Partial Differentiations

$Z = f(x,y)$ or $f(x,y,z) = 0$

$$\left.\begin{array}{l}\dfrac{\partial Z}{\partial x} = Z_x = f_x \\[2ex] \dfrac{\partial Z}{\partial y} = Z_y = f_y\end{array}\right\} \quad 1^{st} \text{ partial derivatives}$$

$$\left.\begin{array}{l}\dfrac{\partial^2 Z}{\partial x^2} = Z_{xx} = f_{xx} \\ \dfrac{\partial^2 Z}{\partial y^2} = Z_{yy} = f_{yy} \\ \dfrac{\partial^2 Z}{\partial y \partial x} = Z_{yx} \\ \dfrac{\partial^2 Z}{\partial x \partial y} = Z_{xy}\end{array}\right\} \quad 2^{nd} \text{ partial derivatives}$$

$$Z_{xy} = Z_{yx}$$

Ex.1

If $Z = x^y$, find $\dfrac{\partial Z}{\partial x}$, $\dfrac{\partial Z}{\partial y}$

$\dfrac{\partial Z}{\partial x} = y\, x^{y-1}$ y constant , $\dfrac{\partial Z}{\partial y} = x^y \cdot \ln x \cdot dy$, x constant $\Rightarrow$ power function

Ex.2

If $Z = \tan^{-1}\dfrac{y}{x}$, show that $Z_{yx} = Z_{xy}$

$$Z_x = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x} = \frac{-\mathbf{y}}{\mathbf{x^2} + \mathbf{y^2}}$$

$$Z_{yx} = \frac{(x^2+y^2)(-1) + y.2y}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \cdots (1)$$

$$Z_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$Z_{xy} = \frac{(x^2+y^2)(1)+x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \cdots (2)$$

(1) & (2) are equal

Properties:

1) If $\omega = f(\upsilon)$, $\upsilon = g(x,y)$

$$\left.\begin{aligned} \frac{\partial\omega}{\partial x} &= \frac{\partial\omega}{\partial\upsilon}\cdot\frac{\partial\upsilon}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial\upsilon}\cdot\frac{\partial\upsilon}{\partial x} \\ \frac{\partial\omega}{\partial y} &= \frac{\partial\omega}{\partial\upsilon}\cdot\frac{\partial\upsilon}{\partial y} \end{aligned}\right\} \text{chain rule}$$

2) If $\omega = f(x,y)$, $x = g(r,s)$, $y = h(r,s)\backslash$

$$\left.\begin{aligned} \frac{\partial\omega}{\partial r} &= \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial r} \\ \frac{\partial\omega}{\partial r}s &= \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial s} \end{aligned}\right\} \text{chain rule}$$

3) Total differential

If $\omega = f(x,y,z,.....)$

$$d\omega = f_x dx + f_y dy + f_z dz + ...$$

$$or \quad d\omega = \omega_x dx + \omega_y dy + \omega_z dz + ...$$

Ex.1

If $\omega = f(x,y,z) = xyz + x^2 + y^2 + z^2$, Find $\frac{d\omega}{dx}$

By property (3)

$$d\omega = \omega_x dx + \omega_y dy + \omega_z dz$$
$$d\omega = (yz + 2x)dx + (xz + 2y)dy + (xy + 2z)dz$$
$$\frac{d\omega}{dx} = (yz + 2x)\frac{dx}{dx} + (xz + 2y)\frac{dy}{dx} + (xy + 2z)\frac{dz}{dx}$$

Ex.2

If $\omega = f(x + ct) + g(x - ct)$ ..... (1) , Show that $\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$

There are two methods to solve this Ex.

First method:

Let $x + ct = r$, $x - ct = s$

Eq.(1) becomes

$$\omega = f(r) + g(s) \qquad ..... (1)$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial g(s)}{\partial s} \cdot \frac{\partial s}{\partial t}$$
$$= f'(r) \cdot c + g'(s) \cdot (-c)$$

$$\frac{\partial^2 \omega}{\partial t^2} = c\left[\frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial t}\right] - c\left[\frac{\partial g(s)}{\partial s} \cdot \frac{\partial s}{\partial t}\right]$$

$$= c\left(f''(r) \cdot c\right) - c\left(g''(s) \cdot (-c)\right)$$

$$\therefore \frac{\partial^2 \omega}{\partial t^2} = c^2\left(f''(r)\right) + \left(g''(s)\right) \qquad ...... \quad (2)$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial g(s)}{\partial s} \cdot \frac{\partial s}{\partial x}$$
$$= f'(r) \cdot 1 + g'(s) \cdot 1$$

$$\frac{\partial^2 \omega}{\partial x^2} = f''(r) \cdot 1 \cdot 1 + g''(r) \cdot 1 \cdot 1$$

In eq.(2)

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$

Second method:

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$$\frac{\partial \omega}{\partial t} = f'(x+ct)\cdot c + g'(x-ct)\cdot(-c)$$

$$\frac{\partial^2 \omega}{\partial t^2} = f''(x+ct)\cdot c\cdot c + g''(x-ct)\cdot(-c)\cdot(-c)$$

$$\therefore \frac{\partial^2 \omega}{\partial t^2} = c^2\left[f'' + g''\right] \qquad ..... \ (2)$$

$$\frac{\partial \omega}{\partial x} = f'(x+ct)\cdot 1 + g'(x-ct)\cdot 1$$

$$\frac{\partial^2 \omega}{\partial x^2} = f''(x+ct)\cdot 1\cdot 1 + g''(x-ct)\cdot 1\cdot 1$$

$$\therefore \frac{\partial^2 \omega}{\partial x^2} = \left[f'' + g''\right] \qquad ..... \ (3)$$

From (2) & (3)

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$

Ex.3

If $z = x^n \ f\left(\frac{y}{x}\right)$, Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$\frac{\partial z}{\partial x} = x^n \cdot f'(\frac{y}{x})\cdot(\frac{-y}{x^2}) + n\,x^{n-1}\cdot f(\frac{y}{x})$$

$$x\cdot\frac{\partial z}{\partial x} = x\cdot\left[-x^n\cdot x^{-2}\cdot y\ f'(\frac{y}{x}) + n\,x^{n-1}\cdot f(\frac{y}{x})\right]$$

$$x\cdot\frac{\partial z}{\partial x} = -x^{n-1}\cdot y\ f'(\frac{y}{x}) + n\,x^n\cdot f(\frac{y}{x}) \qquad ..... \ (1)$$

$$\frac{\partial z}{\partial y} = x^n\cdot f'(\frac{y}{x})\cdot(\frac{1}{x}) + 0$$

$$y\cdot\frac{\partial z}{\partial y} = x^{n-1}\cdot y\ f'(\frac{y}{x}) \qquad ..... \ (2)$$

From (1) & (2)

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=nx^{n}f(\frac{y}{x})$$
$$=nz$$

Ex.4

Express $\frac{\partial\omega}{\partial r}$ and $\frac{\partial\omega}{\partial s}$ in terms of r & s if $\omega=x+2y+z^{2}$,

$x=\frac{r}{s}$, $\mathrm{y}=r^{2}+\ln \mathrm{s}$, $\mathrm{z}=2r$

$$\frac{\partial\omega}{\partial r}=\frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial r}$$
$$=1\cdot\frac{1}{s}+2\cdot2r+2z\cdot2=\frac{1}{s}+4r+4z=8r+\frac{1}{s}$$

$$\frac{\partial\omega}{\partial s}=\frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial s}$$
$$=1\cdot\frac{-r}{s^{2}}+2\cdot\frac{1}{s}+2z\cdot0=\frac{-r}{s^{2}}\frac{2}{s}$$

<u>Problems:</u>

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

1) $f(x,y,z)=z\sin^{-1}\frac{y}{x}$

2) $f(x,y,z)=\frac{\mathrm{x}(2-\cos 2\mathrm{y})}{\mathrm{x}^{2}+\mathrm{y}^{2}}$

3) Find $\frac{\partial\omega}{\partial\upsilon}$ when

$u=0$, $\upsilon=0$ if $\omega=x^{2}+\frac{y}{x}$, $x=u-2\upsilon+1$, $\mathrm{y}=2\mathrm{u}+\upsilon-2$

4) If $\omega=f\left(\frac{xy}{x^{2}+y^{2}}\right)$, show that $x\frac{\partial\omega}{\partial x}+y\frac{\partial\omega}{\partial y}=0$

5) If $\omega=f(x,y)$, and $x=r\cos\theta$, $\mathrm{y}=r\sin\theta$, show that

$$(\frac{\partial\omega}{\partial x})^{2}+\frac{1}{r^{2}}(\frac{\partial\omega}{\partial y})^{2}=f_{x}^{2}+f_{y}^{2}$$

6) If $f(x,y,z)=0$ & $\mathrm{z}=x+y$, find $\frac{\mathrm{dz}}{\mathrm{d}x}$

7) Find the directional derivative of $f(x,y) = x\tan^{-1}\frac{y}{x}$ at (1,1) in the direction of $\vec{A} = 2\vec{i} - \vec{j}$

8) In which direction is the directional derivative of $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$

9) The D.D. of $f(x,y)$ at $p_0(1,2)$ in the direction towards $p_1(2,3)$ is $2\sqrt{2}$ and the D.D. at $p_0(1,2)$ towards $p_2(1,0)$ is -3 , find D.D. at p_0 towards the origin.

References:

1- calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies (أ. رياض احمد عزت)