APPENDIX A

VECTOR ANALYSIS

A.1 GENERAL CURVILINEAR COORDINATES

Let us consider a general orthogonal coordinate system in which a point is located by the intersection of three mutually perpendicular surfaces (of unspecified form or shape),

$$u = \text{constant}$$
$$v = \text{constant}$$
$$w = \text{constant}$$

where u, v, and w are the variables of the coordinate system. If each variable is increased by a differential amount and three more mutually perpendicular surfaces are drawn corresponding to these new values, a differential volume is formed which is closely a rectangular parallelepiped. Since u, v, and w need not be measures of length, such as, for example, the angle variables of the cylindrical and spherical coordinate systems, each must be multiplied by a general function of u, v, and w in order to obtain the differential sides of the parallelepiped. Thus we define the scale factors h_1, h_2, and h_3 each as a function of the three variables u, v, and w and write the lengths of the sides of the differential volume as

$$dL_1 = h_1 du$$
$$dL_2 = h_2 dv$$
$$dL_3 = h_3 dw$$

In the three coordinate systems discussed in Chap. 1, it is apparent that the variables and scale factors are

$$\begin{array}{llll} \text{Cartesian:} & u = x & v = y & w = z \\ & h_1 = 1 & h_2 = 1 & h_3 = 1 \\ \text{Cylindrical:} & u = \rho & v = \phi & w = z \\ & h_1 = 1 & h_2 = \rho & h_3 = 1 \\ \text{Spherical:} & u = r & v = \theta & w = \phi \\ & h_1 = 1 & h_2 = r & h_3 = r\sin\theta \end{array} \tag{A.1}$$

The choice of u, v, and w above has been made so that $\mathbf{a}_u \times \mathbf{a}_v = \mathbf{a}_w$ in all cases. More involved expressions for h_1, h_2, and h_3 are to be expected in other less familiar coordinate systems.[1]

A.2 DIVERGENCE, GRADIENT, AND CURL IN GENERAL CURVILINEAR COORDINATES

If the method used to develop divergence in Secs. 3.4 and 3.5 is applied to the general curvilinear coordinate system, the flux of the vector $\mathbf{D}$ passing through the surface of the parallelepiped whose unit normal is $\mathbf{a}_u$ is

$$D_{u0} dL_2 dL_3 + \frac{1}{2}\frac{\partial}{\partial u}(D_u dL_2 dL_3) du$$

or

$$D_{u0} h_2 h_3 dv\, dw + \frac{1}{2}\frac{\partial}{\partial u}(D_u h_2 h_3 dv\, dw) du$$

and for the opposite face it is

$$-D_{u0} h_2 h_3 dv\, dw + \frac{1}{2}\frac{\partial}{\partial u}(D_u h_2 h_3 dv\, dw) du$$

giving a total for these two faces of

$$\frac{\partial}{\partial u}(D_u h_2 h_3 dv\, dw) du$$

Since u, v, and w are independent variables, this last expression may be written as

[1] The variables and scale factors are given for nine orthogonal coordinate systems on pp. 50–59 in J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Company, New York, 1941. Each system is also described briefly.

$$\frac{\partial}{\partial u}(h_2 h_3 D_u) du\, dv\, dw$$

and the other two corresponding expressions obtained by a simple permutation of the subscripts and of u, v, and w. Thus the total flux leaving the differential volume is

$$\left[\frac{\partial}{\partial u}(h_2 h_3 D_u) + \frac{\partial}{\partial v}(h_3 h_1 D_v) + \frac{\partial}{\partial w}(h_1 h_2 D_w)\right] du\, dv\, dw$$

and the divergence of **D** is found by dividing by the differential volume

$$\nabla \cdot \mathbf{D} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 D_u) + \frac{\partial}{\partial v}(h_3 h_1 D_v) + \frac{\partial}{\partial w}(h_1 h_2 D_w)\right] \qquad \text{(A.2)}$$

The components of the gradient of a scalar V may be obtained (following the methods of Sec. 4.6) by expressing the total differential of V,

$$dV = \frac{\partial V}{\partial u} du + \frac{\partial V}{\partial v} dv + \frac{\partial V}{\partial w} dw$$

in terms of the component differential lengths, $h_1 du$, $h_2 dv$, and $h_3 dw$,

$$dV = \frac{1}{h_1}\frac{\partial V}{\partial u} h_1 du + \frac{1}{h_2}\frac{\partial V}{\partial v} h_2 dv + \frac{1}{h_3}\frac{\partial V}{\partial w} h_3 dw$$

Then, since

$$d\mathbf{L} = h_1 du \mathbf{a}_u + h_2 dv \mathbf{a}_v + h_3 dw \mathbf{a}_w \qquad \text{and} \qquad dV = \nabla V \cdot d\mathbf{L}$$

we see that

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\mathbf{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\mathbf{a}_v + \frac{1}{h_3}\frac{\partial V}{\partial w}\mathbf{a}_w \qquad \text{(A.3)}$$

The components of the curl of a vector **H** are obtained by considering a differential path first in a $u =$ constant surface and finding the circulation of **H** about that path, as discussed for cartesian coordinates in Sec. 8.3. The contribution along the segment in the $\mathbf{a}_v$ direction is

$$H_{v0} h_2 dv - \frac{1}{2}\frac{\partial}{\partial w}(H_v h_2 dv) dw$$

and that from the oppositely directed segment is

$$-H_{v0} h_2 dv - \frac{1}{2}\frac{\partial}{\partial w}(H_v h_2 dv) dw$$

The sum of these two parts is

$$-\frac{\partial}{\partial w}(H_v h_2 dv) dw$$

or

$$-\frac{\partial}{\partial w}(h_2 H_v) dv\, dw$$

and the sum of the contributions from the other two sides of the path is

$$\frac{\partial}{\partial v}(h_3 H_w) dv\, dw$$

Adding these two terms and dividing the sum by the enclosed area, $h_2 h_3 dv\, dw$, we see that the $\mathbf{a}_u$ component of curl $\mathbf{H}$ is

$$(\nabla \times \mathbf{H})_u = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial v}(h_3 H_w) - \frac{\partial}{\partial w}(h_2 H_v)\right]$$

and the other two components may be obtained by cyclic permutation. The result is expressible as a determinant,

$$\nabla \times \mathbf{H} = \begin{vmatrix} \dfrac{\mathbf{a}_u}{h_2 h_3} & \dfrac{\mathbf{a}_v}{h_3 h_1} & \dfrac{\mathbf{a}_w}{h_1 h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 H_u & h_2 H_v & h_3 H_w \end{vmatrix} \tag{A.4}$$

The Laplacian of a scalar is found by using (2) and (3):

$$\nabla^2 V = \nabla \cdot \nabla V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial w}\right)\right] \tag{A.5}$$

Equations (2) to (5) may be used to find the divergence, gradient, curl, and Laplacian in any orthogonal coordinate system for which h_1, h_2, and h_3 are known.

Expressions for $\nabla \cdot \mathbf{D}$, ∇V, $\nabla \times \mathbf{H}$, and $\nabla^2 V$ are given in cartesian, circular cylindrical, and spherical coordinate systems inside the back cover.

A.3 VECTOR IDENTITIES

The vector identities listed below may be proved by expansion in cartesian (or general curvilinear) coordinates. The first two identities involve the scalar and vector triple products, the next three are concerned with operations on sums, the following three apply to operations when the argument is multiplied by a scalar function, the next three apply to operations on scalar or vector products, and the last four concern the second-order operations.

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}\equiv(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}\equiv(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B} \tag{A.6}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\equiv(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{A.7}$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B})\equiv\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B} \tag{A.8}$$

$$\nabla(V+W)\equiv\nabla V+\nabla W \tag{A.9}$$

$$\nabla\times(\mathbf{A}+\mathbf{B})\equiv\nabla\times\mathbf{A}+\nabla\times\mathbf{B} \tag{A.10}$$

$$\nabla\cdot(V\mathbf{A})\equiv\mathbf{A}\cdot\nabla V+V\nabla\cdot\mathbf{A} \tag{A.11}$$

$$\nabla(VW)\equiv V\nabla W+W\nabla V \tag{A.12}$$

$$\nabla\times(V\mathbf{A})\equiv\nabla V\times\mathbf{A}+V\nabla\times\mathbf{A} \tag{A.13}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})\equiv\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B} \tag{A.14}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})\equiv(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) \tag{A.15}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})\equiv\mathbf{A}\nabla\cdot\mathbf{B}-\mathbf{B}\nabla\cdot\mathbf{A}+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B} \tag{A.16}$$

$$\nabla\cdot\nabla V\equiv\nabla^2 V \tag{A.17}$$

$$\nabla\cdot\nabla\times\mathbf{A}\equiv 0 \tag{A.18}$$

$$\nabla\times\nabla V\equiv 0 \tag{A.19}$$

$$\nabla\times\nabla\times\mathbf{A}\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} \tag{A.20}$$

APPENDIX B

UNITS

We shall describe first the International System (abbreviated SI, for Système International d'Unités), which is used in this book and is now standard in electrical engineering and much of physics. It has also been officially adopted as the international system of units by many countries, including the United States.[1]

The fundamental unit of length is the meter, which was defined in the latter part of the 19th century as the distance between two marks on a certain platinum-iridium bar. The definition was improved in 1960 by relating the meter to the wavelength of the radiation emitted by the rare gas isotope krypton 86 under certain specified conditions. This so-called krypton meter was accurate to four parts per billion, a value leading to negligible uncertainties in constructing skyscrapers or building highways, but capable of causing an error greater than one meter in determining the distance to the moon. The meter was redefined in 1983 in terms of the velocity of light. At that time the velocity of light was specified to be an auxiliary constant with an *exact* value of 299 792 458 meters per second. As a result, the latest definition of the meter is the distance light travels in a vacuum

[1] The International System of Units was adopted by the Eleventh General Conference on Weights and Measures in Paris in 1960, and it was officially adopted for scientific usage by the National Bureau of Standards in 1964. It is a metric system which interestingly enough is the only system which has ever received specific sanction from Congress. This occurred first in 1966 and then again in 1975 with the Metric Conversion Act, which provides for "voluntary conversion" to the metric system. No specific time was specified, however, and we can assume that it will still be a few years before the bathroom scale reads mass in kilograms and Miss America is a 90–60–90.

in 1/299 792 458 of a second. If greater accuracy is achieved in measuring c, that value will remain 299 792 458 m/s, but the length of the meter will change.

It is evident that our definition of the meter is expressed in terms of the "second," the fundamental unit of time. The second is defined as 9 192 631 770 periods of the transition frequency between the hyperfine levels $F = 4$, $m_F = 0$, and $F = 3$, $m_F = 0$ of the ground state ${}^2s_{1/2}$ of the atom of cesium 133, unperturbed by external fields. This definition of the second, complex though it may be, permits time to be measured with an accuracy better than one part in 10^{13}.

The standard mass of one kilogram is defined as the mass of an international standard in the form of a platinum-iridium cylinder at the International Bureau of Weights and Measures at Sèvres, France.

The unit of temperature is the kelvin, defined by placing the triple-point temperature of water at 273.16 kelvins.

A fifth unit is the candela, defined as the luminous intensity of an omnidirectional radiator at the freezing temperature of platinum (2042 K) having an area of 1/600 000 square meter and under a pressure of 101 325 newtons per square meter.

The last of the fundamental units is the ampere. Before explicitly defining the ampere, we must first define the newton. It is defined in terms of the other fundamental units from Newton's third law as the force required to produce an acceleration of one meter per second per second on a one-kilogram mass. We now may define the ampere as that constant current, flowing in opposite directions in two straight parallel conductors of infinite length and negligible cross section, separated one meter in vacuum, that produces a repulsive force of 2×10^{-7} newton per meter length between the two conductors. The force between the two parallel conductors is known to be

$$F = \mu_0 \frac{I^2}{2\pi d}$$

and thus

$$2 \times 10^{-7} = \mu_0 \frac{1}{2\pi}$$

or

$$\mu_0 = 4\pi 10^{-7} \qquad (\mathrm{kg \cdot m/A^2 \cdot s^2, \text{ or } H/m})$$

We thus find that our definition of the ampere has been formulated in such a way as to assign an exact simple numerical value to the permeability of free space.

Returning to the International System, the units in which the other electric and magnetic quantities are measured are given in the body of the text at the time each quantity is defined, and all of them can be related to the basic units already defined. For example, our work with the plane wave in Chap. 11 shows that the velocity with which an electromagnetic wave propagates in free space is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

and thus

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{4\pi 10^{-7} c^2} = 8.854\,187\,817 \times 10^{-12} \text{ F/m}$$

It is evident that the numerical value of ϵ_0 depends upon the defined value of the velocity of light in vacuum, 299 792 458 m/s.

The units are also given in Table B.1 for easy reference. They are listed in the same order that they are defined in the text.

Finally, other systems of units have been used in electricity and magnetism. In the electrostatic system of units (esu), Coulomb's law is written for free space,

$$F = \frac{Q_1 Q_2}{R^2} \qquad \text{(esu)}$$

The permittivity of free space is assigned the value of unity. The gram and centimeter are the fundamental units of mass and distance, and the esu system is therefore a cgs system. Units bearing the prefix stat- belong to the electrostatic system of units.

In a similar manner, the electromagnetic system of units (emu) is based on Coulomb's law for magnetic poles, and the permeability of free space is unity. The prefix ab- identifies emu units. When electric quantities are expressed in esu units, magnetic quantities in emu units, and both appear in the same equation (such as Maxwell's curl equations), the velocity of light appears explicitly. This follows from noting that in esu $\epsilon_0 = 1$, but $\mu_0 \epsilon_0 = 1/c^2$, and therefore $\mu_0 = 1/c^2$, and in emu $\mu_0 = 1$, and hence $\epsilon_0 = 1/c^2$. Thus, in this intermixed system known as the gaussian system of units,

$$\nabla \times \mathbf{H} = 4\pi \mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad \text{(gaussian)}$$

Other systems include the factor 4π explicitly in Coulomb's law, and it then does not appear in Maxwell's equations. When this is done, the system is said to be rationalized. Hence the gaussian system is an unrationalized cgs system (when rationalized it is known as the Heaviside-Lorentz system), and the International System we have used throughout this book is a rationalized mks system.

Table B.2 gives the conversion factors between the more important units of the International System (or rationalized mks system) and the gaussian system, and several other assorted units.

Table B.3 lists the prefixes used with any of the SI units, their abbreviations, and the power of ten each represents. Those checked are widely used. Both the prefixes and their abbreviations are written without hyphens, and therefore 10^{-6} F = 1 microfarad = 1 μF = 1000 nanofarads = 1000 nF, and so forth.

TABLE B.1
Names and units of the electric and magnetic quantities in the International System (in the order they appear in the text)

Symbol	Name	Unit	Abbreviation
v	Velocity	meter/second	m/s
F	Force	newton	N
Q	Charge	coulomb	C
r, R	Distance	meter	m
ϵ_0, ϵ	Permittivity	farad/meter	F/m
E	Electric field intensity	volt/meter	V/m
ρ_v	Volume charge density	coulomb/meter^3	C/m^3
v	Volume	meter^3	m^3
ρ_L	Linear charge density	coulomb/meter	C/m
ρ_S	Surface charge density	coulomb/meter^2	C/m^2
Ψ	Electric flux	coulomb	C
D	Electric flux density	coulomb/meter^2	C/m^2
S	Area	meter^2	m^2
W	Work, energy	joule	J
L	Length	meter	m
V	Potential	volt	V
p	Dipole moment	coulomb-meter	C·m
I	Current	ampere	A
J	Current density	ampere/meter^2	A/m^2
μ_e, μ_h	Mobility	meter^2/volt-second	m^2/V·s
e	Electronic charge	coulomb	C
σ	Conductivity	siemens/meter	S/m
R	Resistance	ohm	Ω
P	Polarization	coulomb/meter^2	C/m^2
$\chi_{e,m}$	Susceptibility		
C	Capacitance	farad	F
R_s	Sheet resistance	ohm per square	Ω
H	Magnetic field intensity	ampere/meter	A/m
K	Surface current density	ampere/meter	A/m
B	Magnetic flux density	tesla (or weber/meter^2)	T (or Wb/m^2)
μ_0, μ	Permeability	henry/meter	H/m
Φ	Magnetic flux	weber	Wb
V_m	Magnetic scalar potential	ampere	A
A	Vector magnetic potential	weber/meter	Wb/m
T	Torque	newton-meter	N·m
m	Magnetic moment	ampere-meter^2	A·m^2
M	Magnetization	ampere/meter	A/m
$\mathcal{R}$	Reluctance	ampere-turn/weber	A·t/Wb
L	Inductance	henry	H
M	Mutual inductance	henry	H
ω	Radian frequency	radian/second	rad/s
c	Velocity of light	meter/second	m/s
λ	Wavelength	meter	m
η	Intrinsic impedance	ohm	Ω
k	Wave number	meter^{-1}	m^{-1}
α	Attenuation constant	neper/meter	Np/m
β	Phase constant	radian/meter	rad/m
f	Frequency	hertz	Hz

TABLE B.1
(continued)

Symbol	Name	Unit	Abbreviation
$\mathcal{P}$	Poynting vector	watt/meter^2	W/m^2
P	Power	watt	W
δ	Skin depth	meter	m
Γ	Reflection coefficient		
s	Standing-wave ratio		
γ	Propagation constant	meter^{-1}	m^{-1}
G	Conductance	siemen	S
Z	Impedance	ohm	Ω
Y	Admittance	siemen	S
Q	Quality factor		

TABLE B.2
Conversion of International to gaussian and other units (use $c = 2.997\,924\,58 \times 10^8$)

Quantity	1 mks unit	= gaussian units	= other units
d	1 m	10^2 cm	39.37 in
F	1 N	10^5 dyne	0.2248 lb$_f$
W	1 J	10^7 erg	0.7376 ft-lb$_f$
Q	1 C	$10c$ statC	0.1 abC
ρ_v	1 C/m^3	$10^{-5}c$ statC/cm^3	10^{-7} abC/cm^3
D	1 C/m^2	$4\pi 10^{-3}c$ (esu)	$4\pi 10^{-5}$ (emu)
E	1 V/m	$10^4/c$ statV/cm	10^6 abV/cm
V	1 V	$10^6/c$ statV	10^8 abV
I	1 A	0.1 abA	$10c$ statA
H	1 A/m	$4\pi 10^{-3}$ oersted	$0.4\pi c$ (esu)
V_m	1 A·t	0.4π gilbert	$40\pi c$ (esu)
B	1 T	10^4 gauss	$100/c$ (esu)
Φ	1 Wb	10^8 maxwell	$10^6/c$ (esu)
A	1 Wb/m	10^6 maxwell/cm	
R	1 Ω	10^9 abΩ	$10^5/c^2$ statΩ
L	1 H	10^9 abH	$10^5/c^2$ statH
C	1 F	$10^{-5}c^2$ statF	10^{-9} abF
σ	1 S/m	10^{-11} abS/cm	$10^{-7}c^2$ statS/cm
μ	1 H/m	$10^7/4\pi$ (emu)	$10^3/4\pi c^2$ (esu)
ϵ	1 F/m	$4\pi 10^{-7}c^2$ (esu)	$4\pi 10^{-11}$ (emu)

TABLE B.3
Standard prefixes used with SI units

Prefix	Abbrev.	Meaning	Prefix	Abbrev.	Meaning
atto-	a-	10^{-18}	deka-	da-	10^{1}
femto-	f-	10^{-15}	hecto-	h-	10^{2}
pico-	p-	10^{-12}	kilo-	k-	10^{3}
nano-	n-	10^{-9}	mega-	M-	10^{6}
micro-	μ-	10^{-6}	giga-	G-	10^{9}
milli-	m-	10^{-3}	tera-	T-	10^{12}
centi-	c-	10^{-2}	peta-	P-	10^{15}
deci-	d-	10^{-1}	exa-	E-	10^{18}

APPENDIX C

MATERIAL CONSTANTS

Table C.1 lists typical values of the relative permittivity ϵ'_R or dielectric constant for common insulating and dielectric materials, along with representative values for the loss tangent. The values should only be considered representative for each material, and they apply to normal temperature and humidity conditions, and to very low audio frequencies. Most of them have been taken from "Reference Data for Radio Engineers,"[1] "The Standard Handbook for Electrical Engineers,"[2] and von Hippel,[3] and these volumes may be referred to for further information on these and other materials.

Table C.2 gives the conductivity for a number of metallic conductors, for a few insulating materials, and for several other materials of general interest. The values have been taken from the references listed previously, and they apply at zero frequency and at room temperature. The listing is in the order of decreasing conductivity.

Some representative values of the relative permeability for various diamagnetic, paramagnetic, ferrimagnetic, and ferromagnetic materials are listed in Table C.3. They have been extracted from the references listed above, and the

[1] See Suggested References for Chap. 11.

[2] See Suggested References for Chap. 5.

[3] von Hippel, A. R.: "Dielectric Materials and Applications," The Technology Press of the Massachusetts Institute of Technology, Cambridge, MA and John Wiley and Sons, Inc., New York, 1954.

TABLE C.1
ϵ'_R and ϵ''/ϵ'

Material	ϵ'_R	ϵ''/ϵ'
Air	1.0005	
Alcohol, ethyl	25	0.1
Aluminum oxide	8.8	0.000 6
Amber	2.7	0.002
Bakelite	4.74	0.022
Barium titanate	1200	0.013
Carbon dioxide	1.001	
Ferrite (NiZn)	12.4	0.000 25
Germanium	16	
Glass	4–7	0.002
Ice	4.2	0.05
Mica	5.4	0.000 6
Neoprene	6.6	0.011
Nylon	3.5	0.02
Paper	3	0.008
Plexiglas	3.45	0.03
Polyethylene	2.26	0.000 2
Polypropylene	2.25	0.000 3
Polystyrene	2.56	0.000 05
Porcelain (dry process)	6	0.014
Pyranol	4.4	0.000 5
Pyrex glass	4	0.000 6
Quartz (fused)	3.8	0.000 75
Rubber	2.5–3	0.002
Silica or SiO_2 (fused)	3.8	0.000 75
Silicon	11.8	
Snow	3.3	0.5
Sodium chloride	5.9	0.000 1
Soil (dry)	2.8	0.05
Steatite	5.8	0.003
Styrofoam	1.03	0.000 1
Teflon	2.1	0.000 3
Titanium dioxide	100	0.001 5
Water (distilled)	80	0.04
Water (sea)		4
Water (dehydrated)	1	0
Wood (dry)	1.5–4	0.01

data for the ferromagnetic materials is only valid for very low magnetic flux densities. Maximum permeabilities may be an order of magnitude higher.

Values are given in Table C.4 for the charge and rest mass of an electron, the permittivity and permeability of free space, and the velocity of light.[4]

[4] Cohen, E. R., and B. N. Taylor: "The 1986 Adjustment of the Fundamental Physical Constants," Pergamon Press, Elmsford, NY, 1986.

TABLE C.2
σ

Material	σ, S/m	Material	σ, S/m
Silver	6.17×10^7	Graphite	7×10^4
Copper	5.80×10^7	Silicon	2300
Gold	4.10×10^7	Ferrite (typical)	100
Aluminum	3.82×10^7	Water (sea)	5
Tungsten	1.82×10^7	Limestone	10^{-2}
Zinc	1.67×10^7	Clay	5×10^{-3}
Brass	1.5×10^7	Water (fresh)	10^{-3}
Nickel	1.45×10^7	Water (distilled)	10^{-4}
Iron	1.03×10^7	Soil (sandy)	10^{-5}
Phosphor bronze	1×10^7	Granite	10^{-6}
Solder	0.7×10^7	Marble	10^{-8}
Carbon steel	0.6×10^7	Bakelite	10^{-9}
German silver	0.3×10^7	Porcelain (dry process)	10^{-10}
Manganin	0.227×10^7	Diamond	2×10^{-13}
Constantan	0.226×10^7	Polystyrene	10^{-16}
Germanium	0.22×10^7	Quartz	10^{-17}
Stainless steel	0.11×10^7		
Nichrome	0.1×10^7		

TABLE C.3
μ_R

Material	μ_R
Bismuth	0.999 998 6
Paraffin	0.999 999 42
Wood	0.999 999 5
Silver	0.999 999 81
Aluminum	1.000 000 65
Beryllium	1.000 000 79
Nickel chloride	1.000 04
Manganese sulfate	1.000 1
Nickel	50
Cast iron	60
Cobalt	60
Powdered iron	100
Machine steel	300
Ferrite (typical)	1000
Permalloy 45	2500
Transformer iron	3000
Silicon iron	3500
Iron (pure)	4000
Mumetal	20 000
Sendust	30 000
Supermalloy	100 000

TABLE C.4
Physical Constants

Quantity	Value
Electron charge	$e = (1.602\,177\,33 \pm 0.000\,000\,46) \times 10^{-19}$ C
Electron mass	$m = (9.109\,389\,7 \pm 0.000\,005\,4) \times 10^{-31}$ kg
Permittivity of free space	$\epsilon_0 = 8.854\,187\,817 \times 10^{-12}$ F/m
Permeability of free space	$\mu_0 = 4\pi 10^{-7}$ H/m
Velocity of light	$c = 2.997\,924\,58 \times 10^{8}$ m/s

APPENDIX D

ORIGINS OF THE COMPLEX PERMITTIVITY

As we learned in Chap. 5, a dielectric can be modeled as an arrangement of atoms and molecules in free space, which can be polarized by an electric field. The field forces positive and negative bound charges to separate against their Coulomb attractive forces, thus producing an array of microscopic dipoles. The molecules can be arranged in an ordered and predictable manner (such as in a crystal) or may exhibit random positioning and orientation, as would occur in an amorphous material or a liquid. The molecules may or may not exhibit permanent dipole moments (existing before the field is applied), and if they do, they will usually have random orientations throughout the material volume. As discussed in Sec. 5.7, the displacement of charges in a regular manner, as induced by an electric field, gives rise to a macroscopic polarization, **P**, defined as the dipole moment per unit volume:

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{N\Delta v} \mathbf{p_i} \tag{D.1}$$

where N is the number of dipoles per unit volume and $\mathbf{p_i}$ is the dipole moment of the ith atom or molecule, found through

$$\mathbf{p_i} = Q_i \mathbf{d}_i \tag{D.2}$$

Q_i is the positive one of the two bound charges composing dipole i, and $\mathbf{d}_i$ is the distance between charges, expressed as a vector from the negative to the positive charge. Again, borrowing from Sec. 5.7, the electric field and the polarization are related through

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{D.3}$$

where the electric susceptibility, χ_e, forms the more interesting part of the dielectric constant:

$$\epsilon_R = 1 + \chi_e \tag{D.4}$$

Therefore, to understand the nature of ϵ_R, we need to understand χ_e, which in turn means that we need to explore the behavior of the polarization, **P**.

Here, we consider the added complications of how the dipoles respond to a time-harmonic field that propagates as a wave through the material. The result of applying such a forcing function is that *oscillating* dipole moments are set up, and *these in turn establish a polarization wave that propagates through the material*. The effect is to produce a polarization function, $\mathbf{P}(z, t)$, having the same functional form as the driving field, $\mathbf{E}(z, t)$. The molecules themselves do not move through the material, but their oscillating dipole moments collectively exhibit wave motion, just as waves in pools of water are formed by the up and down motion of the water. From here, the description of the process gets complicated and in many ways beyond the scope of our present discussion. We can form a basic qualitative understanding, however, by considering the classical description of the process, which is that the dipoles, once oscillating, behave as microscopic antennas, re-radiating fields that in turn co-propagate with the applied field. Depending on the frequency, there will be some phase difference between the incident field and the radiated field at a given dipole location. This results in a net field (formed through the superposition of the two) that now interacts with the next dipole. Radiation from this dipole adds to the previous field as before, and the process repeats from dipole to dipole. The accumulated phase shifts at each location are manifested as a net slowing down of the phase velocity of the resultant wave. Attenuation of the field may also occur which, in this classical model, can be accounted for by partial phase cancellation between incident and radiated fields.

In our classical model, the medium is an ensemble of identical fixed electron oscillators, in which the Coulomb binding forces on the electrons are modeled by springs that attach the electrons to the positive nuclei. We consider electrons for simplicity, but similar models can be used for any bound charged particle. Figure D.1 shows a single oscillator, located at position z in the material, and oriented along x. A uniform plane wave, assumed linearly polarized along x, propagates through the material in the z direction. The electric field in the wave displaces the electron of the oscillator in the x direction through a distance represented by the vector **d**; a dipole moment is thus established,

$$\mathbf{p}(z, t) = -e\mathbf{d}(z, t) \tag{D.5}$$

where the electron charge, e, is treated as a positive quantity. The applied force is

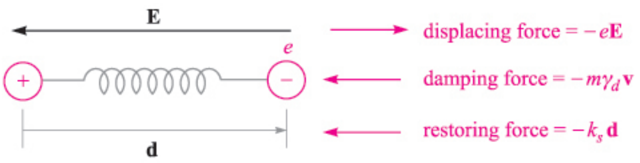

FIGURE D.1
Atomic dipole model, with Coulomb force between positive and negative charge modeled by that of a spring having spring constant, k_s. An applied electric field displaces the electron through distance d, resulting in dipole moment, $\mathbf{p} = -e\mathbf{d}$.

$$\mathbf{F}_a(z, t) = -e\mathbf{E}(z, t) \tag{D.6}$$

We need to remember that $\mathbf{E}(z, t)$ at a given oscillator location is the *net* field, composed of the original applied field plus the radiated fields from all other oscillators. The relative phasing between oscillators is precisely determined by the spatial and temporal behavior of $\mathbf{E}(z, t)$.

The restoring force on the electron, $\mathbf{F}_r$, is that produced by the spring which is assumed to obey Hooke's law:

$$\mathbf{F}_r(z, t) = -k_s\mathbf{d}(z, t) \tag{D.7}$$

where k_s is the spring constant (not to be confused with the propagation constant). If the field is turned off, the electron is released and will oscillate about the nucleus at the *resonant frequency*, given by

$$\omega_0 = \sqrt{k_s/m} \tag{D.8}$$

where m is the mass of the electron. The oscillation, however, will be damped since the electron will experience forces and collisions from neighboring oscillators. We model these as a velocity-dependent damping force:

$$\mathbf{F}_d(z, t) = -m\gamma_d\mathbf{v}(z, t) \tag{D.9}$$

where $\mathbf{v}(z, t)$ is the electron velocity. Associated with this damping is the *dephasing* process among the electron oscillators in the system. Their relative phasing, once fixed by the applied sinusoidal field, is destroyed through collisions and dies away exponentially until a state of totally random phase exists between oscillators. The $1/e$ point in this process occurs at the *dephasing time* of the system, which is inversely proportional to the damping coefficient, γ_d (in fact it is $2/\gamma_d$). We are, of course, driving this damped resonant system with an electric field at frequency ω. We can therefore expect the response of the oscillators, measured through the magnitude of $\mathbf{d}$, to be frequency-dependent in much the same way as an RLC circuit is when driven by a sinusoidal voltage.

We can now use Newton's second law, and write down the forces acting on the single oscillator of Fig. D.1. To simplify the process a little we can use the complex form of the electric field:

$$\mathbf{E}_c = \mathbf{E}_0 e^{-jkz} e^{j\omega t} \tag{D.10}$$

Defining $\mathbf{a}$ as the acceleration vector of the electron, we have

$$m\mathbf{a} = \mathbf{F}_a + \mathbf{F}_r + \mathbf{F}_d$$

or

$$m\frac{\partial^2 \mathbf{d}_c}{\partial t^2} + m\gamma_d \frac{\partial \mathbf{d}_c}{\partial t} + k_s \mathbf{d}_c = -e\mathbf{E}_c \tag{D.11}$$

Note that since we are driving the system with the complex field, $\mathbf{E}_c$, we anticipate a displacement wave, $\mathbf{d}_c$, of the form:

$$\mathbf{d}_c = \mathbf{d}_0 e^{-jkz} e^{-j\omega t} \tag{D.12}$$

With the waves in this form, time differentiation produces a factor of $j\omega$. Consequently (D.11) can be simplified and rewritten in phasor form:

$$-\omega^2 \mathbf{d}_s + j\omega\gamma_d \mathbf{d}_s + \omega_0^2 \mathbf{d}_s = -\frac{e}{m}\mathbf{E}_s \tag{D.13}$$

where (D.4) has been used. We now solve (D.13) for $\mathbf{d}_s$, obtaining

$$\mathbf{d}_s = \frac{-(e/m)\mathbf{E}_s}{(\omega_0^2 - \omega^2) + j\omega\gamma_d} \tag{D.14}$$

The dipole moment associated with displacement $\mathbf{d}_s$ is

$$\mathbf{p}_s = -e\mathbf{d}_s \tag{D.15}$$

The polarization of the medium is then found assuming that all dipoles are identical. Eq. (D.1) thus becomes

$$\mathbf{P}_s = N\mathbf{p}_s$$

which, when using (D.14) and (D.15), becomes

$$\mathbf{P}_s = \frac{Ne^2/m}{(\omega_0^2 - \omega^2) + j\omega\gamma_d}\mathbf{E}_s \tag{D.16}$$

Now, using (D.3) we identify the susceptibility associated with the resonance as

$$\chi_{res} = \frac{Ne^2}{\epsilon_0 m}\frac{1}{(\omega_0^2 - \omega^2) + j\omega\gamma_d} = \chi'_{res} - j\chi''_{res} \tag{D.17}$$

The real and imaginary parts of the permittivity are now found through the real and imaginary parts of χ_{res}: Knowing that

$$\epsilon = \epsilon_0(1 + \chi_{res}) = \epsilon' - j\epsilon''$$

we find

$$\epsilon' = \epsilon_0(1 + \chi'_{res}) \tag{D.18}$$

and

$$\epsilon'' = \epsilon_0 \chi''_{res} \tag{D.19}$$

The above expressions can now be used in Eqs. (35) and (36) in Chap. 11 to evaluate the attenuation coefficient, α, and phase constant, β, for the plane wave as it propagates through our resonant medium.

The real and imaginary parts of χ_{res} as functions of frequency are shown in Fig. D.2 for the special case in which $\omega \doteq \omega_0$. Eq. (D.17) in this instance becomes

$$\chi_{res} \doteq -\frac{Ne^2}{\epsilon_0 m \omega_0 \gamma_d}\left(\frac{j+\delta_n}{1+\delta_n^2}\right) \tag{D.20}$$

where the *normalized detuning* parameter, δ_n, is

$$\frac{2}{\gamma_d}(\omega-\omega_0) \tag{D.21}$$

Key features to note in Fig. D.2 include the symmetric χ_e'' function, whose full-width at its half-maximum amplitude is γ_d. Near the resonant frequency, where χ_{res}'' maximizes, wave attenuation maximizes as seen from Eq. (35), Chap. 11. Additionally, we see that away from resonance, attenuation is relatively weak, and the material becomes transparent. As Fig. D.2 shows, there is still significant variation of χ_{res}' with frequency away from resonance, which leads to a frequency-dependent refractive index; this is expressed approximately as

$$n \doteq \sqrt{1+\chi_{res}'} \qquad \text{(away from resonance)} \tag{D.22}$$

This frequency-dependent n, arising from the material resonance, leads to phase and group velocities that also depend on frequency. Thus, group dispersion, leading to pulse broadening effects as discussed in Chap. 12, can be directly attributable to material resonances.

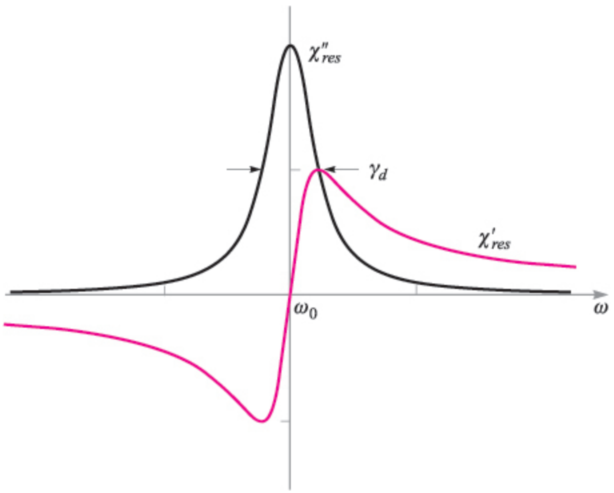

FIGURE D.2
Plots of the real and imaginary parts of the resonant susceptibility, χ_{res}, as given by Eq. (D.20). The full-width at half-maximum of the imaginary part, χ''_{res}, is equal to the damping coefficient, γ_d.

Somewhat surprisingly, the classical ‘‘spring model’’ described here can provide very accurate predictions on dielectric constant behavior with frequency (particularly off-resonance) and can be used to a certain extent to model absorption properties. The model is insufficient, however, when attempting to describe the more salient features of materials; specifically, it assumes that the oscillating electron can assume any one of a continuum of energy states, when in fact energy states in any atomic system are quantized. As a result, the important effects arising from transitions between discrete energy levels, such as spontaneous and stimulated absorption and emission, are not included in our classical spring system. Quantum mechanical models must be used to fully describe the medium polarization properties, but the results of such studies often reduce to those of the spring model when field amplitudes are very low.

Another way that a dielectric can respond to an electric field is through the orientation of molecules that possess permanent dipole moments. In such cases, the molecules must be free to move or rotate, and so the material is typically a liquid or a gas. Figure D.3 shows an arrangement of polar molecules in a liquid (such as water) in which there is no applied field (D.3*a*) and where an electric field is present (D.3*b*). Applying the field causes the dipole moments, previously having random orientations, to line up, and so a net material polarization, **P**, results. Associated with this, of course, is a susceptibility function, χ_e, through which **P** relates to **E**.

Some interesting developments occur when the applied field is time-harmonic. With field periodically reversing direction, the dipoles are forced to follow, but do so against their natural propensity to randomize, owing to thermal motion. Thermal motion thus acts as a ‘‘restoring’’ force, effectively opposing the applied field. We can also think of the thermal effects as viscous forces that introduce some difficulty in ‘‘pushing’’ the dipoles back and forth. One might expect (correctly) that polarizations of greater amplitude in each direction can be attained at lower frequencies, since enough time is given during each cycle for the dipoles to achieve complete alignment. The polarization amplitude will weaken as the frequency increases, since there is no longer enough time for complete alignment during each cycle. This is the basic description of *dipole relaxation* mechanism for the complex permittivity. There is no resonant frequency associated with the process.

The complex susceptibility associated with dipole relaxation is essentially that of an ‘‘overdamped’’ oscillator, and is given by

$$\chi_{rel} = \frac{Np^2/\epsilon_0}{3k_B T(1 + j\omega\tau)} \tag{D.23}$$

where p is the permanent dipole moment magnitude of each molecule, k_B is Boltzmann’s constant, and T is the temperature in degees Kelvin. τ is the thermal randomization time, defined as the time for the polarization, **P**, to relax to $1/e$ of its original value when the field is turned off. χ_{rel} is complex, and so it will possess absorptive and dispersive components (imaginary and real parts) as we

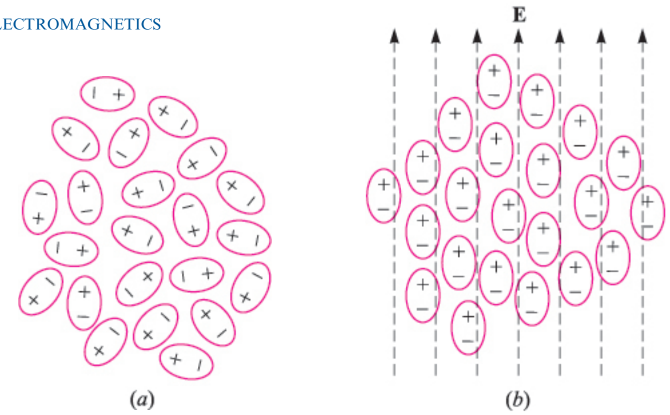

FIGURE D.3
Idealized sketches of ensembles of polar molecules under conditions of (*a*) random orientation of the dipole moments, and (*b*) dipole moments aligned under the influence of an applied electric field. Conditions in (*b*) are greatly exaggerated, since typically only a very small percentage of the dipoles align themselves with the field. But still enough alignment occurs to produce measurable changes in the material properties.

found in the resonant case. The form of Eq. (D.23) is identical to that of the response of a series RC circuit driven by a sinusoidal voltage (where τ becomes RC).

Microwave absorption in water occurs through the relaxation mechanism in polar water molecules, and is the primary means by which microwave cooking is done, as discussed in Chap. 11. Frequencies near 2.5 GHz are typically used, since these provide the optimum penetration depth. The peak water absorption arising from dipole relaxation occurs at much higher frequencies, however.

A given material may possess more than one resonance and may have a dipole relaxation response as well. In such cases, the net susceptibility is found in frequency domain by the direct sum of all component susceptibilities. In general, we may write:

$$\chi_e = \chi_{rel} + \sum_{i=1}^{n} \chi_{res}^{i} \qquad \text{(D.24)}$$

where χ_{res}^{i} is the susceptibility associated with the ith resonant frequency, and n is the number of resonances in the material. The reader is referred to the suggested references for Chap. 11 for further reading on resonance and relaxation effects in dielectrics.

INDEX

DIVERGENCE

CARTESIAN $$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

CYLINDRICAL $$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

SPHERICAL $$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

GRADIENT

CARTESIAN $$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

CYLINDRICAL $$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

SPHERICAL $$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

CURL

CARTESIAN $\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_z}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$

CYLINDRICAL $\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right]\mathbf{a}_z$

SPHERICAL $\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\mathbf{a}_\phi$

LAPLACIAN

CARTESIAN $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

CYLINDRICAL $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$

SPHERICAL $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$