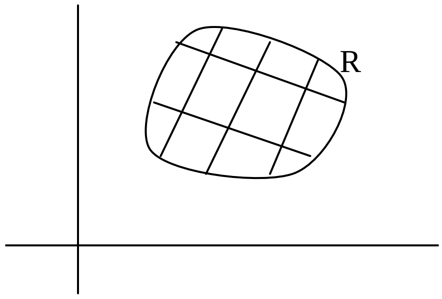
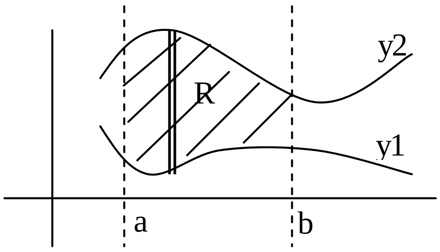
Double Integral

Definition: let R be closed region in the (x, y)- plane. If f is a function of two variables that is define on the region R, then the double integrals on R is written by



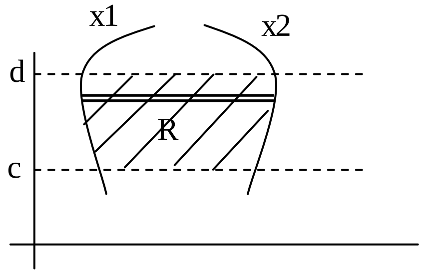
$$\lim_{\substack{n\to\infty \\ \Delta A_r\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\Delta A_r = \iint_R f(x,y)\, dA$$

اذا كانت المنحنيات بهذه الصيغة يؤخذ المقطع شاقولي $dydx$



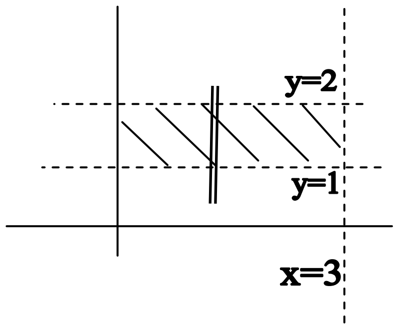
$$\iint_R f(x,y)\, dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\, dy\, dx$$

اما اذا كانت المنحنيات بالشكل التالي يؤخذ المقطع افقيا $dxdy$



$$\iint_R f(x,y)\, dA = \int_c^d \int_{x_1}^{x_2} f(x,y)\, dx\, dy$$

Examples:



1) Evaluate $\int_0^3\int_1^2 (1+8xy)\,dydx$

i) sketch: since dxdy $\Rightarrow$ vertical

$y=1 \quad , \quad y=2$

ii)

$$\int_0^3\int_1^2 (1+8xy)\,dydx = \int_0^3 (y+8x\frac{y^2}{2})\Big|_1^2 dx$$

$$= \int_0^3 \{[2+4x(4)]-[1+4x(1)]\}\,dx$$

$$= \int_0^3 \{[2+16x]-[1+4x]\}\,dx$$

$$= \int_0^3 \{1+12x\}\,dx$$

$$= (x+12\frac{x^2}{2})\Big|_0^3$$

$$= (3+6(9))-(0) = (3+54) = \mathbf{57}$$

2) Evaluate $\iint_R (2x-y^2)\,dA$ over the triangular R enclosed by

$y=1-x \quad , \quad y=1+x \quad , \quad y=3$

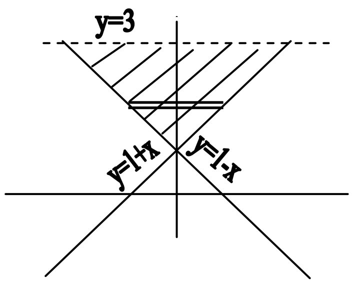
i) sketch:

$y=1-x$, $y=1+x$

$if \quad x=0 \Rightarrow y=1$, $if \quad x=0 \Rightarrow y=1$

$if \quad y=0 \Rightarrow x=1$, $if \quad y=0 \Rightarrow x=-1$

$\Rightarrow (0,1) \,\&\, (1,0)$, $\Rightarrow (0,1) \,\&\, (-1,0)$



$$\left|\iint_R (2x-y^2)dA = \int_1^3 \int_{1-y}^{y-1} (2x-y^2)dxdy\right|$$

$$= \int_1^3 (x^2 - y^2x)\Big|_{1-y}^{y-1} dy \;=\; \int_1^3 \Big\{\big[(y-1)^2 - y^2(y-1)\big] - \big[(1-y)^2 - y^2(1-y)\big]\Big\}dy$$

$$= \int_1^3 \{y^2 - 2y + 1 - y^3 + y^2 - 1 + 2y - y^2 + y^2 - y^3\}dy$$

$$= \int_1^3 (-2y^3 + 2y^2)dy$$

$$= (-2\frac{y^4}{4} + 2\frac{y^3}{3})\Big|_1^3$$

$$= \frac{-18}{2} + 18 - (\frac{-1}{2} + \frac{2}{3}) = 18 - \frac{81}{2} + \frac{1}{2} - \frac{2}{3}$$

$$= \left|18 - \frac{244}{6}\right|$$

3) Evaluate $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\ dx\, dy$

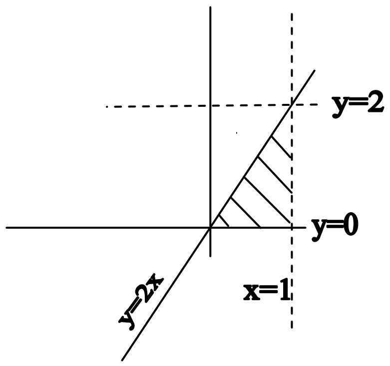
Reverse the order of integration

Since $dxdy$ horizontal

$x = \frac{y}{2} \quad \Rightarrow \quad y = 2x$

$x = 1$

for y from $0 \to 2$



$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\ dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\ dy\, dx \;=\; \int_0^1 e^{x^2} y\Big|_0^{2x} dx$$

$$= \int_0^1 e^{x^2}(2x - 0)dx$$

$$= e^{x^2}\Big|_0^1 = e^1 - e^0 = \mathbf{e - 1}$$

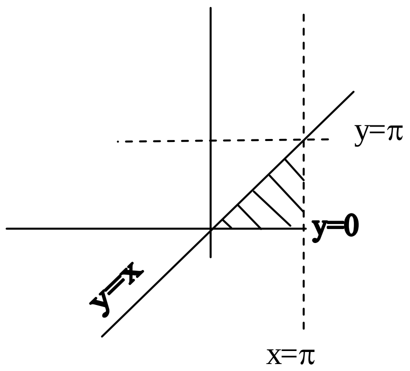
4) Evaluate $\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$

From left $x = y$

From right $x = \pi$

value of y , from $0 \Rightarrow x$

reverse the order



$$\Rightarrow \int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy = \int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^{\pi} \frac{\sin x}{x}\cdot y\Big|_0^{x} dx = \int_0^{\pi} \frac{\sin x}{x}\cdot x\, dx$$

$$= \int_0^{\pi} \sin x\, dx = -\cos x\Big|_0^{\pi} = -(-1-1) = \mathbf{2}$$

5)

$$\int_0^{2}\int_x^{2} 2y^2 \sin xy\, dydx = \int_0^{2}\int_0^{y} 2y^2 \sin xy\, dxdy$$

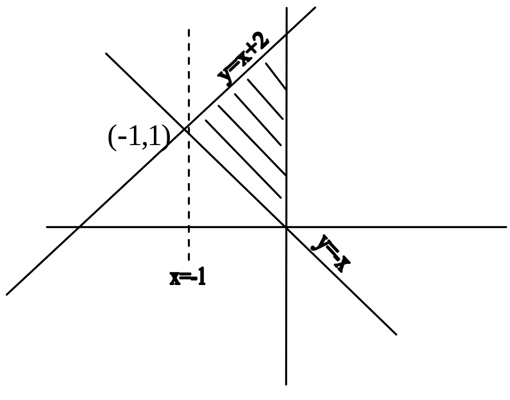
$$= \int_0^{2} [-2y\cos xy]_0^{y}\, dy = \int_0^{2} [-2y\cos y^2 + 2y]\, dy$$

$$= [-\sin y^2 + y^2]_0^{2} = \mathbf{4 - \sin 4}$$

6) Write an equivalent double of integration reversed $\int_{-1}^{0}\int_{-x}^{x+2} (x^2 + y^2)dy\, dx$

$$\int_{-1}^{0}\int_{-x}^{x+2} (x^2 + y^2)dy\, dx$$

$$= \int_0^{1}\int_{-y}^{0} (\mathbf{x^2 + y^2})\mathbf{dxdy} + \int_1^{2}\int_{y-2}^{0} (\mathbf{x^2 + y^2})\mathbf{dxdy}$$



7) Draw the region bounded by $y=e^x$, $y=\sin x$, $x=\pi$, $x=-\pi$ and evaluate its area.

$$\mathbf{A}=\int_{-\pi}^{\pi}\int_{\sin x}^{e^x}\mathbf{dydx}$$

$$=\int_{-\pi}^{\pi}\mathbf{y}\Big|_{\sin x}^{e^x}$$

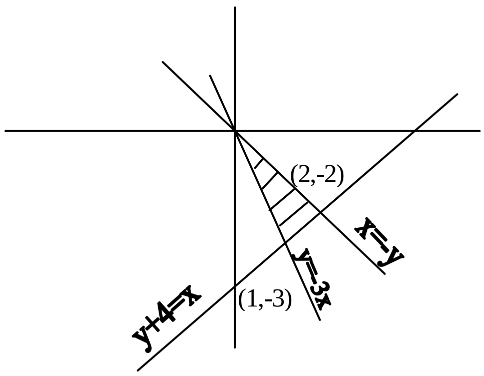
$$=\int_{-\pi}^{\pi}(\mathbf{e^x}-\sin\mathbf{x})\mathbf{dx}$$

$$=\mathbf{e^x}+\cos\mathbf{x}\Big|_{-\pi}^{\pi}$$

$$\mathbf{e}^{\pi}-\mathbf{e}^{-\pi}+\cos\pi-\cos(-\pi)=\mathbf{e}^{\pi}-\mathbf{e}^{-\pi}$$

8) Find the area bounded by $y=-x$, $y=-3x$ and $x=y+4$.

Solution:



$$\mathbf{A}=\int_{-3}^{-2}\int_{\frac{-y}{3}}^{y+4}\mathbf{dxdy}+\int_{-2}^{0}\int_{\frac{-y}{3}}^{-y}\mathbf{dxdy}$$

$$=\int_{-3}^{-2}\mathbf{x}\Big|_{\frac{-4}{3}}^{y+4}\mathbf{dy}+\int_{-2}^{0}\mathbf{x}\Big|_{\frac{-4}{3}}^{-y}\mathbf{dy}$$

$$=\int_{-3}^{-2}(\mathbf{y}+4+\frac{4}{3})\mathbf{dy}+\int_{-2}^{0}(-\mathbf{y}+\frac{\mathbf{y}}{3})\mathbf{dy}$$

$$(\frac{\mathbf{y}^2}{2}+4\mathbf{y}+\frac{\mathbf{y}^2}{6})\Big|_{-3}^{-2}+\frac{-\mathbf{y}^2}{2}+\frac{\mathbf{y}^2}{6}\Big|_{-2}^{0}=2$$

Problems

1) $\int_0^1\int_0^1 \frac{x}{(xy+1)^2}\,dydx$

2) $\int_0^{\ln 2}\int_0^1 xy\,e^{y^2x}\,dydx$

3) $\int_{\frac{\pi}{2}}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\frac{y}{x}\,dydx$

4) $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy$

5) $\int_1^2\int_x^{2x} \frac{x}{y}\,dydx$

6) $\int_0^1\int_0^{\pi} y\cos xy\,dxdy$

7) $\int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dxdy$

8) $\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\cos\theta\,drd\theta$

9) Evaluate $\iint_R dA$ **,** R: 1st quadrant bounded by $y^2 = x$ & $x^2 = y$

10) Evaluate $\iint_R xydA$ **,** R: the region bounded by $y^2 = x$ & $y = x$

11) Evaluate $\iint_R x(1+y^2)^{-\frac{1}{2}}dA$ **,** R: the region in the 1st quadrant enclosed by:

$y = x^2$, $y = 4$, $x = 0$

12) Evaluate $\iint_R \sin(y^3)\,dA$ **,** R: the region bounded by $y = \sqrt{x}$, $y = 2$ & $x = 0$

13) $\int_0^1\int_y^1 x^2 e^{xy}\,dxdy$

References

1- calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**