

## B. Second Order Differential Equations:

The second order linear differential equations with constant coefficient has the general form is:

$$ay'' + by' + cy = F(x) \quad \dots(1),$$

where a, b and c are constants.

If  $F(x) = 0$  then (1) is called homogenous.

If  $F(x) \neq 0$  then (1) is called non homogenous.

**Ex:**

- 1)  $y'' - x^2y' + \sin x y = 0$  is linear, 2<sup>nd</sup> order, homo.
- 2)  $y'' - (y')^2 + y = \sin x$  is non linear, 2<sup>nd</sup> order, non homo.
- 3)  $y'' + 2yy' = \ln x$

**a) Homogeneous.**

**b) Nonhomogeneous.**

- *Undeterminant coefficients.*
- *Variation of parameters.*

**a) The Second order linear homogenous D.Eq. with constant coefficient:**

The general form is

$$ay'' + by' + cy = 0 \quad \dots(2)$$

where a, b and c are constants.

### The general solution

Put  $y' = Dy$  and  $y'' = D^2y$  in eq. (2) (D is an operator)

$$\Rightarrow a D^2y + bDy + cy = 0$$

$$\Rightarrow (aD^2 + bD + c)y = 0 \quad (\text{using } D\text{-operator})$$

now substitute D by r and leave y then

$$ar^2 + br + c = 0$$

is called characteristic equation of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are two values of  $r$  :

- 1- *real (equal and not equal).*
- 2- *complex.*

**Case 1:** If  $b^2 - 4ac > 0$  then  $r_1$  and  $r_2$  are distinct ( $r_1 \neq r_2$ ) and real roots, and the general solution is  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

**Case 2:** If  $b^2 - 4ac = 0$  then  $r_1 = r_2 = r$ , and the general solution is:

$$y = (c_1 + c_2 x) e^{rx}$$

**Case 3:** If  $b^2 - 4ac < 0$  then the roots are two complex conjugate roots  $r = \alpha \pm i\beta$ ,  $i = \sqrt{-1}$ , and the general solution is:

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

**Ex.1:** Solve  $y'' - 2y' - 3y = 0$

**Solution:**

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0 \quad , \quad y = 1 \quad , \quad y' = r \quad , \quad y'' = r^2$$

$$(r + 1)(r - 3) = 0$$

$$r + 1 = 0 \quad \Rightarrow \quad r = -1$$

$$r - 3 = 0 \quad \Rightarrow \quad r = 3$$

the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x}$$

**Ex.2:** Solve  $y'' - 6y' + 9y = 0$

**Solution:**

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r - 3)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 3$$

$$\therefore y = (c_1 + c_2 x) e^{3x}$$

**Ex.3: Solve**  $y'' + y' + y = 0$

**Solution:**

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0 \quad a = 1, b = 1, c = 1$$

$$r = \frac{-b \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i \quad \alpha = \frac{-1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

$$\therefore y = e^{\frac{-1}{2}x} \left( c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right)$$

**Exercise: solve**

1.  $4y'' - 12y' + 5y = 0$  ans:  $y = c_1 e^{(1/2)x} + c_2 e^{(5/2)x}$
2.  $3y'' - 14y' - 5y = 0$  ans:  $y = c_1 e^{5x} + c_2 e^{(-1/3)x}$
3.  $4y'' + y = 0$  ans:  $y = c_1 \cos(x/2) + c_2 \sin(x/2)$
4.  $y'' - 8y' + 16y = 0$  ans:  $y = c_1 e^{4x} + c_2 x e^{4x}$
5.  $y'' + 9y = 0$  ans:  $y = c_1 \cos 3x + c_2 \sin 3x$

**b) The Second order linear non homogenous D.Eq. with constant coefficient:**

The general form is:  $ay'' + by' + cy = F(x) \dots(3)$

where a, b and c are constants.

**The general solution**

If  $y_h$  is the solution of the homo. D.Eq.  $ay'' + by' + cy = 0$ , then the general solution of eq. (3) is:

$$y = y_h + y_p \quad \begin{array}{l} y_h \text{ (complementary function)} \\ y_p \text{ (particular integral)} \end{array}$$

- i)  $y_h$  is y homo.
- ii)  $y_p$  (use the table)

**Methods of finding  $y_p$ :**

There are two methods:

**1) Undetermined coefficients:**

In this method  $y_p$  depends on the roots  $r_1$ , and  $r_2$  of characteristic equation and on the form of  $F(x)$  in eq. (3) as follows:

$F(x)$	Choice of $y_p$	M.R.
$kx^n$ nth degree polynomial	$k_n x^n + k_{n-1} x^{n-1} + k_{n-2} x^{n-2} + \dots + k_0$	0
$ke^{px}$	$ce^{px}$	p
$k \sin \beta x$ or $k \cos \beta x$	$c_1 \cos \beta x + c_2 \sin \beta x$	$\mp i\beta$

Note: For repeated term (root), multiply by  $x$ .

**Ex.1:** Use the table to write  $y_p$

1)  $F(x) = 3x^2$  ,  $k = 3$  ,  $n = 2$

$$y_p = k_2 x^2 + k_1 x + k_0$$

Mathematics: Lecture 2  
Differential Equations:

$$2) F(x) = \frac{-1}{2} e^{-3x} \quad , \quad k = \frac{-1}{2} \Rightarrow c$$

$$y_p = ce^{-3x}$$

$$3) F(x) = 2 \cos 3x \quad , \quad k = 2 \quad , \quad \beta = 3$$

$$y_p = c_1 \cos 3x + c_2 \sin 3x$$

$$4) F(x) = 3x^2 - 3x + 5 - 2e^{3x} \quad , \quad k = -3 \quad , \quad c = -2$$

$$y_p = k_2 x^2 + k_1 x + k_0 + ce^{3x}$$

$$5) F(x) = 2 \cos x - \frac{1}{2} \sin x$$

$$y_p = c_1 \cos x + c_2 \sin x$$

$$6) F(x) = \sin x - \cos 2x$$

$$y_p = c_1 \cos x + c_2 \sin x + A \cos 2x + B \sin 2x$$

**Ex.2:** Solve  $y'' - y' - 2y = 4x^2 \dots (1)$

**Solution:**

$$y'' - y' - 2y = 0$$

the char. Eq.  $r^2 - r - 2 = 0$

$$(r + 1)(r - 2) = 0$$

$$r_1 = -1, r_2 = 2$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

$f(x) = 4x^2$  is polynomial of second degree then

$$y_p = k_2 x^2 + k_1 x + k_0 \dots (2)$$

$$\Rightarrow y'_p = 2k_2 x + k_1 \quad , \quad y''_p = 2k_2$$

Substitution gives

$$2k_2 - (2k_2 x + k_1) - 2(k_2 x^2 + k_1 x + k_0) = 4x^2$$

$$\text{coeff. of } x^2 : -2k_2 = 4 \Rightarrow k_2 = -2$$

$$\text{coeff. of } x : -2k_2 - 2k_1 = 0 \Rightarrow k_1 = 2$$

$$\text{const} : 2k_2 - k_1 - 2k_0 = 0 \Rightarrow k_0 = -3$$

$$y_p = -2x^2 + 2x - 3$$

$$y_g = y_h + y_p = (c_1 e^{-x} + c_2 e^{2x}) - 2x^2 + 2x - 3$$

**Ex.3:**  $y'' - y' - 2y = e^{3x}$

**Solution:**

$$y'' - y' - 2y = e^{3x} \quad \dots (1)$$

$$y'' - y' - 2y = 0$$

$$r^2 - r - 2 = 0$$

$$(r - 2)(r + 1) = 0 \Rightarrow r_1 = 2, r_2 = -1$$

$$y_h = (c_1 e^{2x} + c_2 e^{-x}), \text{ Put}$$

$$y_p = ce^{3x} \quad \dots (2)$$

$$y'_p = 3ce^{3x}, \quad y''_p = 9ce^{3x}$$

Substitute In (1)

$$9ce^{3x} - 3ce^{3x} - 2ce^{3x} = e^{3x}$$

$$9c - 3c - 2c = 1 \Rightarrow 4c = 1 \Rightarrow c = \frac{1}{4}$$

$$\text{In (2)} \Rightarrow y_p = \frac{1}{4} e^{3x}$$

$$y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4} e^{3x}$$

### Modification rule قاعدة التعديل

- (١) إذا كان  $F(x) = kx^n$  وكان احد جذري المعادلة القياسية = ٠ ← يضرب  $y_p$  السابق في  $x$ .  
(٢)  
- a إذا كان  $F(x) = ke^{px}$  وكان احد جذري المعادلة القياسية =  $p$  ← يضرب  $y_p$  السابق في  $x$ .  
- b إذا كان  $F(x) = ke^{px}$  وكان جذري المعادلة القياسية =  $p$  ← يضرب  $y_p$  السابق في  $x^2$ .  
(٣) إذا كان  $F(x) = \begin{cases} k \cos \beta x \\ k \sin \beta x \end{cases}$  وكان  $r = \mp i\beta, \alpha = 0$  ← يضرب  $y_p$  السابق في  $x$ .

**Ex.4:** Solve  $y'' + y = \sin x$

Solution:

$$y'' + y = 0$$

$$r^2 + 1 = 0, r^2 = -1 \Rightarrow r = \pm i, \alpha = 0, \beta = 1$$

$$y_h = c_1 \cos x + c_2 \sin x$$

$$y_p = x(c_3 \cos x + c_4 \sin x), \quad y'_p = x(-c_3 \sin x + c_4 \cos x) + (c_3 \cos x + c_4 \sin x)$$

$$y''_p = x(-c_3 \cos x - c_4 \sin x) + (-c_3 \sin x + c_4 \cos x) + (-c_3 \sin x + c_4 \cos x)$$

Substitution gives

$$-2c_3 \sin x + 2c_4 \cos x = \sin x$$

$$-2c_3 = 1 \Rightarrow c_3 = -1/2, 2c_4 = 0 \Rightarrow c_4 = 0$$

$$y_g = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x$$

**Exercise:** Find the general solution

1)  $y'' = 9x^2 + 2x - 1$

2)  $y'' - y' - 2y = \sin 2x$

3)  $y'' - 5y = 3e^x - 2x + 1$

4)  $y'' + 2y' + y = 3e^{-x}$

5)  $y'' - y' - 2y = x^2 - x$

**2- Variation of parameters.**

Let  $y_h = c_1 u_1 + c_2 u_2$  be the homogenous solution of  $ay'' + by' + cy = F(x)$  and the particular solution has the form  $y_p = u_1 v_1 + u_2 v_2$  where  $v_1$  and  $v_2$  are unknown functions of  $x$  which must be determined, first solve the following linear equations for  $v'_1$  and  $v'_2$ :

$$v'_1 u_1 + v'_2 u_2 = 0$$

$$v'_1 u'_1 + v'_2 u'_2 = F(x)$$

which can be solved with respect to  $v'_1$  and  $v'_2$  by Grammar rule as follows

$$D = \begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & u_2 \\ F(x) & u'_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} u_1 & 0 \\ u'_1 & F(x) \end{vmatrix}$$

$$\text{and } v'_1 = \frac{D_1}{D}, \quad v'_2 = \frac{D_2}{D}$$

by integration of  $v'_1$  and  $v'_2$  with respect to  $x$  we can find  $v_1$  and  $v_2$ .

**Ex.1:**

$$\text{Solve } y'' - y' - 2y = e^{3x} \quad \dots\dots (1)$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}, \text{ hence}$$

$$u_1 = e^{-x} \Rightarrow u'_1 = -e^{-x}$$

$$u_2 = e^{2x} \Rightarrow u'_2 = 2e^{2x}$$

$$y_p = v_1 u_1 + v_2 u_2$$

$$v'_1 u_1 + v'_2 u_2 = 0$$

$$v'_1 u'_1 + v'_2 u'_2 = F(x)$$

$$v'_1 (e^{-x}) + v'_2 (e^{2x}) = 0$$

$$v'_1 (-e^{-x}) + v'_2 (2e^{2x}) = e^{3x}$$

Solving this system by Cramer rule gives

$$D = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x, \quad D_1 = \begin{vmatrix} 0 & e^{2x} \\ e^{3x} & 2e^{2x} \end{vmatrix} = -e^{5x}, \quad D_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{3x} \end{vmatrix} = e^{2x}$$

$$v'_1 = \frac{-e^{5x}}{3e^x} = \frac{-1}{3} e^{4x} \Rightarrow v_1 = \int \frac{-1}{3} e^{4x} = -\frac{1}{12} e^{4x},$$

$$v'_2 = \frac{e^{2x}}{3e^x} = \frac{1}{3} e^x \Rightarrow v_2 = \int \frac{1}{3} e^x = \frac{1}{3} e^x$$



$$\therefore y_p = -\frac{1}{2}e^{4x}e^{-x} + \frac{1}{3}e^xe^{2x} = \frac{1}{4}e^{3x}$$

the general solution is :  $y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$

**Ex.2: solve**

$$y''+y=\sec x$$

Solution:

$$y''+y=0$$

$$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r = \pm i \quad \alpha=0, \beta=1$$

$$y_h=c_1\cos x + c_2\sin x, \quad u_1=\cos x, u_2=\sin x, f(x)=\sec x$$

$$y_p= v_1u_1+v_2u_2$$

$$= v_1\cos x + v_2\sin x \quad \text{then}$$

$$v_1'(\cos x) + v_2'(\sin x) = 0$$

$$v_1'(-\sin x) + v_2'(\cos x) = \sec x$$

$$D = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1,$$

$$D_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\sin x \frac{1}{\cos x} = -\tan x,$$

$$D_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$v_1' = \frac{-\tan x}{1} = -\tan x \Rightarrow v_1 = \int \frac{-\sin x}{\cos x} dx = \ln |\cos x|$$

$$v_2' = 1 \Rightarrow v_2 = \int dx = x$$

$$y_p = \ln |\cos x| \cos x + x \sin x$$

$$y_g = c_1\cos x + c_2\sin x + \ln |\cos x| \cos x + x \sin x$$

**Exercise**

1.  $y''-2y'+y = e^x \ln x$

2.  $y''-2y'+y = \frac{e^x}{x^5}$

3.  $y''+4y=\sin^2 2x$

**Problems:** Find the general solution

1-  $y'' + 2y' + y = 4e^{-x} \ln x$

2-  $y'' + 4y' + 20y = 23\sin t - 15\cos t$        $y(0) = 0$  ,    $y'(0) = -1$

3-  $y'' - 4y' + 3y = 4e^{3x}$        $y(0) = -1$  ,    $y'(0) = 3$

4-  $y'' + y = x^2 + x$

5-  $y'' - 2y' + y = \frac{12e^x}{x^3}$

6-  $y'' + 4y = 4 \sec 2x$

**References:**

1- calculus & Analytic Geometry (Thomas).

2- Calculus (Howard Anton).

3- Advanced Mathematics for Engineering Studies (أ. رياض احمد عزت)

4- Modern Introduction Differential Equations, Schaum's Outline Series.