AMERICAN UNIVERSITY of SHARJAH

Linear Algebra
MTH 221

Class Notes

Fall 2007
Padmapani Seneviratne

28-08-07

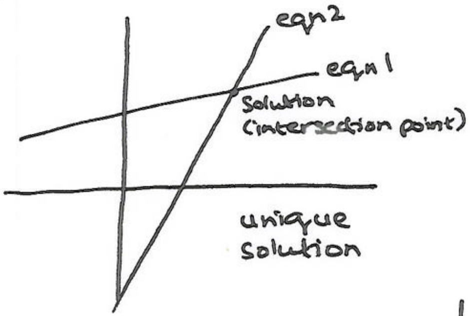
Chapter 1
Linear Equations in Linear Algebra.

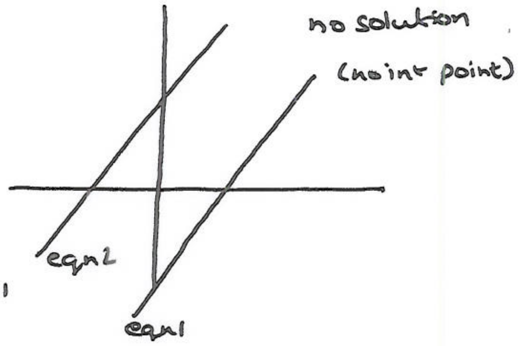
An equation of the type $Ax + By = C$ is called a linear equation.
"Linear" because the graph is a straight line.

Similarly we write $a_1x_1 + a_2x_2 + a_nx_n = b$ is called a linear equation
a system of linear equations is a collection of one or more linear equations

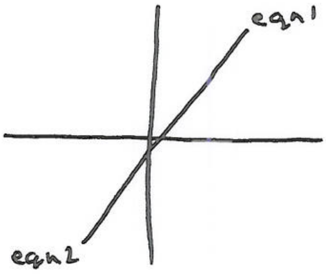
eg: $2x - y = 2$
$3x + 4y = 6$

graph:





eg: $x_1 - 2x_2 = -1$ ①
$-x_1 + 2x_2 = 1$ ②



A system of equations has either
1. No solution (Inconsistant)
2. Exactly one solution } consistent
3. Infinitely many solutions

Matrices:
eg: $x_1 - 2x_2 + x_3 = 0$
$2x_2 - 8x_3 = 8$
$-4x_1 + 5x_2 + 9x_3 = -9$

System of linear equations:
We use a compact notation to identify a linear system.

① coefficient Matrix of the system: $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

② augmented matrix of the system: $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$

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eg: Solve the system:

$x_1 - 2x_2 + x_3 = 0$ – ①

$2x_2 - 8x_3 = 8$ – ②

$-4x_1 + 5x_2 + 9x_3 = -9$ – ③

Augmented matrix: ① $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$

4 × ①: $4x_1 - 8x_2 + 4x_3 = 0$

③: $-4x_1 + 5x_2 + 9x_3 = -9$

$-3x_2 + 13x_3 = -9$

Replace eqn ③ with 4×① + ③ ②

$x_1 - 2x_2 + x_3 = 0$ – ①

$2x_2 - 8x_3 = 8$ – ②

$-3x_2 + 13x_3 = -9$ – ③

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & +13 & -9 \end{bmatrix}$$

Simplify eq②: $x_1 - 2x_2 + x_3 = 0$ – ①

$x_2 - 4x_3 = 4$ – ②

$-3x_2 + 13x_3 = -9$ – ③

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

② × 3: $3x_2 - 12x_3 = 12$ – ②

③: $-3x_2 + 13x_3 = -9$ – ③

$x_3 = 3$

Replace eq ③ with 3×② + ③

$x_1 - 2x_2 + x_3 = 0$ – ①

$x_2 - 4x_3 = 4$ – ②

$x_3 = 3$ – ③

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Eliminate $-4x_3$ from ②:

4 × ③: $4x_3 = 12$ – ③

②: $x_2 - 4x_3 = 4$ – ②

$x_2 = 16$ – ②

Eliminate x_3 from eq①:

-1 × ③: $-x_3 = -3$

①: $x_1 - 2x_2 + x_3 = 0$

$x_1 - 2x_2 = -3$ – ①

$x_1 - 2x_2 = -3$ – ①

$x_2 = 16$ – ②

$x_3 = 3$ – ③

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Next step:

2 × ②: $2x_2 = 32$

①: $x_1 - 2x_2 = -3$

$x_1 = 29$

$x_1 = 29$ – ①

$x_2 = 16$ – ②

$x_3 = 3$ – ③

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Determine if the following system is consistent.

$$\begin{aligned} x_2-4x_3&=8\\ 2x_1-3x_2+2x_3&=1\\ 5x_1-8x_2+7x_3&=1 \end{aligned} \qquad \begin{bmatrix} 0 & 1 & -4 & 8\\ 2 & -3 & 2 & 1\\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Interchange $R_1 \longleftrightarrow R_2$:

$$\begin{bmatrix} 2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$R_3 \longleftrightarrow R_3+(-\frac{5}{2}R_1)$:

$$\begin{bmatrix} 2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 0 & -\frac{1}{2} & 2 & -3/2 \end{bmatrix}$$

$R_3 \longleftrightarrow R_3+(\frac{1}{2}R_2)$:

$$\begin{bmatrix} 2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 0 & 0 & 0 & 5/2 \end{bmatrix} \begin{aligned} &\rightarrow 2x_1-3x_2+2x_3=1\\ &\rightarrow x_2-4x_3=8\\ &\quad 0=5/2 \quad \times \end{aligned}$$

A contradiction has been found therefore the system is not consistent.

1.2 Raw Reduction and Echelon forms

Definition: A rectangular matrix is in "Echelon Form" (or Raw Echelon Form) if it has the following 3 properties.

1) All non zero rows are above any rows of all zeros.
2) Each leading entry of a row is in a column to the right of the leading entry above it.

② $\begin{bmatrix} \textcircled{1} & 0 & -1 & 4\\ 0 & \textcircled{3} & 5 & 6\\ 0 & 0 & \textcircled{-1} & 3 \end{bmatrix}$ ① $\begin{bmatrix} 0 & 0 & 0\\ 1 & 3 & 5\\ 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 5\\ 2 & 1 & 2\\ 0 & 0 & 0 \end{bmatrix}$

3) All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} \textcircled{1} & 2 & 3 & 4\\ 0 & \textcircled{5} & 6 & 7\\ 0 & 0 & \textcircled{8} & 9 \end{bmatrix}$$

all are zeror below leading entry

If a matrix in echelon form satisfies the following additional conditions then it is in a "Reduced Raw Echelon Form" (or reduced echelon form).

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4) The leading entry in each non-zero row is 1

5) Each leading 1 is the only non-zero entry in it's column.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 1: Each matrix is row equivalent to one and only one reduced raw echelon.

Elementary Row operations:

1) Replace one Row by the sum of itself and a multiple of another Row - **Replacement.**

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{Replace } R_2 \text{ by } R_2+(-2)R_1$$

2) Interchange the rows. - **Interchange.**

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{} R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

3) Multiply all entries in a row by a nonzero constant - **Scaling.**

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{} R_1 \leftrightarrow \tfrac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem:

$$\begin{bmatrix} A \end{bmatrix} \overset{\text{Raw operations}}{\sim} \begin{bmatrix} \text{reduced raw Echelon form} \end{bmatrix}$$

Definition: A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the Reduced Raw Echelon form of A. A pivot column is a column of A that contains a pivot position.

PIVOTS

$$\begin{bmatrix} \boxed{2} & 3 & 4 \\ \boxed{4} & \boxed{6} & 0 \\ \boxed{0} & 0 & 1 \end{bmatrix}$$

PIVOT COLUMN

HW: Raw Reduce the matrix A below to echelon form and locate pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

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$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad R_1 \longleftrightarrow R_4$$

$R_2 \longrightarrow R_2 + R_1$

$R_3 \longrightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Simplify row: by $R_2 \longrightarrow \frac{1}{2}R_2$ (scaling)

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + (-5R_2)$

$R_4 \longrightarrow R_4 + (3R_2)$

Replace $R_3 \longrightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

The Row Reduction Algorithm.

Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

① $$\begin{bmatrix} \boxed{0} & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

pivot

Step 1: Begin with the leftmost nonzero column. This is a pivot column.

Step 2: Interchange rows so that the matrix doesnt begin with a 0

② $$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: Use row replacement to create zeros in all positions below the pivot.

③ $R_2 \longrightarrow R_2 + (-R_1)$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

pivot

④ $R_3 \longrightarrow R_3 + \left(-\frac{3}{2}R_2\right)$

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

pivots

pivot

"ECHELON FORM" OR "ROW ECHELON FORM"

To create "Reduced Row echelon form"

Step: Beginning with the rightmost pivot and working upwards and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

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$R_2 \to R_2 + (-2R_3)$
$R_1 \to R_1 + (-6R_3)$:
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_2$:
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 + 9R_2$:
$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Scale R_1 : $R_1 \to \frac{1}{3}R_1$
$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced Row echelon Form.

Pg41 ③ ①
$$\begin{bmatrix} \overset{\text{pivot.}}{\textcircled{1}} & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

$R_2 \to R_2 + (-4R_1)$
$R_3 \to R_3 + (-6R_1)$ ②
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

Scaling ③
$R_2 \to -\frac{1}{3}R_2$
$R_3 \to -\frac{1}{5}R_3$
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 + (-R_2)$ ④
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + (-2R_2)$ ⑤
$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pg41 ④ ①
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$
Need Reduced Echelon.

$R_2 \to R_2 + (-3R_1)$
$R_3 \to R_3 + (-5R_1)$

②
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

$R_2 \to -\frac{1}{4}R_2$ ③
$R_3 \to -\frac{1}{2}R_3$
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 8 & 17 \end{bmatrix}$$

$R_3 \to R_3 + (-4R_2)$

④
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_3 \to \frac{1}{5}R_3$ ⑤
$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + (-3R_3)$
$R_1 \to R_1 + (-7R_3)$

⑥
$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + (-3R_2)$

⑦
$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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(1.2) Solutions of linear Systems.

eg: suppose that the augmented matrix of a linear system has been changed to equivalent reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The associated system of equations:

$$\left.\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}\right\} (*)$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called basic variables.
- The other variable x_3 is called a free variable (meaning that you can select any value for x_3)

eg: $\left.\begin{aligned} &x_3 = 0 \\ &x_1 = 1,\ x_2 = 4 \end{aligned}\right\} (1,4,0)$ particular solution

$\left.\begin{aligned} &x_3 = 10 \\ &x_1 = 51,\ x_2 = -6 \end{aligned}\right\} (51,-6,10)$ particular solution

- Whenever a system is consistent as in (*) the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables

$$\left.\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &= \text{free} \end{aligned}\right\} \text{general solution}$$

* Particular solution:

eg: $x_3 = 0$, $x_1 = 1$, $x_2 = 4$ — $(1,4,0)$ is a particular solution

2.1 (ii) Find the general solution of the system whose augmented matrix is given by:

① $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$ convert to reduced echelon form first.

$R_2 \longrightarrow 3R_1 + R_2$
$R_3 \longrightarrow R_3 + 2R_1$

② $\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Scale: $R_1 \longrightarrow \frac{1}{3} R_1$

③ $\begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\left.\begin{aligned} x_1 - \tfrac{4}{3}x_2 + \tfrac{2}{3}x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\} \text{2 free variables.}$$

general solution:

$$\begin{aligned} x_1 &= \tfrac{4}{3}x_2 - \tfrac{2}{3}x_3 \\ x_2 &= \text{Free} \\ x_3 &= \text{free} \end{aligned}$$

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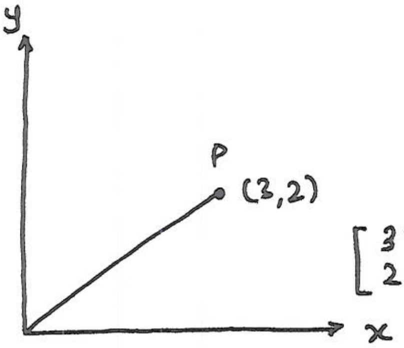
1.3 Vector Equations



$\begin{bmatrix}3\\2\end{bmatrix}$ Column matrix vector.

- a matrix with only one column is called a column vector or simply a vector

eg: $\underline{u} = \begin{bmatrix}3\\-1\end{bmatrix}$, $\underline{v} = \begin{bmatrix}0\\-2\end{bmatrix}$, $\underline{\omega} = \begin{bmatrix}\omega_1\\\omega_2\end{bmatrix}$

- The set of all the vectors with 2 entries is denoted by $\mathbb{R}^2$ ($\mathbb{R}$ - real numbers)
- $\mathbb{R}^2$ - The set of all points in the (x,y) plane.
- Properties: (Let $\underline{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$, $\underline{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$)

① vectors $\underline{u}$ and $\underline{v}$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$

② $\underline{u} + \underline{v} = \begin{bmatrix}u_1 + v_1\\u_2 + v_2\end{bmatrix}$ if $\underline{u} = \begin{bmatrix}1\\-1\end{bmatrix}$ and $\underline{v} = \begin{bmatrix}2\\3\end{bmatrix}$ then $\underline{u} + \underline{v} = \begin{bmatrix}3\\2\end{bmatrix}$

$\underline{u} - \underline{v} = \begin{bmatrix}u_1 - v_1\\u_2 - v_2\end{bmatrix}$

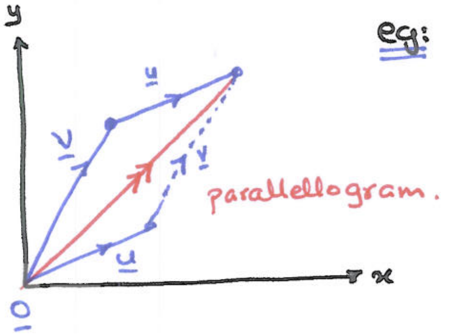
③ let c be a scalar (number);

The scalar multiple of vector $\underline{u}$ by scalar c is:

$c\underline{u} = c\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}cu_1\\cu_2\end{bmatrix}$. if $c = 2$ $\underline{u} = \begin{bmatrix}4\\-3\end{bmatrix}$ then $c\underline{u} = \begin{bmatrix}8\\-6\end{bmatrix}$

- Parallellogram rule:

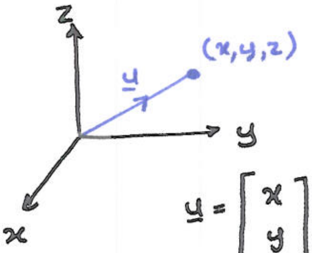
If vector $\underline{u}$ and vector $\underline{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\underline{u} + \underline{v}$ corresponds to the fourth vertex of the parallellogram, whose other vertices are $\underline{u}$, $\underline{0}$ and $\underline{v}$



eg: $\underline{u} = \begin{bmatrix}2\\2\end{bmatrix}$ $\underline{v} = \begin{bmatrix}-6\\1\end{bmatrix}$ $\underline{u} + \underline{v} = \begin{bmatrix}-4\\3\end{bmatrix}$

- Represent the above on a graph in a manner shown to the left and it will give the same solutions.

- Vectors in $\mathbb{R}^3$:



$\underline{u} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ (3×1)

vectors in $\mathbb{R}^n$: $\underline{u} = \begin{bmatrix}u_1\\u_2\\u_3\\\vdots\\u_n\end{bmatrix}$

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Properties for vectors in $\mathbb{R}^n$

- If $\underline{u}, \underline{v}$ and $\underline{w}$ are vectors in $\mathbb{R}^n$ and c and d are scalars,

① $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

② $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

③ $\underline{u} + \underline{0} = \underline{u}$

④ $\underline{u} + (-\underline{u}) = \underline{0}$

⑤ $c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$

⑥ $c(d\underline{u}) = (cd)\underline{u}$

⑦ $1 \cdot \underline{u} = \underline{u}$

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Linear Combinations

Definition: Given vectors, $v_1, v_2, v_3 \ldots\ldots v_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, c_3 \ldots\ldots c_p$, The vector y defined by $y = c_1\underline{v_1} + c_2v_2 + \ldots\ldots + c_p\underline{v_p}$ is called a linear combination of $v_1, v_2, v_3 \ldots\ldots v_p$ with weights $c_1, c_2, c_3 \ldots\ldots c_p$

eg: let $\underline{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Determine whether $\underline{b}$ can be written as a linear combination of $\underline{a}_1$ and $\underline{a}_2$?

ie: determine whether weights x_1 and x_2 exists such that

$$* \boxed{x_1a_1 + x_2a_2 = \underline{b}} \text{ (vector equation)}$$

$$x_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$(*) \quad \begin{aligned} & x_1 + 2x_2 = 7 \\ \Rightarrow\ & -2x_1 + 5x_2 = 4 \\ & -5x_1 + 6x_2 = -3 \end{aligned}$$

The augmented matrix of the system: ①

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

② $R_2 \rightarrow R_2 + 2R_1$, $R_3 \rightarrow R_3 + 5R_1$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \rightarrow$$

Scale: ③ $R_2 \rightarrow \frac{1}{9}R_2$, $R_3 \rightarrow \frac{1}{16}R_3$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

④ $R_3 \rightarrow R_3 + (-R_2)$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

⑤ $R_1 \rightarrow R_1 + (-2R_2)$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 3$
$x_2 = 2$

$\Rightarrow x_1\underline{a}_1 + x_2\underline{a}_2 = \underline{b}$
$3\underline{a}_1 + 2\underline{a}_2 = \underline{b}$

ie: $\underline{b}$ can be written as a linear combination of a_1 and a_2

(*) to start with, many excessive steps can be avoided by using the following format in solving. and here is how the matrix could be written: $\begin{bmatrix} \underline{a}_1, & \underline{a}_2, & \underline{b} \end{bmatrix}$

A vector equation $x_1\underline{a}_1 + x_2\underline{a}_2 + \ldots\ldots + x_n\underline{a}_n = \underline{b}$ has the same solution set as the linear system whose augmented matrix is $[\underline{a}_1, \underline{a}_2 \cdots \underline{a}_n\ \underline{b}]$. NOTE: $\underline{b}$ can be generated by a linear combination of $a_1, a_2, \ldots\ldots, a_n$ if and only if there exists a solution to the linear system corresponding to (*)

09-09-07

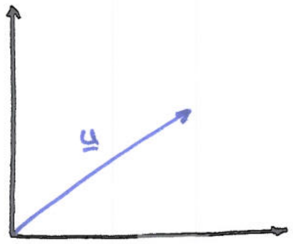
1.3 Linear combinations

Definition

If $v_1, v_2, v_3 \ldots, v_p$ are in $\mathbb{R}^n$. The set of all linear combinations of $v_1, v_2, \ldots v_p$ is denoted by span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3 \ldots, \underline{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned or generated by $v_1, v_2, v_3 \ldots . v_p$.

ie: span $\{v_1, v_2, v_3 \ldots, v_p\}$ is the collection of all vectors that can be written in the form $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + \ldots + c_p\underline{v}_p$

eg: $\underline{u}$ in $\mathbb{R}^n$ span$\{\underline{u}\}$



span$\{\underline{u}\}$ = all the linear combinations of $\underline{u}$

= $\{c\underline{u}\}$ c is any real number.

eg: $2\underline{u}$ in span $\{\underline{u}\}$

$\frac{1}{2}\underline{u}$ in span $\{\underline{u}\}$

$-\underline{u}$ in span $\{\underline{u}\}$

Note: asking whether a vector $\underline{b}$ is in span $\{v_1, v_2, v_3 \ldots, v_p\}$ amounts to asking whether the vector equation $x_1v_1 + x_2v_2 + \ldots + x_pv_p = \underline{b}$ has a solution.

ie: $[v_1, v_2, \ldots v_p, \underline{b}]$ has a solution.

1.3 ⑰ let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is $\underline{b}$ in the plane spanned by $\underline{a}_1$ and $\underline{a}_2$

in other words: does the equation $x_1\underline{a}_1 + x_2\underline{a}_2 = \underline{b}$ has a solution?

solution: $[\underline{a}_1, \underline{a}_2, \underline{b}]$

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & 8+h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & 8+h \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 17+h \end{bmatrix}$$

$0 = 17+h$ $\therefore h = -17$; system is consistent if $h = -17$, ie: $\underline{b}$ is in span $\{a_1, a_2\}$ if $h = -17$.

⑱ $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$: for what value(s) of h is y in the plane generated by v_1 and v_2?

ie: $x_1v_1 + x_2v_2 = \underline{y}$ has a solution?

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{bmatrix} \Rightarrow h = -\frac{7}{2}$$

09-09-07

1.4 the Matrix Equation

Definition: if A is an $m \times n$ matrix, with columns $a_1, a_2 \dots a_n$ and if $\underline{x}$ is in $\mathbb{R}^n$, then the product of matrix A and $\underline{x}$, denoted by $A\underline{x}$ is the linear combination of the columns of A using corresponding entries in $\underline{x}$ as weights.

ie: $A\underline{x} = [\underline{a}_1, \underline{a}_2 \dots \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n\underline{a}_n$

eg: $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Thm: If A is an $m \times n$ matrix, with columns $a_1, a_2, \dots a_n$ and if $\underline{b}$ is in $\mathbb{R}^m$, the matrix equation $A\underline{x} = \underline{b}$ has the same solution set as the vector equation $x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n a_n = \underline{b}$ which, in turn has the same solution set as the system whose augmented matrix is $[\underline{a}_1 \ \underline{a}_2 \dots \underline{a}_n \ \underline{b}]$

NOTE: The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

1.4 ⑭ $\underline{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$: is $\underline{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? why or why not?

write the augmented matrix: (is $\underline{u}$ in span $\{a_1, a_2, a_3\}$)

$\Downarrow$

can it be written as $\underline{u} = x_1 a_1 + x_2 a_2 + x_3 a_3$?

$$\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{bmatrix}$$

inconsistent.
$0 \neq -29$

ie: $\underline{u}$ is not a subset of $\mathbb{R}^3$ spanned by the columns of A.

11-09-07

eg 1.4: Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation

$A\underline{x} = \underline{b}$ consistent for all possible b_1 b_2 and b_3?

Augmented matrix: $\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$

$$\begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_1 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix} \to R_2 \to \tfrac{1}{2} R_2 \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & \frac{4b_1 + b_2}{2} \\ 0 & 7 & 5 & 3b_1 + b_3 - \frac{4b_1 + b_2}{2} \end{bmatrix}$$

For the system to be consistent:

$$0 = 3b_1 + b_3 - \left(\frac{4b_1 + b_2}{2}\right) \text{ — } \circledast$$

Does $\circledast$ remain consistent for all values of b_1, b_2 and b_3?

$b_1 = 0$ $b_2 = 0$ $b_3 = 0$ $\circledast$ is true.

$b_1 = 0$ $b_2 = 0$ $b_3 = 1$

$\circledast = 5$ $0 = 1$ is a contradiction

ie: $A\underline{x} = \underline{b}$ is not consistent for all values of $\underline{b}$

* **Theorem:** Let A be an $m \times n$ matrix. then the following statements are logically equivalent.

ie: for a particular A, either they are all true or they are all false.

(a) For each $\underline{b}$ in $\mathbb{R}^m$, the equation $A\underline{x} = \underline{b}$ has a solution

(b) Each $\underline{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A

(c) The columns of A span $\mathbb{R}^m$

(d) A has a pivot position in every row.

* computation of $A\underline{x}$

compute $A\underline{x}$ where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}_{3\times 1}$$

11-09-07

1.4 ③

$$\underset{3\times2 \longrightarrow 3\times1}{\begin{bmatrix}6&5\\-4&-3\\7&6\end{bmatrix}\underset{\substack{2\times1\\ \downarrow}}{\begin{bmatrix}2\\-3\end{bmatrix}}} = \begin{bmatrix}(6)(2)+(5)(-3)\\(-4)(2)+(-3)(-3)\\(7)(2)+(6)(-3)\end{bmatrix} = \underset{3\times1}{\begin{bmatrix}-3\\1\\-4\end{bmatrix}}$$

1.5 Solution Sets of Linear Systems.

A system of linear equations is said to be homogeneous ($A\underline{x}=0$) if it can be written in the form $A\underline{x}=0$. This system will always have at least one solution ie: $x=0$ which is called the trivial solution

NOTE: The homogeneous equation $A\underline{x}=0$ has a non-trivial solution if and only if the equation has at least one free variable

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1.5 ⑥ Find the solution:

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \text{augmented matrix} \quad \begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + (-R_1);\ R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + (-2R_2)} \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \begin{aligned} x_1 &= -4x_3 \\ x_2 &= 3x_3 \\ x_3 &= \text{free} \end{aligned} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\underline{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \qquad \underline{x} = x_3 \underline{v} \text{ where } \underline{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

eg: describe the solution set:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned} \qquad \text{augmented matrix:} \quad \begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1 \quad R_3 \to R_3 + (-2R_1)$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \longrightarrow \overset{R_3 \to R_3 + 3R_2}{} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \overset{R_1 \to R_1 + (-5R_2)}{\begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\overset{R_1 \to \frac{1}{3}R_1}{\begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \left.\begin{aligned} x_1 - 4/3 x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \begin{aligned} x_1 &= 4/3 x_3 \\ x_2 &= 0 \\ x_3 &= \text{free.} \end{aligned}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{x} = x_3 \underline{u} \text{ where } \underline{u} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

eg: Find the solution set:

$$10x_1 - 3x_2 - 2x_3 = 0 \longrightarrow \begin{bmatrix} 10 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3/10 & -2/10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3/10 x_2 - 2/10 x_3 = 0;$

$x_1 = 3/10 x_2 + 2/10 x_3$

$x_2 = \text{free}$

$x_3 = \text{free}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/10 x_2 + 2/10 x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/10 x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/10 x_3 \\ 0 \\ x_3 \end{bmatrix}$$

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$\underbrace{\underline{x} = x_2\underline{u} + x_3\underline{v}}_{\circledast}$ where $\underline{u} = \begin{bmatrix}3/10\\1\\0\end{bmatrix}$ and $\underline{v} = \begin{bmatrix}2/10\\0\\1\end{bmatrix}$

$\underline{x}$ is a linear combination of vectors $\underline{u}$ and $\underline{v}$. Since x_2 and x_3 are free they can take any value

so: $x = \text{span}\{\underline{u}, \underline{v}\}$ where $\underline{u} = \begin{bmatrix}3/10\\1\\0\end{bmatrix}$, $\underline{v} = \begin{bmatrix}2/10\\0\\1\end{bmatrix}$

Note: sometimes $\circledast$ is written in the form $x = t\underline{u} + s\underline{v}$ to emphasize that the parameters vary overall. real numbers.

$x = t\underline{u} + s\underline{v}$ is called a parametric vector equation.

16-09-07

eg: describe all solutions of $A\underline{x} = \underline{b}$ where $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$ and

$\underline{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$ Augmented matrix: $\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & -4 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left.\begin{array}{l} x_1 - 4/3x_3 = -1 \\ x_2 = 2 \\ x_3 = \text{free} \end{array}\right\} \begin{array}{l} x_1 = -1 + 4/3\,x_3 \\ x_2 = 2 \\ x_3 = x_3 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 4/3\,x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{x} = \underline{p} + x_3\underline{v} \text{ where } \underline{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \underline{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Theorem: suppose the equation $A\underline{x} = \underline{b}$ is consistent for some given $\underline{b}$ and let $\underline{p}$ be a solution. Then the solution set of $A\underline{x} = \underline{b}$ is the set of all vectors of the form $\underline{\omega} = \underline{p} + \underline{v}x$, where $\underline{v}x$ is any solution of the homogeneous equation $A\underline{x} = \underline{0}$

⑮ Describe the set of equations in parametric form:

$$\begin{array}{l} x_1 + 3x_2 + x_3 = 1 \\ -4x_1 - 9x_2 + 2x_3 = -1 \\ -3x_2 - 6x_3 = -3 \end{array} \quad \text{Augmented matrix: } \begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left.\begin{array}{l} x_1 - 5x_3 = -2 \\ x_2 + 2x_3 = 1 \\ x_3 = \text{free} \end{array}\right\}$$

$$\begin{array}{l} x_1 = 5x_3 - 2 \\ x_2 = 1 - 2x_3 \\ x_3 = x_3 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

$\underline{x} = \underline{p} + t\underline{v}_x$ where $\underline{p} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_x = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ and $x_3 = t$ (Parameter).

16-09-07

(1.7) Linear independence.

Definition: An indexed set of vectors $\{v_1, v_2, \ldots\ldots, v_p\}$ in $\mathbb{R}^n$ is said to be Linearly independent if the vector equation $x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots\ldots + x_p\underline{v}_p = \underline{0}$ —① has the trivial solution (ie. $x_1 = 0, x_2 = 0, \ldots\ldots, x_p = 0$)

The set $\{\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_p\}$ is said to be Lineorly dependent if there exists weights $c_1, c_2, \ldots\ldots, c_p$, not all zero, such that $c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_p\underline{v}_p = \underline{0}$ —②
(ie at least one of $c_1, c_2 \ldots c_p$ is not equal to zero)

① let $\underline{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$ Determine if the vectors are linearly dependent.

Solution: consider the linear combination

$$x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 = \underline{0}$$

augmented matrix:
$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left.\begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}\right\} \text{Trivial Soln.}$$

so the given vectors are linearly independent.

③ Check for linear independence

$\underline{u}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\underline{u}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ consider $x_1\underline{u}_1 + x_2\underline{u}_2 = \underline{0}$

Augmented matrix:
$$\begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \left.\begin{matrix} x_1 - 3x_2 = 0 \\ x_2 = \text{free.} \end{matrix}\right\} \begin{matrix} x_1 = 3x_2 \\ x_2 = x_2 \end{matrix}$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ non-trivial Solution so $\underline{u}_1$ and $\underline{u}_2$ are linearly dependent.

18-09-07

⑦ Determine if the columns of the matrix A are Linearly independent

$$A = \begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ 4 & -5 & 7 & 5 \end{bmatrix}$$

Start with the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ 4 & -5 & 7 & 5 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -21 & 19 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 26 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -13 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 0 & -39 & 0 \\ 0 & 1 & 0 & -12 & 0 \\ 0 & 0 & 1 & -13 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & -12 & 0 \\ 0 & 0 & 1 & -13 & 0 \end{bmatrix}$$

$x_1 + 9x_4 = 0$

$x_2 - 12x_4 = 0$

$x_3 - 13x_4 = 0$

$x_4 =$ free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_4 \\ 12x_4 \\ 13x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -9 \\ 12 \\ 13 \\ 1 \end{bmatrix}$$

The system has a free variable ∴ it has a non-trivial solution and the columns of A are linearly dependent.

20-09-07

1.7 cont: sets of one or two vectors:

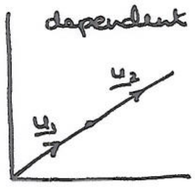
A set containing one vector, say $\{\underline{v}\}$ is linearly independent if and only if $\underline{v}$ is not the zero vector $\underline{0}$

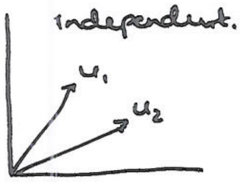
$\{\underline{v}\}: x\underline{v} = 0$

$x = 0 \iff v \neq 0$

Note: The zero vector is linearly dependent.

$\{\underline{0}\}$ $c\underline{0} = \underline{0}$ for any $c \neq 0$.





eg: $\underline{u_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{u_2} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\underline{u_2} = -2\underline{u_1}$ $\Rightarrow \{\underline{u_1}, \underline{u_2}\}$ are linearly dependent.

$2\underline{u_1} + \underline{u_2} = 0$

A set of 2 vectors $\{\underline{u_1}, \underline{u_2}\}$ is linearly dependent if at least one of the vectors is a multiple of the other. the set is linearly independent if and only if neither of the vectors is a multiple of the other.

* Theorem 7:

Characterization of linearly dependent sets.

An indexed set $S = \{\underline{v_1}, \underline{v_2}, \ldots\ldots \underline{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S, is a linear combination of the others.

eg: $\underset{u_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{v}{\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}}$ $\quad v = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

⑨ For what values of h is v_3 in span $\{v_1, v_2\}$ and for what values of h is $\{v_1, v_2, v_3\}$ Linearly dependent?

$\underline{v_1} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \underline{v_2} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \underline{v_3} = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

Solution: If v_3 is in span $\{v_1, v_2\}$

$v_3 = x_1 v_1 + x_2 v_2$ — ①

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

$0 = 8$ × a contradiction because we assumed that v_3 is in span $\{v_1, v_2\}$

So $\underline{v_3}$ is not in span $\{\underline{v_1}, \underline{v_2}\}$ for any h. So $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is linearly independent for all h.

* Theorem 8:

If a set contains more vectors than there are entries then the set is linearly dependent.

$S = \{u_1, u_2 \ldots\ldots u_p\}$ in $\mathbb{R}^n$

#vectors $= p$ #entries $= n$

$p > n$ then the set $\{u_1, u_2 \ldots\ldots u_p\}$ is linearly dependent.

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eg: $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 100 \end{bmatrix}$ Linearly dependent $\#V > \#F$

Thm: If a set contains 0 vector then linearly dependent.

eg: $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

23-09-07

① 1.8 Introduction to Linear transformations.

$A\underline{x}=\underline{b}$

Consider $A\underline{x}=\underline{b}$. A matrix equation can arise in linear algebra in a way that is not directly connected with linear combinations of vectors.

eg: $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ then $A\underline{x}=\underline{b}$ $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

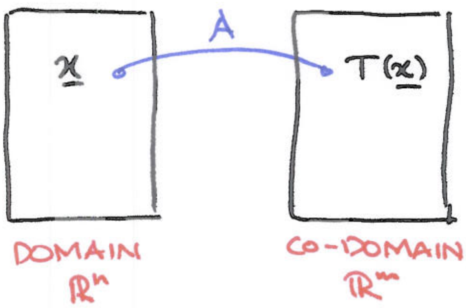
$$= \begin{bmatrix} 4-3+1+3 \\ 2+0+5+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \qquad \Rightarrow \underline{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$$

$$A\underline{u} = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4-12-1+9 \\ 2+0-5+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So solving equation $A\underline{x}=\underline{b}$ amounts to finding all vectors $\underline{x}$ in $\mathbb{R}^n$ that are transformed into the vector $\underline{b}$ in $\mathbb{R}^m$ under the action of multiplication by A.

Definition: A transformation (or function mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\underline{x}$ in $\mathbb{R}^n$ a vector $T(\underline{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the domain of T. $\mathbb{R}^m$ is called the co-domain of T.



$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\underline{x} \longmapsto T(\underline{x})$$

* For $\underline{x}$ in $\mathbb{R}^n$, the vector $T(\underline{x})$ in $\mathbb{R}^n$ is called the image of $\underline{x}$ under T.

* The set of all images $T(\underline{x})$ is called the Range of T.

① let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and define $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\underline{x}) = A\underline{x}$. Find the images under T of $\underline{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

Solution: find $T(\underline{u})$ and $T(\underline{v})$

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$$T(\underline{u}) = A\underline{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(\underline{v}) = A\underline{v} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

eg: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $\underline{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $\underline{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

and define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by
$T(\underline{x}) = A\underline{x}$

a) Find $T(\underline{u})$, The images of $\underline{u}$ under the transformation T.
b) Find an $\underline{x}$ in $\mathbb{R}^2$ whose image under T is $\underline{b}$
c) is there more than one $\underline{x}$ whose image under T is $\underline{b}$
d) Determine if $\underline{c}$ is in the range of the transformation T.

Solution:

a) $T(\underline{u}) = A\underline{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$

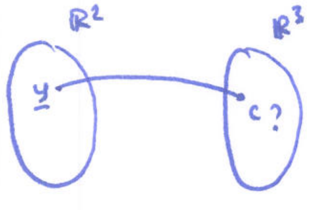
b) Find an $\underline{x}$ so that $T(\underline{x}) = \underline{b}$ is valid.
or solve for $\underline{x}$... augmented matrix: $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = 3/2 \quad x_2 = -1/2$$

$$\to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \quad \text{so} \quad T\left(\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

c) since $\underline{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$ is a uniquesolution. You cant find other $\underline{x}$'s so that $T(\underline{x}) = \underline{b}$

d)



is c in range ?

c is the range of T if there exists a $\underline{u}$ in $\mathbb{R}^2$ (domain) so that $T(\underline{u}) = \underline{c}$

ie can you find $\underline{u}$ in $\mathbb{R}^2$ so that $A\underline{u} = \underline{c}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$0 \neq 5$ contradiction.
There's nosolution
ie c is not in the range of T.

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③ Define T by $T(\underline{x}) = A\underline{x}$

where $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

* Find a vector $\underline{x}$, whose image under T is $\underline{b}$ and determine whether $\underline{x}$ is unique

Solution: Find an $\underline{x}$ such that $T(\underline{x}) = \underline{b}$

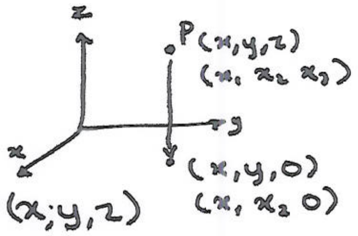
Find an $\underline{x}$ such that $A\underline{x} = \underline{b}$

augmented matrix: $\begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & 2 & -5 & -3 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$ $\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $x_1 = 3$, $x_2 = 1$, $x_3 = 2$ $\underline{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ ie: $T\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

eg: projections.

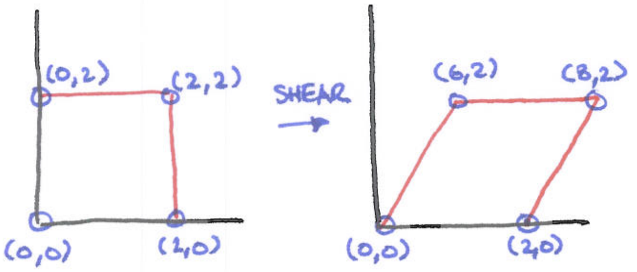


Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Then the transformation $T(\underline{x}) = A\underline{x}$ projects points in $\mathbb{R}^3$ (x, y, z) onto the xy plane $(\mathbb{R}^2)$

For example take a general (arbitrary) point. $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\underline{x} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

eg: shear transformations:



let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ then the transformation $T(\underline{x}) = A\underline{x}$ is a shear transformation.

let $\underline{u}_1 = (0,0)$
$\underline{u}_2 = (2,0)$
$\underline{u}_3 = (2,2)$
$\underline{u}_4 = (0,2)$

$$T(\underline{u}_1) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(\underline{u}_2) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T(\underline{u}_3) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$T(\underline{u}_4) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

25-09-07

Definition: (Linear transformation)

A transformation T is linear if (i) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$ for all $\underline{u},\underline{v}$ in the domains of t

(ii) $T(c\underline{u}) = cT(\underline{u})$ for all vectors $\underline{u}$ in the domain of T and all scalars c

NOTE: If T is a linear transformation then

$$T(\underline{0}) = 0$$

$$T(\underline{x}) = A\underline{x}$$

$$\begin{aligned} T(\underline{x}+\underline{y}) &= A(\underline{x}+\underline{y}) \\ &= A\underline{x} + A\underline{y} \\ &= T(\underline{x}) + T(\underline{y}) \end{aligned}$$

$$\begin{aligned} T(c\underline{x}) &= A(c\underline{x}) \\ &= cA\underline{x} = cT(\underline{x}) \end{aligned}$$

FACT: You can combine (i) and (ii) and write:

T is linear if and only if

$$T(c\underline{u}+d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$$

for all vectors $\underline{u},\underline{v}$ in the domain of T and for all scalars c,d.

27-09-07

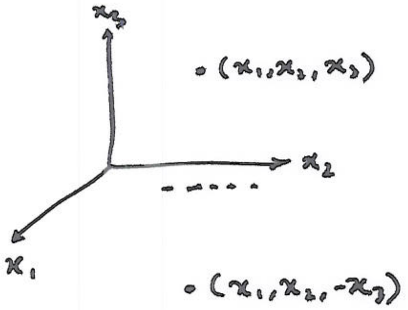
Recap:
Linear transformation:

$*\ T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\left.\begin{array}{l}\text{(i) } T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v}) \\ \text{(ii) } T(c\underline{u}) = cT(\underline{u})\end{array}\right\} \text{combining both: } T(c\underline{u}+d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$$

(35) Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the transformation that reflects each vector $\underline{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\underline{x}) = (x_1, x_2, -x_3)$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\boxed{T(x_1, x_2, x_3) = (x_1, x_2, -x_3)}$ — ⊛



Solution: T is linear if $T(c\underline{u}+d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$ for all vectors, $\underline{u}, \underline{v}$ and scalars c, d.

Let $\underline{u} = (u_1, u_2, u_3)$

$\underline{v} = (v_1, v_2, v_3)$ be arbitrary vectors in $\mathbb{R}^3$

Let c, d be two scalars.

$$c\underline{u} + d\underline{v} = c(u_1, u_2, u_3) + d(v_1, v_2, v_3)$$

$$= (\underbrace{cu_1 + dv_1}_{x_1}, \underbrace{cu_2 + dv_2}_{x_2}, \underbrace{cu_3 + dv_3}_{x_3})$$

applying ⊛ gives: $(cu_1 + dv_1, cu_2 + dv_2, -(cu_3 + dv_3))$

$$= (cu_1 + cu_2 - cu_3) + (dv_1 + dv_2 - dv_3)$$

$$= c(u_1 + u_2 - u_3) + d(v_1 + v_2 - v_3)$$

$$= cT(\underline{u}) + dT(\underline{v})$$ Satisfies requirement of proof

∴ it is linear.

eg: Is the function $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $T(x,y) = (x - 2y, 4x + 3y)$ ⊛ a linear transformation?

Sol: ① let $\underline{u} = (u_1 + u_2)$ and $\underline{v} = (v_1 + v_2)$ be arbitrary vectors and c, d be scalars, Then T is linear if $T(c\underline{u} + d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$

$$c\underline{u} + d\underline{v} = c(u_1, u_2) + d(v_1, v_2) = (cu_1 + dv_1, cu_2 + dv_2)$$

$$T(c\underline{u} + d\underline{v}) = T(\underbrace{cu_1 + dv_1}_{x}, \underbrace{cu_2 + dv_2}_{y})$$

applying ⊛: $= (cu_1 + dv_1 - 2(cu_2 + dv_2), 4(cu_1 + dv_1) + 3(cu_2 + dv_2))$

$$= (cu_1 - 2cu_2, 4cu_1 + 3cu_2) + (dv_1 - 2dv_2, 4dv_1 + 3dv_2)$$

$$= c(u_1 - 2u_2, 4u_1 + 3u_2) + d(v_1 - 2v_2, 4v_1 + 3v_2)$$

$$= cT(\underline{u}) + dT(\underline{v})$$ linear because proof is satisfied.

$$\left(\begin{array}{l} T(\underline{u}) = T(u_1, u_2) = (u_1 - 2u_2, 4u_1 + 3u_2) \\ T(\underline{v}) = T(v_1, v_2) = (v_1 - 2v, 4v_1 + 3v_2) \end{array}\right)$$

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(31) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a Linear Transformation and let $\{v_1, v_2, v_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\underline{v}_1), T(\underline{v}_2), T(\underline{v}_3)\}$ is linearly dependent.

Solution: since $\{v_1, v_2, v_3\}$ is linearly dependent.

Then there exists scalars c_1, c_2, c_3 not all zero

such that $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$

Apply transformation: $T(c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3) = T(\underline{0})$

Since T is linear $c_1 T(\underline{v}_1) + c_2 T(\underline{v}_2) + c_3 T(\underline{v}_3) = \underline{0}$

c_1, c_2, c_3 are not all zero

ie: the set $\{T(\underline{v}_1) + T(\underline{v}_2) + T(\underline{v}_3)\}$ is linearly dependent.

(32) Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ ⊛ is not linear.

To show that T is linear $T(c\underline{u} + d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$ for all $\underline{u}, \underline{v}$

To show that T is not linear, you can simply give one example so that

$$T(c\underline{u} + d\underline{v}) \neq cT(\underline{u}) + dT(\underline{v})$$

Counter example: $c = 1 \quad d = 1$

Take $\underline{u} = (0, 1)$ $\qquad c\underline{u} + d\underline{v} = 1(0,1) + 1(0,-1) = (0,0)$

$\underline{v} = (0, -1)$

Apply to ⊛: $T(c\underline{u} + d\underline{v}) = T(0,0)$

$\overset{⊛}{=} (0,0)$ — ②

$T(\underline{u}) = T(0,1) \overset{⊛}{=} ((4)(0) - (2)(1), 3|1|) = (-2, 3)$

$T(\underline{v}) = T(0,-1) \overset{⊛}{=} ((4)(0) - 2(-1), 3|1|) = (2, 3)$

$cT(\underline{u}) + dT(\underline{v}) = (-2,3) + (2,3) = (0,6)$ — ①

So $(0,0) \neq (0,6)$ ie ① $\neq$ ② $\therefore$ T is not linear.

30-09-07

1.9 The matrix of a Linear transformation.

$I_3 =$

eg: let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the identity matrix of size 2.

let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Suppose T is a linear transformation such that $T(e_1) = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} 6 \\ -16 \\ 4 \end{bmatrix}$, Find a formula for the image of an arbitrary vector $x = (x_1, x_2)$ in $\mathbb{R}^2$.

let $\underline{x} = (x_1, x_2)$ be an arbitrary vector in $\mathbb{R}^2$

$$\underline{x} = (x_1, 0) + (0, x_2)$$
$$\underline{x} = x_1(1,0) + x_2(0,1)$$
$$= \underline{x_1} e_1 + \underline{x_2} e_2$$

Since T is linear:

$$T(\underline{x}) = T(x_1 \underline{e_1} + x_2 \underline{e_2})$$
$$T(\underline{x}) = x_1 T(\underline{e_1}) + x_2 T(\underline{e_2}) = x_1 \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ -16 \\ 4 \end{bmatrix}$$

$$T(\underline{x}) = \begin{bmatrix} 3x_1 + 6x_2 \\ -5x_1 - 16x_2 \\ 0x_1 + 4x_2 \end{bmatrix} \Rightarrow T(\underline{x}) = \begin{bmatrix} -3 & 6 \\ -5 & -16 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow T(\underline{x}) = Ax.$$

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, then there exists a unique matrix A such that $T(\underline{x}) = Ax$.

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

1.9 ① Assume T is a linear transformation. Find the standard matrix of T.

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$

$T(e_1) = (3,1,3,1)$

$T(e_2) = (-5,2,0,0)$

$e_1 = (1,0)$ $e_2 = (0,1)$

$T(\underline{x}) = A\underline{x}$ where the standard matrix

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

⑤ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vertical shear transformation that maps $\underline{e_1}$ into $\underline{e_1} - 2e_2$ but leaves the vector $\underline{e_2}$ unchanged

$T(\underline{e_1}) = \underline{e_1} - 2\underline{e_2}$ standard matrix: $A = [T(\underline{e_1})\ T(\underline{e_2})] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$T(\underline{e_3}) = e_2$

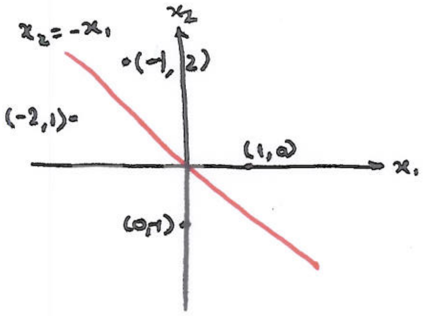
30-09-07

⑨ $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ first performs a horizontal shear that transforms $\underline{e}_2$ into $\underline{e}_2 - 2\underline{e}_1$ (leaving $\underline{e}_1$ unchanged) and then reflects through the line $x_2 = -x_1$.

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step 1: $T(\underline{e}_1) = \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad T(\underline{e}_2) = \underline{e}_2 - 2\underline{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Step 2:

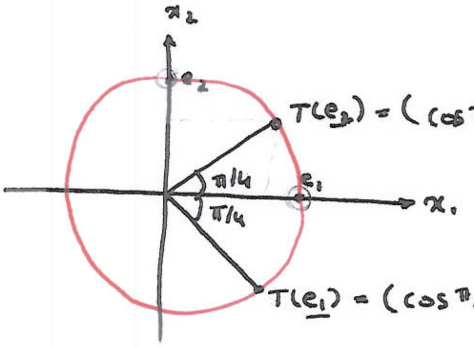


$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

std matrix: $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

④ $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ Rotates points (about the origin) through $-\pi/4$ radians (clockwise.)



$T(\underline{e}_2) = (\cos \pi/4, \sin \pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$

STD matrix $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$T(\underline{e}_1) = (\cos \pi/4, -\sin \pi/4) = (1/\sqrt{2}, -1/\sqrt{2})$

02-10-07

Standard matrix:

㉒ Let $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that

$T(x_1,x_2) = (x_1-2x_2, -x_1+3x_2, 3x_1-2x_2) \rightarrow \circledast$

Find $\underline{x}$ such that $T(\underline{x}) = (-1,4,9)$

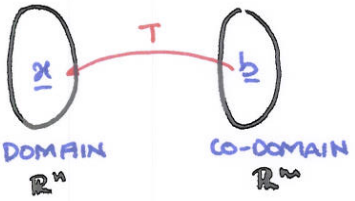
Soln: we need $x = (x_1,x_2)$ such that $T(x_1,x_2) = (-1,4,9)$

$\circledast\ (x_1-2x_2, -x_1+3x_2, 3x_1-2x_2) = (-1,4,9)$

$$\left.\begin{aligned} x_1-2x_2 &= -1 \\ -x_1+3x_2 &= 4 \\ 3x_1-2x_2 &= 9 \end{aligned}\right\} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 5 \\ x_2 &= 3 \end{aligned}$$

Definition (Mapping onto):



- A mapping $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\underline{b}$ in $\mathbb{R}^m$ is the image of at least one $\underline{x}$ in $\mathbb{R}^n$. (ie range = codomain).

- or it can be said as follows: for each $\underline{b}$ in $\mathbb{R}^n$ (there exists an $\underline{x}$ in $\mathbb{R}^n$ such that $T(x) = \underline{b}$

- or as follows: for each $\underline{b}$ in the co-domain $\mathbb{R}^m$ there exists at least one solution of $T(\underline{x}) = \underline{b}$

- or in another form as a negative statement:

The mapping T is not onto when there is some $\underline{b}$ in $\mathbb{R}^m$ for which $T(\underline{x}) = \underline{b}$ has no solution.

Theorem: Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix of T, then:

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$

$A = [\underline{a}_1\ \underline{a}_2 \ldots\ldots \underline{a}_n]$ $\underline{v}$ in $\mathbb{R}^m$ may be any vector ($\underline{v} = x_1\underline{a}_1 + x_2\underline{a}_2 \cdots + x_n\underline{a}_n$)

Recall: Sec 1.4 thm 4: Let A be a $m \times n$ matrix then the following are equivalent:

a) for each $\underline{b}$ in $\mathbb{R}^m$, the equation $A\underline{x} = \underline{b}$ has a solution
b) each $\underline{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A
c) The columns of A span $\mathbb{R}^m$
d) A has a pivot position in every row.

02-10-07

㉕ let $T(x_1, x_2, x_3, x_4) = (0, x_1+x_2, x_2+x_3, x_3+x_4)$ Determine whether T is onto.

Standard matrix of T: $A = [T(\underline{e}_1)\ T(\underline{e}_2)\ T(\underline{e}_3)\ T(\underline{e}_4)]$

$$e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$A = \begin{array}{c}\begin{bmatrix}0 & 0 & 0 & 0\\1 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{bmatrix}\\ \begin{array}{cccc}T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) & T(\underline{e}_4)\end{array}\end{array} \rightarrow \begin{bmatrix}1 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

- Since the last row doesn't have a pivot position, the columns of A do not span $\mathbb{R}^4$ $\therefore$ T is not onto.

04-10-07

⓵.⑨ One to one

Definition: A mapping $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be one to one if each $\underline{b}$ in $\mathbb{R}^m$ is the image of at most one $\underline{x}$ in $\mathbb{R}^n$.

Theorem: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is 1-1 if and only if the equation $T(\underline{x}) = \underline{0}$ has only the trivial solution.

eg: let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$

Show that T is 1-1

Solution: let $T(\underline{x}) = \underline{0}$

$(3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2) = (0,0,0)$

$3x_1 + x_2 = 0$
$5x_1 + 7x_2 = 0$
$x_1 + 3x_2 = 0$

augmented matrix: $\begin{bmatrix} 3 & 1 & 0 \\ 5 & 7 & 0 \\ 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 5 & 7 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -8 & 0 \\ 0 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$ Trivial solution

∴ T is 1-1.

㉗ $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

$\mathbb{R}^3 \rightarrow \mathbb{R}^2$ is T one to one?

consider $T(\underline{x}) = 0$ $T(\underline{x}) = (x_1 - 5x_2 + 4x_3, x_2 - 6x) = (0,0)$

$\left.\begin{matrix} x_1 - 5x_2 + 4x_3 = 0 \\ x_2 - 6x_3 = 0 \end{matrix}\right\}$ $\begin{bmatrix} 1 & -5 & 4 & 0 \\ 0 & 1 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -26 & 0 \\ 0 & 1 & -6 & 0 \end{bmatrix}$

$x_1 = 26x_3$
$x_2 = 6x_3$
$x_3 =$ free.

There is a non trivial solution Therefore T is not 1-1.

Theorem: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T.

(b) T is one to one if and only if the columns of A are linearly independent.

Note: (Recall sec 1.7) The columns of a matrix A are linearly independent if and only if the equation $A\underline{x} = \underline{0}$ has only the trivial soln.

㉒ $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$

$\mathbb{R}^2 \rightarrow \mathbb{R}^3$

Standard matrix: $A = [T(\underline{e}_1) \; T(\underline{e}_2)] \Rightarrow T(\underline{e}_1) = (1, -1, 3)$

$T(\underline{e}_2) = (-2, 3, -2)$

consider $A\underline{x} = \underline{0}$

augmented matrix: $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 3 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$

∴ Trivial soln. T is 1-1.

07-10-07

Review sheet

⑧ $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

Standard matrix: $A = [T(e_1) \;\; T(e_2) \;\; T(e_3)]$

$T(1,0,0) = (1,0)$

$T(0,1,0) = (-5,1)$

$T(0,0,1) = (4,-6)$

$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -26 \\ 0 & 1 & -6 \end{bmatrix}$ standard matrix A has pivot positions in both rows, so T is onto.

c) $T(\underline{x}) = \underline{0}$ has the trivial soln.

$A\underline{x} = \underline{0}$ has the trivial soln.

$$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -5 & 4 & 0 \\ 0 & 1 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -26 & 0 \\ 0 & 1 & -6 & 0 \end{bmatrix}$$

$\left.\begin{array}{l} x_1 = 26x_3 \\ x_2 = 6x_3 \\ x_3 = \text{free.} \end{array}\right\}$ $A\underline{x} = \underline{0}$ has a nontrivial solotion so it is $\boxed{\text{not } 1-1.}$

② For what values of h is b in the span of v_1, v_2

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} h \\ -5 \\ 3 \end{bmatrix}$$

in other words, can you write $\underline{b}$ as a linear combination of $\underline{v}_1, \underline{v}_2$

$\underline{b} = x_1\underline{v}_1 + x_2\underline{v}_2$

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & h \\ 0 & -1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$$

for the system to be consistent

$\boxed{h = -7/2}$

23-10-07

2.1 Matrix Operations

For information... refer to textbook sec 2.1

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 2 & 6 & 7 \end{bmatrix}$$ 2D array.

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1j} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2j} \cdots\cdots a_{2n} \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1} & a_{i2} \cdots\cdots a_{ij} \cdots\cdots a_{in} \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mj} \cdots\cdots a_{mn} \end{bmatrix}_{m\times n}$$

$1 \leq i \leq m$

$1 \leq j \leq n$

$A = [a_{ij}]_{m\times n}$

$B = [b_{ij}]_{m\times n}$

Note: to add two matrices A and B

The size A = size B.

Theorem:

(i) $A+B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B+A$ Commutative

(ii) $A+(B+C) = (A+B)+C$

(iii) $A+O = A$

(iv) Suppose r is a scalar:

$r(A+B) = rA + rB$

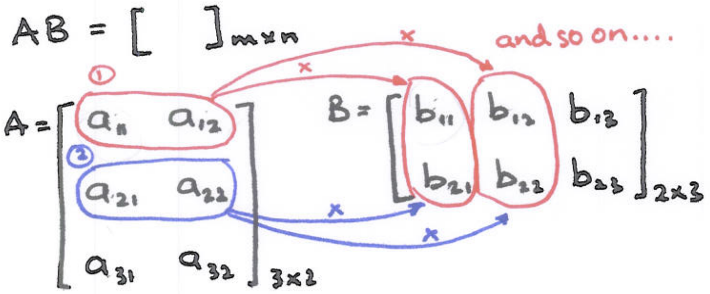
(v) If r,s are scalars: $(r+s)A = rA + sA$

(vi) $r(sA) = (rs)A$

Matrix multiplication:

$A = [a_{ij}]_{m\times p}$, $B = [b_{ij}]_{r\times n}$ p should be equal to r to multiply the matrices.

$AB = [\quad]_{m\times n}$ and so on....



$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2\times 3}$$

$$AB = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}), (a_{11}b_{12} + a_{12}b_{22}), \cdots \\ (a_{21}b_{11} + a_{22}b_{21}), \cdots\cdots\cdots\cdots, \cdots \\ (a_{31}b_{11} + a_{32}b_{21}), \cdots\cdots\cdots\cdots, \cdots \end{bmatrix}_{3\times 3}$$

$$BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} (b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31}), (b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{32}) \\ (b_{21}a_{11} + b_{22}a_{21} + b_{23}a_{31}), (b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32}) \end{bmatrix}_{2\times 2}$$

23-10-07

⑤ $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}_{3\times 2}$ $B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}_{2\times 2}$ $AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

$BA = B_{2\times\textcircled{2}} A_{\textcircled{3}\times 2}$ $2 \neq 3$ ∴ can't multiply.

Note: in general $AB \neq BA$

Identity matrices:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $AI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

in general: $I_{n\times n} = I_n$ Called a square matrix.

- Identity matrices are square matrices, with all diagonals 1 and rest of the numbers 0.

(i) $A_{m\times n} I_n = A_{m\times n}$

(ii) $I_m A_{m\times n} = A_{m\times n}$

Example: $\underset{3\times 2}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $AI_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Power of a matrix:

$a^5 = a \times a \times a \times a \times a$

suppose A is a matrix, $A^2 = A \times A$, and suppose A is a square $n\times n$.

$A^k = \underbrace{A \times A \times \ldots . \times A}_{k \text{ times.}}$

Transpose (A^T):

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ Rows get turned into columns.

25-10-07

Transpose:

eg: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}_{2\times5}$ $\quad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \end{bmatrix}_{5\times2}$

- The matrix constructed from A by interchanging rows with columns.

$A = [a_{ij}]_{m\times n}$

$A^T = [a_{ij}]_{n\times m}$

Theorem:

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$ proof: $A = [a_{ij}] \quad B = [b_{ij}]$

$A+B = [a_{ij} + b_{ij}]$

$(A+B)^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}]$

$= A^T + B^T$

(iii) If r is a scalar:

$(rA)^T = rA^T$: proof: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad (4A)^T = \begin{bmatrix} 4 & 12 \\ 8 & 16 \end{bmatrix}$

$r=4 : 4A = \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix} \quad = 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4A^T$

(iv) $(AB)^T = B^T A^T$: $A = [a_{ij}] \quad B = [b_{ij}]$

$AB = [c_{ij}]$

$(AB)^T = [c_{ij}]^T = [c_{ji}] = B^T A^T$

$(ABC)^T = [(AB)(C)]^T = C^T (AB)^T = C^T B^T A^T$

Note: If $A = [a_{ij}]_{m\times n}$

$B = [\,b_1 \; b_2 \; \dots \; b_p\,]_{n\times p}$

columns.

$AB = [\,\underset{\text{1st col}}{Ab_1} \quad \underset{\text{2nd col}}{Ab_2} \; \dots \; \underset{\substack{\text{3rd col}\\ \text{pth col.}}}{Ab_p}\,]_{m\times p}$

eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ $\quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}_{3\times2}$ $\quad AB = \begin{bmatrix} -2 & 3 \\ -2 & 9 \\ -2 & 15 \end{bmatrix}_{3\times2}$

⑰ $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}_{2\times2}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}_{2\times3}$ Find the first 2 cols of B.

Sol: $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ $\quad AB = \left[A\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \; A\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \; A\begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \right] = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$

25-10-07

$$A\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Augmented matrix: $\begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$ $b_{11} = 7$, $b_{21} = 4$

$A\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$ Augmented matrix: $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}$

$b_{12} = -8$

$b_{22} = -5$

2.2 Inverse of A Matrix

- $ab = 1$ ∴ a is inverse of b
  b is inverse of a.
- in matrices if $AB = I$, B is the inverse of A.
  A is the inverse of B.
- Inverses can only be found for square matrices.

DEFINITION:

* An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix B such that $AB = I_n$ and $BA = I_n$

Then B is called the inverse of A and is written as $B = A^{-1}$

ie: $AA^{-1} = I$; $A^{-1}A = I$

- 2x2 matrices:

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$ with $ad - bc \neq 0$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

① let $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ $A^{-1} = \frac{1}{(8)(4)-(6)(5)}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

Check! $AA^{-1} = I$.

eg: $A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ because $ad - bc = 0$ No inverse.

30-10-07

* An algorithm to find inverse of any $n\times n$ matrix

- let A be an $n\times n$ matrix, consider the matrix

$$[A \,\vdots\, I_n] \xrightarrow[\text{Row operat...}]{\text{Transform}} [I_n \,\vdots\, B] \quad \text{Then } B = A^{-1}$$

eg: find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

→ consider $[A : I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + (-3R_3)]{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & -1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & \textcircled{0} & \textcircled{3} & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$[I : A^{-1}]$

Finally, check $AA^{-1} = I$

eg: find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

consider $[A : I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_3 \to R_3 + (-5R_1)} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to (-\frac{1}{4}R_3)} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + (-R_3)]{R_2 \to R_2 + (-3R_3)} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 2 & 0 & -15/4 & 1 & 3/4 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + (-R_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] = A^{-1},$$

check!

$AA^{-1} = I$

30-10-07

eg: find the inverse of $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

consider $[A \vdots I]$: $\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + (-R_1)$
$R_3 \to R_3 + (-5R_1)$ $\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right]$

$R_3 \to R_2 + (-3R_2)$ $\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & -4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 3 & 1 \end{array}\right]$

Row of zeros indicates that there is no inverse (called a singular matrix).

Theorem: If A is an $n \times n$ matrix (invertible) ie, it has an inverse, Then for each b in $\mathbb{R}^n$, the equation $A\underline{x} = b$ has a unique solution, and the solution

$$A^{-1}A\,\underline{x} = A^{-1}b$$
$$I\underline{x} = A^{-1}b$$
$$\underline{x} = A^{-1}b$$

Theorem: If A, B are invertible matrices:

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$ (like with transpose covered earlier)

(iii) $(AB)(B^{-1}A^{-1}) = I$

$$\Rightarrow ABB^{-1}A^{-1}$$
$$= AIA^{-1}$$
$$= AA^{-1} = I$$

(iv) $(A^T)^{-1} = (A^{-1})^T$ **proof:** $AA^{-1} = I$

$$(AA^{-1})^T = I^T$$
$$(A^{-1})^T A^T = I$$
$$(A^T)^{-1} = (A^{-1})^T$$

01-11-07

2.2 Elementary Matrices.

Definition:

An elementary matrix is one that is obtained by performing a single row operation on an identity matrix.

Elementary matrices $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Interchanging rows type 1:

$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$

Type 2: $R_2 \longrightarrow KR_2$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Type 3: $R_3 \longrightarrow R_3 + 3R_1$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

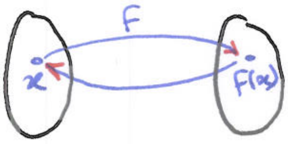
$$E_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4k & 5k & 6k \\ 7 & 8 & 9 \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7+3 & 8\overset{2}{+}(3) & 9+(3)3 \end{bmatrix}$$

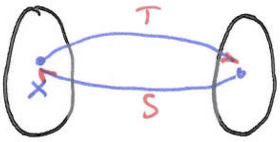
2.3 Characterization of Invertible Matrices.

Read the textbook for theorems and information.



$g = f^{-1}$

$g(f(x)) = x$
$f(g(x)) = x$



$S(T(x)) = x$
$T(S(x)) = x$

Definition: A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be invertible if there exists a function $S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$S(T(x)) = x$ for all x in $\mathbb{R}^n$ —①
$T(S(x)) = x$ for all x in $\mathbb{R}^n$ —②

01-11-07

Theorem: Suppose $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T(x) = Ax$ where A is the standard matrix, then T is invertible if and only if A is an invertible matrix, and $T^{-1}(x) = A^{-1}x$

04-11-07

Invertible L.T.

$T(x)=Ax$ is a L.T. A standard matrix T is invertible if and only if A is invertible $T^{-1}(x)=A^{-1}x$

㉝ $T(x_1,x_2)=(-5x_1+9x_2, 4x_1-7x_2)$ is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$
Show that t is invertible and find a formula for T^{-1}

Sol: $T(x)=Ax$

$$A=\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix} \qquad T(e_1)=\begin{bmatrix} -5 \\ 4 \end{bmatrix} \quad T(e_2)=\begin{bmatrix} 9 \\ -7 \end{bmatrix}$$

$$A^{-1}=\frac{1}{(-5)(-7)-(9)(4)}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = -1\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}}_{\text{ie T is invertible.}}$$

$$T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$$

$$T^{-1}(x_1,x_2)=A^{-1}\underline{x} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T^{-1}(x_1,x_2)=\underline{(7x_1+9x_2, 4x_1+5x_2)}$$

㉞ $T(x_1,x_2)=(6x_1-8x_2, -5x_1+7x_2)$ is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$
Show that T is invertible and find a formula for T^{-1}

$T(x)=A\underline{x}$

$$A=\begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} \qquad A^{-1}=\frac{1}{(6)(7)-(-8)(-5)}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix}}_{\text{ie T is invertible}}$$

$$T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2 \qquad T^{-1}(x)=A^{-1}(x) = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\left(\frac{7}{2}x_1+4x_2, \frac{5}{2}x_1+3x_2\right)}$$

HW: Read The Invertible matrix theorem.

Let A be a non-square matrix. Then the following are equivalent

(1) A is an invertible matrix
(2) A is row equivalent to the $n \times n$ identity matrix.
(3) A has n-pivot positions.
(4) The equation $Ax=0$ has only a trivial solution.
(5) The columns of A form a linearly independent set.
(6) The linear transformation $T(x)=A\underline{x}$ is one-to-one
⋮
(12) in the book.

04-11-07

3.1 Determinants

2x2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ det. A = ad−bc

or

$|A|$

Finding a determinant is also known as expanding a determinant.

3x3: $A = \begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{bmatrix}$

Expand using first row

det. A =

or

$|A|$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left[a_{22}a_{33} - a_{23}a_{32}\right] - a_{12}\left[a_{21}a_{33} - a_{23}a_{31}\right] + a_{13}\left[a_{21}a_{32} - a_{22}a_{31}\right]$$

⑤ $A = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$

$$\det A = 2\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} - 3\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + (-4)\begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix}$$

$$= 2[0-5] - 3[24-25] - 4[4-0]$$

$$= -23.$$

Choosing sign for an entry: $(-1)^{i+j}$

06-11-07

Determinants:

⑨ $$\rightarrow\begin{vmatrix} \overset{+}{6} & \overset{-}{1} & \overset{+}{0} & \overset{-}{5} \\ \overset{+}{1} & \overset{+}{7} & \overset{-}{2} & \overset{+}{-5} \\ \overset{+}{2} & \overset{-}{0} & \overset{+}{0} & \overset{-}{0} \\ \overset{-}{8} & \overset{+}{3} & \overset{-}{1} & \overset{+}{8} \end{vmatrix}$$

It is best to select the row with the most zeros to calculate the determinant, since this will give us the simplest calculation. (The rest of the "sub-determinants" will be multiplied by zero)

$$= 2\begin{vmatrix} 1 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = 2\times1\begin{vmatrix} 2 & -5 \\ 1 & 8 \end{vmatrix} + 2\times5\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 2\times(16+5)+10\times(7-6)$$
$$= 42+10 = 52$$

Co factors:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$(i, j)^{th}$ co factor.

$j^{th} \leftarrow$ row

$i^{th} \leftarrow$ col.

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$C_{32} = (-1)^{3+2} \det A_{32} = -1\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

eg: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ expand using 2nd row:

$$\det A = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$ is called the co-factor expansion along row 2.

⑪ $$A = \begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} \quad \det A = 3\begin{vmatrix} -2 & 3 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 3\left[-2\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix}\right] =$$

Find determinant using columns this time instead of using the rows.

The above matrix is known as an upper triangular matrix, due to the configuration of numbers and zeros.

Theorem: If A is a triangular matrix, then $\det A = a_{11}a_{22}a_{33}a_{44}$ (the product of diagonal entries).

Triangular matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

upper Triangular — Lower triangular.

⑲ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det A = ad - bc$

a) row operations: $R_1 \longleftrightarrow R_2$

$$E_1 = \begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - ad = -(ad-bc)$$
$$= -\det A$$

b) $R_2 \longleftrightarrow kR_2$

$$E_2 = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} = kad - kbc = k(ad-bc)$$
$$= k\det A.$$

c) $R_2 \rightarrow R_2 + kR_1$

$$E_3\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$$
$$= a(d+kb) - b(c+ka)$$
$$= ad - bc = \det A.$$

06-11-07.

3.2 Properties of Determinants.

Theorem: Row operations:

Let A be an $n \times n$ square matrix:

① If two rows of A are interchanged to produce matrix B then Det B = - det A

② If one row of A is multiplied by k to produce matrix B, then det B = k det A

③ If a multiple of one row of A is added to another row to produce B, then det B = det A.

①
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} c & d \\ a & b \end{bmatrix} = B$$

$$\det A = ad - bc \qquad \det B = cb - ad$$

$$\det A = -\det B.$$

②
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} ak & bk \\ c & d \end{bmatrix}$$

$$\det A = adk - bck$$
$$= k(ad - bc)$$
$$= k(\det A)$$

$$\det A = K(\det A)$$

③
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} a & b \\ 3a+c & 3b+d \end{bmatrix}$$

$$\det A = a(3b+d) - b(3a+c)$$
$$= 3\cancel{ab} + ad - 3\cancel{ab} - bc$$
$$= ad - bc$$

08-10-07

Theorem: $\det(A) = \det(A^T)$

eg: $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

2nd Row: $\det A = 0\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 2[-1-3] - [1+3]$

$= -8-4 = -12$

$A^T = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ **1st Row:** $\det A^T = 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$

$= [-2-1] + 3[-1-2] = -3-9 = -12$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ **1st Row:** $\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$

Theorem: If A and B are both $n \times n$ square matrices

$\det(AB) = (\det A)(\det B)$

$\det(A^T B^T) = (\det A^T)(\det B^T) = (\det A)(\det B) = \det(AB)$

(29) Find $\det B^5$ where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

1st row: $\det B = 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (1-4) + (2-1) = -2$

$\det B^5 = (-2)^5 = -32.$

(31) Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

Solution: $AA^{-1} = I$

$\det(AA^{-1}) = \det(I)$

$(\det A)(\det A^{-1}) = 1$

$(\det A^{-1}) = \dfrac{1}{(\det A)}$

(34) Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$

Soln: $\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1})$

$= (\det P)(\det A)\left(\dfrac{1}{(\det P)}\right)$

$= \det A.$

08-11-07

4.1 Vector Spaces.

V = set of objects (eg $V = \mathbb{R}^3$)

R = real numbers:

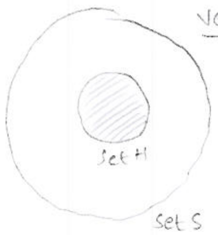
let u, v, w be any vectors in V $\left(\begin{matrix} \underline{u} = (x_1, y_1, z_1) \\ \underline{v} = (x_2, y_2, z_2) \end{matrix} \right)$

let α, β be any two scalars from $\mathbb{R}$.

1. $u + v = v + u$ – commutative.
2. $u + (v + w) = (u + v) + w$ – associative.
3. There is a zero vector $\underline{0}$ in V such that $\underline{u} + \underline{0} = \underline{u}$
4. The sum of $u + v$ is in V.
5. For each u in V there is a $(-\underline{u})$ in V, such that $\underline{u} + (-\underline{u}) = \underline{0}$
6. The scalar multiple $\alpha \underline{u}$ is in V.
7. $\alpha(u + v) = \alpha u + \alpha v$
8. $(\alpha + \beta) u = \alpha u + \beta u$
9. $\alpha(\beta u) = (\alpha\beta) u$.

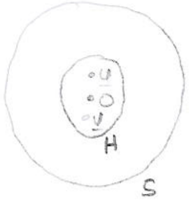
11-11-07

Subspaces:



Venn Diagram

H is a subset of S.



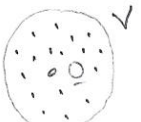
H subset:

1. $\underline{0}$ is in H
2. $\underline{u}+\underline{v}$ in H
3. $\alpha\underline{u}$ is in H

Definition: A subspace of a vector space V is a subset H of V that satisfies:

1. The zero vector $\underline{0}$ of V is in H.
2. H is closed under addition.
   ie. for all $\underline{u},\underline{v}$ under H $\underline{u}+\underline{v}$ is in H
3. H is closed under scalar multiplication.
   ie. for all $\underline{u}$ in H and for all α in $\mathbb{R}$, αu is in H.

eg:



$H = \{\underline{0}\}$

many other elements are also present.

(21) Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times2}$

* $M_{2\times2}$ = set of all 2x2 matrices:

$$M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \text{ are in } \mathbb{R} \right\} \text{ is a vector space.}$$

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a,b,d \text{ are in } \mathbb{R} \right\}$$

Solution: (1) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in H

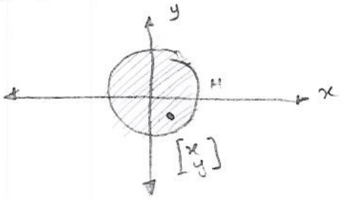
(2) $\underline{u} = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, \underline{v} = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ in H.

$\underline{u}+\underline{v} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & d_1+d_2 \end{bmatrix}$ is in H.

(3) α in $\mathbb{R}$, $\underline{u} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ in H

$\alpha\underline{u} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & \alpha d \end{bmatrix}$ is in H So H is a subspace of $M_{2\times2}$

(3) Let H be the set of points inside and on the unit circle in the xy-plane. ie:



$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x^2+y^2 \leq 1 \right\}$$

is H a subspace of $\mathbb{R}^2$?

11-11-07

eg: $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $1^2+0^2=1\leq 1$ ✓

$\underline{u}$ is in H

$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $0^2+1^2=1\leq 1$ ✓

$\underline{v}$ is in H

$\underline{u}+\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $1^2+1^2 \not\leq 1$ ×

$\underline{u}+\underline{v}$ is not in H

ie: H is not a subspace of $\mathbb{R}^2$

eg: let V be a vectorspace and let $\underline{u}, \underline{v}$ be a vector in V.

Span $\{\underline{u}, \underline{v}\}$ = set of all linear combinations of $\underline{u}$ and $\underline{v}$

$H = \text{span}\{\underline{u}, \underline{v}\} = \{\alpha\underline{u} + \beta\underline{v} \mid \alpha, \beta \text{ in } \mathbb{R}\}$

Show that span $\{\underline{u}, \underline{v}\}$ is a subspace of $\underline{V}$

Soln: 1). $\underline{0} = 0\underline{u} + 0\underline{v} \Rightarrow \underline{0}$ is in H.

2). Let

$\underline{a} = \alpha_1\underline{u} + \beta_1\underline{v}$

$\underline{b} = \alpha_2\underline{u} + \beta_2\underline{v}$ be in H.

$$\underline{a}+\underline{b} = (\alpha_1\underline{u} + \beta_1\underline{v}) + (\alpha_2\underline{u} + \beta_2\underline{v})$$
$$= (\alpha_1+\alpha_2)\underline{u} + (\beta_1+\beta_2)\underline{v}$$

So $\underline{a}+\underline{b}$ is in H

3). Take ~~aray~~ any λ in $\mathbb{R}$ and let $\underline{a} = \alpha\underline{u} + \beta\underline{v}$ in H

$\lambda\underline{a} = \lambda(\alpha\underline{u} + \beta\underline{v}) = (\lambda\alpha)\underline{u} + (\lambda\beta)\underline{v}$ is in H

$\Rightarrow H = \text{Span}\{\underline{u}, \underline{v}\}$ is a subspace of V.

e.g $u, v, \in V$ P.① 13/11/07

span$\{u, v\}$ is a subset of V

Thm: Let $v_1, v_2, v_3, \dots v_p$ be in a vector space V.

Then span$\{v_1, v_2, \dots v_p\}$ is a subspace of V.

(10) Let (H) be the set of vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that (H) is a subspace of $\mathbb{R}^3$:

Solution: $H = \left\{ \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} : t \text{ is in } \mathbb{R} \right]$

Let $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$ be any vector in H

Then $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t \underset{\overset{\uparrow}{u}}{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}} = (t\,u)$

explanation: ie any vector in H can be written as a linear combination of u

$H = \text{span}\{u\} \Rightarrow H$ is a subspace of $\mathbb{R}^3$.

(12)

Let (W) be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$

Show that (W) is a subspace of $\mathbb{R}^4$

slution: $W := \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

Take any $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$ in (W)

$$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = \begin{bmatrix} s \\ s \\ 2s \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ -t \\ -t \\ 4t \end{bmatrix} \Rightarrow s \underset{\overset{\uparrow}{u}}{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}} + t \underset{\overset{\uparrow}{v}}{\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}}$$

$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s\,u + t\,v$ $\Rightarrow W = \text{span}\{u, v\}$

$\Rightarrow W$ is a ~~subset~~ subspace of $\mathbb{R}^4$

4.2 Null space of a matrix:

13/11/07

Homogenous equation: $A\underline{x} = \underline{0}$

$$\text{Nul } A = \{ x \mid \underset{x \text{ in } \mathbb{R}^n}{Ax = 0} \}$$

Def: The null space of an $m \times n$ matrix, written Nul A, is the set of all Solutions, to the homogenous equation $A\underline{x} = \underline{0}$

ie $\text{Nul } A = \{ x \mid \substack{x \text{ is in } \mathbb{R}^n \\ A\underline{x} = \underline{0}} \}$

* To check wether $\underline{w}$ is in (Nul A) check $A\underline{w} = \underline{0}$ if $A\underline{w} = \underline{0}$ $\Rightarrow$ $\underline{w}$ is in Null A

problem 1: Determine if $\underline{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

Check $A\underline{w} = 0 \Rightarrow \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 - 15 + 12 = 0 \\ 6 - 6 + 0 = 0 \\ -8 + 12 - 4 = 0 \end{bmatrix} = \boxed{0}$

$\Rightarrow$ $\underline{w}$ is in Null A

Thm: The Null space, Nul A of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

problem (a) Determine if the given set

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a - 2b = 4c \\ 2a = c + 3d \end{matrix} \right\}$$

a vector space?

Solution: consider

$a - 2b - 4c = 0$
$2a - c - 3d = 0$

$$A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{matrix} a - 2b - 4c = 0 \\ 2a - c - 3d = 0 \end{matrix}$$

Take any $\underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ in W

then $A\underline{x} = \underline{0}$

ie $\underline{x} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is in Nul A

ie W = Null A

$\Rightarrow$ W is a subspace of $\mathbb{R}^4$

$\Rightarrow$ W is a Vector space.

problem(8):

$\underline{W} = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$

is $\underline{W}$ a vector space?

Solution: Check for $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$5(0) - 1 = 0 + 2(0) \neq 0$

$\underline{0}$ does not satisfy then relationship

ie 0 is not in $\underline{W}$ then $A\underline{x} \neq \underline{0}$

$\Rightarrow$ $\underline{W}$ is not a vector space

Find a spanning set for the Null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

Step (1): consider $A\underline{x} = \underline{0}$

$$[A \,|\, 0] = \begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$$

$x_1 - 7x_3 + 6x_4 = 0$ $\qquad$ $x_1 = 7x_3 - 6x_4$

$x_2 - 4x_3 - 2x_4 = 0$ $\quad \Rightarrow \quad$ $x_2 = -4x_3 + 2x_4$

$x_3 =$ free $\qquad$ $x_3 = x_3$

$x_4 =$ free $\qquad$ $x_4 = x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 7x_3 \\ -4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -6x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overset{s}{x_3} \underset{u}{\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}} + \overset{t}{x_4} \underset{\underline{v}}{\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

$$\underline{x} = s\underline{u} + t\underline{v}$$

$$\text{Nul } A = \text{span}\{u, v\} = \text{span}\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

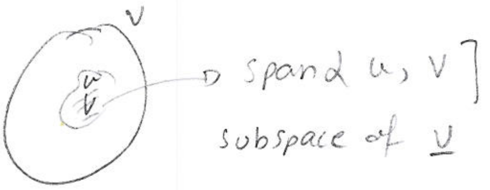
MTH 221 15/11/07 (1)

Column space of a matrix A :

Def : The column space of an m×n matrix A, written col A, is the set of all linear combinations of the columns of A.

If $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

$\text{Col } A = \text{span}\{a_1, a_2, \ldots a_n\}$



Theorm : The column space col A of an m×n matrix(A), is a subspace of $\mathbb{R}^m$

A m×n

problem 24 :

Let $A = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array} \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ Determine if $\underline{w}$ is in col A ?

$\underline{w}$ is in col A iff $\underline{w}$ is in span$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$

$\underline{w} = x_1 v_1 + x_2 v_2 + x_3 v_3$

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = x_1 \begin{bmatrix} -8 \\ 6 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -9 \\ 8 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix} \xrightarrow[\text{using row operations}]{\text{check for consistance}} \begin{bmatrix} 0 & -2 & -1 & -2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix}$$

MTH 221

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1/2 \\ 0 & -2 & -1 & 2 \\ 6 & 4 & 8 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1/2 \\ 0 & -2 & -1 & -2 \\ 0 & 4 & 2 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1/2 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$ The system of equations is consistent

$\Rightarrow$ $\underline{w}$ is a Col A

problem 16:

Find a matrix A such that W = Col A where

$$W = \left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} : b, c, d \text{ are in } \mathbb{R} \right\}$$

W = Col A

W = span$\{v_1, v_2, \ldots, v_p\}$ $\Rightarrow$ $W\left\{ \begin{bmatrix} b \\ 2b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -c \\ c \\ 5c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ -4d \\ d \end{bmatrix} \middle| : b, c, d \in \mathbb{R} \right\}$

$$W = \left\{ b \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + c \underbrace{\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}}_{\underline{v}_2} + d \underbrace{\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}}_{\underline{v}_3} : b, c, d \in \mathbb{R} \right\}$$

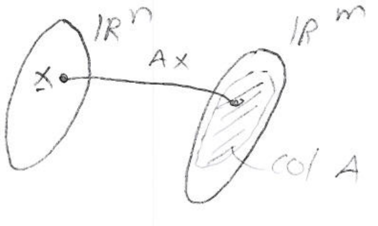
$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} \right\}$$

S.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Col A := span$\{\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n\}$

Col A := $\{\underline{b} \mid A\underline{x} = \underline{b}$ for some x in $\mathbb{R}^n\}$



18-11-07

4.3 Bases.

$S = \{V_1, V_2 \dots V_k\}$

$\underline{u} = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 \dots + \alpha_k V_k$

Linear independence /dependence:

$V_1, V_2, V_3 \dots V_k$ are all LI, if the system has the trivial soln...

ie $x_1 = 0,\ x_2 = 0,\ x_3 = 0 \dots x_k = 0$

Definition: Let H be a subspace of a vector space V. an indexed set of vectors $B = \{b_1, b_2 \dots, b_p\}$ in V is a basis for H if:

(i) B is linearly independent.

(ii) $H = \text{span}\{b_1, b_2 \dots b_p\}$

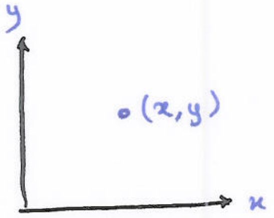
eg: $V = \mathbb{R}^n$

$$V = \mathbb{R}^2 \quad \begin{bmatrix} x \\ y \end{bmatrix} = x \overset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + y \overset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= x e_1 + y e_2$$

$\{\underline{e}_1\ \underline{e}_2\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$



$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \underline{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

$B = \{e_1\ e_2\ e_3 \dots e_n\}$ is called the standard basis for $\mathbb{R}^n$

② let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ V_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $B = \{V_1, V_2, V_3\}$ a basis for $\mathbb{R}^3$?

No! because V_2 is a zerovector and that makes V_1, V_2 and V_3 linearly dependent.

④ Is the set $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix} \right\}$ a basis for $\mathbb{R}_3$?

using the invertible matrix theorem from 2.3

① If the colums are combined into an augmented matrix and the matrix is invertible, it is independent. If det $A \neq 0$ then this is true.

② A is row equivalent to the identity I_3

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} = A$$

$$|A| = 2 \begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -2 & 5 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} = -24 \neq 0$$

A is invertible ∴ columns of A are LI and columns of A span $\mathbb{R}^3$

18-11-07

Method 2:

$$A = \begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -7 \\ -2 & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 0 & 1 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

ie A is row equivalent to I_3, columns of A form a LI set. and columns of A Span $\mathbb{R}^3$.

⑥ Does $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$?

20-11-07

Bases for Nul A and col A:

(a) Find a basis for nul A, where $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$

$\text{Nul} A = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

Find a general solution to $A\underline{x} = 0$

$$[A \vdots 0] = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3x_3 + 2x_4 = 0$
$x_2 - 5x_3 + 4x_4 = 0$
$x_3 = x_4 = \text{free}.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

Nul A : span$\{\underline{u}, \underline{v}\}$

Need to check: are $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ LI?

$$E = \begin{bmatrix} 3 & -2 \\ 5 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -2 \\ 5 & -4 \end{bmatrix}}_{\text{Row operations}} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 0 & -4 \end{bmatrix} \rightarrow \begin{array}{cc} e_1 & e_2 \\ \left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right. & \left.\right] \end{array}$$

ie... They are LI and they form a basis for Nul A.

Column space:

Theorem: The pivot columns of a matrix A form a basis for col A.

eg: Assume A is row equivalent to B. Find bases for Col A and Nul A.

$$A = \begin{array}{c} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \\ \text{①} \quad \text{②} \qquad\qquad \end{array} \overset{\text{row equiv}}{\sim} B = \begin{array}{c} \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{①} \ \text{②} \qquad\quad \end{array}$$

since col ① and ② are pivot columns,

$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$ is a basis for col Ⓐ

Null space:

$Ax = 0$

$[A \vdots 0] \sim [B \vdots 0]$

$$[B \vdots 0] = \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & 1 & 5/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -6x_2 - 5x_4$
$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$
$x_3 = x_3$
$x_4 = x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

20-11-07

$\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$ are LI, and they form a basis for nul A.

⑮ Find a basis for the space spanned by the vectors v_1, v_2, v_3, v_4 and v_5

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix} \quad v_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \quad v_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Soln: The problem is equivalent to finding a basis for col A, where.

$$A = [v_1\ v_2\ v_3\ v_4\ v_5] = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{RO} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ie cols ①, ②, and ④ will form a basis for col A.

$\{\underline{v_1}, \underline{v_2}, \underline{v_4}\}$

Theorem: Spanning set theorem: Read for HW.

22-11-07

6.1 Inner Products Length orthogonality.

Inner Products:

$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$

Inner product $\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$

(Dot products) eg: $\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \underline{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

$\underline{u} \cdot \underline{v} = (1)(2) + (-1)(3) + (2)(1) = 1$

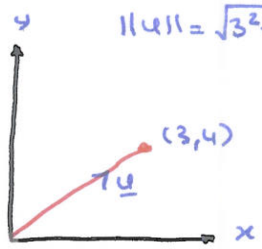
$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

Length:

$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbb{R}^n$ $\quad \|\underline{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

norm

eg:



$\|u\| = \sqrt{3^2+4^2} = 5$

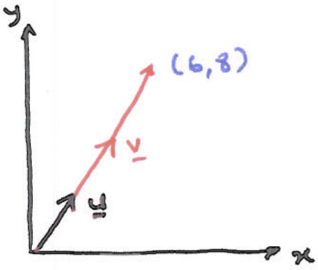
$\underline{u} \cdot \underline{u} = u_1^2 + u_2^2 + \dots + u_n^2$ and $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$

Unit Vectors:

let $\underline{v}$ be in $\mathbb{R}^n$.

A unit vector $\underline{u}$ in the direction of v is given by $\underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|}$

eg:



$\underline{v} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \|\underline{v}\| = \sqrt{6^2+8^2} = 10$

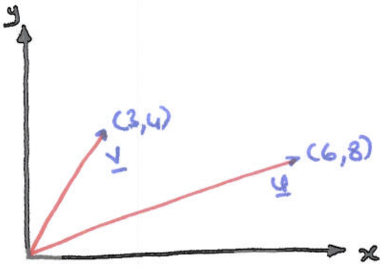
$\underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|} = \dfrac{\begin{bmatrix} 6 \\ 8 \end{bmatrix}}{10} = \dfrac{1}{10}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 6/10 \\ 8/10 \end{bmatrix}$

To check that $\underline{u}$ has a magnitude 1, $\|\underline{u}\| = \sqrt{\left(\frac{6}{10}\right)^2 + \left(\frac{8}{10}\right)^2} = \sqrt{\frac{100}{100}} = 1.$

Distance Between Vectors:

Let $\underline{u}, \underline{v}$ be in $\mathbb{R}^n$, then the distance between $\underline{u}$ and $\underline{v}$ is given by

$\text{dist}(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$



$\text{dist}(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\| = \left\| \begin{bmatrix} 6-3 \\ 8-4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\|$

$= \sqrt{3^2+4^2} = 5$

22-11-07.

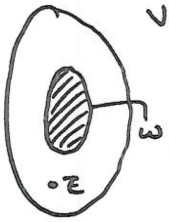
Orthogonal Vectors:

Two vectors $\underline{u}, \underline{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\underline{u} \cdot \underline{v} = 0$

eg: $\underline{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \underline{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \underline{u} \cdot \underline{v} = (2)(-1) + (3)(1) + (-1)(1) = 0$

(Product of 2 ⊥ vectors = 0)

Orthogonal Compliments:



Vector space $\mathbb{R}^n$

W is a Subspace of $\mathbb{R}^n$

z in $\mathbb{R}^n$

* Z in $\mathbb{R}^n$ is said to be orthogonal to W if $\underline{z} \cdot \underline{\omega} = 0$ for every $\underline{\omega}$ in W

* The set of all vectors z that are orthogonal to W is called the orthogonal complement of W and we write $W^{\perp}$

$$W^{\perp} := \left\{ \underline{z} \text{ in } \mathbb{R}^n \;\middle|\; \underline{z} \cdot \underline{\omega} = 0 \text{ for every } \underline{\omega} \text{ in } W \right\}.$$

Note: $W^{\perp}$ is a subspace of $\mathbb{R}^n$

Theorem: Let A be an m×n matrix

(i) $(\text{col } A)^{\perp} = \text{Nul } A^T$

(ii) $(\text{Row} A)^{\perp} = \text{Nul } A$

Sec 4.2

25/11/07

problem (28)

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 9x_1 + x_2 - 6x_3 &= 9 \end{aligned}$$

A [illegible] has a solution

$$\left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ -9 & 2 & 5 & 1 \\ 4 & 1 & -6 & 9 \end{array}\right]$$

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ -9 & 2 & 5 & 5 \\ 4 & 1 & -6 & 45 \end{array}\right]$$

$$A\underline{y} = 5\underline{b}$$

Things to study: For mid Term (2)

2.1, 2.2 ← matrices / Definitions & properties.

2.3 $AA^{-1} = I$

Computation $\left[A \mid I\right] = \left[I \mid A^{-1}\right]$

3.1 Determinants

3.2 properties

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$$

$R_2 \leftrightarrow 2R_2$

$$\det \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix} = 2(\det A)$$

$R_2 \leftrightarrow R_2 + KR_1$

$$\det \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} = 5$$

- 4.1, 4.2, 4.3
- Subspaces

$\text{Null } A = \{x \in \mathbb{R}^n \mid Ax = 0]$

$\text{Col } A := \text{Span}\{a_1 \ldots a_n)$

04-12-07

4.5 Dimension of a vector space.

$B = \{b_1, b_2, \ldots . b_n\}$ is a basis for V

Definition: The dimension of a vector space V is the # of vectors in a basis and we write dim(V)

* if V is spanned by a finite set, then V is said to be finite-dimensional, otherwise infinite-dimensional.

eg: $\mathbb{R}^2$ $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ dimension = 2

$\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ be any vector

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = xe_1 + ye_2$$

③ Find a basis and state the dimension for the subspace.

$$H = \left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$$

we define it to be zero dimensional

$$\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

$$H = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\} \leftarrow \boxed{\text{A basis for the subspace}}$$

dim (H) = 3

Check for LI

if it is ... then

* **Theorem:** If a vector space V has a basis of n vectors, then every basis of V must contain n vectors.

* **Theorem:** If a vector space V has a basis, $B = \{b_1, b_2 \ldots . b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

eg: ⑬ Determine the dimensions for nul A and col A for

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix} \right\}$$

if LI.... then proceed.

$$\begin{bmatrix} 1 & -6 & 9 & 0 & -2 & 0 \\ 0 & 1 & 2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{RO} \begin{bmatrix} 1 & 0 & 21 & 0 & 164/5 & 0 \\ 0 & 1 & 2 & 0 & 29/5 & 0 \\ 0 & 0 & 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Dim (col A)} = 3}$$

$x_1 = -21x_3 - \frac{164}{5}x_5$

$x_2 = -2x_3 - \frac{29}{5}x_5$

$x_3 = x_3$

$x_4 = -1/5\, x_5$

$x_5 = x_5$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -21 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -164/5 \\ -29/5 \\ 0 \\ -1/5 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{span} \left\{ \begin{bmatrix} -21 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -164/5 \\ -29/5 \\ 0 \\ -1/5 \\ 1 \end{bmatrix} \right\} \quad \text{if LI then} \quad \boxed{\dim(\text{Nul } A) = 2}$$

$\boxed{\text{LI}}$

Theorem: the basis theorem.

Let V be a P-dimensional vectorspace, where $P \geq 1$. Any linearly independent set of exactly P elements in V is automatically a basis for V. Any set of exactly P elements that spans V is automatically a basis for V.

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4.6 Rank.

Row Space: $A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}_{m\times n}$ $\quad \text{Row}A = \text{span}\{v_1, v_2, \dots . v_m\}$

$A^T = \{r_1\ r_2 \dots . r_m\}$ $\quad \text{span}\{r_1, r_2 \dots . r_m\} = \text{col}A^T$

Definition: The set of all linear combinations of the row vectors is called the row space of a matrix A, and written row A

* $\text{Row } A = \text{col } A^T$

Theorem: If two matrices A and B are row equivalent, then their row spaces are the same.

If B is in echelon form, the non-zero rows of B form a basis for row A as well as Row B.

Definition: The rank of A is the dimension of the col A.

eg: Assume A is row equivalent to B, without calculations, list rank A, dim(Nul A). Then find basis for col A, row A and Nul A.

① $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$ $\quad$ Pivots $\quad B = \begin{bmatrix} \textcircled{1} & 0 & -1 & 5 \\ 0 & \textcircled{-2} & 5 & -6 \\ 0 & 0 & \textcircled{0} & \textcircled{0} \end{bmatrix}$

Free variables.

$\text{Rank}A = \dim(\text{col}A) = 2$ (because 2 Pivot columns.)

$\dim(\text{nul }A) = 2$ (because 2 free vars) Also known as Nullity.

(Basis) $\text{col } A = \left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$ $\quad$ (Basis) $\text{Row } A = \left\{ \left[(1\ 0\ -1\ 5), (0\ -2\ 5\ -6) \right] \right\}$

(Basis) Nul A = HW!!

② $A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \sim B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad$ rank (A) = 3

Nullity(A) = dim(Nul A)

(Basis) $\text{Row } A := \{(1 -3\ 0\ 5\ -7), (0\ 0\ 2\ -3\ 8), (0\ 0\ 0\ 0\ 5)\}$

(Basis) $\text{Col } A = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$

Rank theorem: let A be an m×n matrix then, $\boxed{\text{Rank}(A) + \text{Nullity}(A) = n.}$

Polynomials.

$P(t) = a_0 + a_1 t + a_2 t^2 + \dots . + a_n t^n + \dots .$

$a_0 . a_1, \dots . a_n \in \mathbb{R}$

degree: highest power of function. (variable t).

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Polynomial of degree at most n:

ie: $p(t) = a_0 + a_1t + \dots\dots + a_nt^n$

$\mathbb{P}_n$ = set of all polynomials of degree at most n.

$\mathbb{R}_n = \{a_0 + a_1t + \dots\dots + a_nt^n \,|\, a_i \in \mathbb{R}\}$

3 conditions for being a vector space:

① $\underline{0} = 0 + 0t + 0t^2 + 0t^3 + \dots\dots + 0t^n$ in $\mathbb{P}_n$

② let $\left.\begin{matrix} p(t) = a_0 + a_1t + \dots + a_nt^n \\ q(t) = b_0 + b_1t + \dots + b_nt^n \end{matrix}\right\}$ be in $\mathbb{P}_n$ (ie the sum of both). $p(t) + q(t)$

③ let $\alpha \in \mathbb{R}$

$\alpha p(t) = (\alpha a_0) + (\alpha a_1)t + \dots + (\alpha a_n)t^n$ be in $\mathbb{P}_n$

$S = \{1, t, t^2, \dots, t^n\}$ you can write any polynomial $p(t) = a_1t + a_0 + \dots\, a_nt^n$ as a linear combination of S

S spans $\mathbb{P}_n$

S is LI for:

$\alpha_0 1 + \alpha_1 t + \alpha_2 t^2 + \dots\dots + a_nt^n = \underline{0} = 0 + 0t + 0t^2 + \dots + 0t^n$

$\Rightarrow \alpha_0 = 0, \alpha_1 = 0 \dots\, \alpha_n = 0$

S is called the standard basis for $\mathbb{P}_n$

$\dim \mathbb{P}_n = n+1$

09-12-07

5.1 Eigen Values and Eigen Vectors.

Definition: An eigen vector of an $n \times n$ matrix A, is a non-zero vector $\underline{x}$ such that $A\underline{x} = \lambda \underline{x}$ for some scalar λ

A scalar λ is called an Eigen-value of A if there is a non-trivial solution $\underline{x}$ of $A\underline{x} = \lambda \underline{x}$. Such an $\underline{x}$ is called an eigen vector corresponding to λ

$A\underline{x} = \lambda \underline{x}$ to find eigen values consider this equation $\boxed{\det(A - \lambda I) = 0}$

$A\underline{x} - \lambda \underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = 0$

eg: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ Find the eigen values and their associated eigen vectors:

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}$$

- now take det and solve for λ:

$\det(A - \lambda I) = 0$

$(1-\lambda)(4-\lambda) + 2 = 0$

$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda_1 = 2$

$\lambda_2 = 3$ are the eigenvalues of A.

- now find Eigen Vectors:

$\lambda_1 = 2$: consider $A\underline{x} = \lambda \underline{x}$

$A\underline{x} = 2\underline{x}$ $\Rightarrow \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 + x_2 = 2x_1$

$-2x_1 + 4x_2 = 2x_2$ $\Rightarrow$ $-x_1 + x_2 = 0$

$-2x_1 + 2x_2 = 0$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0$ $\quad$ $x_1 = x_2$

x_2 = free $\quad$ $x_2 = x_2$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen Vector corresponding to $\lambda_1 = 2$

$\left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| x_2 \text{ is in } \mathbb{R} \right\}$ eigenspace associated with $\lambda_1 = 2$.

* Note $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for this Eig.. space.

Then repeat for $\lambda_2 = 3$.

eg: $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ Find eigen values.

Theorem: The eigen values for a Δ matrix are the entries on the diagonal.

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = \det \begin{bmatrix} 3-\lambda & 4 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & -\lambda \end{bmatrix} = 0$$

$(3-\lambda)(2-\lambda)(-\lambda) = 0$

$\lambda = 0, \lambda = 2, \lambda = 3$

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eg: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ An eigen value of A is 2. Find a basis for the corresponding eigen space.

Consider $A\underline{x} = \lambda \underline{x}$

$\lambda = 2 \quad A\underline{x} = 2\underline{x}$

$A\underline{x} - 2\underline{x} = 0$

$(A - 2I)\underline{x} = 0$

$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \underline{x} = \underline{0}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \underline{x} = 0$$

Now consider: $\left[\begin{array}{ccc:c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right]$

Reduced to: $\left[\begin{array}{ccc:c} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\left.\begin{array}{c} x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{array}\right\} \begin{array}{l} x_1 = \frac{1}{2}x_2 - 3x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Eigen space $\left\{ x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| \begin{array}{c} x_2, x_3 \\ \text{in } \mathbb{R} \end{array} \right\}$

a basis for Eigen space corresponding to $\lambda = 2$ is $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

11-12-07

5.2 Characteristic Equation.

The scalar equation $\det(A - \lambda I) = 0$ is called the characteristic equation (Polynomial) of A.

Note: The Scalar λ is an eigen value of A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$

eg: Find the characteristic equation of

$$A = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & 2 & -8 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3-\lambda & 2 & -1 & 1 \\ 0 & 2-\lambda & -8 & 0 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 5-\lambda \end{bmatrix}$$

characteristic polynomial

$$P(\lambda) = \det(A - \lambda I) = (3-\lambda)(2-\lambda)(3-\lambda)(5-\lambda)$$

$$= (2-\lambda)(5-\lambda)(3-\lambda)^2$$

$$\underbrace{\lambda = 2}_{}, \underbrace{\lambda = 5}_{}, \underbrace{\lambda = 3, \lambda = 3}_{}$$

multiplicity =1 multiplicity of $\lambda = 3$ is two.

Theorem: If $v_1, v_2, \ldots v_r$ are eigen vectors that correspond to distinct, eigen values $\lambda_1, \lambda_2 \ldots \lambda_r$ of an $n \times n$ matrix A, Then the set $\{v_1, v_2, \ldots v_r\}$ is linearly independent.

Similarity: If A, B are $n \times n$ matrices, then A is similar to B if there exists an invertible matrix P such that $P^{-1}AP = B$ or $A = PBP^{-1}$

5.3 Diagonalization:

A square matrix A is said to be Diagonalizable if A is similar to a Diagonal matrix ie $A = PDP^{-1}$ For some invertible matrix P and a diagonal matrix D

Theorem: An $n \times n$ matrix A is diagonalizable if and only if A has n - linearly independent eigen vectors.

eg: diagonalise $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

step 1: Find eigen values:

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$= (5-\lambda)(4-\lambda)(5-\lambda) = 0$$

$$\lambda = 5, \lambda = 4.$$

step 2: Find n (here n=3) linearly independent Eigen vectors:

$(\lambda = 5)$

$$A\underline{x} = \lambda \underline{x}$$

$$A\underline{x} - \lambda \underline{x} = \underline{0}$$

$$(A - 5I)\underline{x} = \underline{0}$$

$$A - 5I : \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{①} \text{Augmented matrix} \begin{bmatrix} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0$

$x_2 = x_2$

$x_3 = x_3$

②

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + x_3 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

③

v_1 and v_2 are the corresponding eigen vectors associated with $\lambda = 5$, ie $\{v_1, v_2\}$ is a basis for eigen space.

— Repeat for $(\lambda = 4)$ —

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$\lambda=4$: $(A-4I)\underline{x}=\underline{0}$ Augmented matrix: $\begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1+\frac{1}{2}x_2=0$
$x_3=0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underset{v_3}{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}$$

by earlier thm: $\{v_1, v_2, v_3\}$ are LI.

Step 3: Construct P. (from vectors in step 2.)

$$P = [\underline{v_1}, \underline{v_2}, \underline{v_3}] = \begin{bmatrix} 0 & -2 & -1/2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

step 4: Construct D from the corresponding Eigen values. (order of the eigen values should match the order chosen for the columns of P.) $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Then $A = PDP^{-1}$ if to check

$\underset{\smile}{A}P = P\underset{\smile}{D}$

13-12-07

⑰ Diagonalize $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ if possible.

(S1) $\underline{\det(A-\lambda I)=0}$: $\det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$

$(4-\lambda)^2(5-\lambda)=0$

$\lambda = 4$; $\lambda = 5$

mult=2 mult=1

(S2) $\underline{\lambda=4:}$ $A\underline{x} = \lambda \underline{x}$ $(A-4I)\underline{x}=0$

$A\underline{x} = 4\underline{x}$ $[A-4I \,\vdots\, 0]$ $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A\underline{x} = 4\underline{x}$

$(A-4I)\underline{x}=0$

$x_1 = 0$

$x_3 = 0$

$x_2 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underset{\underset{\underline{v}_1}{\uparrow}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\underline{\{\underline{v}_1\}}$ is a basis for the eigen space for $\underline{\lambda=4}$

(S3) $\underline{\lambda=5:}$ $(A-5I)\underline{x}=\underline{0}$ $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$, $x_3 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \underset{\underline{v}_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$\underline{\{\underline{v}_2\}}$ is a basis for the eigen space for $\lambda=5$

Since we only have two linearly independent eigen vectors and since $n=3$, we cannot diagonalize A.

Example for section 4.1

① $M_{2\times2}$ = Set of all 2×2 matrices: $= \left\{ \begin{Bmatrix} a & b \\ c & d \end{Bmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$

$M_{2\times2}$ is a vector space over $\mathbb{R}$ $H = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \middle| a,b,c \in \mathbb{R} \;\; a+b-c=0 \right\}$

① $\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ (is $\underline{0}$ in H?)

② $\underline{u} = \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} a_2 & b_2 \\ c_1 & 0 \end{bmatrix} \in H$ (is $\underline{u}+\underline{v}$ in H)

$\underline{u}+\underline{v} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & 0 \end{bmatrix}$

since $\underline{u} \in H \Rightarrow a_1+b_1-c_1=0$ — ①

$\underline{v} \in H \Rightarrow a_2+b_2-c_2=0$ — ②

check $(a_1+a_2)+(b_1+b_2)-(c_1+c_2)=0$

$\underbrace{(a_1+b_1-c_1)}_{① = 0} + \underbrace{(a_2+b_2-c_2)}_{② = 0} = 0$ $\therefore$ $\underline{u}+\underline{v}$ is in H.

③ let α be a scalar (is $\alpha\underline{u}$ in H?)

$\underline{u} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ in H $\Rightarrow a+b-c=0$

$\alpha\underline{u} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & 0 \end{pmatrix}$ $\alpha a + \alpha b - \alpha c$

$= \alpha(a+b-c) = \alpha 0 = 0$

$\therefore \alpha\underline{u} \in H$

H is a subspace of $M_{2\times2}$

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$$M_{2\times2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\underset{E_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + b\underset{E_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + c\underset{E_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} + d\underset{E_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

$x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4 = 0$

$$x_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + x_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\{E_1\ E_2\ E_3\ E_4\}$ basis for $M_{2\times2}$

dimension $=4$.

$x_1=0, x_2=0, x_3=0, x_4=0$

② $V = \{$set of all real values, functions from $[a,b]$ to $\mathbb{R}\}$

$V = \{F \mid F:[a,b] \longrightarrow \mathbb{R}\}$

$(F+g)(t) = F(t)+g(t)$

$(\alpha F)(t) = \alpha F(t)$

$\underline{0}$ = zerofn   V is a vector space of over $\mathbb{R}$.

$C[a,b] = \{$set of all continuous functions from $[a,b]$ to $\mathbb{R}\}$

$= \{f \mid f:[a,b] \to \mathbb{R},\ f$ is cts$\}$

- zerofn is $\underline{0} = (a) = 0$ const $\Rightarrow$ continuous.
- $f, g \in C[a,b]$; $f+g$ is in $C[a,b]$
- $x \in \mathbb{R}$ $f \in C[a,b]$ αf is in $C[a,b]$

$\Rightarrow C[a,b]$ is a subspace of V.

③ Kernel of a Linear transformation

$T: V \longrightarrow W$ T is a LT of from vector space V to vector space W

$T(x) = A\underline{x}$ standard matrix.

Ker T: $= \alpha\{ \underline{x}$ in $V \mid \underset{Ax}{T(x)} = 0\}$ "Null space"

Nul $A = \{x \mid Ax = 0\}$