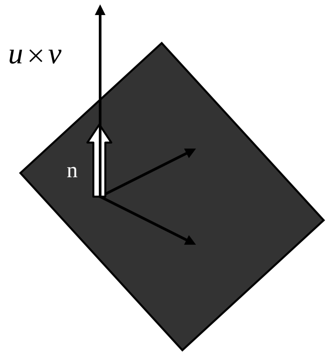
<u>Vectors:</u>

2) The Cross product

The cross product is also called ***vector*** product because the product results a vector.

Def.: The cross product $u \times v = (|u||v| \sin\theta)\, n$, **n** unit vector (***normal***) ***perpendicular to the plane.***

Note: *The vector* $u \times v$ *is orthogonal to both* u and v



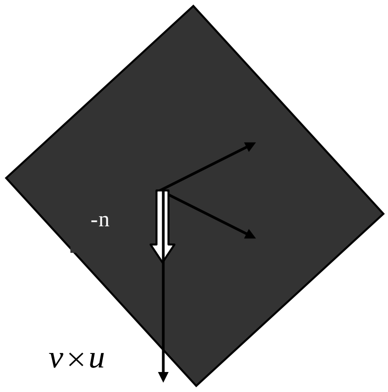
Parallel vectors

Nonzero vectors u and v are parallel if and only if $u \times v = 0$.

Properties of the cross product

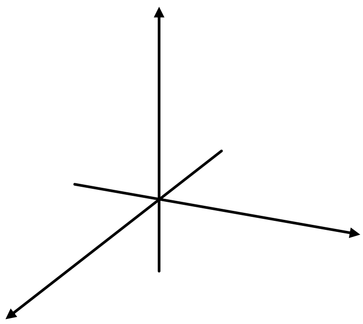
If u , v and w are any vectors and r , s are scalars, then

1) $(ru) \times (sv) = (rs)(v \times u)$
2) $u \times (v + w) = u \times v + u \times w$
3) $(v + w) \times u = v \times u + w \times u$
4) $v \times u = -(u \times v)$
5) $0 \times u = 0$



Notes:

$$i \times j = -(j \times i) = k$$
$$j \times k = -(k \times j) = i$$
$$k \times i = -(i \times k) = j$$



$$\left.\begin{array}{l} i \times i \\ j \times j \\ k \times k \end{array}\right] = 0$$

Vectors:

Calculating Cross product using determinants

If $u = u_1 i + u_2 j + u_3 \mathrm{k}$ and $v = v_1 i + v_2 j + v_3 \mathrm{k}$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex.:

Find $u \times v$ and $v \times u$ if $u = 2i + j + \mathrm{k}$ and $v = -4i + 3j + \mathrm{k}$

Solution

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= -2\mathrm{i} - 6\mathrm{j} + 10\mathrm{k}$$

$$v \times u = -(u \times v) = 2\mathrm{i} + 6\mathrm{j} - 10\mathrm{k}$$

Ex.: Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors.

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= 6\mathrm{i} + 6\mathrm{k}$$

Ex.: Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution

Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction **n** is a unit vector perpendicular to the plane

Vectors:

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{6\mathrm{i} + 6\mathrm{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k$$

Calculating the Triple scalar product (volume): also called **Box product**

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Ex.:

Find the volume of the box determined by $u = i + 2j - \mathrm{k}$, $v = -2i + 3\mathrm{k}$ and $w = 7j - 4\mathrm{k}$.

Solution

$$(u \times v) \cdot w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23$$

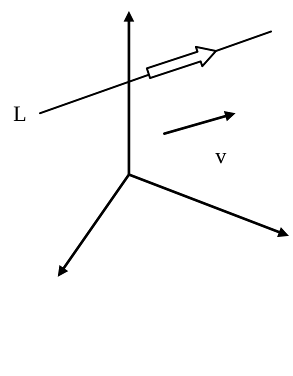
The volume is $|(u \times v) \cdot w| = 23$ units cubed.

Lines and Planes in Space

In the **plane**, ***a line*** is determined by a ***point*** and a ***number giving the slope*** of the line. In **space** a ***line*** is determined by a ***point*** and a ***vector*** giving the direction of the line.

Equations for a line

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $v = v_1 i + v_2 j + v_3 \mathrm{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0 P}$ is parallel to v.



Vectors:

The standard equation of the line through $P_0(x_0, y_0, z_0)$ ***parallel*** to $v = v_1 i + v_2 i + v_3 k$ is:

$$x = x_0 + tv \quad , \quad y = y_0 + tv \quad , \quad z = z_0 + tv \quad , \quad -\infty \prec t \prec \infty$$

and $(x, y, z) = (x_0 + tv \;,\; y_0 + tv \;,\; z_0 + tv)$

Ex.:

Find the equations for the line through $(-2,0,4)$ parallel to $v = 2i + 4j - 2k$.

Solution

With $P_0(x_0, y_0, z_0)$ equal to $(-2,0,4)$ and $v = v_1 i + v_2 j + v_3 k$ equal to $v = 2i + 4j - 2k$

$$x = -2 + 2t \quad , \quad y = 4t \quad , \quad z = 4 - 2t$$

Ex.: Find the equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution

The vector $\overrightarrow{PQ} = 4i - 3j + 7k$ is parallel to the line and equation with $(x_0, y_0, z_0) = (-3,2,-3)$ give

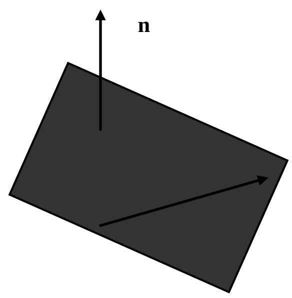
$$x = -3 + 4t \quad , \quad y = 2 - 3t \quad , \quad z = -3 + 7t$$

We could have choose $Q(1,-1,4)$

$$x = 1 + 4t \quad , \quad y = -1 - 3t \quad , \quad z = 4 + 7t$$

An equation for a Plane in space

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is ***normal*** to the nonzero vector $n = Ai + Bj + Ck$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is ***orthogonal*** to **n.**



Thus, the plane through $P_0(x_0, y_0, z_0)$ ***normal*** to r

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or $\quad Ax + By + Cz = D \quad , \quad$ where $D = Ax_0 + By_0 + Cz_0$

Ex.:

Find an equation for the plane through $P_0(-3,0,7)$ perpendicular to $n = 5i + 2j - \mathrm{k}$.

Solution

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$
$$5(x-(-3))+2(y-0)+(-1)(z-7)=0$$
$$5x+15+2y-z+7=0$$
$$5x+2y-z=-22$$

Notice in this example how the components of $n = 5i + 2j - \mathrm{k}$ become the coefficients of x , y *and* z in equation $5x+2y-z=-22$. The vector $n = Ai + Bj + C\mathrm{k}$ is normal to the plane $Ax + By + Cz = D$.

Ex.:

Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Solution

We find a vector ***normal*** to the plane and use it with one of the point to write an equation for the plane.

The cross product:

$$AB \times AC = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k$$ is ***normal*** to the plane.

$$3(x-0)+2(y-0)+6(z-1)=0$$
$$3x+2y+6z=6$$

Lines of intersection

- Two lines are parallel if and only if they have the same direction.
- Two planes are parallel if and only if their normals are parallel.
- The planes that are not parallel ***intersect in a line.***

Ex.:

Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$

and $2x + y - 2z = 5$.

Solution

The line of intersection of two planes is perpendicular to both ***planes' normal*** vectors n_1 and n_2 and therefore parallel to $n_1 \times n_2$. i.e. $n_1 \times n_2$ is a vector parallel to the planes' line of intersection.

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14i + 2j + 15k$$

Ex.: Find the point where the line $\mathbf{x} = \frac{8}{3} + 2\mathbf{t}$, $\mathbf{y} = -2\mathbf{t}$, $\mathbf{z} = 1 + \mathbf{t}$ intersects the plane $3x + 2y + 6z = 6$

Solution

The point $\left(\frac{8}{3} + 2t\,,\ -2t\ ,\ 1 + t\right)$

$$3\left(\frac{8}{3} + 2\mathbf{t}\right) + 2\,(-2\mathbf{t}) + 6(1 + \mathbf{t}) = 6$$

$$8 + 6\text{t} - 4\text{t} + 6 + 6\text{t} = 6$$

$$8\text{t} = -8$$

$$\text{t} = -1$$

The point of intersection is $(x, y, z)\Big|_{t=-1} = (\frac{2}{3}\ ,\ 2\,, 0)$

Angles between planes

The angle between two intersecting planes is defined to be the angle determined by ***their normal vectors***.

Ex.:

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

Solution

The vectors $n_1 = 3i - 6j - 2\text{k}$ and $n_2 = 2i + j - 2\text{k}$

Vectors:

are normals to the planes. The angle between them is
$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$$
$$= \cos^{-1}\left(\frac{4}{21}\right)$$

Problems:

1) Sketch the coordinate axes and then include the vectors u , v and $u \times v$ as vectors starting at the origin

   a. $u = \mathrm{i}$, $v = \mathrm{j}$

   b. $u = \mathrm{i} - \mathrm{k}$, $v = \mathrm{j} + \mathrm{k}$

   c. $u = 2\mathrm{i} - \mathrm{j}$, $v = \mathrm{i} + 2\mathrm{j}$

   d. $u = \mathrm{i} + \mathrm{j}$, $v = \mathrm{i} - \mathrm{j}$

2) In the triangle that determined by the points P , Q and R , find a unite vector perpendicular to plane PQR .

   a. $P(1,1,1)$, $Q(2,1,3)$ and $R(3,-1,1)$

   b. $P(-2,2,0)$, $Q(0,1,-1)$ and $R(-1,2,-2)$

3) Let $u = 5i - j + \mathrm{k}$, $v = j - 5\mathrm{k}$ and $w = -15i + 3j - 3\mathrm{k}$. Which vectors, if any, are:

   a. Perpendicular?

   b. Parallel?

4) Find equations for the lines:

   a. The line through the point $P(3,-4,-1)$ parallel to the vector $i + j + \mathrm{k}$.

   b. The line through $P(1,2,-1)$ and $Q(-1,0,1)$.

   c. The line through the origin parallel to the vector $2j + \mathrm{k}$.

   d. The line through the point $(3,-2,1)$ parallel to the line

      $x = 1 + 2t$, $y = 2 - t$, $z = 3t$

   e. The line through $(1,1,1)$ parallel to the z-axis.

   f. The line through $(2,4,5)$ perpendicular to the plane $3x + 7y - 5z = 21$

Vectors:

g. The line through $(0,-7,0)$ perpendicular to the plane $x+2y+2z=13$

h. The line through $(2,3,0)$ perpendicular to the vectors $u=i+2j+3k$ and $v=3i+4j+5k$

i. The x - axis.

j. The z - axis.

5) Find equations for the planes:

a. The plane through $P_0(0,2,-1)$ normal to $n=3i-2j-k$

b. The plane through $(1,-1,3)$ parallel to the plane $3x+y+z=7$

c. The plane through $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$

d. The plane through $P_0(2,4,5)$ perpendicular to the line

$x=5+t$, $y=1+3t$, $z=4t$

e. The plane through $A(1,-2,1)$ perpendicular to the vector from the origin to A .

6) Find the plane determined by the intersecting lines:

$L1:$ $x=-1+t$, $y=2+t$, $z=1-t$ $-\infty \prec t \prec \infty$

$L2:$ $x=1-4s$, $y=1+2s$, $z=2-2s$ $-\infty \prec s \prec \infty$

7) Find a plane through $P_0(2,1,-1)$ perpendicular to the line of intersection of the planes $2x+y-z=3$, $x+2y+z=2$.

8) Find a plane through the points $P_1(1,2,3)$, $P_2(3,2,1)$ perpendicular to the plane $4x-y+2z=7$.

9) Find the angles between the planes:

a. $x+y=1$, $2x+y-2z=2$

b. $5x+y-z=10$, $x-2y+3z=-1$

10) Find the point in which the line meets the plane.

a. $x=1-t$, $y=3t$, $z=1+t$, $2x-y+3z=6$

b. $x=2$, $y=3+2t$, $z=-2-2t$, $6x+3y-4z=-12$

References:

1- Calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**