad \Rightarrow \quad r\ \mathrm{d}r\ \mathrm{d}\theta$$

Cartesian Coordinates

$y = f(x)$

Polar Coordinates

$r = f(\theta)$

Graphing polar equations

Sketch

i) symmetric about x-axis if replacing θ by $(-\theta)$ does not change the function.
ii) Symmetric about y-axis if replacing θ by $(\pi-\theta)$ does not change the function.
iii) Symmetric about the origin if replacing r by $(-r)$ does not change the function.
iv)

$$\theta = 0$$
$$\frac{\pi}{2}$$
$$\pi$$
$$\vdots$$

<u>Ex.1</u>: Converting an equation from Cartesian form to polar form

$x^2 + y^2 - 4y = 0$

Since $x^2 + y^2 = r^2$ and $y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 - 4y = 0$

$r^2 - 4r\sin\theta = 0$

$r(r - 4\sin\theta) = 0$

$r = 0 \quad$ or $\quad r = 4\sin\theta$

the graph of $r = 0$ is the pole. because the pole is included in the graph of $r - 4\sin\theta = 0$, we can discared $r = 0$ and keep only $r = 4\sin\theta$

<u>Ex</u> 2: Converting an equation from polar form to Cartesian form

$r = -3\cos\theta$

$r^2 = -3r\cos\theta$ Multiply both sides by r

$\Rightarrow \quad x^2 + y^2 = -3x$

$\Rightarrow \quad x^2 + y^2 + 3x = 0$

<u>Ex</u> 3: Converting an equation from polar form to Cartesian form

$r\cos(\theta - \pi/3) = 3$

$r(\cos\theta\cos(\pi/3) + \sin\theta\sin(\pi/3)) = 3$

$\frac{1}{2} r\cos\theta + \frac{\sqrt{3}}{2} r\sin\theta = 3$

$\frac{1}{2} x + \frac{\sqrt{3}}{2} y = 3 \Rightarrow x + \sqrt{3}y = 6$

<u>Ex</u> 4: Converting an equation from polar form to Cartesian form

$r = 4\cos\theta$

$r^2 = 4r\cos\theta \Rightarrow x^2 + y^2 = 4x$

Polar Integral:

Some important curves

$r = a$, $r = a\sin\theta$, $r = a\cos\theta \}$ *circle*

$r = a(1-\cos\theta)$, $r = a(1+\cos\theta)$
$r = a(1-\sin\theta)$, $r = a(1+\sin\theta)$ } cordioid

$r = a\sin 3\theta$, $r = a\cos 3\theta \}$ 3 Leafed rose

$r = a\sin 2\theta$, $r = a\cos 2\theta \}$ 4 Leafed rose

$r^2 = a^2\cos 2\theta$

Standard Polar Graphs

1) Circle

a) $r = a$

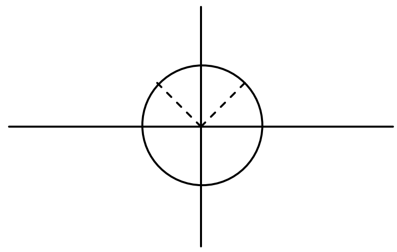
$r = 2$

$\theta = 0 \Rightarrow r = 2$

$\theta = \frac{\pi}{4} \Rightarrow r = 2$

$\theta = \frac{\pi}{2} \Rightarrow r = 2$

$\vdots$



b) $r = a\sin\theta$

i) replace θ by -θ

$$\therefore r = a\sin(-\theta) \Rightarrow r = -a\sin\theta$$

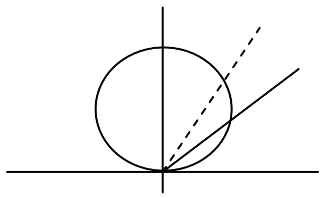
Not symmetric about x-axis

ii) replace θ by π-θ

$$\therefore r = a\sin(\pi-\theta) \Rightarrow r = a\sin\theta$$

symmetric about y-axis

iii) Not symmetric about the origin.



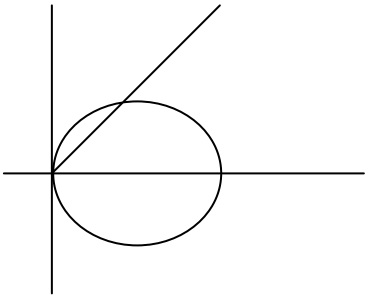
θ	r
0	0
$\frac{\pi}{2}$	a
$\frac{\pi}{6}$	$\frac{a}{2}$

c) $r = a\cos\theta$

i) replace θ by -θ

$$\therefore r = a\cos(-\theta) \quad \Rightarrow \quad r = a\cos\theta$$

symmetric about x-axis

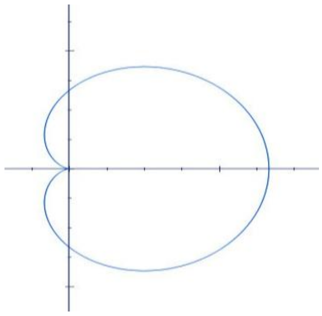


θ	r
0	a
$\frac{\pi}{2}$	0
$\frac{\pi}{3}$	$\frac{a}{2}$

2) Cardioids

a) $r = a(1+\cos\theta)$ Symmetric about x-axis

θ	r
0	2a
$\frac{\pi}{2}$	a
π	0
$\frac{\pi}{3}$	$\frac{3a}{2}$
$\frac{2\pi}{3}$	$\frac{a}{2}$

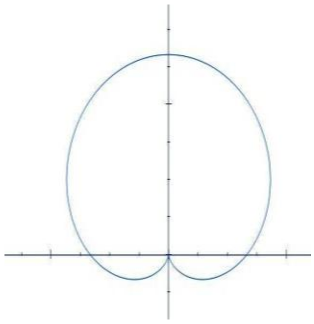


Rapid polar sketching

Ex: Sketch $r = 4(1+\cos\theta)$

θ varies from	Cos θ varies from	4 cos θ varies from	$r = 4(1+\cos\theta)$ varies from
0 to π/2	1 to 0	4 to 0	8 to 4
π/2 to π	0 to -1	0 to -4	4 to 0
π to 3π/2	-1 to 0	-4 to 0	0 to 4
3π/2 to 2π	0 to 1	0 to 4	4 to 8

b) $r = a(1+\sin\theta)$



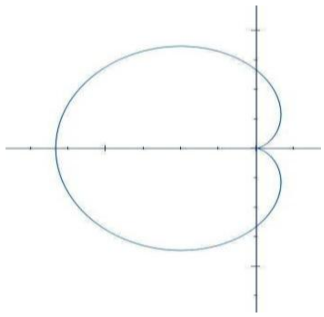
H.W

$r = a(1-\cos\theta)$

$r = a(1-\sin\theta)$

EX.: Find the area of the region enclosed by the cardioids

$r = a(1-\cos\theta)$



$$A = 2\int_0^{\pi}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi} \left.\frac{r^2}{2}\right|_0^{1-\cos\theta} d\theta = \int_0^{\pi}(1-\cos\theta)^2 d\theta$$

$$= \int_0^{\pi}(1-2\cos\theta+\cos^2\theta)d\theta = \int_0^{\pi}(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta))d\theta$$

$$= (\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\Big|_0^{\pi} = \frac{3\pi}{2}$$

Problems

1) Converting equations from Cartesian form to polar form

a) $x^2 + y^2 - 6x = 0$

b) $y^2 = 5y - x^2$

c) $y^2 = 4x$

d) $2xy = 1$

2) Converting an equation from polar form to Cartesian form

a) $r + 2\sin\theta = 0$

b) $r(3\cos\theta - 4\sin\theta = -1$

c) $r = 4$

d) $\theta = \frac{\pi}{4}$

3) a) sketch $r = 5(1 + \sin\theta)$

b) sketch $r = 8\ \cos 2\theta$

4) change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral

a) $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

b) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx$

c) $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2 + y^2)dxdy$

d) $\int_{0}^{6}\int_{0}^{y} x\,dydx$

5) Use polar coordinate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \frac{\mathrm{dydx}}{(1+\mathrm{x}^2+\mathrm{y}^2)^{3/2}}$

6) Find the area of the region R that lies inside the cardioid $r = (1 + \cos\theta)$ and outside the circle $r = 1$.

7) Find the area of the region R that lies inside the cardioid $r = 2(1 + \cos\theta)$ and outside the circle $r = 3$.

8) Find the area of the region R that lies inside the circle $r = 4(\sin\theta)$ and outside the circle $r = 2$.

9) Find the area of the region R cut from the first quadrant by the cardioid $r = (1 + \sin\theta)$.

10) Find the area of the region common to the $r = (1 + \cos\theta)$ and $r = (1 - \cos\theta)$.

References:

1- calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**