C. Higher order Differential Equations:

How to find roots of an equation:

Let $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$ be an eq. of degree n, we denote this eq. by $f(x) = 0$ then:

1) r is a root of the eq. $f(x) = 0$ if $f(r) = 0$.
2) r is repeated root of the eq. $f(x) = 0$ if $f'(r) = 0$.
3) If r is a root of the eq. $f(x) = 0$, then r must be a factor of a_n.
4) If r is a root of the eq. $f(x) = 0$, then $(x - r)$ divides $f(x)$.

<u>Ex.:</u> Find all roots of $x^3 + 4x^2 - 3x - 18 = 0$

Solution: $a_n = 18 :\ \mp 1, \mp 2, \mp 3, \mp 6, \mp 9, \mp 18$

$f(2) = 8 + 16 - 6 - 18 = 0$, 2 is a root of the eq.

There are two methods to factorize f(x): *long division* & *fast division*.

First method: Fast division

	1	4	-3	-18
2	↓	2	12	18
	1	6	9	0

$\Rightarrow\ x^2 + 6x + 9 = 0$

$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$x_1 = 2, \quad x_2 = -3, \quad x_3 = -3$

The roots are 2, -3, -3

Second method: long division

$x^3 + 4x^2 - 3x - 18 = 0$

$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$$
\begin{array}{r}
x^2 + 6x + 9 \\
(x-2)\overline{)\,x^3 + 4x^2 - 3x - 18} \\
\underline{\mp x^3 \pm 2x^2} \qquad\qquad \\
6x^2 - 3x \qquad \\
\underline{\mp 6x^2 \pm 12x} \qquad \\
9x - 18 \\
\underline{\mp 9x \pm 18} \\
\mathbf{0} \quad
\end{array}
$$

Higher order linear Differential Equations:

The general form with constant coefficient is:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = F(x) \qquad ...... (1)$$

If $F(x) = 0$ then (1) is called *homogenous, otherwise* (1) is called *nonhomogenous.*

<u>The general solution</u>

The methods of solving second order homogenous D.Eqs. with constant coefficients can be extended to solve higher order homogenous and nonhomogenous D.Eq. with constant coefficients.

a) Homogenous: the characteristic equation of nth order homogenous D. Eq.:
$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = 0$ is:
$r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$

Let r_1, r_1 , r_2 , ... , r_n be the roots of characteristic equation then:

1) If r_1 , r_2 , ... , r_n are all distinct then the solution is:
$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$

2) If r_1 repeated *m* times, then y_h will contain the terms:
$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + ... + c_m x^{m-1} e^{r_1 x}$

3) If some of roots are complex $(r = \alpha \mp i\beta)$ then y_h will contain
$(c_1 \cos \beta x + c_2 \sin \beta x) e^{\alpha x}$

Ex.1: solve y''' -3y''+2y'=0

Solution:

$r^3 - 3r^2 + 2r = 0 \quad \Rightarrow \quad r(r^2 - 3r + 2) = 0$
$r(r - 2)(r - 1) = 0$
$\Rightarrow \quad r_1 = 0 \;\;,\;\; r_2 = 2 \;\;,\;\; r_3 = 1$

are all distinct

$\Rightarrow \quad y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}$
$y_h = c_1 + c_2 e^{2x} + c_3 e^x$

Ex.2:

$y^{(4)}-3y'''+3y''-y'=0$

$r^4-3r^3+3r^2-r=0 \quad \Rightarrow \quad r(r^3-3r^2+3r-1)=0$

$r(r-1)^3=0$

$\Rightarrow \; r_1=0 \; , \; r_2=r_3=r_4=1 \qquad \Rightarrow \; \mathrm{m}=3$

$\Rightarrow y_{\mathrm{h}}=c_1e^{r_1x}+(c_2x^{m-3}+c_3x^{m-2}+c_4x^{m-1})e^{rx}$

$y_{\mathrm{h}}=c_1+c_2e^x+c_3xe^x+c_4x^2e^x$

Ex.3:

$y^{(4)}-3y'''-2y''+2y'+12y=0$

$r^4-3r^3-2r^2+2r+12=0$

$r=2$ is a root $\Rightarrow$ $(r-2)$ is a factor

	1	-3	-2	2	12
2	↓	2	-2	-8	-12
	1	-1	-4	-6	0

$\Rightarrow \; r^3-r^2-4r-6=0$

$\Rightarrow (r-2)(r^3-r^2-4r-6)=0 \quad , \quad r=3$ root $\Rightarrow$ $(r-3)$ is a factor

	1	-1	-4	-6
3	↓	3	6	6
	1	2	2	0

$\Rightarrow \; r^2+2r+2=0$

$(r-2)(r-3)\,(r^2+2r+2)=0$

$r_1=2, \quad r_2=3, \quad r=-1\mp i \quad \alpha=-1 \; , \; \beta=1$

$\Rightarrow \quad y_{\mathrm{h}}=c_1e^{2x}+c_2e^{3x}+(c_3\cos x+c_4\sin x)e^{-x}$

b) Nonhomogenous: the general form of *n*th order nonhomogenous differential equation is:

$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y+a_ny=F(x) \qquad \text{......} \; (1)$

The general solution is $\mathrm{y_g}=\mathrm{y_h}+y_p$

<u>Methods of finding y_p:</u>

1) Undetermined coefficients

We can extend the methods of solving *second order* non homogenous D.Eqs. with constant coefficients to solve higher order nonhomogenous D.Eq. with constant coefficients.

Ex.1: $y^{(4)}-8y''+16y=-18\sin x$

Solution:

$$y_g = y_h + y_p$$

$y^{(4)}-8y''+16y=0$

$r^4-8r^2+16=0 \Rightarrow (r^2-4)^2=0 \Rightarrow r^2=4 \Rightarrow r=\pm 2$

$y_h=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x}$

let $y_p=A\cos x+B\sin x$, $y'_p=-A\sin x+B\cos x$, $y''_p= -A\cos x-B\sin x$

$y'''_p= A\sin x-B\cos x$, $y^{(4)}_p= A\cos x+B\sin x$

$A\cos x+B\sin x+8A\cos x+8B\sin x+16A\cos x+16B\sin x=-18\sin x$

$25A\cos x+25B\sin x=-18\sin x$

$25A=0 \Rightarrow A=0$

$25B=-18 \Rightarrow B=-18/25$

$$y_g = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x} - \frac{18}{25}\sin x$$

2) Variation of parameters

In this method, the particular solution yp has the form $y_p=v_1u_1+v_2u_2+\dots+v_nu_n$
Where $u_1, u_2, \dots, u_n$ are taken from $y_h=c_1u_1+c_2u_2+\dots+c_nu_n$.
To find $v_1, v_2, \dots, v_n$, we must solve the following linear eqs. For $v'_1, v'_2, \dots, v'_n$:

$$\begin{aligned}
v'_1u_1 + v'_2u_2 + \dots + v'_nu_n &= 0\\
v'_1u'_1 + v'_2u'_2 + \dots + v'_nu'_n &= 0\\
&\vdots\\
v'_1u_1^{(n-2)} + v'_2u_2^{(n-2)} + \dots + v'_nu_n^{(n-2)} &= 0\\
v'_1u_1^{(n-1)} + v'_2u_2^{(n-1)} + \dots + v'_nu_n^{(n-1)} &= f(x)
\end{aligned}$$

Ex2: solve y'''+y'=secx

Solution:

Let y'''+y'=0

$r^3+r=0 \Rightarrow r(r^2+1)=0 \Rightarrow r=0,\ r^2=-1 \Rightarrow r1=0,\ r=\pm i$

$y_h=c_1+c_2\cos x+c_3\sin x$

$u_1=1,\ u_2=\cos x,\ u_3=\sin x,\ f(x)=\sec x$

$$v_1' + v_2' \cos x + v_3' \sin x = 0$$
$$v_1'(0) + v_2'(-\sin x) + v_3(\cos x) = 0$$
$$v_1'(0) + v_2'(-\cos x) - v_3'(\sin x) = \sec x$$

$$D = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

$$D_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x & -\cos x & -\sin x \end{vmatrix} = \sec x(\sin^2 x + \cos^2 x) = \sec x$$

$$D_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x & -\sin x \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix} = -\cos x \sec x = -1$$

$$D_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \sec x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$v_1' = \frac{D_1}{D} = \sec x \Rightarrow v_1 = \int \sec x dx = \ln(\sec x + \tan x)$$

$$v_2' = \frac{D_2}{D} = -1 \Rightarrow v_2 = \int -1\, dx = -x$$

$$v_3' = \frac{D_3}{D} = -\tan x \Rightarrow v_3 = -\int \tan x\, dx = \ln \cos x$$

$y_p= \ln(\sec x+\tan x)-x\ \cos x-\ln\cos x\ \sin x$

$y_g= c_1+c_2\cos x+c_3\sin x+\ \ln(\sec x+\tan x)-x\ \cos x - \ln\cos x\ \sin x$

Exercise: Solve

1) $y'''-6y''+12y'-8y=0$
2) $y'''-y=0$
3) $y^{(5)}-2y^{(4)}+y'''=0$
4) $y'''-6y''+2y'+36y=0$
5) $y^{(4)}+8y'''+24y''+32y'+16y=0$
6) $y^{(4)}-4y''+4y=0$

Problems: Find the general solution $y'''-6y''+12y'-8y=0$

1) $y^{(4)}+8y''+16y=0$
2) $y^{(4)}+y=x+1$
3) $y'''-3y'+2y=e^x$
4) $y^{(4)}-16y=0$
5) $y'''-y'=4x^3+6x^2$

References:

1- Calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies (أ. رياض احمد عزت)
4- Modern Introduction Differential Equations, Schaum's Outline Series.