CHAPTER

8

SYNCHRONOUS MACHINE TRANSIENT ANALYSIS

8.1 INTRODUCTION

The steady state performance of the synchronous machine was described in Chapter 3. Under balanced steady state operations, the rotor mmf and the resultant stator mmf are stationary with respect to each other. As a result, the flux linkages with the rotor circuit do not change with time, and no voltages are induced in the rotor circuits. The per phase equivalent circuit then becomes a constant generated emf in series with a simple impedance. In Chapter 3, for steady state operation the generator was represented with a constant emf behind the synchronous reactance X_s. For salient-pole rotor, because of the nonuniformity of the air gap, the generator was modeled with direct axis reactance X_d and the quadrature axis reactance X_q.

Under transient conditions, such as short circuits at the generator terminals, the flux linkages with the rotor circuits change with time. This result in transient currents in all the rotor circuits, which in turn reacts on the armature. For the transient analysis, the idealized synchronous machine is represented as a group of magnetically coupled circuits with inductances which depend on the angular position of the rotor. The resulting differential equations describing the machine have time-varying coefficients, and a closed form of solution in most cases is not feasible. A

great simplification can be made by transformation of stator variables from phases a, b, and c into new variables the frame of reference of which moves with the rotor. The transformation is based on the so-called *two-axis theory*, which was pioneered by Blondel, Doherty, Nickle, and Park [20, 61]. The transformed equations are linear provided that the speed is assumed to be constant.

In this chapter, the voltage equation of a synchronous machine is first established. Reference frame theory is then used to establish the machine equations with the stator variables transformed to a reference frame fixed in the rotor (Park's equations). The Park's equations are solved numerically during balanced three-phase short circuit. If the speed deviation is taken into account, transformed equations become nonlinear and must be solved by numerical integration. In *MATLAB*, the nonlinear differential equations of the synchronous machine in matrix form can be simulated with ease. Also, there is the additional advantage that the original voltage equations can be used without the need for any transformations. In particular, the numerical solution is obtained for the line-to-line and the line-to-ground short circuits using direct-phase quantities.

Another objective of this chapter is to develop simple network models of the synchronous generator for the power system fault analysis and transient stability studies. For this purpose, the generator behavior is divided into three periods: the *subtransient period*, lasting only for the first few cycles; the *transient period* covering a relatively longer time; and, finally, the *steady state period*. Thus, the generator equivalent circuits during transient state are obtained.

8.2 TRANSIENT PHENOMENA

To better understand the synchronous machine transient phenomena, we first study the transient behavior of a simple RL circuit. Consider a sinusoidal voltage $v(t) = V_m \sin(\omega t+\alpha)$ applied to a simple RL circuit at time $t = 0$, as shown in Figure 8.1.

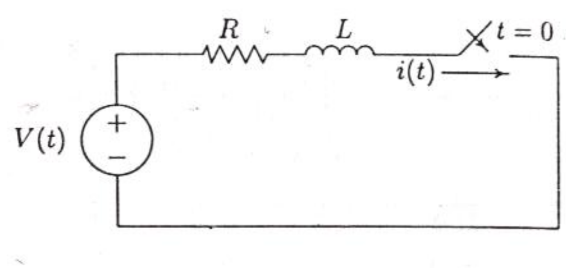


FIGURE 8.1
A simple series circuit with constant R and L.

The circuit consists of R in series with a constant L. The instantaneous voltage equation for the circuit is

$$Ri(t) + L\frac{di(t)}{dt} = V_m \sin(\omega t + \alpha) \tag{8.1}$$

The solution for the current may be shown to be

$$i(t) = I_m \sin(\omega t + \alpha - \gamma) - I_m e^{-t/\tau} \sin(\alpha - \gamma) \tag{8.2}$$

where $I_m = V_m/Z$, $\tau = L/R$, $\gamma = \tan^{-1} \omega L/R$, and $Z = \sqrt{R^2 + X^2}$. The first term is the steady state sinusoidal component. The second term is a dc transient component known as *dc offset* which decays exponentially. The dc and sinusoidal components are equal and opposite when $t = 0$, so that the condition for zero initial current is satisfied. The magnitude of the dc component depends on the instant of application of the voltage to the circuit, as defined by the angle α. The dc component is zero when $(\alpha = \gamma)$. This current waveform is shown in Figure 8.2(a). Similarly, the dc component will have a maximum initial value of V_m/Z which is the peak value of the alternating component, if the circuit is closed when $\alpha = \gamma - \pi/2$ radians. The current waveform with maximum dc offset is shown in Figure 8.2(b). If $\omega L \gg R$, then $\gamma \simeq \pi/2$, so that circuit closure at voltage maximum would give no dc component, and closure at voltage zero would cause the maximum dc transient current to flow.

Example 8.1

In the circuit of Figure 8.1, let $R = 0.125\ \Omega$, $L = 10$ mH, and the source voltage be given by $v(t) = 151\sin(377t + \alpha)$. Determine the current response after closing the switch for the following cases.

(a) No dc offset.

(b) For maximum dc offset.

$$Z = 0.125 + j(377)(0.01) = 0.125 + j3.77 = 3.772\angle 88.1^\circ$$

$$I_m = \frac{151}{3.772} = 40 \text{ A}$$

and

$$\tau = \frac{L}{R} = 0.08 \text{ sec}$$

From (8.2) the response is

$$i(t) = 40\sin(\omega t + \alpha - 88.1^\circ) - 40e^{-t/0.08}\sin(\alpha - 88.1^\circ)$$

The response has no dc offset if switch is closed when $\alpha = 88.1^\circ$, and it has the maximum dc offset when $\alpha = 88.1^\circ - 90^\circ = -1.9^\circ$. The following commands produce the responses shown in Figures 8.2(a) and 8.2(b).


```
alf1 = 88.1*pi/180;
alf2 = -1.9*pi/180;
gamma = 88.1*pi/180;
t = 0:.001:.3;
i1 = 40*sin(377*t+alf1-gamma)-40*exp(-t/.08).*sin(alf1-gamma);
i2 = 40*sin(377*t+alf2-gamma)-40*exp(-t/.08).*sin(alf2-gamma);
subplot(2,1,1), plot(t, i1)
xlabel('t, sec'), ylabel('i(t)')
subplot(2,1,2), plot(t, i2)
xlabel('t, sec'), ylabel('i(t)')
subplot(111)
```

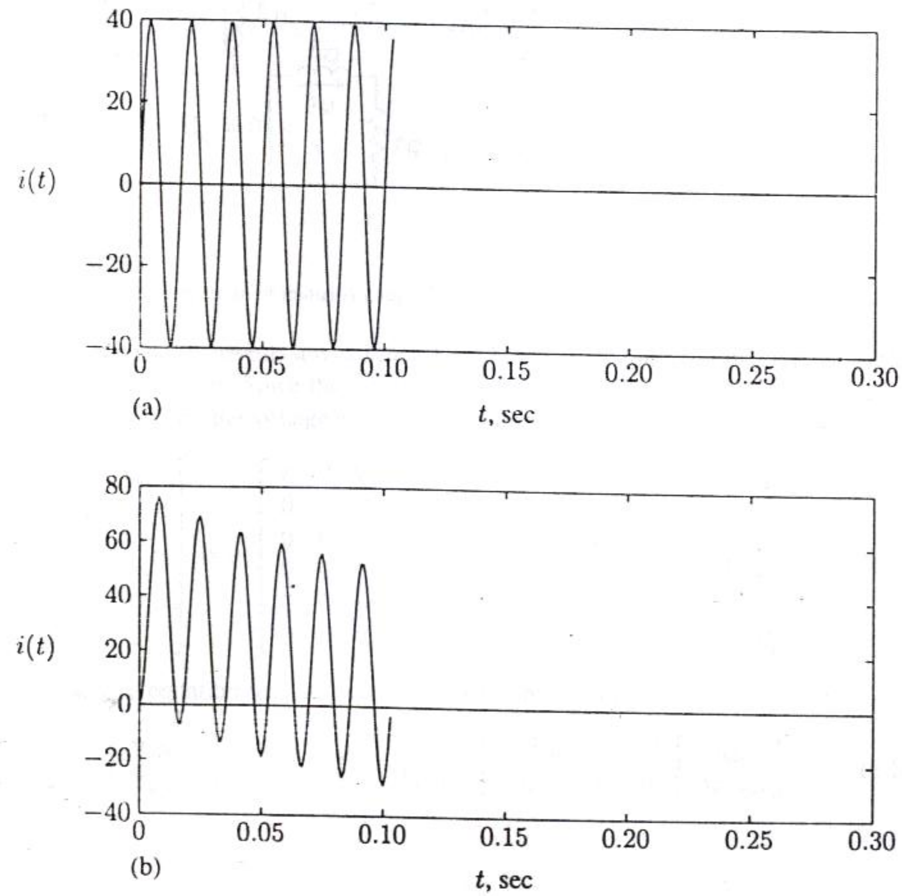


FIGURE 8.2
Current waveform, (a) with no dc offset, (b) with maximum dc offset.

8.3 SYNCHRONOUS MACHINE TRANSIENTS

The synchronous machine consists of three stator windings mounted on the stator, and one field winding mounted on the rotor. Two additional fictitious windings could be added to the rotor, one along the direct axis and one along the quadrature axis, which model the short-circuited paths of the damper windings. These windings are shown schematically in Figure 8.3.

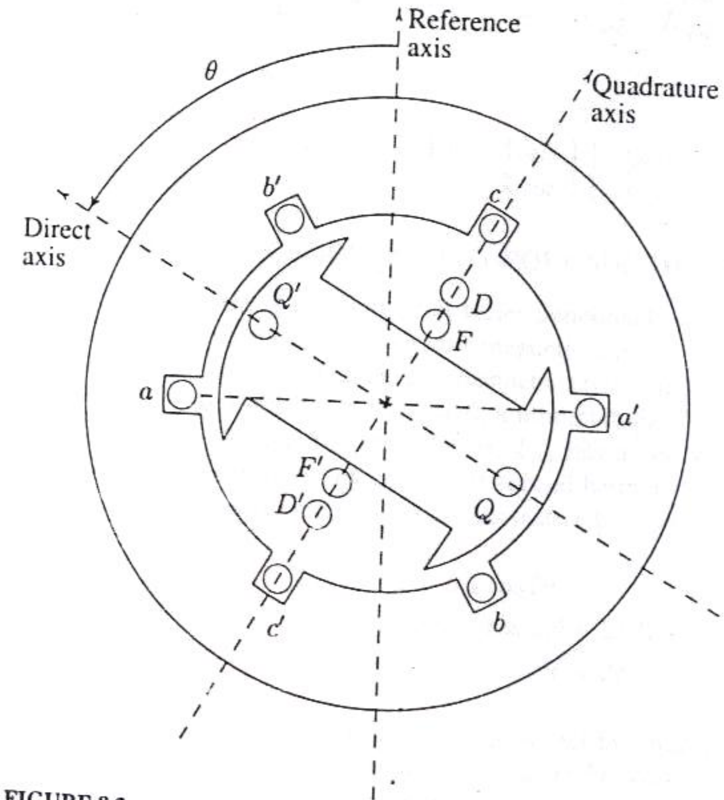


FIGURE 8.3
Schematic representation of a synchronous machine.

We shall assume a synchronously rotating reference frame (axis) rotating with the synchronous speed ω which will be along the axis of phase a at $t = 0$. If θ is the angle by which rotor direct axis is ahead of the magnetic axis of phase a, then

$$\theta = \omega t + \delta + \frac{\pi}{2} \tag{8.3}$$

where δ is the displacement of the quadrature axis from the synchronously rotating reference axis and $(\delta + \frac{\pi}{2})$ is the displacement of the direct axis.

In the classical method, the idealized synchronous machine is represented as a group of magnetically coupled circuits with inductances which depend on the angular position of the rotor. In addition, saturation is neglected and spatial distribution of armature mmf is assumed sinusoidal. The circuits are shown schematically in Figure 8.4.

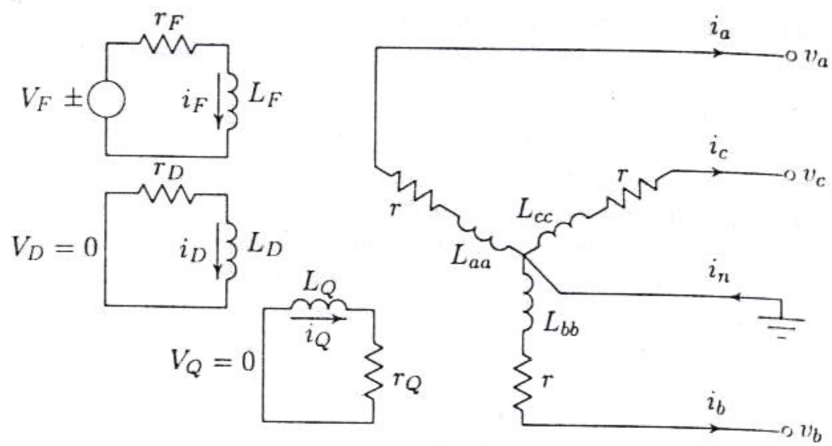


FIGURE 8.4
Schematic representation of mutually coupled circuits.

The stator currents are assumed to have a positive direction flowing out of the machine terminals. Since the machine is a generator, following the circuit passive sign convention, the voltage equation becomes

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ -v_F \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix} - \frac{d}{dt} \begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_D \\ \lambda_Q \end{bmatrix} \tag{8.4}$$

The above equation may be written in partitioned form as

$$\begin{bmatrix} \mathbf{v}_{abc} \\ \mathbf{v}_{FDQ} \end{bmatrix} = -\begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} - \frac{d}{dt} \begin{bmatrix} \boldsymbol{\lambda}_{abc} \\ \boldsymbol{\lambda}_{FDQ} \end{bmatrix} \tag{8.5}$$

where

$$\mathbf{v}_{FDQ} = \begin{bmatrix} -v_F \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{i}_{FDQ} = \begin{bmatrix} i_F \\ i_D \\ i_Q \end{bmatrix} \quad \boldsymbol{\lambda}_{FDQ} = \begin{bmatrix} \lambda_F \\ \lambda_D \\ \lambda_Q \end{bmatrix} \quad \text{etc.} \tag{8.6}$$

The flux linkages are functions of self- and mutual inductances given by

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_F \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} & L_{aF} & L_{aD} & L_{aQ} \\ L_{ba} & L_{bb} & L_{bc} & L_{bF} & L_{bD} & L_{bQ} \\ L_{ca} & L_{cb} & L_{cc} & L_{cF} & L_{cD} & L_{cQ} \\ L_{Fa} & L_{Fb} & L_{Fc} & L_{FF} & L_{FD} & L_{FQ} \\ L_{Da} & L_{Db} & L_{Dc} & L_{DF} & L_{DD} & L_{DQ} \\ L_{Qa} & L_{Qb} & L_{Qc} & L_{QF} & L_{QD} & L_{QQ} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix} \tag{8.7}$$

or in compact form we have

$$\begin{bmatrix} \lambda_{abc} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} \tag{8.8}$$

8.3.1 INDUCTANCES OF SALIENT-POLE MACHINES

The self-inductance of any stator coil varies periodically from a maximum (when the direct axis coincides with the coil magnetic axis) to a minimum (when the quadrature axis is in line with the coil magnetic axis). The self-inductance L_{aa}, for example, will be a maximum for $\theta = 0$, a minimum for $\theta = 90°$ and maximum, again for $\theta = 180°$, and so on. That is, L_{aa} has a period of 180° and can be represented approximately by cosines of second harmonics. Because of the rotor symmetry, the diagonal elements of the submatrix $\mathbf{L}_{SS}$ are represented as

$$\begin{aligned} L_{aa} &= L_s + L_m \cos 2\theta \\ L_{bb} &= L_s + L_m \cos 2(\theta - 2\pi/3) \\ L_{cc} &= L_s + L_m \cos 2(\theta + 2\pi/3) \end{aligned} \tag{8.9}$$

where θ is the angle between the direct axis and the magnetic axis of phase a, as shown in Figure 8.3. The mutual inductances between any two stator phases are also periodic functions of rotor angular position because of the rotor saliency. We can conclude from the symmetry considerations that the mutual inductance between phase a and b should have a negative maximum when the pole axis is lined up 30° behind phase a or 30° ahead of phase b, and a negative minimum when it is midway between the two phases. Thus, the variations of stator mutual inductances, i.e., the off-diagonal elements of the submatrix $\mathbf{L}_{SS}$ can be represented as follows.

$$\begin{aligned} L_{ab} = L_{ba} &= -M_s - L_m \cos 2(\theta + \pi/6) \\ L_{bc} = L_{cb} &= -M_s - L_m \cos 2(\theta - \pi/2) \\ L_{ca} = L_{ac} &= -M_s - L_m \cos 2(\theta + 5\pi/6) \end{aligned} \tag{8.10}$$

All the rotor self-inductances are constant since the effects of stator slots and saturation are neglected. They are represented with single subscript notation.

$$L_{FF} = L_F \quad L_{DD} = L_D \quad L_{QQ} = L_Q \tag{8.11}$$

The mutual inductance between any two circuits both in direct axis (or both in quadrature axis) is constant. The mutual inductance between any rotor direct axis circuit and quadrature axis circuit vanishes. Thus, we have

$$L_{FD} = L_{DF} = M_R \quad L_{FQ} = L_{QF} = 0 \quad L_{DQ} = L_{QD} = 0 \tag{8.12}$$

Finally, let us consider the mutual inductances between stator and rotor circuits, which are periodic functions of rotor angular position. Because only the space-fundamental component of the produced flux links the sinusoidally distributed stator, all stator-rotor mutual inductances vary sinusoidally, reaching a maximum when the two coils in question align. Thus, their variations can be written as follows.

$$\begin{aligned} L_{aF} = L_{Fa} &= M_F \cos\theta \\ L_{bF} = L_{Fb} &= M_F \cos(\theta - 2\pi/3) \\ L_{cF} = L_{Fc} &= M_F \cos(\theta + 2\pi/3) \\ L_{aD} = L_{Da} &= M_D \cos\theta \\ L_{bD} = L_{Db} &= M_D \cos(\theta - 2\pi/3) \\ L_{cD} = L_{Dc} &= M_D \cos(\theta + 2\pi/3) \\ L_{aQ} = L_{Qa} &= M_Q \sin\theta \\ L_{bQ} = L_{Qb} &= M_Q \sin(\theta - 2\pi/3) \\ L_{cQ} = L_{Qc} &= M_Q \sin(\theta + 2\pi/3) \end{aligned} \tag{8.13}$$

The resulting differential equations (8.4) describing the behavior of the machine have time-varying coefficients given by (8.9)–(8.13), and we are not able to use Laplace transforms directly to obtain a closed form of solution.

8.4 THE PARK TRANSFORMATION

A great simplification can be made by transformation of stator variables from phases a, b, and c into new variables the frame of reference of which moves with the rotor. The transformation is based on the so called *two-axis theory*, which was pioneered by Blondel, Doherty, Nickle, and Park [20, 61].

The transformed quantities are obtained from the projection of the actual variables on three axes; one along the direct axis of the rotor field winding, called the direct axis; a second along the neutral axis of the field winding, called the quadrature axis; and the third on a stationary axis. For example, the three armature currents i_a, i_b, and i_c are replaced by three fictitious currents with the symbols i_d, i_q, and i_0. They are found such that, in a balanced condition, when $i_a + i_b + i_c = 0$, they produce the same flux, at any instant, as the actual phase currents in the armature. The third fictitious current i_0 is needed to make the transformation possible when the sum of the three-phase current is not zero.

The Park transformation for currents is as follows

$$\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix} = \sqrt{2/3} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta - 2\pi/3) & \cos(\theta + 2\pi/3) \\ \sin\theta & \sin(\theta - 2\pi/3) & \sin(\theta + 2\pi/3) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \tag{8.14}$$

or, in matrix notation

$$\mathbf{i}_{0dq} = \mathbf{P}\mathbf{i}_{abc} \tag{8.15}$$

Similarly for voltages and flux linkages, we have

$$\mathbf{v}_{0dq} = \mathbf{P}\mathbf{v}_{abc} \tag{8.16}$$

$$\lambda_{0dq} = \mathbf{P}\lambda_{abc} \tag{8.17}$$

The Park transformation matrix is orthogonal, i.e., $\mathbf{P}^{-1} = \mathbf{P}^T$ and thus, it is a power invariant transformation matrix. For the inverse Park transformation matrix we get

$$\mathbf{P}^{-1} = \sqrt{2/3} \begin{bmatrix} 1/\sqrt{2} & \cos\theta & \sin\theta \\ 1/\sqrt{2} & \cos(\theta - 2\pi/3) & \sin(\theta - 2\pi/3) \\ 1/\sqrt{2} & \cos(\theta + 2\pi/3) & \sin(\theta + 2\pi/3) \end{bmatrix} \tag{8.18}$$

We now wish to transform the time-varying inductances to a rotor frame of reference with the original rotor quantities unaffected. Thus, in (8.17) we augment the $\mathbf{P}$ matrix with a 3×3 identity matrix $\mathbf{U}$ to get

$$\begin{bmatrix} \lambda_{0dq} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{abc} \\ \lambda_{FDQ} \end{bmatrix} \tag{8.19}$$

or

$$\begin{bmatrix} \lambda_{abc} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{odq} \\ \lambda_{FDQ} \end{bmatrix} \tag{8.20}$$

Substituting in (8.8), we get

$$\begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{odq} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{odq} \\ \mathbf{i}_{FDQ} \end{bmatrix} \tag{8.21}$$

or

$$\begin{bmatrix} \lambda_{odq} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{odq} \\ \mathbf{i}_{FDQ} \end{bmatrix} \tag{8.22}$$

Substituting for $\mathbf{P}$, $\mathbf{P}^{-1}$ and the inductances given by (8.9)–(8.13), the above equation reduces to

$$\begin{bmatrix} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \lambda_F \\ \lambda_D \\ \lambda_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & KM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix} \tag{8.23}$$

where we have introduced the following new parameters

$$L_0 = L_s - 2M_s \tag{8.24}$$

$$L_d = L_s + M_s + \frac{3}{2}L_m \tag{8.25}$$

$$L_q = L_s + M_s - \frac{3}{2}L_m \tag{8.26}$$

and $k = \sqrt{3/2}$.

Transforming the stator-based currents ($\mathbf{i}_{abc}$) into rotor-based currents ($\mathbf{i}_{0dq}$), with rotor currents unaffected, we obtain

$$\begin{bmatrix} \mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} \tag{8.27}$$

or

$$\begin{bmatrix} \mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{odq} \\ \mathbf{i}_{FDQ} \end{bmatrix} \tag{8.28}$$

and similarly for voltages, we get

$$\begin{bmatrix} \mathbf{v}_{abc} \\ \mathbf{v}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{odq} \\ \mathbf{v}_{FDQ} \end{bmatrix} \tag{8.29}$$

Substituting (8.20), (8.28), and (8.29) into (8.5), we get

$$\begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{0dq} \\ \mathbf{v}_{FDQ} \end{bmatrix} = -\begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} - \frac{d}{dt} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{0dq} \\ \lambda_{FDQ} \end{bmatrix} \tag{8.30}$$

or

$$\begin{bmatrix} \mathbf{v}_{0dq} \\ \mathbf{v}_{FDQ} \end{bmatrix} = -\begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} - \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{0dq} \\ \lambda_{FDQ} \end{bmatrix} \tag{8.31}$$

Evaluating the first term, and obtaining the derivative of the second term in (8.31), yields

$$\begin{bmatrix} \mathbf{v}_{0dq} \\ \mathbf{v}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} - \begin{bmatrix} \mathbf{P}\frac{d}{dt}\mathbf{P}^{-1} & 0 \\ 0 & \mathbf{U} \end{bmatrix} \begin{bmatrix} \lambda_{0dq} \\ \lambda_{FDQ} \end{bmatrix} - \frac{d}{dt} \begin{bmatrix} \lambda_{0dq} \\ \lambda_{FDQ} \end{bmatrix} \tag{8.32}$$

Next, the expression for $\mathbf{P}\frac{d}{dt}\mathbf{P}^{-1}$ can be written as

$$\mathbf{P}\frac{d}{dt}\mathbf{P}^{-1} = \mathbf{P}\frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{P}^{-1} = \omega\mathbf{P}\frac{d}{d\theta}\mathbf{P}^{-1} \tag{8.33}$$

Substituting for $\mathbf{P}$ from (8.14), and for the derivative of $\mathbf{P}^{-1}$ from (8.18), we get

$$\begin{aligned} \mathbf{P}\frac{d}{dt}\mathbf{P}^{-1} &= 2/3\omega \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta - 2\pi/3) & \cos(\theta + 2\pi/3) \\ \sin\theta & \sin(\theta - 2\pi/3) & \sin(\theta + 2\pi/3) \end{bmatrix} \begin{bmatrix} 0 & -\sin\theta & \cos\theta \\ 0 & -\sin(\theta - 2\pi/3) & \cos(\theta - 2\pi/3) \\ 0 & -\sin(\theta + 2\pi/3) & \cos(\theta + 2\pi/3) \end{bmatrix} \\ &= \omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned} \tag{8.34}$$

Substituting (8.23) and (8.34) into (8.32), the machine equation in the rotor frame of reference becomes

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \\ -v_F \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & \omega L_q & 0 & 0 & \omega kM_Q \\ 0 & -\omega L_d & r & -\omega kM_F & -\omega kM_D & 0 \\ 0 & 0 & 0 & r_F & 0 & 0 \\ 0 & 0 & 0 & 0 & r_D & 0 \\ 0 & 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$- \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & kM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_0 \\ i_d \\ i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix} \tag{8.35}$$

We now make some observations regarding the nature of the above equations. The most important one is that they have constant coefficients provided that speed is assumed constant. Also, the first equation

$$v_0 = -ri_0 - L_0\frac{di_0}{dt}$$

is not coupled to the other equations. Therefore, it can be treated separately. The variables v_0, L_0, and i_0 are known as the *zero-sequence variables*. The name originally comes from the theory of symmetrical components, as discussed in Chapter 10. Finally, we note that while the transformation technique is a mathematical process, it provides valuable insight into internal phenomena and gives the effects of transients. Furthermore, it provides physical meaning to the new quantities.

8.5 BALANCED THREE-PHASE SHORT CIRCUIT

Consider a three-phase synchronous generator operating at synchronous speed with constant excitation. We will explore the nature of the three armature currents and the field current following a three-phase short circuit at the armature terminals. The machine is assumed to be initially unloaded, i.e.,

$$i_a(0^+) = i_b(0^+) = i_c(0^+) = 0$$

With reference to (8.15), this condition results in

$$i_0(0^+) = i_d(0^+) = i_q(0^+) = 0$$

The initial value of the field current is

$$i_F(0^+) = \frac{V_F}{r_F}$$

For balanced three-phase short circuit at the terminals of the machine

$$v_a = v_b = v_c = 0$$

With reference to (8.16), this condition results in

$$v_0 = v_d = v_q = 0$$

Since $i_0 = 0$, the machine equation in the rotor reference frame following a three-phase short circuit becomes

$$\begin{bmatrix} v_d \\ -v_F \\ 0 \\ v_q \\ 0 \end{bmatrix} = - \begin{bmatrix} r & 0 & 0 & \omega L_q & \omega k M_Q \\ 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 \\ -\omega L_d & -\omega k M_F & -\omega k M_D & r & 0 \\ 0 & 0 & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \end{bmatrix} - \begin{bmatrix} L_d & kM_F & kM_D & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \end{bmatrix} \tag{8.36}$$

This equation is in the state-space form and can be written in compact form as

$$\mathbf{v} = -\mathbf{R}\mathbf{i} - \mathbf{L}\frac{d}{dt}\mathbf{i} \tag{8.37}$$

or

$$\frac{d}{dt}\mathbf{i} = -\mathbf{L}^{-1}\mathbf{R}\mathbf{i} - \mathbf{L}^{-1}\mathbf{v} \tag{8.38}$$

If speed is assumed constant, the resulting state-space equation is linear and an analytical solution can be obtained by the Laplace transform technique. However, the availability of powerful simulation packages make it possible to simulate the nonlinear differential equations of the synchronous machine easily in matrix form. To consider the speed variation we need to include the dynamic equation of the machine. This is a second-order differential equation known as the *swing equation* which is described in Chapter 11. The swing equation can be expressed in the state-space form as two first-order differential equation and can easily be augmented with (8.36). Since the speed variation has very little effect in the momentary current immediately following the fault, speed variation may be neglected.

Once a solution is obtained for the direct axis and quadrature axis currents, the phase currents are obtained through the inverse Park transformation, i.e.,

$$\mathbf{i}_{abc} = \mathbf{P}^{-1}\mathbf{i}_{0dq} \tag{8.39}$$

Substituting for $\mathbf{P}^{-1}$ from (8.18), and noting $i_0 = 0$, the phase currents are

$$\begin{aligned} i_a &= i_d \cos\theta + i_q \sin\theta \\ i_b &= i_d \cos(\theta - 2\pi/3) + i_q \sin(\theta - 2\pi/3) \\ i_c &= i_d \cos(\theta + 2\pi/3) + i_q \sin(\theta + 2\pi/3) \end{aligned} \tag{8.40}$$

MATLAB provides two M-files named **ode23** and **ode45** for numerical solution of differential equations employing the Runge-Kutta-Fehlberg integration method. **ode23** uses a simple second and third order pair of formulas for medium accuracy and **ode45** uses a fourth and fifth order pair for higher accuracy. Synchronous machine simulation during balanced three-phase fault is demonstrated in the following example.

Example 8.2

A 500-MVA, 30-kV, 60-Hz synchronous generator is operating at no-load with a constant excitation voltage of 400 V. A three-phase short circuit occurs at the armature terminals. Use **ode45** to simulate (8.36), and obtain the transient waveforms for the current in each phase and the field current. Assume the short circuit is applied at the instant when the rotor direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$. Also, assume that the rotor speed remains constant at the synchronous value. The machine parameters are

Generator Parameters for Example 8.2		
L_d = 0.0072 H	L_q = 0.0070 H	L_F = 2.500 H
L_D = 0.0068 H	L_Q = 0.0016 H	M_F = 0.100 H
M_D = 0.0054 H	M_Q = 0.0026 H	M_R = 0.125 H
r = 0.0020 Ω	r_F = 0.4000 Ω	r_D = 0.015 Ω
r_Q = 0.0150 Ω	L_0 = 0.0010 H	

The dc field voltage is $V_F = 400$ V. The derivatives of the state equation given by (8.38), together with the coefficient matrices in (8.36), are defined in a function file named **symshort.m**, which returns the state derivatives. The initial value of the field current is

$$i_F(0^+) = \frac{V_F}{r_F} = \frac{400}{0.4} = 1000 \text{ A}$$

and since the machine is initially on no-load

$$i_0(0^+) = i_d(0^+) = i_q(0^0) = 0$$

The following file **chp8ex2.m** uses **ode45** to simulate the differential equations defined in **symshort** over the desired interval. The periodic nature of currents necessitates a very small step size for integration. The currents i_d and i_q are substituted in (8.40) and the phase currents are determined.

```
VF = 400; rF = 0.4; iF0 = VF/rF;
f = 60;  w=2.*pi*f;
```

```
d = 0;    d=d*pi/180;
t0 = 0 ; tfinal = 0.80;
tspan =[t0, tfinal];
i0 = [0; iF0; 0; 0; 0 ];          % Initial currents
[t,i] = ode45('symshort',tspan,i0);
theta = w*t + d + pi/2;
id = i(:,1), iq = i(:,4), iF = i(:,2);
ia = sqrt(2/3)*(id.*cos(theta) + iq.*sin(theta));
ib = sqrt(2/3)*(id.*cos(theta-2*pi/3)+ iq.*sin(theta-2*pi/3));
ic = sqrt(2/3)*(id.*cos(theta+2*pi/3)+ iq.*sin(theta+2*pi/3));
figure(1), plot(t,ia),  xlabel('Time - sec.'), ylabel('ia, A')
title(['Three-phase short circuit ia, ','delta =',num2str(d)])
figure(2), plot(t,ib),  xlabel('Time - sec.'), ylabel('ib, A')
title(['Three-phase short circuit ib, ','delta =',num2str(d)])
figure(3), plot(t,ic),  xlabel('Time - sec.'), ylabel('ic, A')
title(['Three-phase short circuit ic, ','delta =',num2str(d)])
figure(4), plot(t,iF),  xlabel('Time - sec.'), ylabel('iF, A')
title(['Three-phase short circuit iF,','delta = ',num2str(d)])
```

Results of the simulations are shown in Figure 8.5.

Armature currents in the various phases vary with time in a rather complicated way. Analysis of the waveforms show that they consist of

- A fundamental-frequency component.
- A dc component.
- A double-frequency component.

The fundamental-frequency component is symmetrical with respect to the time axis. Its superposition on the dc component will give an unsymmetrical waveform. The degree of asymmetry depends upon the point of the voltage cycle at which the short circuit takes place. The field current shown in Figure 8.5, like the stator current, consists of dc and ac components. The ac component is decaying and is comprised of a fundamental and a second harmonic. The second harmonic components in the field current as well as the armature currents are relatively small and are usually neglected. Furthermore, in Section 8.7 we see that during short circuit, the effective reactance of the machine may be assumed only along the direct axis and very simple models are obtained for power system fault studies and transient stability analysis. Before we obtain these simplified models, we consider the unbalanced short circuit of synchronous machine.

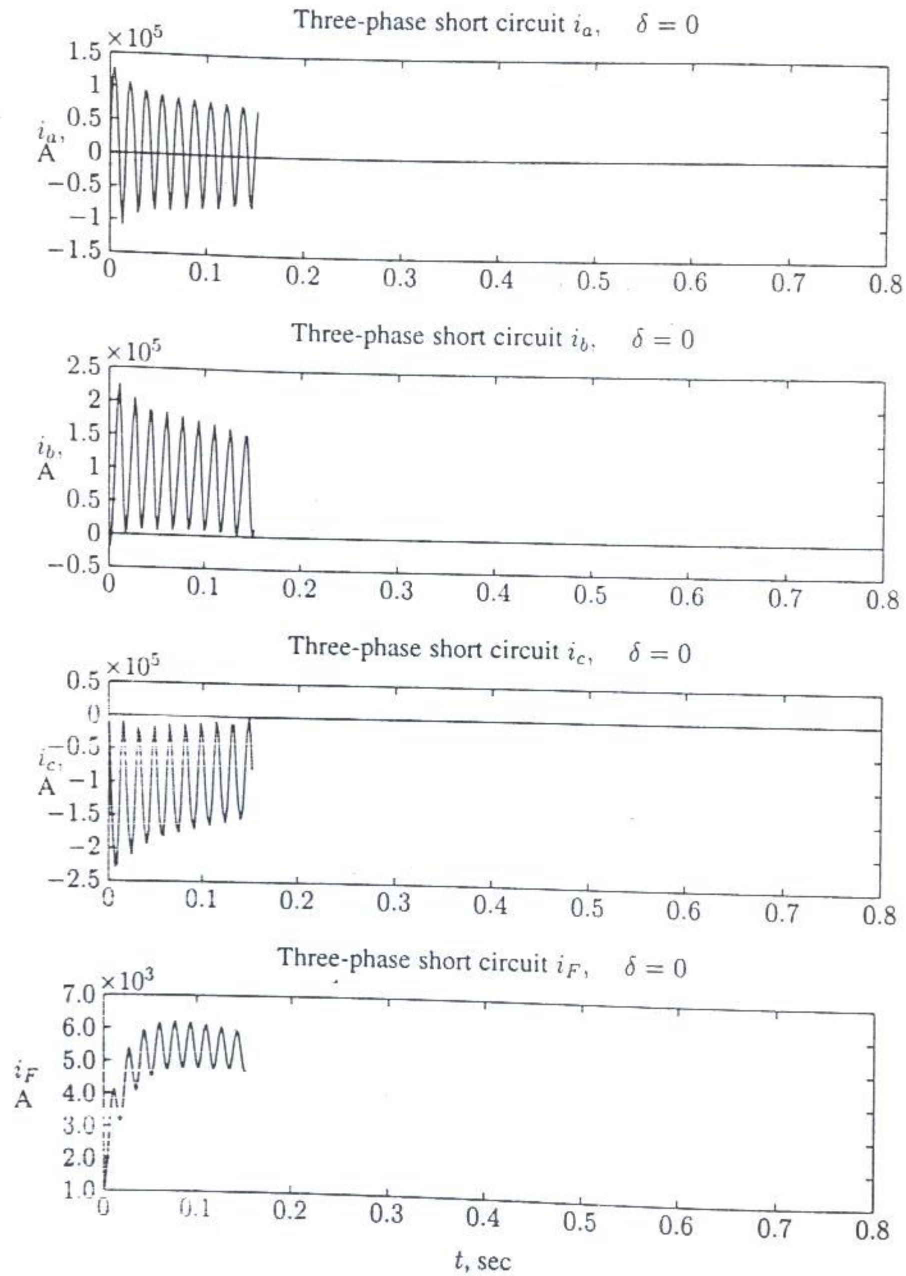


FIGURE 8.5
Balanced three-phase short-circuit current waveforms.

8.6 UNBALANCED SHORT CIRCUITS

Most frequent faults on synchronous machines are phase-to-phase and phase-to-neutral short circuits. These unbalanced faults are most difficult to analyze. The d-q-0 model is not well suited for the study of unbalanced fault and requires further transformation. The analytical solution becomes exceptionally complicated and at the end of it all the solutions are still only approximate. In the numerical solution the original voltage equations can be used without the need for any transformations. In the following section the machine equations are developed in direct-phase quantities for simulation of the synchronous machine for the line-to-line and the line-to-ground short circuits.

8.6.1 LINE-TO-LINE SHORT CIRCUIT

For a solid short circuit between phases a and b,

$$v_b = v_c = 0$$

and

$$i_b = -i_c$$

Since phase a is not involved in the short circuit and the generator is initially on no-load, $i_a = 0$, thus

$$i_0 = i_a + i_b + i_c = 0$$

and from (8.35), $v_0 = 0$. Substituting the above conditions into (8.15) and (8.16) yields

$$v_d \sin\theta - v_q \cos\theta = 0 \tag{8.41}$$

and

$$i_d = \sqrt{2}\, i_b \sin\theta \tag{8.42}$$

$$i_q = \sqrt{2}\, i_b \cos\theta \tag{8.43}$$

Derivatives of the direct axis and the quadrature axis currents are

$$\frac{di_d}{dt} = \sqrt{2}\,\frac{di_b}{dt}\sin\theta + \sqrt{2}\,\omega i_b \cos\theta \tag{8.44}$$

$$\frac{di_q}{dt} = \sqrt{2}\,\frac{di_b}{dt}\cos\theta - \sqrt{2}\,\omega i_b \sin\theta \tag{8.45}$$

Substituting (8.42)–(8.45) into (8.36) and applying (8.41) to the first and fourth equations in (8.36), the voltage equation for a line-to-line fault in direct-phase quantities is obtained.

$$\begin{bmatrix} -v_F \\ 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} \sqrt{2}k\omega M_F\cos\theta & r_F & 0 & 0 \\ \sqrt{2}k\omega M_D\cos\theta & r_D & 0 & 0 \\ \sqrt{2}k\omega M_Q\sin\theta & 0 & 0 & r_Q \\ \sqrt{2}[r+\omega(L_d-L_q)]\sin 2\theta & k\omega M_F\cos\theta & k\omega M_D\cos\theta & k\omega M_Q\sin\theta \end{bmatrix} \begin{bmatrix} i_b \\ i_F \\ i_D \\ i_Q \end{bmatrix} - \begin{bmatrix} \sqrt{2}kM_F\sin\theta & L_F & M_R & 0 \\ \sqrt{2}kM_D\sin\theta & M_R & L_D & 0 \\ -\sqrt{2}kM_Q\cos\theta & 0 & 0 & L_Q \\ \sqrt{2}(L_d\sin^2\theta + L_q\cos^2\theta) & kM_F\sin\theta & kM_D\sin\theta & -kM_Q\cos\theta \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_b \\ i_F \\ i_D \\ i_Q \end{bmatrix} \tag{8.46}$$

This equation is in the state-space form and is written in compact form as (8.37). The state derivatives is given by (8.38), which is suitable for numerical integration.

Example 8.3

The synchronous generator in Problem 8.2 is operating at no-load with a constant excitation voltage of 400 V. A line-to-line short circuit occurs between phases b and c at the armature terminals. Use **ode45** to simulate (8.46), and obtain the waveforms for current in phase b and the field current. Assume the short circuit is applied at the instant when the rotor direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$. Also, assume that the rotor speed remains constant at the synchronous value.

The dc field voltage is $V_F = 400$ V. The derivatives of the state equation given by (8.38), together with the coefficient matrices in (8.46) are defined in a function file named **llshort.m** which returns the state derivatives. The following file **chp8ex3.m** uses **ode45** to simulate the differential equation defined in **llshort** over the desired interval. The current in phase b and the field current are determined and their plots are shown in Figure 8.6.

```
VF = 400; rF = 0.4; iF0 = VF/rF;
f = 60;     w = 2.*pi*f;
d = 0;      d = d*pi/180;
t0 = 0 ; tfinal = 0.80;
tspan = [t0, tfinal];
```

```
i0 = [0; iF0; 0; 0;];              % Initial currents
[t,i] = ode45('llshort', tspan, i0);
ib=i(:,1);  iF=i(:,2);
figure(1), plot(t,ib), xlabel('t, sec'), ylabel('ib, A')
title(['Line-line short circuit ib, ','delta = ', num2str(d)])
figure(2), plot(t,iF), xlabel('t, sec.'),ylabel('iF, A')
title(['Line-line short circuit iF,  ','delta = ',num2str(d)])
```

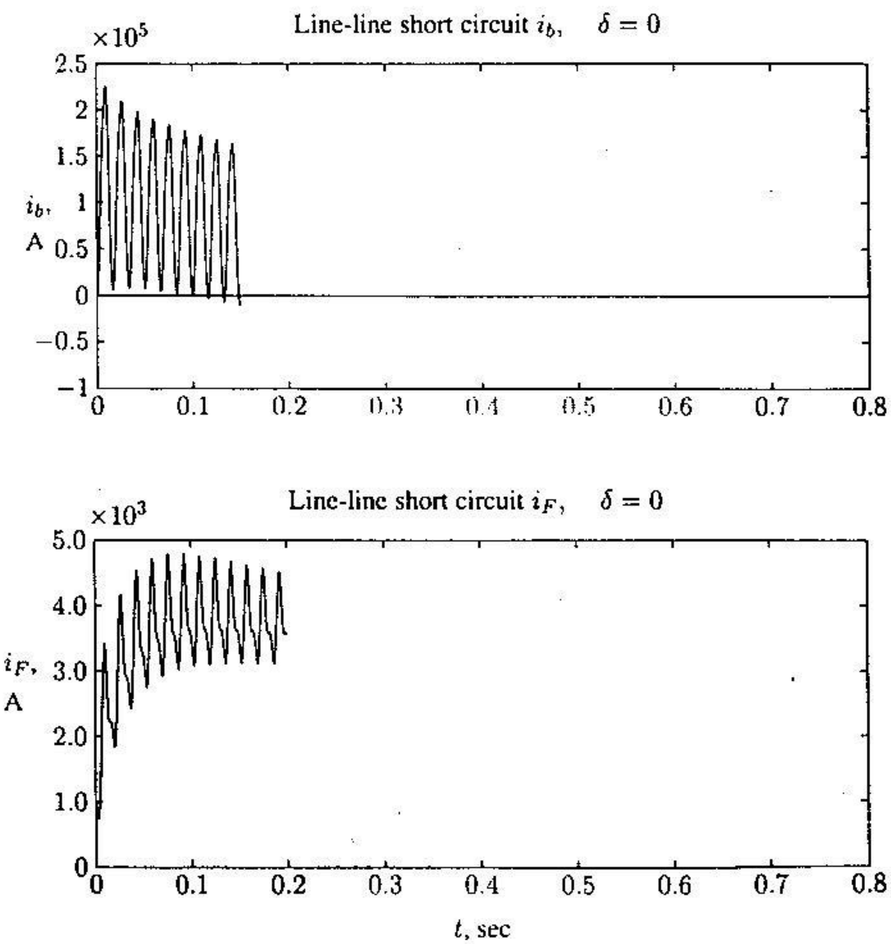


FIGURE 8.6
Line-to-line short-circuit current waveforms.

8.6.2 LINE-TO-GROUND SHORT CIRCUIT

For a solid short circuit between phases a and ground

$$v_a = 0$$

and with the machine initially on no-load

$$i_b = i_c = 0$$

A convenient way to obtain the voltage equation for line-to-ground short circuit is to start with (8.4), i.e., the machine voltage equation in direct phase quantities. Applying the short circuit condition to this equation and expressing the inductances in terms of the more commonly d-q-0 reactances, the following equation is obtained for the line-to-ground fault on phase a.

$$\begin{bmatrix} 0 \\ -v_F \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} r - 2\omega L_m \sin 2\theta & -\omega M_F \sin\theta & -\omega M_D \sin\theta & \omega M_Q \cos\theta \\ -\omega M_F \sin\theta & r_F & 0 & 0 \\ -\omega M_D \sin\theta & 0 & r_D & 0 \\ \omega M_Q \cos\theta & 0 & 0 & r_Q \end{bmatrix} \begin{bmatrix} i_a \\ i_F \\ i_D \\ i_Q \end{bmatrix} - \begin{bmatrix} L_s + L_m \cos 2\theta & M_F \cos\theta & M_D \cos\theta & M_Q \sin\theta \\ M_F \cos\theta & L_F & M_R & 0 \\ M_D \cos\theta & M_R & L_D & 0 \\ M_Q \sin\theta & 0 & 0 & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_a \\ i_F \\ i_D \\ i_Q \end{bmatrix} \tag{8.47}$$

where

$$L_s = \frac{1}{3}(L_0 + L_d + L_q) \tag{8.48}$$

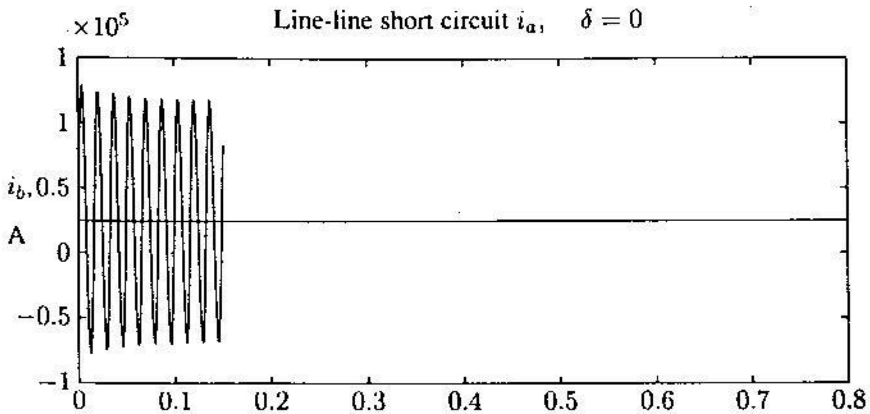
$$L_m = \frac{1}{3}(L_d - L_q) \tag{8.49}$$

Equation (8.47) is in the state-space form and is written in compact form as (8.37). The state derivatives is given by (8.38) which is suitable for numerical integration.

Example 8.4

The synchronous generator in Problem 8.2 is operating at no-load with a constant excitation voltage of 400 V. A line-to-ground short circuit occurs on phase a of the armature terminals. Use **ode45** to simulate (8.47), and obtain the waveforms for the current in phase a and the field current. Assume the short circuit is applied at the instant when the rotor direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$. Also, assume that the rotor speed remains constant at the synchronous value.

The dc field voltage is $V_F = 400$ V. The derivative of the state equation given by (8.38), together with the coefficient matrices in (8.47), are defined in a function file named **lgshort.m** which returns the state derivatives. The following file **chp8ex4.m** uses **ode45** to simulate the differential equation defined in **lgshort** over the desired interval. The phase and the field currents are determined and their plots are shown in Figure 8.7.



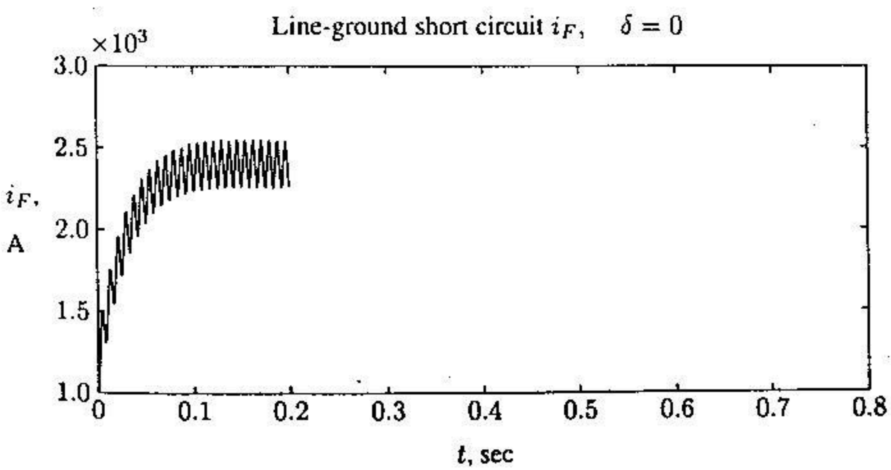


FIGURE 8.7
Line-to-ground short-circuit current waveforms.


```
VF = 400; rF = 0.4; iF0 = VF/rF;
f = 60;     w = 2.*pi*f;
d = 0;      d = d*pi/180;
t0 = 0 ; tfinal = 0.80; tspan = [t0, tfinal];
i0 = [0; iF0; 0; 0;];                   % Initial currents
tol =  0.0001;                          % accuracy
[t,i] = ode45('lgshort', tspan, i0, tol);
ia=i(:,1);  iF=i(:,2);
figure(1), plot(t,ia), xlabel('t, sec'), ylabel('ia, A')
title(['Line-ground short circuit ia,','delta =', num2str(d)])
figure(2), plot(t,iF), xlabel('t, sec'), ylabel('iF, A')
title(['Line-ground short circuit iF,','delta = ',num2str(d)])
```

A three-phase model, which uses direct physical parameters, is well suited for simulation on a computer, and it is not necessary to go through complex transformations. The analysis can easily be extended to take the speed variation into account by including the dynamic equations of the machine.

8.7 SIMPLIFIED MODELS OF SYNCHRONOUS MACHINES FOR TRANSIENT ANALYSES

In Chapter 3, for steady-state operation, the generator was represented with a constant emf behind a synchronous reactance X_s. For salient-pole rotor, because of the nonuniformity of the air gap, the generator was modeled with direct axis reactance X_d and the quadrature axis reactance X_q. However, under short circuit conditions, the circuit reactance is much greater than the resistance. Thus, the stator current lags nearly $\pi/2$ radians behind the driving voltage, and the armature reaction mmf is centered almost on the direct axis. Therefore, during short circuit, the effective reactance of the machine may be assumed only along the direct axis.

The three-phase short circuit waveform shown in Figure 8.5 shows that the ac component of the armature current decays from a very high initial value to the steady state value. This is because the machine reactance changes due to the effect of the armature reaction. At the instant prior to short circuit, there is some flux on the direct axis linking both stator and rotor, due only to rotor mmf if the machine is on open circuit, or due to the resultant of rotor and stator mmf if some stator current is flowing. When there is a sudden increase of stator current on short circuit, the flux linking stator and rotor cannot change instantaneously due to eddy currents flowing in the rotor and damper circuits, which oppose this change. Since, the stator mmf is unable at first to establish any armature reaction, the reactance of armature reaction is negligible, and the initial reactance is very small and similar in value to the leakage reactance. As the eddy current in the damper circuit and eventually in the field circuit decays, the armature reaction is fully established. The armature reaction which is produced by a nearly zero power factor current provides mostly demagnetizing effect and the machine reactance increases to the direct axis synchronous reactance.

For purely qualitative purposes, a useful picture can be obtained by thinking of the field and damper windings as the secondaries of a transformer whose primary is the armature winding. During normal steady state conditions, there is no transformer action between stator and rotor windings of the synchronous machine as the resultant field produced by the stator and rotor both revolve with the same synchronous speed. This is similar to a transformer with open-circuited secondaries. For this condition, its primary may be described by the synchronous reactance X_d. During disturbance, the rotor speed is no longer the same as that of the revolving field produced by stator windings resulting in the transformer action. Thus, field and damper circuits resemble much more nearly as short-circuited secondaries. The equivalent circuit for this condition, referred to the stator side, is shown in Figure 8.8. Ignoring winding resistances, the equivalent reactance of Figure 8.8,

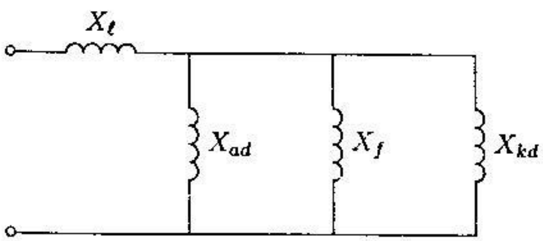


FIGURE 8.8
Equivalent circuit for the subtransient period.

known as the *direct axis subtransient reactance*, is

$$X''_d = X_\ell + \left(\frac{1}{X_{ad}} + \frac{1}{X_f} + \frac{1}{X_{kd}}\right)^{-1} \tag{8.50}$$

If the damper winding resistance R_k is inserted in Figure 8.8 and the Thévenin's inductance seen at the terminals of R_k is obtained, the circuit time constant, known as the *direct axis short circuit subtransient time constant*, becomes

$$\tau''_d = \frac{X_{kd} + (\frac{1}{X_\ell} + \frac{1}{X_f} + \frac{1}{X_{ad}})^{-1}}{R_k} \tag{8.51}$$

In (8.51) reactances are assumed in per unit and, therefore, they have the same numerical values as inductances in per unit. For a 2-pole, turbo-alternators X''_d may be between 0.07 and 0.12 per unit, while for water-wheel alternators the range may be 0.1 to 0.35 per unit. The direct axis subtransient reactance X''_d is only used

in calculations if the effect of the initial current is important, as for example, when determining the circuit breaker short-circuit rating.

Typically, the damper circuit has relatively high resistance and the direct axis short circuit subtransient time constant is very small, around 0.035 second. Thus, this component of current decays quickly. It is then permissible to ignore the branch of the equivalent circuit which takes account of the damper windings, and the equivalent circuit reduces to that of Figure 8.9.

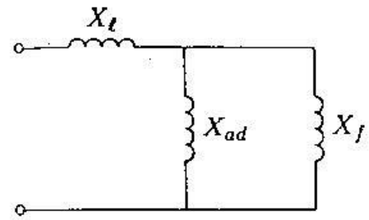


FIGURE 8.9
Equivalent circuit for the transient period.

Ignoring winding resistances, the equivalent reactance of Figure 8.9, known as the *direct axis short circuit transient reactance*, is

$$X'_d = X_\ell + \left(\frac{1}{X_{ad}} + \frac{1}{X_f}\right)^{-1} \tag{8.52}$$

If the field winding resistance R_f is inserted in Figure 8.9, and the Thévenin's inductance seen at the terminals of R_f is obtained, the circuit time constant, known as the *direct axis short circuit transient time constant*, becomes

$$\tau'_d = \frac{X_f + (\frac{1}{X_\ell} + \frac{1}{X_{ad}})^{-1}}{R_f} \tag{8.53}$$

The direct axis transient short circuit reactance X'_d may lie between 0.10 to 0.25 per unit. The short circuit transient time constant τ'_d is usually in order of 1 to 2 seconds.

The field time constant which characterizes the decay of transients with the armature open-circuited is called the *direct axis open circuit transient time constant*. This is given by

$$\tau'_{d0} = \frac{X_f}{R_f} \tag{8.54}$$

Typical values of the direct axis open circuit transient time constant are about 5 seconds. τ'_d is related to τ'_{d0} by

$$\tau'_d = \frac{X'_d}{X_d}\tau'_{d0} \tag{8.55}$$

Finally, when the disturbance is altogether over, there will be no hunting of the rotor, and, hence there will not be any transformer action between the stator and the rotor, and the circuit reduces to that of Figure 8.10.

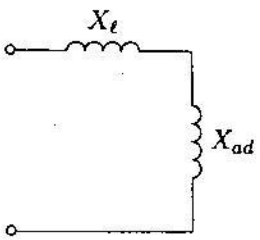


FIGURE 8.10
Equivalent circuit for the steady state.

The equivalent reactance becomes the direct axis synchronous reactance, given by

$$X_d = X_\ell + X_{ad} \tag{8.56}$$

Similar equivalent circuits are obtained for reactances along the quadrature axis. These reactances X_q'', X_q', and X_q may be considered for cases when the circuit resistance results in a power factor appreciably above zero and the armature reaction is not necessarily totally on the direct axis.

The fundamental-frequency component of armature current following the sudden application of a short circuit to the armature of an initially unloaded machine can be expressed as

$$i_{ac}(t) = \sqrt{2}E_0\left[\left(\frac{1}{X_d''} - \frac{1}{X_d'}\right)e^{-t/\tau_d''} + \left(\frac{1}{X_d'} - \frac{1}{X_d}\right)e^{-t/\tau_d'} + \frac{1}{X_d}\right]\sin(\omega t + \delta) \tag{8.57}$$

A typical symmetric trace of the short circuit waveform obtained for the data of Example 8.5 is shown in Figure 8.11 (page 340).

It should be recalled that in the derivation of the above results, resistance was neglected except in consideration of the time constant. In addition, in the above treatment the dc and the second harmonic components corresponding to the decay of the trapped armature flux has been neglected. It should also be emphasized that the representation of the short-circuited paths of the damper windings and the solid iron rotor by a single equivalent damper circuit is an approximation to the actual situation. However, this approximation has been found to be quite valid in many cases. The synchronous machine reactances and time constants are provided by the manufacturers. These values can be obtained by a short circuit test described in the next section.

Example 8.5

A three-phase, 60-Hz synchronous machine is driven at constant synchronous speed by a prime mover. The armature windings are initially open-circuited and field voltage is adjusted so that the armature terminal voltage is at the rated value (i.e., 1.0 per unit). The machine has the following per unit reactances and time constants.

$$X_d'' = 0.15 \text{ pu} \qquad \tau_d'' = 0.035 \text{ sec}$$
$$X_d' = 0.40 \text{ pu} \qquad \tau_d' = 1.0 \text{ sec}$$
$$X_d = 1.20 \text{ pu}$$

a) Determine the steady state, transient and subtransient short circuit currents.
b) Obtain the fundamental-frequency waveform of the armature current for a three-phase short circuit at the terminals of the generator. Assume the short circuit is applied at the instant when the rotor direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$.

$$I_d = \frac{E_0}{X_d} = \frac{1.0}{1.2} = 0.8333 \text{ pu}$$
$$I_d' = \frac{E_0}{X_d'} = \frac{1.0}{0.4} = 2.5 \text{ pu}$$
$$I_d'' = \frac{E_0}{X_d''} = \frac{1.0}{0.15} = 6.666 \text{ pu}$$

To obtain the short circuit waveform, we write the following commands.

```
w0 = 2*pi*60;
E0 = 1.0;       delta = 0;
Xd2dash = 0.15;
Xddash = 0.4;
Xd = 1.2;
tau2dash = 0.035; taudash = 1.0;
t=0:1/(4*240):1.0;
iac = sqrt(2)*E0*((1/Xd2dash-1/Xddash)*exp(-t/tau2dash)+...
(1/Xddash-1/Xd)*exp(-t/taudash) + 1/Xd).*sin(w0*t + delta);
plot(t, iac), xlabel('t, sec'), ylabel('iac, A')
end
```

The result is shown in Figure 8.11.

The trace is obtained up to 1 second. If the simulation period is extended to about $5\tau_d' = 5.0$ seconds, the short circuit will reach to its steady state with a peak value of $I_{d(max)} = \sqrt{2}\,E_0/X_d = \sqrt{2}\,(1.0)/1.2 = 1.1785$ per unit.

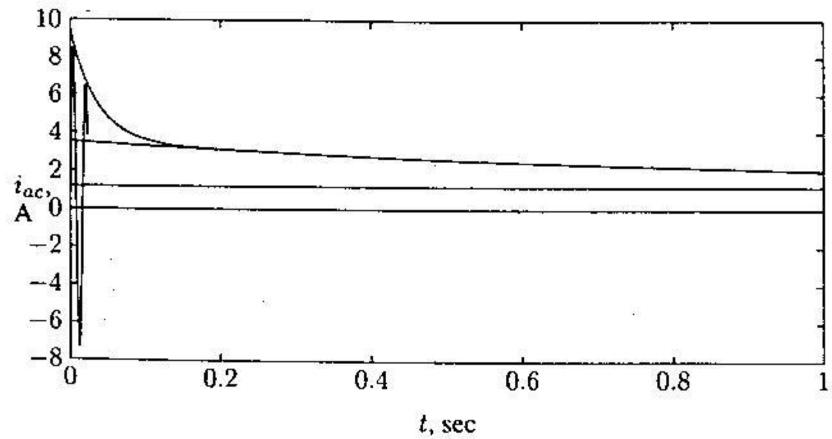


FIGURE 8.11
The 60-Hz component of the short-circuit current of a synchronous generator.

8.8 DC COMPONENTS OF STATOR CURRENTS

In the expression for the armature current as given by (8.57), the unidirectional transient component has not been taken into account. As seen from consideration of the simple RL circuit of Figure 8.1, there will in general be a dc offset depending on when the voltage is applied. Similarly, in the synchronous machine, the dc offset component depend on the instantaneous value of the stator voltage at the time of the short circuit. The rotor position is given by $\theta = \omega t + \delta + \pi/2$. The dc component depends on the rotor position δ when the short circuit occurs at time $t = 0$. The time constant associated with the decay of the dc component of the stator current is known as the *armature short circuit time constant*, τ_a. Most of the decay of the dc component occurs during the subtransient period. For this reason the average value of the direct axis and quadrature axis subtransient reactances is used for finding τ_a. It is approximately given by

$$\tau_a = \frac{X_d'' + X_q''}{2R_a} \tag{8.58}$$

Typical values of the armature short circuit time constant is around 0.05 to 0.17 second.

Since the three-phase voltages are each separated by $2\pi/3$ radians, the amount of the dc component of the armature current is different in each phase and depends upon the point of the voltage cycle at which the short circuit occurs. The dc component for phase a is given by

$$I_{dc} = \sqrt{2}\frac{E_0}{X_d''}\sin\delta\, e^{-t/\tau_a} \tag{8.59}$$

The superposition of the dc component on the fundamental frequency component will give an asymmetrical waveform.

$$i_{asy}(t) = \sqrt{2}E_0\left[\left(\frac{1}{X_d''} - \frac{1}{X_d'}\right)e^{-t/\tau_d''} + \left(\frac{1}{X_d'} - \frac{1}{X_d}\right)e^{-t/\tau_d'} + \frac{1}{X_d}\right]\sin(\omega t + \delta)$$
$$+\sqrt{2}\frac{E_0}{X_d''}\sin\delta e^{-t/\tau_a} \tag{8.60}$$

The degree of asymmetry depends upon the point of the voltage cycle at which the fault takes place. The worst possible transient condition is $\delta = \pi/2$. The maximum possible initial magnitude of the dc component is

$$I_{dc_{(max)}} = \sqrt{2}\,\frac{E_0}{X_d''} \tag{8.61}$$

Therefore, the maximum rms current (ac plus dc) at the beginning of the short circuit is

$$I_{asy} = \sqrt{I_d''^2 + I_{dc}^2} = \sqrt{\left(\frac{E_0}{X_d''}\right)^2 + \left(\sqrt{2}\frac{E_0}{X_d''}\right)^2} \tag{8.62}$$

from which

$$I_{asy} = \sqrt{3}\,\frac{E_0}{X_d''} \tag{8.63}$$
$$= \sqrt{3}I_d''$$

In practice, the *momentary duty* of a circuit breaker is given in terms of the asymmetrical short circuit current.

Example 8.6

For the machine in Example 8.5, assume that a three-phase short circuit is applied at the instant when the rotor quadrature axis is along the magnetic axis of phase a, i.e., $\delta = \pi/2$ radians. Obtain the asymmetrical waveform of the armature current for phase a. The armature short circuit time constant is $\tau_a = 0.15$ sec.

In the *MATLAB* program of Example 8.5, we make $\delta = \pi/2$ and use (8.60) in place of the i_{ac} statement. Running this example results in the waveform shown in Figure 8.12.

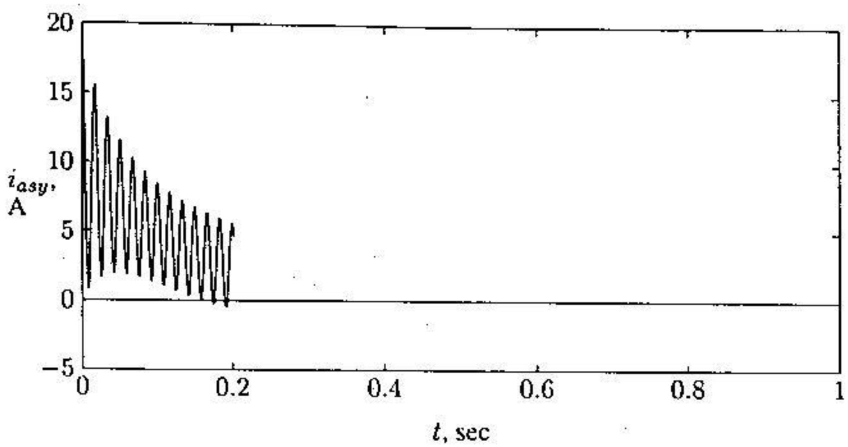


FIGURE 8.12
Synchronous generator asymmetrical short-circuit current $\delta = \pi/2$.

8.9 DETERMINATION OF TRANSIENT CONSTANTS

A sudden three-phase short circuit is applied to the terminals of an unloaded generator and the oscillogram of the current in one phase is obtained. The test is repeated until a symmetric waveform is obtained which does not contain the dc offset. This occurs when the voltage is near maximum at the instant of fault. The machine reactances X''_d, X'_d, and X_d and the time constants τ''_d and τ'_d are determined by analyzing the oscillogram waveform as follow.

The waveform is divided into three periods: the subtransient period, lasting only for the first two cycles, during which the current decrement is very rapid; the transient period, covering a relatively longer time, during which the current decrement is more moderate; and finally, the steady state period.

The no-load generated voltage E_0 is obtained by measuring the phase voltage and expressing it in per unit. The direct axis synchronous reactance X_d is determined from the part of the oscillogram where the envelope of the current has become constant. Denoting this amplitude with $I_{d_{(max)}}$, the rms value of the steady short circuit is $I_d = I_{d_{(max)}}/\sqrt{2}$. From this the direct axis synchronous reactance is found

$$X_d = \frac{E_0}{I_d} \tag{8.64}$$

The peak steady short circuit current is subtracted from two points after approximately the 10th cycle where the subtransient component has decayed. Dividing these values by $\sqrt{2}$ results in the following term

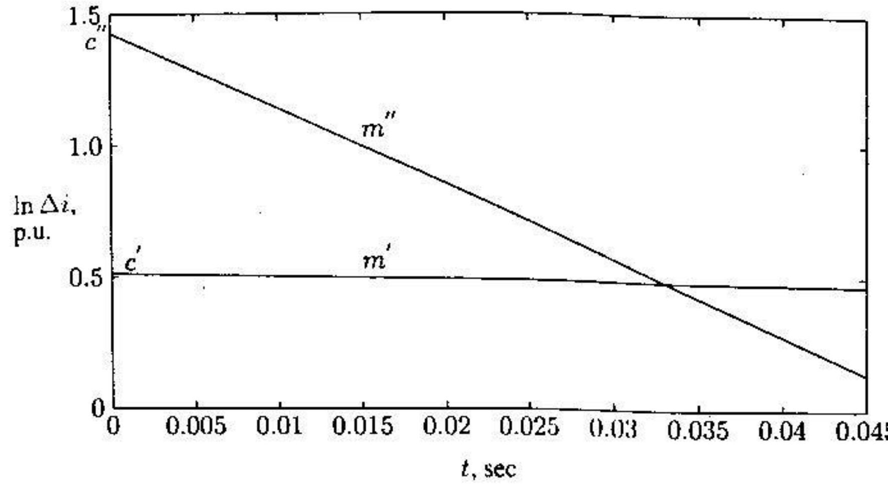


FIGURE 8.13
Current difference logarithm, $\ln \Delta i'$ and $\ln \Delta i''$.

$$\Delta i' = (I'_d - I_d)e^{-t/\tau'_d}$$

or

$$\begin{aligned}\ln \Delta i' &= \ln(I'_d - I_d) - t/\tau'_d \\ &= c' - m't\end{aligned}$$

If the points given by $\ln \Delta i'$ are plotted against a linear time scale, a straight line is obtained with y-intercept $c' = \ln(I'_d - I_d)$ and slope $-m'$, as shown in Figure 8.13. The rms transient component of current is obtained from

$$I'_d = e^{c'} + I_d \tag{8.65}$$

Transient reactance and time constant are then obtained by

$$X'_d = \frac{E_0}{I'_d} \tag{8.66}$$

and

$$\tau'_d = \frac{1}{m'} \tag{8.67}$$

To find the subtransient components, the peak current of the first two cycles are divided by $\sqrt{2}$. Subtracting the steady short circuit current and the rms transient currents found earlier from these points results in

$$\Delta i'' = (I''_d - I'_d)e^{-t/\tau''_d}$$

or

$$\ln \Delta i'' = \ln(I''_d - I'_d) - t/\tau''_d$$
$$= c'' - m''t$$

If the points given by $\ln \Delta i''$ are plotted against a linear time scale, a straight line is obtained with y-intercept $c'' = \ln(I''_d - I'_d)$ and slope $-m''$, shown in Figure 8.13. The rms subtransient component of current is given by

$$I''_d = e^{c''} + I'_d \tag{8.68}$$

Subtransient reactance and time constant are

$$X''_d = \frac{E_0}{I''_d} \tag{8.69}$$

and

$$\tau''_d = \frac{1}{m''} \tag{8.70}$$

The above procedure is demonstrated in the following example.

Example 8.7

A three-phase, 60-Hz synchronous machine is driven at constant synchronous speed by a prime mover. The armature windings are initially open-circuited and field voltage is adjusted so that the armature terminal voltage is at the rated value (i.e., 1.0 per unit). The generator is suddenly subjected to a symmetrical three-phase short circuit at the instant when direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$. An oscillogram of the short-circuited current is obtained. The peak values at the first two cycles, at the 20th and 21st cycles, and the steady value after a long time were recorded as tabulated in the following table.

I_{max}, pu	8.7569	6.7363	···	2.8893	2.8608	···	1.1785
Time, sec	0.0042	0.0208	···	0.3208	0.3375	···	5.0000

Determine the transient and the subtransient reactances and time constants.

The following statements are written with reference to the above procedure.

```
E0 = 1.0;
Im=[8.7569   6.7363  2.8893  2.8608   1.1785];
 t=[0.0042   0.0208  0.3208  0.3375   5.0000];
I = Im/sqrt(2);                 % The rms value of the above envelope
id=I(5);                       % rms value of the steady short circuit
```



```
Dt2=[t(3) t(4)];                % Time for 20th and 21st cycles
Di2=[I(3)-id I(4)-id];%Diff. between transient envelope and id
LDi2= log(Di2);                %Natural log of the above two points
c2=polyfit(Dt2, LDi2, 1);
                       %Finds coefficients of a 1st-order polynomial
                   % i.e. the slope and intercept of a straight line
iddash=(exp(c2(2))+id)    % rms value of the transient current
Xddash=E0/iddash                % Direct axis transient reactance
taudash=abs(1/c2(1))   %Direct axis sc transient time constant
Di=(iddash-id)*[exp(-t(1)/taudash)  exp(-t(2)/taudash)];
Di1=[I(1)-Di(1)-id   I(2)-Di(2)-id];   % Subtransient envelope
LDi1=log(Di1);
Dt1 =[t(1)  t(2)];            % Natural log of the first two points
c1=polyfit(Dt1, LDi1, 1);
                       %Finds coefficients of a 1st-order polynomial
                   % i.e. the slope and intercept of a straight line

id2dash=exp(c1(2))+iddash  %rms value of  subtransient current
Xd2dash= E0/id2dash        % Direct axis subtransient reactance
tau2dash=abs(1/c1(1))% direct axis sc subtransient time const.

t=0:.005:.045;
fit2 = polyval(c2, t); % line C2 evaluated for all values of t
fit1 = polyval(c1, t); % line C1 evaluated for all values of t
plot(t, fit1, t, fit2),grid % Logarithmic plot of id'' and id'
ylabel('ln(I)  pu' )       % intercepts are ln(Id'') and ln(Id')
xlabel('t, sec')        %slopes are reciprocal of time constants
```

The result is

$I'_d = 2.5038$ pu $X'_d = 0.3994$; pu $\tau'_d = 0.9941$ sec
$I''_d = 6.6728$ pu $X''_d = 0.1499$; pu $\tau''_d = 0.0348$ sec

Example 8.8

A 100-MVA, 13.8-kV, 60-Hz, Y-connected, three-phase synchronous generator is connected to a 13.8-kV/220-kV, 100-MVA, Δ-Y connected transformer. The reactances in per unit to the machine's own base are

$$X_d = 1.0 \text{ pu} \quad X'_d = 0.25 \text{ pu} \quad X''_d = 0.12 \text{ pu}$$

and its time constants are

$$\tau_a = 0.25 \text{ sec} \quad \tau''_d = 0.4 \text{ sec} \quad \tau'_d = 1.1 \text{ pu}$$

The transformer reactance is 0.20 per unit on the same base. The generator is operating at the rated voltage and no-load when a three-phase fault occurs at the secondary terminals of the transformer as shown in Figure 8.14.

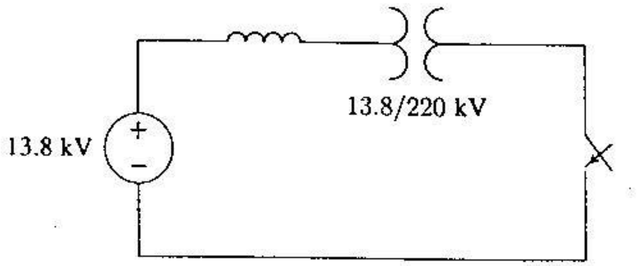


FIGURE 8.14
One-line diagram for Example 8.8.

(a) Find the subtransient, transient, and the steady state short circuit currents in per unit and actual amperes on both sides of the transformer.
(b) What is the maximum rms current (ac plus dc) at the beginning of the fault?
(c) Obtain the instantaneous expression for the short circuit current including the dc component. Assume $\delta = \pi/2$ radians.

(a) The base current on the generator side is

$$I_{B1} = \frac{S_B}{\sqrt{3}\, V_{B1}} = \frac{100 \times 10^3}{\sqrt{3}\, 13.8} = 4184 \text{ A}$$

The base current on the secondary side of the transformer is

$$I_{B2} = \frac{13.8}{220}(4184) = 262.4 \text{ A}$$

the subtransient, transient and the steady state short circuit currents are

$$I''_d = \frac{1.0}{0.12 + 0.2} = 3.125 \text{ pu} = 13,075 \text{ A} \quad \text{on the generator side}$$
$$= 820 \text{ A} \quad \text{on the 220-kV side}$$
$$I'_d = \frac{1.0}{0.25 + 0.2} = 2.22 \text{ pu} = 9,288.5 \text{ A} \quad \text{on the generator side}$$
$$= 582.5 \text{ A} \quad \text{on the 220-kV side}$$
$$I_d = \frac{1.0}{1.0 + 0.2} = 0.833 \text{ pu} = 3,486.6 \text{ A} \quad \text{on the generator side}$$
$$= 218.6 \text{ A} \quad \text{on the 220-kV side}$$

(b) The maximum rms current (ac plus dc) at the beginning of the fault is

$$I_{asy} = \sqrt{3}\, I''_d = \sqrt{3}\,(3.125) = 5.4 \text{ pu} = 22,646 \text{ A} \quad \text{on the generator side}$$

(c) The instantaneous short circuit current including the dc component from (8.60), for $\delta = \pi/2$ is

$$i(t) = \sqrt{2}[(I''_d - I'_d)e^{-t/0.4} + (I'_d - I_d)e^{-t/1.1} + I_d]\sin(377t + \pi/2) + \sqrt{2} I''_d e^{-t/0.25}$$

or

$$i(t) = [1.28e^{-2.5t} + 1.96e^{-0.91t} + 1.18]\sin(377t + \pi/2) + 4.42e^{-4t} \text{ pu}$$

Use *MATLAB* to obtain a plot of $i(t)$.

8.10 EFFECT OF LOAD CURRENT

If the fault occurs when the generator is delivering a prefault load current, two methods might be used in the solution of three-phase symmetrical fault currents.

(a) Use of internal voltages behind reactances

When there is a prefault load current, three fictitious internal voltages E'', E', and E_0 may be considered to be effective during the subtransient, transient, and the steady state periods, respectively. These voltages are known as the *voltage behind subtransient reactance, voltage behind transient reactance*, and *voltage behind synchronous reactance*. Consider the one-line diagram of a loaded generator shown in Figure 8.15(a). The internal voltages shown by the phasor diagram in Figure 8.15(b) are given by

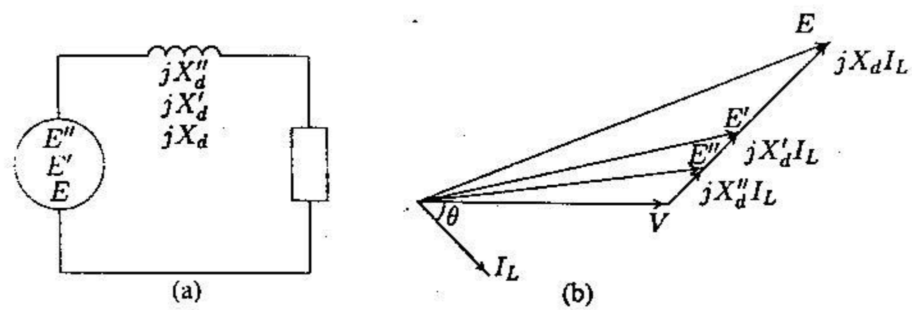


FIGURE 8.15
(a) One-line diagram of a loaded generator, (b) phasor diagram.

$$E'' = V + jX''_d I_L \tag{8.71}$$
$$E' = V + jX'_d I_L$$
$$E = V + jX_d I_L$$

Example 8.9

In Example 8.8, a three-phase load of 100 MVA, 0.8 power factor lagging is connected to the transformer secondary side as shown in Figure 8.16. The line-to-line voltage at the load terminals is 220 kV. A three-phase short circuit occurs at the load terminals. Find the generator transient current including the load current.

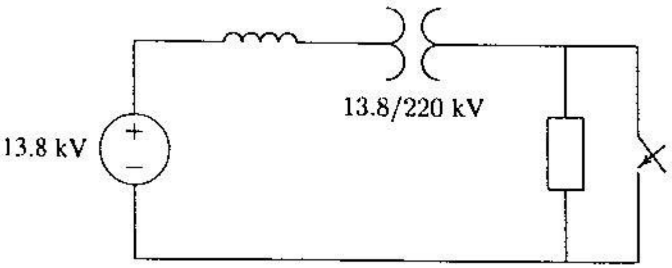


FIGURE 8.16
One-line diagram for Example 8.9.

The load may be represented by a per unit impedance as shown in Figure 8.16.

$$S_L = \frac{100\angle 36.87^\circ}{100} = 1\angle 36.87^\circ \text{ pu}$$
$$V = \frac{220}{220} = 1\angle 0^\circ \text{ pu}$$
$$Z_L = \frac{|V^2|}{S^*} = \frac{1}{1\angle -36.87^\circ} = 0.8 + j0.6 \text{ pu}$$

Before fault, the load current is

$$I_L = \frac{V}{Z_L} = \frac{1.0\angle 0^\circ}{0.8 + j0.6} = 0.8 - j0.6 = 1\angle -36.87^\circ \text{ pu}$$

The emf behind transient reactance is

$$E' = V + j(X'_d + X_t)I_L$$
$$= 1.0\angle 0^\circ + j(0.25 + 0.2)(0.8 - j0.6) = 1.27 + j0.36 = 1.32\angle 15.83^\circ \text{ pu}$$

When the fault is applied by closing switch S, the generator short circuit transient current is

$$I'_g = \frac{E'}{j(X'_d + X_t)} = \frac{1.32\angle 15.83^\circ}{j(0.25 + 0.2)} = 0.8 - j2.822 = 2.93\angle -74.17^\circ \text{ pu}$$

(b) Using Thévenin's theorem and superposition with load current

The fault current is found in the absence of the load by obtaining the Thévenin's equivalent circuit to the point of fault. The total short circuit current is then given by superimposing the fault current with the load current.

Example 8.10

Find the generator transient current in Problem 8.9 using Thévenin's method.

The one-line diagram of Example 8.10 without the load is shown in Figure 8.17(a). The circuit for the Thévenin's equivalent impedance with respect to the point of fault is shown in Figure 8.17(b). The Thévenin's voltage is the prefault

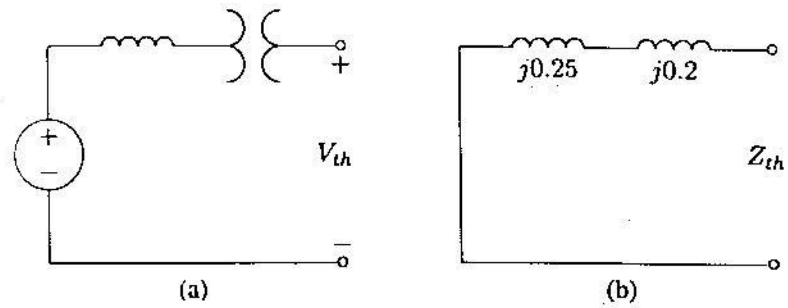


FIGURE 8.17
(a) One-line diagram for Example 8.10 without the load and (b) Thévenin's equivalent impedance to the point of fault.

terminal voltage, i.e.,

$$V_{th} = \frac{220}{220} = 1\angle 0^\circ \text{ pu}$$

and the Thévenin's impedance is

$$Z_{th} = j(0.25 + 0.2) = j0.45$$

The fault contribution is

$$I'_f = \frac{V_{th}}{Z_{th}} = \frac{1.0\angle 0^\circ}{j0.45} = -j2.222 \text{ pu}$$

Now superimposing the load current with the fault current results in

$$I'_g = I'_f + I_L = -j2.222 + (0.8 - j0.6) = 0.8 - j2.822 = 2.93\angle -74.17^\circ \text{ pu}$$

which checks with the result in Example 8.9.

PROBLEMS

8.1. A sinusoidal voltage given by $v(t) = 390\sin(315t + \alpha)$ is suddenly applied to a series RL circuit. $R = 32\ \Omega$ and $L = 0.4$ H.
(a) The switch is closed at such a time as to permit no transient current. What value of α corresponds to this instant of closing the switch? Obtain the instantaneous expression for $i(t)$. Use *MATLAB* to plot $i(t)$ up to 80 ms in steps of 0.01 ms.
(b) The switch is closed at such a time as to permit maximum transient current. What value of α corresponds to this instant of closing the switch? Obtain the instantaneous expression for $i(t)$. Use *MATLAB* to plot $i(t)$ up to 80 ms in steps of 0.01 ms.
(c) What is the maximum value of current in part (b) and at what time does this occur after the switch is closed?

8.2. Consider the synchronous generator in Example 8.2. A three-phase short circuit is applied at the instant when the rotor direct axis position is at $\delta = 30°$. Use **ode45** to simulate (8.36), and obtain and plot the transient waveforms for the current in each phase and the field current.

8.3. Consider the synchronous generator in Example 8.2. A line-to-line short circuit occurs between phases b and c at the instant when the rotor direct axis position is at $\delta = 30°$. Use **ode45** to simulate (8.46), and obtain the transient waveforms for the current in phase b and the field current.

8.4. Consider a line-to-ground short circuit between phase a and ground in a synchronous generator. Apply the short circuit conditions

$$v_a = 0$$
$$i_b = i_c = 0$$

to the voltage equation of the synchronous machine given by (8.4). Substitute for all the flux linkages in terms of the inductances given by (8.9)–(8.13) and verify Equation (8.47).

8.5. Consider the synchronous generator in Example 8.2. A line-to-ground short circuit occurs between phase a and ground at the instant when the rotor direct axis position is at $\delta = 30°$. Use **ode45** to simulate (8.47), and obtain the transient waveforms for the current in phase a and the field current.

8.6. A three-phase, 60-Hz synchronous machine is driven at constant synchronous speed by a prime mover. The armature windings are initially open-circuited and field voltage is adjusted so that the armature terminal voltage is at the

rated value (i.e., 1.0 per unit). The machine has the following per unit reactances and time constants.

$$X''_d = 0.25 \text{ pu} \qquad \tau''_d = 0.04 \text{ sec}$$
$$X'_d = 0.45 \text{ pu} \qquad \tau'_d = 1.4 \text{ sec}$$
$$X_d = 1.50 \text{ pu}$$

(a) Determine the steady state, transient, and subtransient short circuit currents.
(b) Obtain and plot the fundamental-frequency waveform of the armature current for a three-phase short circuit at the terminals of the generator. Assume the short circuit is applied at the instant when the rotor direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$.

8.7. For the machine in Problem 8.6, assume that a three-phase short circuit is applied at the instant when the rotor quadrature axis is along the magnetic axis of phase a, i.e., $\delta = \pi/2$ radians. Obtain and plot the asymmetrical waveform of the armature current for phase a. The armature short circuit time constant is $\tau_a = 0.3$ sec.

8.8. A three-phase, 60-Hz synchronous machine is driven at constant synchronous speed by a prime mover. The armature windings are initially open-circuited and field voltage is adjusted so that the armature terminal voltage is at the rated value (i.e., 1.0 per unit). The generator is suddenly subjected to a symmetrical three-phase short circuit at the instant when direct axis is along the magnetic axis of phase a, i.e., $\delta = 0$. An oscillogram of the short-circuited current is obtained. The peak values at the first two cycles, at the 20th and 21st cycles, and the steady value after a long time were recorded as tabulated in the following table.

I_{max}, pu	5.4016	4.6037	$\cdots$	2.6930	2.6720	$\cdots$	0.9445
Time, sec	0.0042	0.0208	$\cdots$	0.3208	0.3375	$\cdots$	10.004

Determine the transient and the subtransient reactances and time constants.

8.9. A 100-MVA, three-phase, 60-Hz generator driven at constant speed has the following per unit reactances and time constants

$$X''_d = 0.20 \text{ pu} \qquad \tau''_d = 0.04 \text{ sec}$$
$$X'_d = 0.30 \text{ pu} \qquad \tau'_d = 1.0 \text{ sec}$$
$$X_d = 1.20 \text{ pu} \qquad \tau_a = 0.25 \text{ sec}$$

The armature windings are initially open-circuited and field voltage is adjusted so that the armature terminal voltage is at the rated value (i.e., 1.0

per unit). The generator is suddenly subjected to a symmetrical three-phase short circuit at the instant when $\delta = \pi/2$. Obtain and plot the asymmetrical waveform of the armature current for phase a. Determine

(a) The rms value of the ac component in phase a at $t = 0$.
(b) The dc component of the current in phase a at $t = 0$.
(c) The rms value of the asymmetrical current in phase a at $t = 0$.

8.10. A 100-MVA, 20-kV, 60-Hz three-phase synchronous generator is connected to a 100-MVA, 20/400 kV three-phase transformer. The machine has the following per unit reactances and time constants.

$$X''_d = 0.15 \text{ pu} \qquad \tau''_d = 0.035 \text{ sec}$$
$$X'_d = 0.25 \text{ pu} \qquad \tau'_d = 0.50 \text{ sec}$$
$$X_d = 1.25 \text{ pu} \qquad \tau_a = 0.3 \text{ sec}$$

The transformer reactance is 0.25 per unit. The generator is operating at the rated voltage and no-load when a three-phase short circuit occurs at the secondary terminals of the transformer.
(a) Find the subtransient, transient, and the steady state short circuit currents in per unit and actual amperes on both sides of the transformer.
(b) What is the maximum asymmetrical rms current (ac plus dc) at the beginning of the short circuit?
(c) Obtain and plot the instantaneous expression for the short circuit current including the dc component. Assume $\delta = \pi/2$ radians.

8.11. In Problem 8.10, a three-phase load of 80-MVA, 0.8 power factor lagging is connected to the transformer secondary side. The line-to-line voltage at the load terminals is 400 kV. A three-phase short circuit occurs at the load terminals. Find the generator transient current including the load current.

8.12. A 100-MVA, 20-kV synchronous generator is connected through a transmission line to a 100-MVA, 20-kV synchronous motor. The per unit transient reactances of the generator and motor are 0.25 and 0.20, respectively. The line reactance on the base of 100 MVA is 0.1 per unit. The motor is taking 50 MW at 0.8 power factor leading at a terminal voltage of 20 kV. A three-phase short circuit occurs at the generator terminals. Determine the transient currents in each of the two machines and in the short circuit.

CHAPTER 9

BALANCED FAULT

9.1 INTRODUCTION

Fault studies form an important part of power system analysis. The problem consists of determining bus voltages and line currents during various types of faults. Faults on power systems are divided into *three-phase balanced faults* and *unbalanced faults*. Different types of unbalanced faults are *single line-to-ground fault*, *line-to-line fault*, and *double line-to-ground fault*, which are dealt with in Chapter 10. The information gained from fault studies are used for proper relay setting and coordination. The three-phase balanced fault information is used to select and set phase relays, while the line-to-ground fault is used for ground relays. Fault studies are also used to obtain the rating of the protective switchgears.

The magnitude of the fault currents depends on the internal impedance of the generators plus the impedance of the intervening circuit. We have seen in Chapter 8 that the reactance of a generator under short circuit condition is not constant. For the purpose of fault studies, the generator behavior can be divided into three periods: the *subtransient period*, lasting only for the first few cycles; the *transient period*, covering a relatively longer time; and, finally, the *steady state period*. In this chapter, three-phase balanced faults are discussed. The bus impedance matrix by the *building algorithm* is formulated and is employed for the systematic computation of bus voltages and line currents during the fault. Two functions are developed for the formation of the bus impedance matrix. These function are **Zbus = zbuild(zdata)** and **Zbus = zbuildpi(linedata, gendata, yload)**. The latter one is compatible with power flow input/output files. A program named **symfault** is developed for systematic computation of three-phase balanced faults for a large interconnected power system.

9.2 BALANCED THREE-PHASE FAULT

This type of fault is defined as the simultaneous short circuit across all three phases. It occurs infrequently, but it is the most severe type of fault encountered. Because the network is balanced, it is solved on a per-phase basis. The other two phases carry identical currents except for the phase shift.

In Chapter 8 it was shown that the reactance of the synchronous generator under short-circuit conditions is a time-varying quantity, and for network analysis three reactances were defined. The subtransient reactance X_d'', for the first few cycles of the short circuit current, transient reactance X_d', for the next (say) 30 cycles, and the synchronous reactance X_d, thereafter. Since the duration of the short circuit current depends on the time of operation of the protective system, it is not always easy to decide which reactance to use. Generally, the subtransient reactance is used for determining the interrupting capacity of the circuit breakers. In fault studies required for relay setting and coordination, transient reactance is used. Also, in typical transient stability studies, transient reactance is used.

A fault represents a structural network change equivalent with that caused by the addition of an impedance at the place of fault. If the fault impedance is zero, the fault is referred to as the *bolted fault* or the *solid fault*. The faulted network can be solved conveniently by the Thévenin's method. The procedure is demonstrated in the following example.

Example 9.1

The one-line diagram of a simple three-bus power system is shown in Figure 9.1. Each generator is represented by an emf behind the transient reactance. All impedances are expressed in per unit on a common 100 MVA base, and for simplicity, resistances are neglected. The following assumptions are made.

(*i*) Shunt capacitances are neglected and the system is considered on no-load.

(*ii*) All generators are running at their rated voltage and rated frequency with their emfs in phase.

Determine the fault current, the bus voltages, and the line currents during the fault when a balanced three-phase fault with a fault impedance $Z_f = 0.16$ per unit occurs on

(a) Bus 3.
(b) Bus 2.
(c) Bus 1.

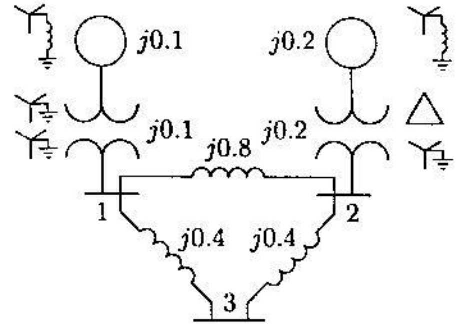


FIGURE 9.1
The impedance diagram of a simple power system.

The fault is simulated by switching on an impedance Z_f at bus 3 as shown in Figure 9.2(a). Thévenin's theorem states that the changes in the network voltage caused by the added branch (the fault impedance) shown in Figure 9.2(a) is equivalent to those caused by the added voltage $V_3(0)$ with all other sources short-circuited as shown in Figure 9.2(b).

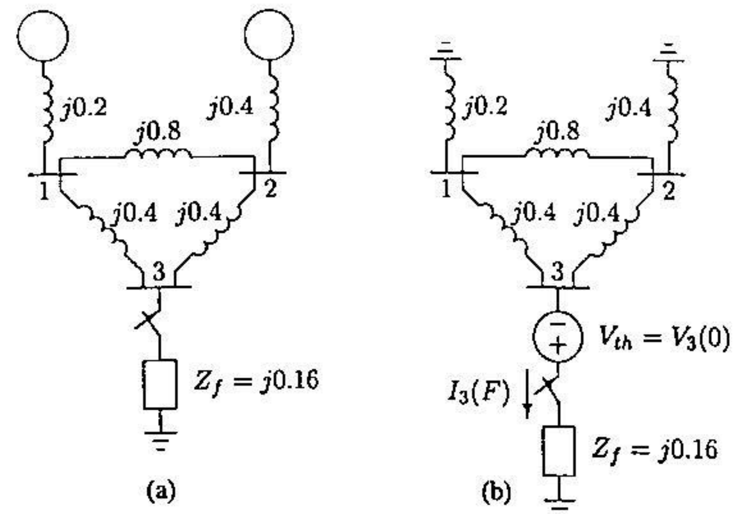


FIGURE 9.2
(a) The impedance network for fault at bus 3. (b) Thévenin's equivalent network.

(a) From 9.2(b), the fault current at bus 3 is

$$I_3(F) = \frac{V_3(0)}{Z_{33} + Z_f}$$

where $V_3(0)$ is the Thévenin's voltage or the prefault bus voltage. The prefault bus voltage can be obtained from the results of the power flow solution. In this example, since the loads are neglected and generator's emfs are assumed equal to the rated value, all the prefault bus voltages are equal to 1.0 per unit, i.e.,

$$V_1(0) = V_2(0) = V_3(0) = 1.0 \quad \text{pu}$$

Z_{33} is the Thévenin's impedance viewed from the faulted bus.

To find the Thévenin's impedance, we convert the Δ formed by buses 123 to an equivalent Y as shown in Figure 9.3(a).

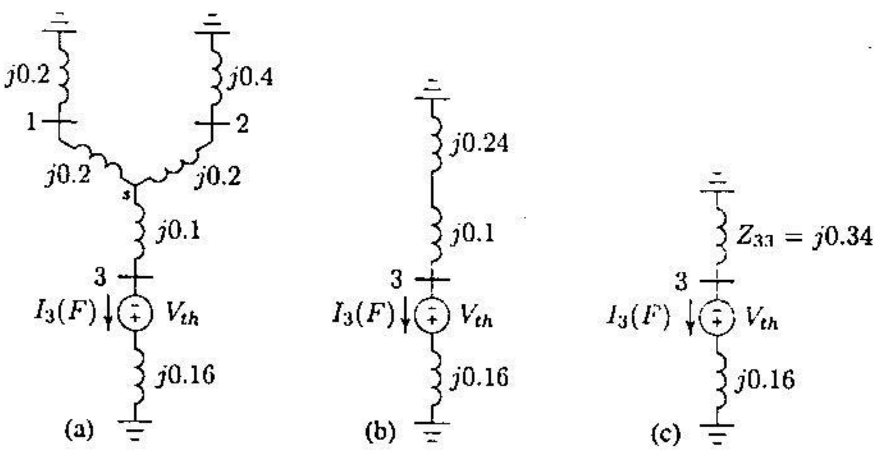


FIGURE 9.3
Reduction of Thévenin's equivalent network.

$$Z_{1s} = Z_{2s} = \frac{(j0.4)(j0.8)}{j1.6} = j0.2 \qquad Z_{3s} = \frac{(j0.4)(j0.4)}{j1.6} = j0.1$$

Combining the parallel branches, Thévenin's impedance is

$$\begin{aligned} Z_{33} &= \frac{(j0.4)(j0.6)}{j0.4 + j0.6} + j0.1 \\ &= j0.24 + j0.1 = j0.34 \end{aligned}$$

From Figure 9.3(c), the fault current is

$$I_3(F) = \frac{V_3(F)}{Z_{33} + Z_f} = \frac{1.0}{j0.34 + j0.16} = -j2.0 \quad \text{pu}$$

With reference to Figure 9.3(a), the current divisions between the two generators are

$$I_{G1} = \frac{j0.6}{j0.4 + j0.6} I_3(F) = -j1.2 \quad \text{pu}$$
$$I_{G2} = \frac{j0.4}{j0.4 + j0.6} I_3(F) = -j0.8 \quad \text{pu}$$

For the bus voltage changes from Figure 9.3(b), we get

$$\begin{aligned} \Delta V_1 &= 0 - (j0.2)(-j1.2) = -0.24 \quad \text{pu} \\ \Delta V_2 &= 0 - (j0.4)(-j0.8) = -0.32 \quad \text{pu} \\ \Delta V_3 &= (j0.16)(-j2) - 1.0 = -0.68 \quad \text{pu} \end{aligned}$$

The bus voltages during the fault are obtained by superposition of the prefault bus voltages and the changes in the bus voltages caused by the equivalent emf connected to the faulted bus, as shown in Figure 9.2(b), i.e.,

$$\begin{aligned} V_1(F) &= V_1(0) + \Delta V_1 = 1.0 - 0.24 = 0.76 \quad \text{pu} \\ V_2(F) &= V_2(0) + \Delta V_2 = 1.0 - 0.32 = 0.68 \quad \text{pu} \\ V_3(F) &= V_3(0) + \Delta V_3 = 1.0 - 0.68 = 0.32 \quad \text{pu} \end{aligned}$$

The short circuit-currents in the lines are

$$I_{12}(F) = \frac{V_1(F) - V_2(F)}{z_{12}} = \frac{0.76 - 0.68}{j0.8} = -j0.1 \quad \text{pu}$$
$$I_{13}(F) = \frac{V_1(F) - V_3(F)}{z_{13}} = \frac{0.76 - 0.32}{j0.4} = -j1.1 \quad \text{pu}$$
$$I_{23}(F) = \frac{V_2(F) - V_3(F)}{z_{23}} = \frac{0.68 - 0.32}{j0.4} = -j0.9 \quad \text{pu}$$

(b) The fault with impedance Z_f at bus 2 is depicted in Figure 9.4(a), and its Thévenin's equivalent circuit is shown in Figure 9.4(b). To find the Thévenin's impedance, we combine the parallel branches in Figure 9.4(b). Also, combining parallel branches from ground to bus 2 in Figure 9.5(a), results in

$$Z_{22} = \frac{(j0.6)(j0.4)}{j0.6 + j0.4} = j0.24$$

From Figure 9.5(b), the fault current is

$$I_2(F) = \frac{V_2(0)}{Z_{22} + Z_f} = \frac{1.0}{j0.24 + j0.16} = -j2.5 \quad \text{pu}$$

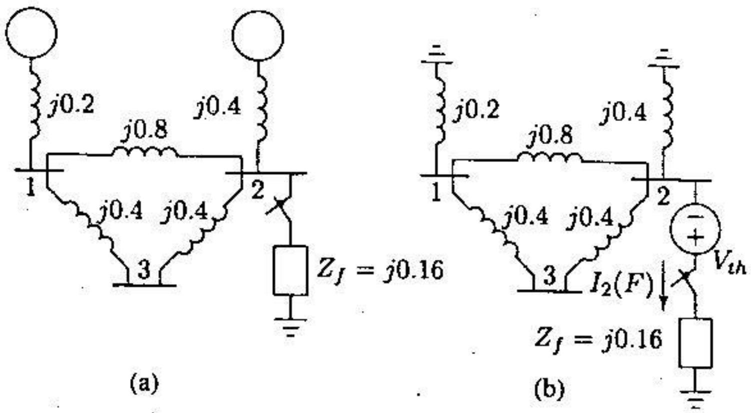


FIGURE 9.4
(a) The impedance network for fault at bus 2. (b) Thévenin's equivalent network.

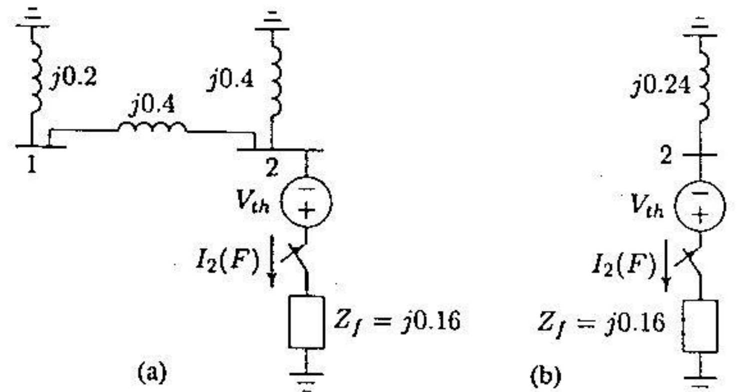


FIGURE 9.5
Reduction of Thévenin's equivalent network.

With reference to Figure 9.5(a), the current divisions between the generators are

$$I_{G1} = \frac{j0.4}{j0.4 + j0.6} I_2(F) = -j1.0 \quad \text{pu}$$
$$I_{G2} = \frac{j0.6}{j0.4 + j0.6} I_2(F) = -j1.5 \quad \text{pu}$$

For the bus voltage changes from Figure 9.4(a), we get

$$\Delta V_1 = 0 - (j0.2)(-j1.0) = -0.2 \quad \text{pu}$$
$$\Delta V_2 = 0 - (j0.4)(-j1.5) = -0.6 \quad \text{pu}$$
$$\Delta V_3 = -0.2 - (j0.4)(\frac{-j1.0}{2}) = -0.4 \quad \text{pu}$$

The bus voltages during the fault are obtained by superposition of the prefault bus voltages and the changes in the bus voltages caused by the equivalent emf connected to the faulted bus, as shown in Figure 9.4(b), i.e.,

$$V_1(F) = V_1(0) + \Delta V_1 = 1.0 - 0.2 = 0.8 \quad \text{pu}$$
$$V_2(F) = V_2(0) + \Delta V_2 = 1.0 - 0.6 = 0.4 \quad \text{pu}$$
$$V_3(F) = V_3(0) + \Delta V_3 = 1.0 - 0.4 = 0.6 \quad \text{pu}$$

The short circuit-currents in the lines are

$$I_{12}(F) = \frac{V_1(F) - V_2(F)}{z_{12}} = \frac{0.8 - 0.4}{j0.8} = -j0.5 \quad \text{pu}$$
$$I_{13}(F) = \frac{V_1(F) - V_3(F)}{z_{13}} = \frac{0.8 - 0.6}{j0.4} = -j0.5 \quad \text{pu}$$
$$I_{32}(F) = \frac{V_3(F) - V_2(F)}{z_{32}} = \frac{0.6 - 0.4}{j0.4} = -j0.5 \quad \text{pu}$$

(c) The fault with impedance Z_f at bus 1 is depicted in Figure 9.6(a), and its Thévenin's equivalent circuit is shown in Figure 9.6(b).

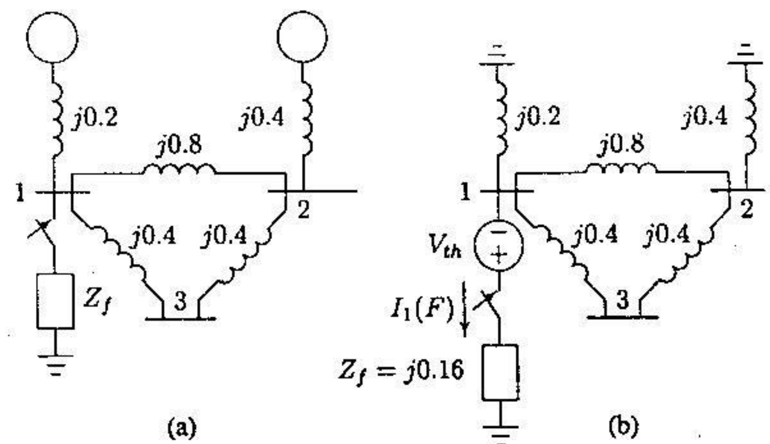


FIGURE 9.6
(a) The impedance network for fault at bus 1. (b) Thévenin's equivalent network.

To find the Thévenin's impedance, we combine the parallel branches in Figure 9.6(b). Also, combining parallel branches from ground to bus 1 in Figure 9.7(a),

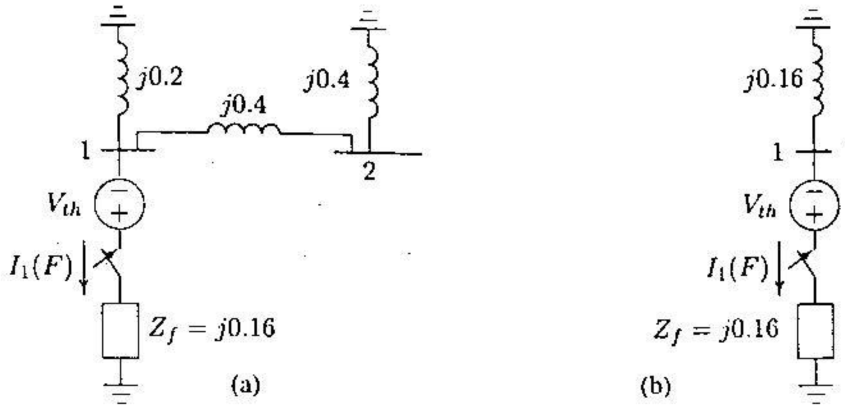


FIGURE 9.7
Reduction of Thévenin's equivalent network.

results in

$$Z_{11} = \frac{(j0.2)(j0.8)}{j0.2 + j0.8} = j0.16$$

From Figure 9.7(b), the fault current is

$$I_1(F) = \frac{V_1(0)}{Z_{11} + Z_f} = \frac{1.0}{j0.16 + j0.16} = -j3.125 \quad \text{pu}$$

With reference to Figure 9.7(a), the current divisions between the two generators are

$$I_{G1} = \frac{j0.8}{j0.2 + j0.8} I_2(F) = -j2.50 \quad \text{pu}$$
$$I_{G2} = \frac{j0.2}{j0.2 + j0.8} I_2(F) = -j0.625 \quad \text{pu}$$

For the bus voltage changes from Figure 9.6(b), we get

$$\Delta V_1 = 0 - (j0.2)(-j2.5) = -0.50 \quad \text{pu}$$
$$\Delta V_2 = 0 - (j0.4)(-j0.625) = -0.25 \quad \text{pu}$$
$$\Delta V_3 = -0.5 + (j0.4)(\frac{-j0.625}{2}) = -0.375 \quad \text{pu}$$

Bus voltages during the fault are obtained by superposition of the prefault bus voltages and the changes in the bus voltages caused by the equivalent emf connected

to the faulted bus, as shown in Figure 9.6(b), i.e.,

$$V_1(F) = V_1(0) + \Delta V_1 = 1.0 - 0.50 = 0.50 \quad \text{pu}$$
$$V_2(F) = V_2(0) + \Delta V_2 = 1.0 - 0.25 = 0.75 \quad \text{pu}$$
$$V_3(F) = V_3(0) + \Delta V_3 = 1.0 - 0.375 = 0.625 \quad \text{pu}$$

The short-circuit currents in the lines are

$$I_{21}(F) = \frac{V_2(F) - V_1(F)}{z_{21}} = \frac{0.75 - 0.5}{j0.8} = -j0.3125 \quad \text{pu}$$
$$I_{31}(F) = \frac{V_3(F) - V_1(F)}{z_{31}} = \frac{0.625 - 0.5}{j0.4} = -j0.3125 \quad \text{pu}$$
$$I_{23}(F) = \frac{V_2(F) - V_3(F)}{z_{23}} = \frac{0.75 - 0.625}{j0.4} = -j0.3125 \quad \text{pu}$$

In the above example the load currents were neglected and all prefault bus voltages were assumed to be equal to 1.0 per unit. For more accurate calculation, the prefault bus voltages can be obtained from the power flow solution. As we have seen in Chapter 6, in a power system, loads are specified and the load currents are unknown. One way to include the effects of load currents in the fault analysis is to express the loads by a constant impedance evaluated at the prefault bus voltages. This is a very good approximation which results in linear nodal equations. The procedure is summarized in the following steps.

- The prefault bus voltages are obtained from the results of the power flow solution.
- In order to preserve the linearity feature of the network, loads are converted to constant admittances using the prefault bus voltages.
- The faulted network is reduced into a Thévenin's equivalent circuit as viewed from the faulted bus. Applying Thévenin's theorem, changes in the bus voltages are obtained.
- Bus voltages during the fault are obtained by superposition of the prefault bus voltages and the changes in the bus voltages computed in the previous step.
- The currents during the fault in all branches of the network are then obtained.

9.3 SHORT-CIRCUIT CAPACITY (SCC)

The short-circuit capacity at a bus is a common measure of the strength of a bus. The short-circuit capacity or the short-circuit MVA at bus k is defined as the product of the magnitudes of the rated bus voltage and the fault current. The short-circuit MVA is used for determining the dimension of a bus bar, and the *interrupting* capacity of a circuit breaker. The interrupting capacity is only one of many ratings of a circuit breaker and should not be confused with the *momentary duty* of the breaker described in (8.63).

Based on the above definition, the short-circuit capacity or the short-circuit MVA at bus k is given by

$$SCC = \sqrt{3}\,V_{Lk} I_k(F) \times 10^{-3} \quad \text{MVA} \tag{9.1}$$

where the line-to-line voltage V_{Lk} is expressed in kilovolts and $I_k(F)$ is expressed in amperes. The symmetrical three-phase fault current in per unit is given by

$$I_k(F)_{pu} = \frac{V_k(0)}{X_{kk}} \tag{9.2}$$

where $V_k(0)$ is the per unit prefault bus voltage, and X_{kk} is the per unit reactance to the point of fault. System resistance is neglected and only the inductive reactance of the system is allowed for. This gives minimum system impedance and maximum fault current and a pessimistic answer. The base current is given by

$$I_B = \frac{S_B \times 10^3}{\sqrt{3}\,V_B} \tag{9.3}$$

where S_B is the base MVA and V_B is the line-to-line base voltage in kilovolts. Thus, the fault current in amperes is

$$\begin{aligned} I_k(F) &= I_k(F)_{pu} I_B \\ &= \frac{V_k(0)}{X_{kk}} \frac{S_B \times 10^3}{\sqrt{3}\,V_B} \end{aligned} \tag{9.4}$$

Substituting for $I_k(F)$ from (9.4) into (9.1) results in

$$SCC = \frac{V_k(0) S_B}{X_{kk}} \frac{V_L}{V_B} \tag{9.5}$$

If the base voltage is equal to the rated voltage, i.e., $V_L = V_B$

$$SCC = \frac{V_k(0) S_B}{X_{kk}} \tag{9.6}$$

The prefault bus voltage is usually assumed to be 1.0 per unit, and we therefore obtain from (9.6) the following approximate formula for the short-circuit capacity or the short-circuit MVA.

$$SCC = \frac{S_B}{X_{kk}} \quad \text{MVA} \tag{9.7}$$

9.4 SYSTEMATIC FAULT ANALYSIS USING BUS IMPEDANCE MATRIX

The network reduction used in the preceding example is not efficient and is not applicable to large networks. In this section a more general fault circuit analysis using nodal method is obtained. We see that by utilizing the elements of the bus impedance matrix, the fault current as well as the bus voltages during fault are readily and easily calculated.

Consider a typical bus of an n-bus power system network as shown in Figure 9.8. The system is assumed to be operating under balanced condition and a per phase circuit model is used. Each machine is represented by a constant voltage source behind proper reactances which may be X_d'', X_d', or X_d. Transmission lines are represented by their equivalent π model and all impedances are expressed in per unit on a common MVA base. A balanced three-phase fault is to be applied at bus k through a fault impedance Z_f. The prefault bus voltages are obtained from the power flow solution and are represented by the column vector

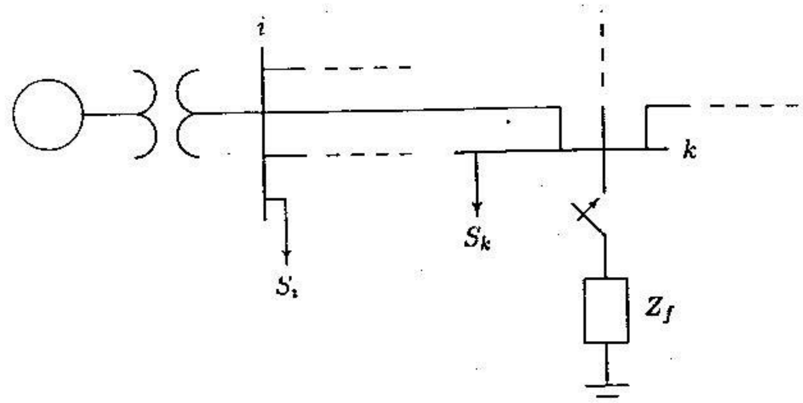


FIGURE 9.8
A typical bus of a power system.

$$\mathbf{V}_{bus}(0) = \begin{bmatrix} V_1(0) \\ \vdots \\ V_k(0) \\ \vdots \\ V_n(0) \end{bmatrix} \tag{9.8}$$

As already mentioned, short circuit currents are so much larger than the steady-state values that we may neglect the latter. However, a good approximation is to represent the bus load by a constant impedance evaluated at the prefault bus voltage, i.e.,

$$Z_{iL} = \frac{|V_i(0)|^2}{S_L^*} \tag{9.9}$$

The changes in the network voltage caused by the fault with impedance Z_f is equivalent to those caused by the added voltage $V_k(0)$ with all other sources short-circuited. Zeroing all voltage sources and representing all components and loads by their appropriate impedances, we obtain the Thévenin's circuit shown in Figure 9.9. The bus voltage changes caused by the fault in this circuit are represented by the column vector

$$\Delta\mathbf{V}_{bus} = \begin{bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_k \\ \vdots \\ \Delta V_n \end{bmatrix} \tag{9.10}$$

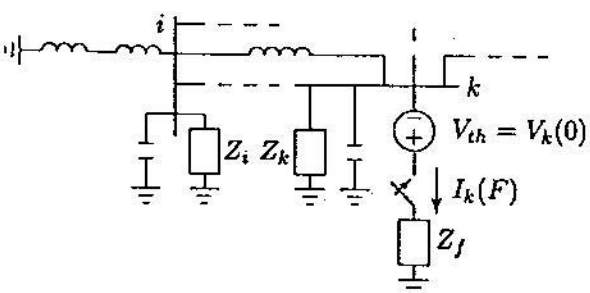


FIGURE 9.9
A typical bus of a power system.

From Thévenin's theorem bus voltages during the fault are obtained by superposition of the prefault bus voltages and the changes in the bus voltages given by

$$\mathbf{V}_{bus}(F) = \mathbf{V}_{bus}(0) + \Delta\mathbf{V}_{bus} \tag{9.11}$$

In Section 6.2, we obtained the node-voltage equation for an n-bus network. The injected bus currents are expressed in terms of the bus voltages (with bus 0 as reference), i.e.,

$$\mathbf{I}_{bus} = \mathbf{Y}_{bus}\mathbf{V}_{bus} \tag{9.12}$$

where $\mathbf{I}_{bus}$ is the bus current vector entering the bus and $\mathbf{Y}_{bus}$ is the bus admittance matrix. The diagonal element of each bus is the sum of admittances connected to it, i.e.,

$$Y_{ii} = \sum_{j=0}^{m} y_{ij} \qquad j \neq i \tag{9.13}$$

The off-diagonal element is equal to the negative of the admittance between the buses, i.e.,

$$Y_{ij} = Y_{ji} = -y_{ij} \tag{9.14}$$

where y_{ij} (lower case) is the actual admittance of the line i-j. For more details refer to Section 6.2.

In the Thévenin's circuit of Figure 9.9, current entering every bus is zero except at the faulted bus. Since the current at faulted bus is leaving the bus, it is taken as a negative current entering bus k. Thus the nodal equation applied to the Thévenin's circuit in Figure 9.9 becomes

$$\begin{bmatrix} 0 \\ \vdots \\ -I_k(F) \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_{11} & \cdots & y_{1k} & \cdots & y_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{k1} & \cdots & y_{kk} & \cdots & y_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{n1} & \cdots & y_{nk} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_k \\ \vdots \\ \Delta V_n \end{bmatrix} \tag{9.15}$$

or

$$\mathbf{I}_{bus}(F) = \mathbf{Y}_{bus}\Delta\mathbf{V}_{bus} \tag{9.16}$$

Solving for $\Delta\mathbf{V}_{bus}$, we have

$$\Delta\mathbf{V}_{bus} = \mathbf{Z}_{bus}\mathbf{I}_{bus}(F) \tag{9.17}$$

where $\mathbf{Z}_{bus} = \mathbf{Y}_{bus}^{-1}$ is known as the *bus impedance matrix*. Substituting (9.17) into (9.11), the bus voltage vector during the fault becomes

$$\mathbf{V}_{bus(F)} = \mathbf{V}_{bus}(0) + \mathbf{Z}_{bus}\mathbf{I}_{bus}(F) \tag{9.18}$$

Writing the above matrix equation in terms of its elements, we have

$$\begin{bmatrix} V_1(F) \\ \vdots \\ V_k(F) \\ \vdots \\ V_n(F) \end{bmatrix} = \begin{bmatrix} V_1(0) \\ \vdots \\ V_k(0) \\ \vdots \\ V_n(0) \end{bmatrix} + \begin{bmatrix} Z_{11} & \cdots & Z_{1k} & \cdots & Z_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{k1} & \cdots & Z_{kk} & \cdots & Z_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{n1} & \cdots & Z_{nk} & \cdots & Z_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ -I_k(F) \\ \vdots \\ 0 \end{bmatrix} \tag{9.19}$$

Since we have only one single nonzero element in the current vector, the kth equation in (9.19) becomes

$$V_k(F) = V_k(0) - Z_{kk}I_k(F) \tag{9.20}$$

Also from the Thévenin's circuit shown in Figure 9.9, we have

$$V_k(F) = Z_f I_k(F) \tag{9.21}$$

For bolted fault, $Z_f = 0$ and $V_k(F) = 0$. Substituting for $V_k(F)$ from (9.21) into (9.20) and solving for the fault current, we get

$$I_k(F) = \frac{V_k(0)}{Z_{kk} + Z_f} \tag{9.22}$$

Thus for a fault at bus k we need only the Z_{kk} element of the bus impedance matrix. This element is indeed the Thévenin's impedance as viewed from the faulted bus. Also, writing the ith equation in (9.19) in terms of its element, we have

$$V_i(F) = V_i(0) - Z_{ik}I_k(F) \tag{9.23}$$

Substituting for $I_k(F)$, bus voltage during the fault at bus i becomes

$$V_i(F) = V_i(0) - \frac{Z_{ik}}{Z_{kk} + Z_f}V_k(0) \tag{9.24}$$

With the knowledge of bus voltages during the fault, we can calculate the fault current in all the lines. For the line connecting buses i and j with impedance z_{ij}, the short circuit current in this line (defined positive in the direction $i \rightarrow j$) is

$$I_{ij}(F) = \frac{V_i(F) - V_j(F)}{z_{ij}} \tag{9.25}$$

We note that with the knowledge of the bus impedance matrix, the fault current and bus voltages during the fault are readily obtained for any faulted bus in the network. This method is very simple and practical. Thus, all fault calculations are formulated in the bus frame of reference using bus impedance matrix Z_{bus}.

One way to find $\mathbf{Z}_{bus}$ is to formulate Y_{bus} matrix for the system and then find its inverse. The matrix inversion for a large power system with a large number of buses is not feasible. A computationally attractive and efficient method for finding Z_{bus} matrix is "building" or "assembling" the impedance matrix by adding one network element at a time. In effect, this is an indirect matrix inversion of the bus admittance matrix. The algorithm for building the bus impedance matrix is described in the next section.

Example 9.2

A three-phase fault with a fault impedance $Z_f = j0.16$ per unit occurs at bus 3 in the network of Example 9.1. Using the bus impedance matrix method, compute the fault current, the bus voltages, and the line currents during the fault.

In this example the bus impedance matrix is obtained by finding the inverse of the bus admittance matrix. In the next section, we describe an efficient method of finding the bus impedance matrix by the method of building algorithm.

To find the bus admittance matrix, the Thévenin's circuit in Figure 9.2(b) is redrawn with impedances converted to admittances as shown in Figure 9.10. The ith diagonal element of the bus admittance matrix is the sum of all admittances connected to bus i, and the ijth off-diagonal element is the negative of the admittance between buses i and j. Referring to Figure 9.10, the bus admittance matrix by inspection is

$$\mathbf{Y}_{bus} = \begin{bmatrix} -j8.75 & j1.25 & j2.5 \\ j1.25 & -j6.25 & j2.5 \\ j2.5 & j2.5 & -j5.0 \end{bmatrix}$$

Using *MATLAB* inverse function **inv**, the bus impedance matrix is obtained

$$\mathbf{Z}_{bus} = \begin{bmatrix} j0.16 & j0.08 & j0.12 \\ j0.08 & j0.24 & j0.16 \\ j0.12 & j0.16 & j0.34 \end{bmatrix}$$

From (9.22), for a fault at bus 3 with fault impedance $Z_f = j0.16$ per unit, the fault current is

$$I_3(F) = \frac{V_3(0)}{Z_{33} + Z_f} = \frac{1.0}{j0.34 + j0.16} = -j2.0 \quad \text{pu}$$

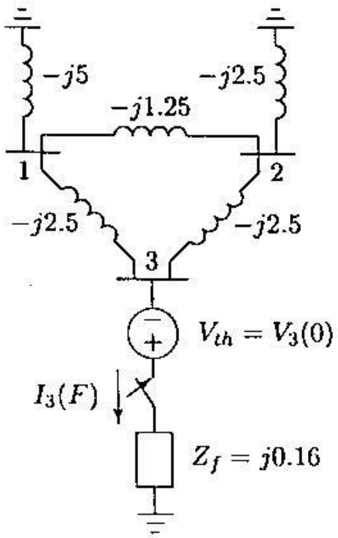


FIGURE 9.10
The admittance diagram for system of Figure 9.2 (b).

From (9.23), bus voltages during the fault are

$$V_1(F) = V_1(0) - Z_{13}I_3(F) = 1.0 - (j0.12)(-j2.0) = 0.76 \quad \text{pu}$$
$$V_2(F) = V_2(0) - Z_{23}I_3(F) = 1.0 - (j0.16)(-j2.0) = 0.68 \quad \text{pu}$$
$$V_3(F) = V_3(0) - Z_{33}I_3(F) = 1.0 - (j0.34)(-j2.0) = 0.32 \quad \text{pu}$$

From (9.25), the short circuit currents in the lines are

$$I_{12}(F) = \frac{V_1(F) - V_2(F)}{z_{12}} = \frac{0.76 - 0.68}{j0.8} = -j0.1 \quad \text{pu}$$
$$I_{13}(F) = \frac{V_1(F) - V_3(F)}{z_{13}} = \frac{0.76 - 0.32}{j0.4} = -j1.1 \quad \text{pu}$$
$$I_{23}(F) = \frac{V_2(F) - V_3(F)}{z_{23}} = \frac{0.68 - 0.32}{j0.4} = -j0.9 \quad \text{pu}$$

The results are exactly the same as the values found in Example 9.1(a). The reader is encouraged to repeat the above calculations for fault at buses 2 and 1, and compare the results with those obtained from parts (b) and (c) in Example 9.1.

Note that the values of the diagonal elements in the bus impedance matrix are the same as the Thévenin's impedances found in Example 9.1, thus eliminating

the repeated need for network reduction for each fault location. Furthermore, the off-diagonal elements are utilized in (9.24) to obtain bus voltages during the fault. Therefore, the bus impedance matrix method is an indispensable tool for fault studies.

9.5 ALGORITHM FOR FORMATION OF THE BUS IMPEDANCE MATRIX

Before we present the building algorithm for the bus impedance matrix, a few definitions from the discipline of the graph theory are introduced. The *graph* of a network describes the geometrical structure of the network. The graph consists of redrawing the network, with a line representing each element of the network. The graph of the network for Figure 9.2(a) before the fault application is shown in Figure 9.11(a). The buses are represented by *nodes* or *vertices* and impedances by

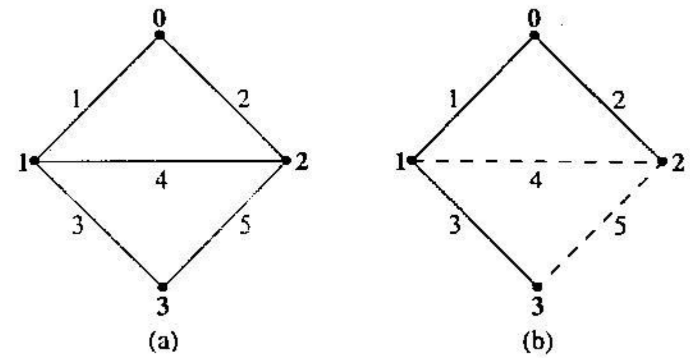


FIGURE 9.11
Graph, a selected tree, and a cotree for the network of Figure 9.2(b).

line segments called *elements* or *edges*. A *tree* of a connected graph is a connected subgraph connecting all the nodes without forming a loop. The elements of a tree are called *branches*. In general, a graph contains multiple trees. The number of branches in any selected tree denoted by b is always one less than the nodes, i.e.,

$$b = n - 1 \tag{9.26}$$

where n is the number of nodes including the reference node 0. Once a tree for a graph has been defined, the remaining elements are referred to as *links*. The collection of links is called a *cotree*. If e is the total number of elements in a graph, the number of links in a cotree is

$$l = e - b = e - n + 1 \tag{9.27}$$

A loop that contains one link is called a *basic loop*. The number of basic loops is unique; it equals the number of links and is the number of independent loop equations. A *cut set* is a minimal set of branches that, when cut, divides the graph into two connected subgraphs. A *fundamental cut set* is a cut set that contains only one branch. The number of fundamental cut sets is unique; it equals the number of branches and is the number of independent node equations. Figure 9.11(b) shows a tree of a graph with the tree branches highlighted by heavy lines and the cotree links by dashed lines.

The bus impedance matrix can be built up starting with a single element and the process is continued until all nodes and elements are included. Let us assume that $\mathbf{Z}_{bus}$ matrix exists for a partial network having m buses and a reference bus 0 as shown in Figure 9.12.

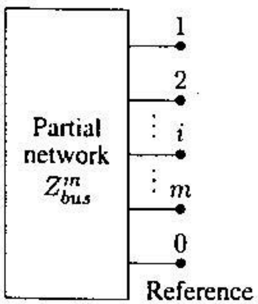


FIGURE 9.12
Partial network.

The corresponding network equation for this partial network is

$$\mathbf{V}_{bus} = \mathbf{Z}_{bus}\mathbf{I}_{bus} \tag{9.28}$$

For an n-bus system, m buses are included in the network and $\mathbf{Z}_{bus}$ is of order $m \times m$. We shall add one element at a time from the remaining portion of the network until all elements are included. The added element may be a branch or a link described as follows.

ADDITION OF A BRANCH

When the added element is a branch, a new bus is added to the partial network creating a new row and a column, and the new bus impedance matrix is of order $(m+1) \times (m+1)$. Let us add a branch with impedance z_{pq} from an existing bus p to a new bus q as shown in Figure 9.13(a). The network equation becomes

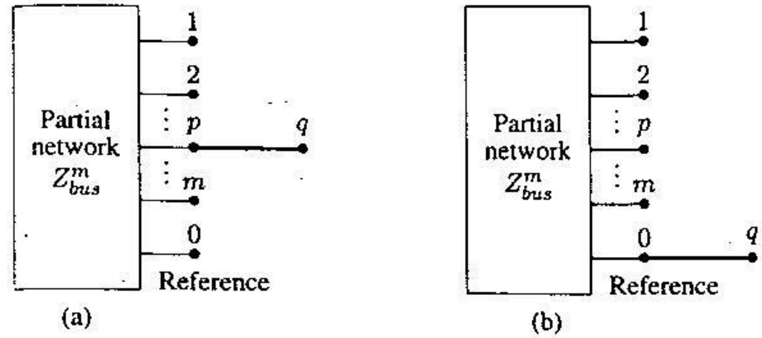


FIGURE 9.13
Addition of a branch p-q.

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \\ \vdots \\ V_m \\ V_q \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1q} \\ Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & Z_{2q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pq} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{mq} \\ Z_{q1} & Z_{q2} & \cdots & Z_{qp} & \cdots & Z_{qm} & Z_{qq} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_p \\ \vdots \\ I_m \\ I_q \end{bmatrix} \tag{9.29}$$

The addition of branch does not affect the original matrix, but requires the calculation of the elements in the q row and column. Since the elements of the power system network are linear and bilateral, $Z_{qi} = Z_{iq}$, for $q = 1, \ldots, m$.

First, let us compute the elements Z_{qi} for $i = 1, \ldots, m$ and $i \neq q$ (i.e., excluding diagonal element Z_{qq}). To calculate these elements we will apply a current source of 1 per unit at the ith bus, i.e., $I_i = 1$ pu, and keep remaining buses open-circuited, i.e., $I_k = 0$, $k = 1, \ldots, m$ and $k \neq i$. From (9.29), we get

$$\begin{aligned} V_1 &= Z_{1i} \\ V_2 &= Z_{2i} \\ &\vdots \\ V_p &= Z_{pi} \\ &\vdots \\ V_m &= Z_{mi} \\ V_q &= Z_{qi} \end{aligned} \tag{9.30}$$

From Figure 9.13(a)

$$V_q = V_p - v_{pq} \tag{9.31}$$

where v_{pq} is the voltage across the added branch with impedance z_{pq}, and is given by

$$v_{pq} = z_{pq} i_{pq} \tag{9.32}$$

Since added element p-q is a branch, $i_{pq} = 0$, thus $v_{pq} = 0$ and (9.31) reduces to

$$Z_{qi} = Z_{pi} \qquad i = 1, \ldots, m \qquad i \neq q \tag{9.33}$$

To calculate the diagonal element Z_{qq}, we will inject a current source of 1 per unit at the qth bus, i.e., $I_q = 1$ pu, and keep other buses open-circuited. From (9.29), we have

$$V_q = Z_{qq} \tag{9.34}$$

Since at the qth bus, the injected current flows from the bus q towards the bus p, $i_{pq} = -I_q = -1$. Hence, (9.32) reduces to

$$v_{pq} = -z_{pq} \tag{9.35}$$

Substituting for v_{pq} in (9.31), we get

$$V_q = V_p + z_{pq} \tag{9.36}$$

Now, since from (9.30) for $i = q$, $V_q = Z_{qq}$ and $V_p = Z_{pq}$, (9.36) becomes

$$Z_{qq} = Z_{pq} + z_{pq} \tag{9.37}$$

If node p is the reference node as shown in Figure 9.13(b), $V_p = 0$ and we obtain

$$Z_{qi} = Z_{pi} = V_p = 0 \qquad i = 1, \ldots, m \qquad i \neq q \tag{9.38}$$

From (9.37), the diagonal element becomes

$$Z_{qq} = z_{pq} \tag{9.39}$$

ADDITION OF A LINK

When the added element is a cotree link between the bus p and q, no new bus is created. The dimension of the $\mathbf{Z}_{bus}$ matrix remains the same but all the elements are required to be calculated. Let us add a link with impedance z_{pq} between two existing buses p and q as shown in Figure 9.14(a). If I_ℓ is the current through the added link in the direction shown in Figure 9.14(a), we have

$$z_{pq} I_\ell = V_p - V_q \tag{9.40}$$

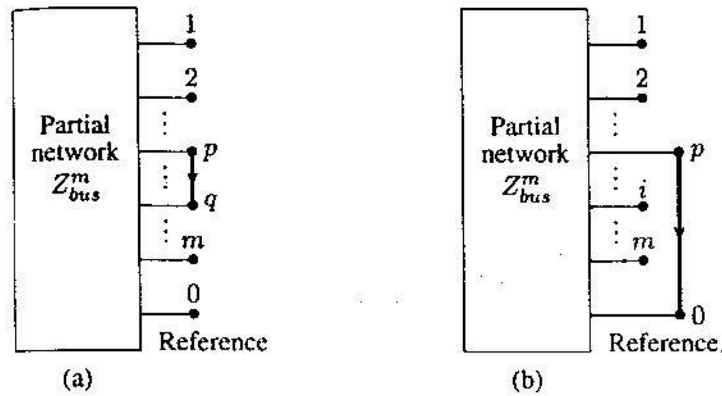


FIGURE 9.14
Addition of a link p-q.

or

$$V_q - V_p + z_{pq} I_\ell = 0 \tag{9.41}$$

The added link modifies the old current I_p to $(I_p - I_\ell)$ and the old current I_q to $(I_q + I_\ell)$ as shown in Figure 9.14(a), and the network equation becomes

$$\begin{aligned}
V_1 &= Z_{11}I_1 + \cdots + Z_{1p}(I_p - I_\ell) + Z_{1q}(I_q + I_\ell) + \cdots + Z_{1m}I_m \\
&\vdots \\
V_p &= Z_{p1}I_1 + \cdots + Z_{pp}(I_p - I_\ell) + Z_{pq}(I_q + I_\ell) + \cdots + Z_{pm}I_m \\
V_q &= Z_{q1}I_1 + \cdots + Z_{qp}(I_p - I_\ell) + Z_{qq}(I_q + I_\ell) + \cdots + Z_{qm}I_m \\
&\vdots \\
V_m &= Z_{m1}I_1 + \cdots + Z_{mp}(I_p - I_\ell) + Z_{mq}(I_q + I_\ell) + \cdots + Z_{mm}I_m
\end{aligned} \tag{9.42}$$

Substituting for V_p and V_q from (9.42) into (9.41) results in

$$\begin{aligned}
&(Z_{q1} - Z_{p1})I_1 + \cdots + (Z_{qp} - Z_{pp})I_p + \cdots + (Z_{qq} - Z_{pq})I_q + \cdots + \\
&(Z_{qm} - Z_{pm})I_m + (z_{pq} + Z_{pp} + Z_{qq} - 2Z_{pq})I_\ell = 0
\end{aligned} \tag{9.43}$$

Equations in (9.42) plus (9.43) result in $m+1$ simultaneous equations, which is written in matrix form as

$$\begin{bmatrix} V_1 \\ \vdots \\ V_p \\ V_q \\ \vdots \\ V_m \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1p} & Z_{1q} & \cdots & Z_{1m} & Z_{1\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{p1} & \cdots & Z_{pp} & Z_{pq} & \cdots & Z_{pm} & Z_{p\ell} \\ Z_{q1} & \cdots & Z_{qp} & Z_{qq} & \cdots & Z_{qm} & Z_{q\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & \cdots & Z_{mp} & Z_{mq} & \cdots & Z_{mm} & Z_{m\ell} \\ Z_{\ell 1} & \cdots & Z_{\ell p} & Z_{\ell q} & \cdots & Z_{\ell m} & Z_{\ell\ell} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_p \\ I_q \\ \vdots \\ I_m \\ I_\ell \end{bmatrix} \tag{9.44}$$

where

$$Z_{\ell i} = Z_{i\ell} = Z_{iq} - Z_{ip} \tag{9.45}$$

and

$$Z_{\ell\ell} = z_{pq} + Z_{pp} + Z_{qq} - 2Z_{pq} \tag{9.46}$$

Now the link current I_ℓ can be eliminated. Equation (9.44) can be partitioned and rewritten in compact form as

$$\begin{bmatrix} \mathbf{V}_{bus} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{bus}^{old} & \Delta\mathbf{Z} \\ \Delta\mathbf{Z}^T & Z_{\ell\ell} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{bus} \\ I_\ell \end{bmatrix} \tag{9.47}$$

where

$$\Delta\mathbf{Z} = [\, Z_{1\ell} \ \cdots \ Z_{p\ell} \ \ Z_{q\ell} \ \cdots \ Z_{m\ell} \,]^T \tag{9.48}$$

Expanding (9.47), we get

$$\mathbf{V}_{bus} = \mathbf{Z}_{bus}^{old}\mathbf{I}_{bus} + \Delta\mathbf{Z} I_\ell \tag{9.49}$$

and

$$0 = \Delta\mathbf{Z}^T\mathbf{I}_{bus} + Z_{\ell\ell}I_\ell \tag{9.50}$$

or

$$I_\ell = -\frac{\Delta\mathbf{Z}^T}{Z_{\ell\ell}}\mathbf{I}_{bus} \tag{9.51}$$

Substituting from (9.51) for I_ℓ in (9.49), we have

$$\mathbf{V}_{bus} = \left[\mathbf{Z}_{bus}^{old} - \frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{\ell\ell}}\right]\mathbf{I}_{bus} \tag{9.52}$$

or

$$\mathbf{V}_{bus} = \mathbf{Z}_{bus}^{new}\mathbf{I}_{bus} \tag{9.53}$$

where

$$\mathbf{Z}_{bus}^{new} = \mathbf{Z}_{bus}^{old} - \frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{\ell\ell}} \tag{9.54}$$

Note that (9.54) reduces the matrix to its original size. The reason for this is that we have not added a new node but only linked two existing nodes.

The bus impedance matrix can be constructed with addition of branches and links in any sequence. However, it is best to select a tree that contains the elements connected to the reference node. If more than one element is connected between a given node and the reference node, only one element can be selected as a branch placing other elements in the cotree. The step-by-step procedure for building the bus impedance matrix which takes us from a given bus impedance matrix $\mathbf{Z}_{bus}^{old}$ to a new $\mathbf{Z}_{bus}^{new}$ is summarized below.

Rule 1: Addition of a Tree Branch to the Reference

Start with the branches connected to the reference node. Addition of a branch z_{q0} between a new node q and the reference node 0 to the given $\mathbf{Z}_{bus}^{old}$ matrix of order $(m \times m)$, results in the $\mathbf{Z}_{bus}^{new}$ matrix of order $(m+1) \times (m+1)$. From the results of (9.38) and (9.39), we have

$$\mathbf{Z}_{bus}^{new} = \begin{bmatrix} Z_{11} & \cdots & Z_{1m} & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & Z_{mm} & 0 \\ 0 & \cdots & 0 & z_{q0} \end{bmatrix} \tag{9.55}$$

This matrix is diagonal with the impedance values of the branches on the diagonal.

Rule 2: Addition of a Tree Branch from a New Bus to an Old Bus

Continue with the remaining branches of the tree connecting a new node to the existing node. Addition of a branch z_{pq} between a new node q and the existing node p to the given $\mathbf{Z}_{bus}^{old}$ matrix of order $(m \times m)$, results in the $\mathbf{Z}_{bus}^{new}$ matrix of order $(m+1) \times (m+1)$. From the results of (9.33) and (9.37), we have

$$\mathbf{Z}_{bus}^{new} = \begin{bmatrix} Z_{11} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{p1} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{mp} \\ Z_{p1} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pp} + z_{pq} \end{bmatrix} \tag{9.56}$$

Rule 3: Addition of a Cotree Link between two existing Buses

When a link with impedance z_{pq} is added between two existing nodes p and q, we augment the $\mathbf{Z}_{bus}^{old}$ matrix with a new row and a new column, and from (9.44) and (9.45) we have

$$\mathbf{Z}_{bus}^{new} = \begin{bmatrix} Z_{11} & Z_{1p} & Z_{1q} & \cdots & Z_{1m} & Z_{1q}-Z_{1p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{p1} & Z_{pp} & Z_{pq} & \cdots & Z_{pm} & Z_{pq}-Z_{pp} \\ Z_{q1} & Z_{qp} & Z_{qq} & \cdots & Z_{qm} & Z_{qq}-Z_{qp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & Z_{mp} & Z_{mq} & \cdots & Z_{mm} & Z_{mq}-Z_{mp} \\ Z_{q1}-Z_{p1} & Z_{qp}-Z_{pp} & Z_{qq}-Z_{pq} & \cdots & Z_{qm}-Z_{pm} & Z_{\ell\ell} \end{bmatrix} \quad (9.57)$$

where

$$Z_{\ell\ell} = z_{pq} + Z_{pp} + Z_{qq} - 2Z_{pq} \quad (9.58)$$

The new row and column is eliminated using the relation in (9.54), which is repeated here

$$\mathbf{Z}_{bus}^{new} = \mathbf{Z}_{bus}^{old} - \frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{\ell\ell}} \quad (9.59)$$

and $\Delta\mathbf{Z}$ is defined as

$$\Delta\mathbf{Z} = \begin{bmatrix} Z_{1q} - Z_{1p} \\ \vdots \\ Z_{pq} - Z_{pp} \\ Z_{qq} - Z_{qp} \\ \vdots \\ Z_{mq} - Z_{mp} \end{bmatrix} \quad (9.60)$$

When bus q is the reference bus, $Z_{qi} = Z_{iq} = 0$ (for $i = 1, m$), and (9.57) reduces to

$$\mathbf{Z}_{bus}^{new} = \begin{bmatrix} Z_{11} & \cdots & Z_{1p} & \cdots & Z_{1m} & -Z_{1p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{p1} & \cdots & Z_{pp} & \cdots & Z_{pm} & -Z_{pp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & \cdots & Z_{mp} & \cdots & Z_{mm} & -Z_{mp} \\ -Z_{p1} & \cdots & -Z_{pp} & \cdots & -Z_{pm} & Z_{\ell\ell} \end{bmatrix} \quad (9.61)$$

where $Z_{\ell\ell} = z_{pq} + Z_{pp}$, and

$$\Delta\mathbf{Z} = \begin{bmatrix} -Z_{1p} \\ \vdots \\ -Z_{pp} \\ \vdots \\ -Z_{mp} \end{bmatrix} \quad (9.62)$$

The algorithm to construct the $\mathbf{Z}_{bus}$ matrix by adding one element at a time can be used to remove lines or generators from the network. The procedure is identical to that of adding elements, except that the removed element is considered as negative impedance, in order to cancel the effect of the element.

Based on the above algorithm, two functions named **Zbus = zbuild(zdata)** and **Zbus = zbuild(linedata, gendata, yload)** are developed for the formation of the bus impedance matrix. These functions are described in Section 9.6. Before demonstrating this program, for the sake of better understanding the building algorithm, we shall demonstrate the hand calculation procedure for the simple three-bus network of Example 9.1.

Example 9.3

Construct the bus impedance matrix for the network in Example 9.1. The one-line impedance diagram is shown in Figure 9.15(a).

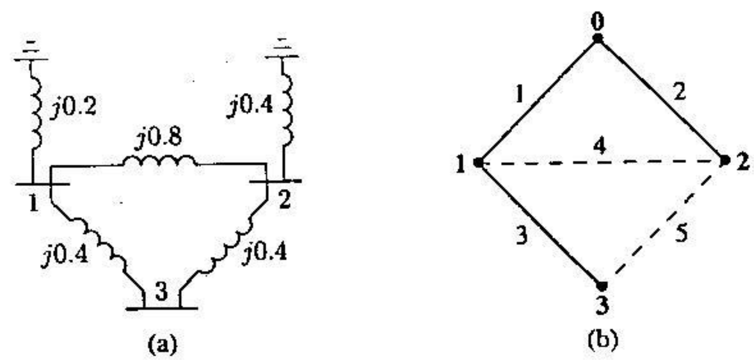


FIGURE 9.15
Impedance diagram of Example 9.1 and a proper tree.

The elements connected to the reference node are included in the proper tree as shown in Figure 9.15(b). We start with those branches of the tree connected to the reference node. Add branch 1, $z_{10} = j0.2$ between node $q = 1$ and reference node 0. According to rule 1, we have

$$\mathbf{Z}_{bus}^{(1)} = Z_{11} = z_{10} = j0.20$$

Next, add branch 2, $z_{20} = j0.4$ between node $q = 2$ and reference node 0

$$\mathbf{Z}_{bus}^{(2)} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix} = \begin{bmatrix} j0.2 & 0 \\ 0 & j0.4 \end{bmatrix}$$

Note that the off-diagonal elements of the bus impedance matrix are zero. This is because there is no connection between these buses other than to the reference. In this example, there are no more branches from a new bus to the reference. We continue with the remaining branches of the tree. Add branch 3, $z_{13} = j0.4$ between the new node $q = 3$ and the existing node $p = 1$. According to rule 2, we get

$$\mathbf{Z}_{bus}^{(3)} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{11} \\ Z_{21} & Z_{22} & Z_{21} \\ Z_{11} & Z_{12} & Z_{11} + z_{13} \end{bmatrix} = \begin{bmatrix} j0.2 & 0 & j0.2 \\ 0 & j0.4 & 0 \\ j0.2 & 0 & j0.6 \end{bmatrix}$$

All tree branches are in place. We now proceed with the links. Add link 4, $z_{12} = j0.8$ between node $q = 2$ and node $p = 1$. From (9.57), we have

$$\mathbf{Z}_{bus}^{(4)} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{12} - Z_{11} \\ Z_{21} & Z_{22} & Z_{23} & Z_{22} - Z_{21} \\ Z_{31} & Z_{32} & Z_{33} & Z_{32} - Z_{31} \\ Z_{21} - Z_{11} & Z_{22} - Z_{21} & Z_{23} - Z_{13} & Z_{44} \end{bmatrix}$$

$$= \begin{bmatrix} j0.2 & 0 & j0.2 & -j0.2 \\ 0 & j0.4 & 0 & j0.4 \\ j0.2 & 0 & j0.6 & -j0.2 \\ -j0.2 & j0.4 & -j0.2 & Z_{44} \end{bmatrix}$$

From (9.58)

$$Z_{44} = z_{12} + Z_{11} + Z_{22} - 2Z_{12} = j0.8 + j0.2 + j0.4 - 2(j0) = j1.4$$

and

$$\frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{44}} = \frac{1}{j1.4}\begin{bmatrix} -j0.2 \\ j0.4 \\ -j0.2 \end{bmatrix}\begin{bmatrix} -j0.2 & j0.4 & -j0.2 \end{bmatrix}$$

$$= \begin{bmatrix} j0.02857 & -j0.05714 & j0.02857 \\ -j0.05714 & j0.11428 & -j0.05714 \\ j0.02857 & -j0.05714 & j0.02857 \end{bmatrix}$$

From (9.59), the new bus impedance matrix is

$$\mathbf{Z}_{bus}^{(4)} = \begin{bmatrix} j0.2 & 0 & j0.2 \\ 0 & j0.4 & 0 \\ j0.2 & 0 & j0.6 \end{bmatrix} - \begin{bmatrix} j0.02857 & -j0.05714 & j0.02857 \\ -j0.05714 & j0.11428 & -j0.05714 \\ j0.02857 & -j0.05714 & j0.02857 \end{bmatrix}$$

$$= \begin{bmatrix} j0.17143 & j0.05714 & j0.17143 \\ j0.05714 & j0.28571 & j0.05714 \\ j0.17143 & j0.05714 & j0.57143 \end{bmatrix}$$

Finally, we add link 5, $z_{23} = j0.4$ between node $q = 3$ and node $p = 2$. From (9.57), we have

$$\mathbf{Z}_{bus}^{(5)} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{13} - Z_{12} \\ Z_{21} & Z_{22} & Z_{23} & Z_{23} - Z_{22} \\ Z_{31} & Z_{32} & Z_{33} & Z_{33} - Z_{32} \\ Z_{31} - Z_{21} & Z_{32} - Z_{22} & Z_{33} - Z_{23} & Z_{44} \end{bmatrix}$$

$$= \begin{bmatrix} j0.17143 & j0.05714 & j0.17143 & j0.11429 \\ j0.05714 & j0.28571 & j0.05714 & -j0.22857 \\ j0.17143 & j0.05714 & j0.57143 & j0.51429 \\ j0.11429 & -j0.22857 & j0.51429 & Z_{44} \end{bmatrix}$$

From (9.58)

$$Z_{44} = z_{23} + Z_{22} + Z_{33} - 2Z_{23} = j0.4 + j0.28571 + j0.57143 - 2(j0.05714) = j1.14$$

and

$$\frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{44}} = \frac{1}{j1.4}\begin{bmatrix} j0.11429 \\ -j0.22857 \\ j0.51429 \end{bmatrix}\begin{bmatrix} j0.11429 & -j0.22857 & j0.51429 \end{bmatrix}$$

$$= \begin{bmatrix} j0.01143 & -j0.02286 & j0.05143 \\ -j0.02286 & j0.04571 & -j0.10286 \\ j0.05143 & -j0.10286 & j0.23143 \end{bmatrix}$$

From (9.59), the new bus impedance matrix is

$$\mathbf{Z}_{bus} = \begin{bmatrix} j0.17143 & j0.05714 & j0.17143 \\ j0.05714 & j0.28571 & j0.05714 \\ j0.17143 & j0.05714 & j0.57143 \end{bmatrix} - \begin{bmatrix} j0.01143 & -j0.02286 & j0.05143 \\ -j0.02286 & j0.04571 & -j0.10286 \\ j0.05143 & -j0.10286 & j0.23143 \end{bmatrix}$$

$$= \begin{bmatrix} j0.16 & j0.08 & j0.12 \\ j0.08 & j0.24 & j0.16 \\ j0.12 & j0.16 & j0.34 \end{bmatrix}$$

This is the desired bus impedance matrix $\mathbf{Z}_{bus}$, which is the same as the one obtained by inverting the $\mathbf{Y}_{bus}$ matrix in Example 9.2.

Example 9.4

The bus impedance matrix for the network shown in Figure 9.16 is found to be

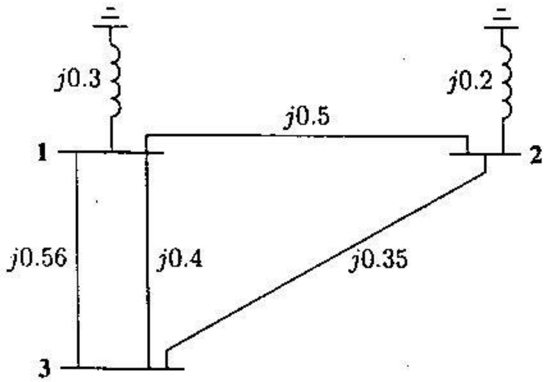


FIGURE 9.16
Impedance diagram for Example 9.4.

$$\mathbf{Z}_{bus} = \begin{bmatrix} j0.183 & j0.078 & j0.141 \\ j0.078 & j0.148 & j0.106 \\ j0.141 & j0.106 & j0.267 \end{bmatrix}$$

The line between buses 1 and 3 with impedance $Z_{13} = j0.56$ is removed by the simultaneous opening of breakers at both ends of the line. Determine the new bus impedance matrix.

The removal of an element is equivalent to connecting a link having an impedance equal to the negated value of the original impedance. Therefore, we add link $z_{13} = -j0.56$ between node $q = 3$ and node $p = 1$. From (9.57), we have

$$\mathbf{Z}_{bus} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{13}-Z_{11} \\ Z_{21} & Z_{22} & Z_{23} & Z_{23}-Z_{21} \\ Z_{31} & Z_{32} & Z_{33} & Z_{33}-Z_{31} \\ Z_{31}-Z_{11} & Z_{32}-Z_{12} & Z_{33}-Z_{13} & Z_{44} \end{bmatrix}$$

Thus, we get

$$Z_{bus} = \begin{bmatrix} j0.183 & j0.078 & j0.141 & -j0.042 \\ j0.078 & j0.148 & j0.106 & j0.028 \\ j0.141 & j0.106 & j0.267 & j0.126 \\ -j0.042 & j0.028 & j0.126 & Z_{44} \end{bmatrix}$$

From (9.58)

$$Z_{44} = z_{13} + Z_{11} + Z_{33} - 2Z_{13} = -j0.56 + j0.183 + j0.267 - 2(j0.141) = -j0.392$$

and

$$\frac{\Delta\mathbf{Z}\,\Delta\mathbf{Z}^T}{Z_{44}} = \frac{1}{-j0.392}\begin{bmatrix} -j0.042 \\ j0.028 \\ j0.126 \end{bmatrix}\begin{bmatrix} -j0.042 & j0.028 & j0.126 \end{bmatrix}$$
$$= \begin{bmatrix} -j0.0045 & j0.0030 & j0.0135 \\ j0.0030 & -j0.0020 & -j0.0090 \\ j0.0135 & -j0.0090 & -j0.0405 \end{bmatrix}$$

From (9.59), the new bus impedance matrix is

$$\mathbf{Z}_{bus} = \begin{bmatrix} j0.183 & j0.078 & j0.141 \\ j0.078 & j0.148 & j0.106 \\ j0.141 & j0.106 & j0.267 \end{bmatrix} - \begin{bmatrix} -j0.0045 & j0.0030 & j0.0135 \\ j0.0030 & -j0.0020 & -j0.0090 \\ j0.0135 & -j0.0090 & -j0.0405 \end{bmatrix}$$
$$= \begin{bmatrix} j0.1875 & j0.0750 & j0.1275 \\ j0.0750 & j0.1500 & j0.1150 \\ j0.1275 & j0.1150 & j0.3075 \end{bmatrix}$$

9.6 ZBUILD AND SYMFAULT PROGRAMS

Two functions are developed for the formation of the bus impedance matrix. One function is named **Zbus = zbuild(zdata)**, where the argument **zdata** is an $e \times 4$ matrix containing the impedance data of an e-element network. Columns 1 and 2 are the element bus numbers and columns 3 and 4 contain the element resistance and reactance, respectively, in per unit. Bus number 0 to generator buses contain generator impedances. These may be the subtransient, transient, or synchronous reactances. Also, any other shunt impedances such as capacitors and load impedance to ground (bus 0) may be included in this matrix.

The other function for the formation of the bus impedance matrix is **zbus = zbuildpi(linedata, gendata, yload)**, which is compatible with the power flow programs. The first argument **linedata** is consistent with the data required for the power flow solution. Columns 1 and 2 are the line bus numbers. Columns 3 through 5 contain line resistance, reactance, and one-half of the total line charging susceptance in per unit on the specified MVA base. The last column is for the transformer tap setting; for lines, 1 must be entered in this column. The lines may be entered in any sequence or order. The generator reactances are not included in the **linedata** of the power flow program and must be specified separately as required by the **gendata** in the second argument. **gendata** is an $n_g \times 4$ matrix, where each row contains bus 0, generator bus number, resistance and reactance. The last argument, **yload** is optional. This is a two-column matrix containing bus number and the complex load admittance. This data is provided by any of the power flow programs **lfgauss**, **lfnewton** or **decouple**. **yload** is automatically generated following the execution of any of the above power flow programs.

The **zbuild** and **zbuildpi** functions obtain the bus impedance matrix by the building algorithm method. These functions select a tree containing elements to the reference node. First, all branches connected to the reference node are processed. Then the remaining branches of the tree are connected, and finally the cotree links are added.

The program **symfault(zdata, Zbus, V)** is developed for the balanced three-phase fault studies. The function requires the **zdata** and the **Zbus** matrices. The third argument **V** is optional. If it is not included, the program sets all the prefault bus voltages to 1.0 per unit. If the variable **V** is included, the prefault bus voltages must be specified by the array **V** containing bus numbers and the complex bus voltage. The voltage vector **V** is automatically generated following the execution of any of the power flow programs. The use of the above functions are demonstrated in the following examples. When **symfault** is executed, it prompts the user to enter the faulted bus number and the fault impedance. The program computes the total fault current and tabulates the magnitude of the bus voltages and line currents during the fault.

Example 9.5

Use the function **zbus = zbuild(zdata)** to obtain the bus impedance matrix for the network in Example 9.3.

The network configuration containing resistances and reactances are specified and the **zbuild** function is used as follows.

```
zdata = [ 0   1   0   0.2
          0   2   0   0.4
          1   2   0   0.8
          1   3   0   0.4
          2   3   0   0.4];

Zbus = zbuild(zdata)
```

The result is


```
Zbus =
      0 + 0.16i      0 + 0.08i      0 + 0.12i
      0 + 0.08i      0 + 0.24i      0 + 0.16i
      0 + 0.12i      0 + 0.16i      0 + 0.34i
```

Example 9.6

A three-phase fault with a fault impedance $Z_f = j0.16$ per unit occurs at bus 3 in the network of Example 9.1. Use the **symfault** function to compute the fault current, the bus voltages and line currents during the fault.

In this example all shunt capacitances and loads are neglected and all the prefault bus voltages are assumed to be unity. The impedance diagram in Figure 9.2(b) is described by the variable **zdata** and the following commands are used.

```
zdata =    [ 0   1   0    0.2
             0   2   0    0.4
             1   2   0    0.8
             1   3   0    0.4
             2   3   0    0.4];

Zbus = zbuild(zdata)
symfault(zdata, Zbus)
```

The result is

```
Zbus =
       0 + 0.1600i      0 + 0.0800i      0 + 0.1200i
       0 + 0.0800i      0 + 0.2400i      0 + 0.1600i
       0 + 0.1200i      0 + 0.1600i      0 + 0.3400i

Enter Faulted Bus No.-> 3
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = j*0.16

Balanced three-phase fault at bus No. 3
Total fault current =   2.0000 Per unit

Bus Voltages during the fault in per unit
     Bus      Voltage        Angle
     No.      Magnitude      Degree
      1        0.7600        0.0000
      2        0.6800        0.0000
      3        0.3200        0.0000
```

```
Line currents for fault at bus No. 3
      From        To      Current        Angle
      Bus         Bus     Magnitude      Degree
       G           1       1.2000      -90.0000
       1           2       0.1000      -90.0000
       1           3       1.1000      -90.0000
       G           2       0.8000      -90.0000
       2           3       0.9000      -90.0000
       3           F       2.0000      -90.0000
```

Example 9.7

The 11-bus power system network of an electric utility company is shown in Figure 9.17.

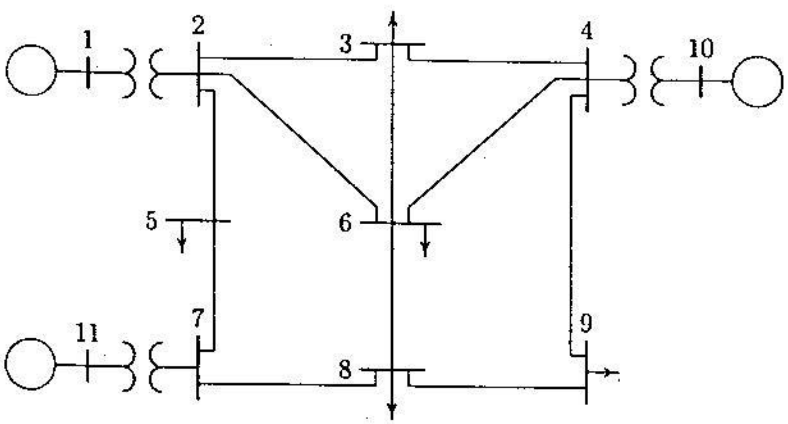


FIGURE 9.17
One-line diagram for Example 9.7

The transient impedance of the generators on a 100-MVA base are given below.

GEN. TRANSIENT IMPEDANCE PU		
Gen. No.	R_a	X'_d
1	0	0.20
10	0	0.15
11	0	0.25

The line and transformer data containing the series resistance and reactance in per unit, and one-half of the total capacitance in per unit susceptance on a 100-MVA base is tabulated below.

LINE AND TRANSFORMER DATA				
Bus No.	Bus No.	R, PU	X, PU	$\frac{1}{2}$ B, PU
1	2	0.00	0.06	0.0000
2	3	0.08	0.30	0.0004
2	5	0.04	0.15	0.0002
2	6	0.12	0.45	0.0005
3	4	0.10	0.40	0.0005
3	6	0.04	0.40	0.0005
4	6	0.15	0.60	0.0008
4	9	0.18	0.70	0.0009
4	10	0.00	0.08	0.0000
5	7	0.05	0.43	0.0003
6	8	0.06	0.48	0.0000
7	8	0.06	0.35	0.0004
7	11	0.00	0.10	0.0000
8	9	0.052	0.48	0.0000

Neglecting the shunt capacitors and the loads, use **zbuild(zdata)** function to obtain the bus impedance matrix. Assuming all the prefault bus voltages are equal to $1\angle 0°$, use **symfault** function to compute the fault current, bus voltages, and line currents for a bolted fault at bus 8. When using **zbuild** function, the generator reactances must be included in the impedance data with bus zero as the reference bus. The impedance data and the required commands are as follows.

```
%          Bus    Bus      R        X
%          No.    No.      pu       pu
zdata =   [0       1      0.00     0.20
           0      10      0.00     0.15
           0      11      0.00     0.25
           1       2      0.00     0.06
           2       3      0.08     0.30
           2       5      0.04     0.15
           2       6      0.12     0.45
           3       4      0.10     0.40
           3       6      0.04     0.40
           4       6      0.15     0.60
           4       9      0.18     0.70
           4      10      0.00     0.08
           5       7      0.05     0.43
           6       8      0.06     0.48
           7       8      0.06     0.35
           7      11      0.00     0.10
           8       9      0.052    0.48
```

```
Zbus = zbuild(zdata)
symfault(zdata, Zbus)
```

The bus impedance matrix is displayed on the screen, and the three-phase short circuit result is

```
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Balanced three-phase fault at bus No. 8
Total fault current =   3.3319 per unit

Bus Voltages during the fault in per unit
     Bus     Voltage       Angle
     No.     Magnitude     Degree
      1       0.8082       -1.8180
      2       0.7508       -2.5443
      3       0.6882       -1.5987
      4       0.7491       -2.4902
      5       0.7007       -2.3762
      6       0.5454       -1.0194
      7       0.5618       -3.8128
      8       0.0000        0.0000
      9       0.3008        2.4499
     10       0.8362       -1.4547
     11       0.6866       -2.2272
Line currents for fault at bus No. 8
     From      To      Current      Angle
     Bus       Bus     Magnitude    Degree
      G         1       0.9697     -82.4034
      1         2       0.9697     -82.4034
      2         3       0.2053     -87.8751
      2         5       0.3230     -79.9626
      2         6       0.4427     -81.6497
      3         6       0.3556     -88.0987
      4         3       0.1503     -88.4042
      4         6       0.3305     -82.3804
      4         9       0.6229     -81.3672
      5         7       0.3230     -79.9626
      6         8       1.1274     -83.8944
      7         8       1.5820     -84.0852
      8         F       3.3319     -83.5126
      9         8       0.6229     -81.3672
      G        10       1.1029     -82.6275
     10         4       1.1029     -82.6275
      G        11       1.2601     -85.1410
     11         7       1.2601     -85.1410
```


Example 9.8

In Example 9.7 consider the shunt capacitors and neglect the loads. Use **zbuildpi** function to obtain the bus impedance matrix. Assuming all the prefault bus voltages are equal to $1\angle 0°$, use **symfault** function to compute the fault current, bus voltages, and line currents for a bolted fault at bus 8.

The **zbuildpi(linedata, gendata, yload)** is designed to be compatible with the power flow programs. The first argument **linedata** is consistent with the data required for the power flow program. The generator reactances are not included in the **linedata** and must be specified separately by the **gendata**. The optional argument **yload** contains the load admittance which is generated from the power flow solution. The loads are neglected in this example, therefore, the argument **yload** is omitted. The impedance data and the required commands are as follows:

```
%          Bus    Bus     R        X        1/2B
%          No.    No.     pu       pu        pu
linedata=[1        2     0.00     0.06     0.0000
           2       3     0.08     0.30     0.0004
           2       5     0.04     0.15     0.0002
           2       6     0.12     0.45     0.0005
           3       4     0.10     0.40     0.0005
           3       6     0.04     0.40     0.0005
           4       6     0.15     0.60     0.0008
           4       9     0.18     0.70     0.0009
           4      10     0.00     0.08     0.0000
           5       7     0.05     0.43     0.0003
           6       8     0.06     0.48     0.0000
           7       8     0.06     0.35     0.0004
           7      11     0.00     0.10     0.0000
           8       9     0.052    0.48     0.0000];
%          Gen.   Ra      Xd'
gendata=[ 1        0      0.20
          10       0      0.15
          11       0      0.25];
Zbus=zbuildpi(linedata, gendata)
symfault(linedata, Zbus)
```

The bus impedance matrix is displayed on the screen, and the three-phase short circuit result is

```
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Balanced three-phase fault at bus No. 8
```

```
     Total fault current =   3.3301 per unit

     Bus Voltages during the fault in per unit

          Bus      Voltage        Angle
          No.      Magnitude      Degree
           1        0.8080       -1.8188
           2        0.7506       -2.5456
           3        0.6879       -1.5986
           4        0.7489       -2.4915
           5        0.7006       -2.3774
           6        0.5451       -1.0185
           7        0.5617       -3.8137
           8        0.0000        0.0000
           9        0.3005        2.4564
          10        0.8361       -1.4553
          11        0.6866       -2.2276

Line currents for fault at bus No. 8

     From       To       Current        Angle
     Bus        Bus      Magnitude      Degree
      1          2        0.9704       -82.4068
      2          3        0.2056       -87.7898
      2          5        0.3230       -79.9386
      2          6        0.4429       -81.6055
      3          6        0.3556       -88.0454
      4          3        0.1505       -88.2647
      4          6        0.3308       -82.2823
      4          9        0.6232       -81.3096
      5          7        0.3228       -79.9261
      6          8        1.1269       -83.8935
      7          8        1.5818       -84.0781
      8          F        3.3301       -83.5110
      9          8        0.6224       -81.3606
     10          4        1.1038       -82.6316
     11          7        1.2604       -85.1416
```

Example 9.9

Repeat the symmetrical three-phase short circuit analysis for Example 9.7 considering the prefault bus voltages and the effect of load currents. The load data is as follows:

LOAD DATA					
Bus No.	Load MW	Load Mvar	Bus No.	Load MW	Load Mvar
1	0.0	0.0	7	0.0	0.0
2	0.0	0.0	8	110.0	90.0
3	150.0	120.0	9	80.0	50.0
4	0.0	0.0	10	0.0	0.0
5	120.0	60.0	11	0.0	0.0
6	140.0	90.0			

Voltage magnitude, generation schedule and the reactive power limits for the regulated buses are tabulated below. Bus 1, whose voltage is specified as $V_1 = 1.04\angle 0°$, is taken as the slack bus.

GENERATION DATA				
Bus No.	Voltage Mag.	Generation, MW	Mvar Limits Min.	Mvar Limits Max.
1	1.040			
10	1.035	200.0	0.0	180.0
11	1.030	160.0	0.0	120.0

Anyone of the power flow programs can be used to obtain the prefault bus voltages and the load admittance. The **lfnewton** program is used which returns the prefault bus voltage array **V** and the bus load admittance array **yload**. The required commands are as follows.

```
clear        % clears all variables from workspace.
basemva = 100;  accuracy = 0.0001;  maxiter = 10;
%       Bus Bus Voltage Angle --Load--  ---Generator---Injected
%        No code Mag. Degree MW   Mvar   MW  Mvar Qmin Qmax Mvar
busdata=[1  1   1.06   0      0.0  0.0    0.0  0.0   0    0   0
         2  0   1.0    0      0.0  0.0    0.0  0     0    0   0
         3  0   1.0    0    150.0 120.0   0.0  0     0    0   0
         4  0   1.0    0      0.0  0.0    0.0  0     0    0   0
         5  0   1.0    0    120.0  60.0   0.0  0     0    0   0
         6  0   1.0    0    140.0  90.0   0.0  0     0    0   0
         7  0   1.0    0      0.0  0.0    0.0  0     0    0   0
         8  0   1.0    0    110.0  90.0   0.0  0     0    0   0
         9  0   1.0    0     80.0  50.0   0.0  0     0    0   0
        10  2   1.035  0      0.0  0.0  200.0  0     0  180   0
        11  2   1.03   0      0.0  0.0  160.0  0     0  120   0];

%        Bus    Bus     R        X        1/2B
%        No.    No.     pu       pu       pu
```

```
linedata=[1     2    0.00     0.06     0.0000     1
          2     3    0.08     0.30     0.0004     1
          2     6    0.12     0.45     0.0005     1
          3     4    0.10     0.40     0.0005     1
          3     6    0.04     0.40     0.0005     1
          4     6    0.15     0.60     0.0008     1
          4     9    0.18     0.70     0.0009     1
          4    10    0.00     0.08     0.0000     1
          5     7    0.05     0.43     0.0003     1
          6     8    0.06     0.48     0.0000     1
          7     8    0.06     0.35     0.0004     1
          7    11    0.00     0.10     0.0000     1
          8     9    0.052    0.48     0.0000     1];
%         Gen.  Ra    Xd'
gendata=[ 1     0    0.20
          10    0    0.15
          11    0    0.25];
lfybus                          % Forms the bus admittance matrix
lfnewton          % Power flow solution by Newton-Raphson method
busout            % Prints the power flow solution on the screen
Zbus=zbuildpi(linedata, gendata, yload) %  Zbus including load
symfault(linedata,Zbus,V)%3-phase fault including load current
```

The result is

```
            Power Flow Solution by Newton-Raphson Method
               Maximum Power Mismatch = 0.0000533178
                     No. of Iterations = 3

Bus  Voltage  Angle      ----Load----     --Generation-- Injected
No.  Mag.     Degree     MW      Mvar      MW       Mvar     Mvar

1    1.040    0.000     0.000    0.000   248.622  149.163    0.0
2    1.031   -0.797     0.000    0.000     0.000    0.000    0.0
3    0.997   -2.619   150.000  120.000     0.000    0.000    0.0
4    1.024   -1.737     0.000    0.000     0.000    0.000    0.0
5    0.981   -7.414   120.000   60.000     0.000    0.000    0.0
6    0.992   -3.336   140.000   90.000     0.000    0.000    0.0
7    1.014   -4.614     0.000    0.000     0.000    0.000    0.0
8    0.981   -5.093   110.000   90.000     0.000    0.000    0.0
9    0.977   -4.842    80.000   50.000     0.000    0.000    0.0
10   1.035   -0.872     0.000    0.000   200.000  144.994    0.0
11   1.020   -3.737     0.000    0.000   160.000  161.121    0.0

Total                 600.000  410.000   608.622  455.278    0.0
```



```
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Balanced three-phase fault at bus No. 8
Total fault current =    3.3571 per unit

Bus Voltages during the in per unit
     Bus       Voltage         Angle
     No.       Magnitude       Degree
      1         0.8876        -0.9467
      2         0.8350        -2.0943
      3         0.7321        -2.5619
      4         0.7866        -3.1798
      5         0.5148        -8.3043
      6         0.5792        -2.4214
      7         0.5179        -8.2563
      8         0.0000         0.0000
      9         0.3156         0.9877
     10         0.8785        -1.7237
     11         0.6631        -5.7789

Line currents for fault at bus No. 8
     From      To      Current       Angle
     Bus       Bus     Magnitude     Degree
      1         2       0.9219      -73.3472
      2         3       0.3321      -73.7856
      2         6       0.5494      -76.3804
      3         6       0.3804      -87.3283
      4         3       0.1336      -87.2217
      4         6       0.3357      -81.1554
      4         9       0.6537      -81.4818
      6         8       1.1974      -85.2964
      7         5       0.0073      -82.5471
      7         8       1.4585      -88.5207
      8         F       3.3571      -85.4214
      9         8       0.6538      -82.8293
     10         4       1.1787      -79.4854
     11         7       1.4733      -87.0395
```

PROBLEMS

9.1. The system shown in Figure 9.18 is initially on no load with generators operating at their rated voltage with their emfs in phase. The rating of the generators and the transformers and their respective percent reactances are marked on the diagram. All resistances are neglected. The line impedance is $j160\ \Omega$. A three-phase balanced fault occurs at the receiving end of the transmission line. Determine the short-circuit current and the short-circuit MVA.

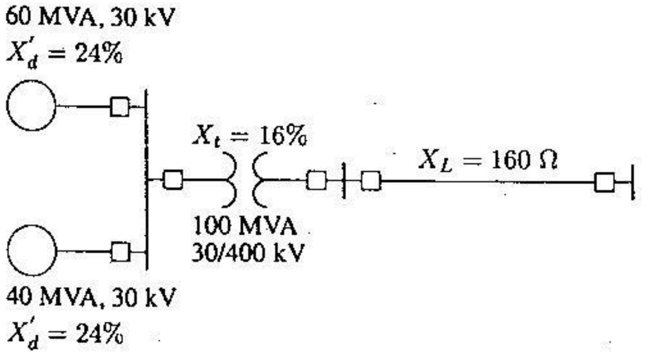


FIGURE 9.18
One-line diagram for Problem 9.1.

9.2. The system shown in Figure 9.19 shows an existing plant consisting of a generator of 100 MVA, 30 kV, with 20 percent subtransient reactance and a generator of 50 MVA, 30 kV with 15 percent subtransient reactance, connected in parallel to a 30-kV bus bar. The 30-kV bus bar feeds a transmission line via the circuit breaker C which is rated at 1250 MVA. A grid supply is connected to the station bus bar through a 500-MVA, 400/30-kV transformer with 20 percent reactance. Determine the reactance of a current limiting reactor in ohm to be connected between the grid system and the existing bus bar such that the short-circuit MVA of the breaker C does not exceed.

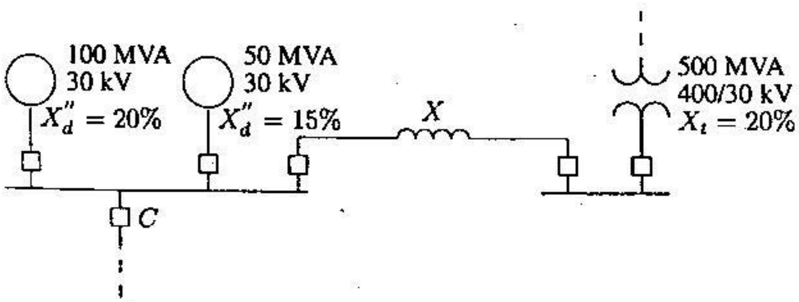


FIGURE 9.19
One-line diagram for Problem 9.2.

9.3. The one-line diagram of a simple power system is shown in Figure 9.20. Each generator is represented by an emf behind the transient reactance. All impedances are expressed in per unit on a common MVA base. All resistances and shunt capacitances are neglected. The generators are operating on no load at their rated voltage with their emfs in phase. A three-phase fault occurs at bus 1 through a fault impedance of $Z_f = j0.08$ per unit.
(a) Using Thévenin's theorem obtain the impedance to the point of fault and the fault current in per unit.
(b) Determine the bus voltages and line currents during fault.

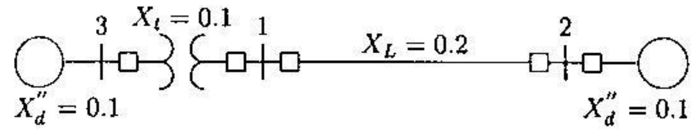


FIGURE 9.20
One-line diagram for Problem 9.3.

9.4. The one-line diagram of a simple three-bus power system is shown in Figure 9.21 Each generator is represented by an emf behind the subtransient reactance. All impedances are expressed in per unit on a common MVA base. All resistances and shunt capacitances are neglected. The generators are operating on no load at their rated voltage with their emfs in phase. A three-phase fault occurs at bus 3 through a fault impedance of $Z_f = j0.19$ per unit.
(a) Using Thévenin's theorem obtain the impedance to the point of fault and the fault current in per unit.
(b) Determine the bus voltages and line currents during fault.

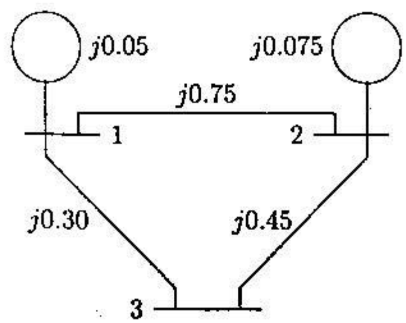


FIGURE 9.21
One-line diagram for Problem 9.4.

9.5. The one-line diagram of a simple four-bus power system is shown in Figure 9.22 Each generator is represented by an emf behind the transient reactance.

All impedances are expressed in per unit on a common MVA base. All resistances and shunt capacitances are neglected. The generators are operating on no load at their rated voltage with their emfs in phase. A bolted three-phase fault occurs at bus 4.
(a) Using Thévenin's theorem obtain the impedance to the point of fault and the fault current in per unit.
(b) Determine the bus voltages and line currents during fault.
(c) Repeat (a) and (b) for a fault at bus 2 with a fault impedance of $Z_f = j0.0225$.

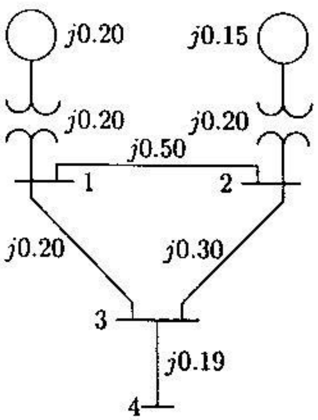


FIGURE 9.22
One-line diagram for Problem 9.5.

9.6. Using the method of building algorithm find the bus impedance matrix for the network shown in Figure 9.23.

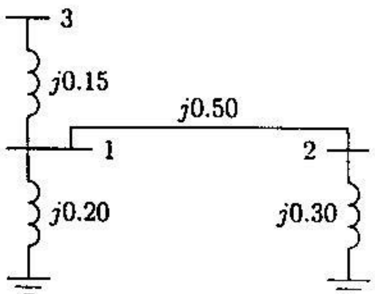


FIGURE 9.23
One-line diagram for Problem 9.6.

9.7. Obtain the bus impedance matrix for the network of Problem 9.3.

9.8. Obtain the bus impedance matrix for the network of Problem 9.4.

9.9. The bus impedance matrix for the network shown in Figure 9.24 is given by

$$Z_{bus} = j\begin{bmatrix} 0.300 & 0.200 & 0.275 \\ 0.200 & 0.400 & 0.250 \\ 0.275 & 0.250 & 0.41875 \end{bmatrix}$$

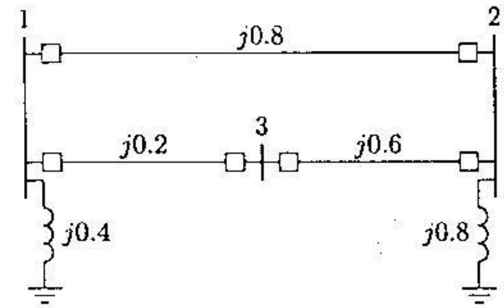


FIGURE 9.24
One-line diagram for Problem 9.9.

There is a line outage and the line from bus 1 to 2 is removed. Using the method of building algorithm determine the new bus impedance matrix.

9.10. The per unit bus impedance matrix for the power system of Problem 9.4 is given by

$$Z_{bus} = j\begin{bmatrix} 0.0450 & 0.0075 & 0.0300 \\ 0.0075 & 0.06375 & 0.0300 \\ 0.0300 & 0.0300 & 0.2100 \end{bmatrix}$$

A three-phase fault occurs at bus 3 through a fault impedance of $Z_f = j0.19$ per unit. Using the bus impedance matrix calculate the fault current, bus voltages, and line currents during fault. Check your result using the **Zbuild** and **symfault** programs.

9.11. The per unit bus impedance matrix for the power system of Problem 9.5 is given by

$$Z_{bus} = j\begin{bmatrix} 0.240 & 0.140 & 0.200 & 0.200 \\ 0.140 & 0.2275 & 0.175 & 0.175 \\ 0.200 & 0.175 & 0.310 & 0.310 \\ 0.200 & 0.1750 & 0.310 & 0.500 \end{bmatrix}$$

(a) A bolted three-phase fault occurs at bus 4. Using the bus impedance matrix calculate the fault current, bus voltages, and line currents during fault.
(b) Repeat (a) for a three-phase fault at bus 2 with a fault impedance of $Z_f = j0.0225$.
(c) Check your result using the **Zbuild** and **symfault** programs.

9.12. The per unit bus impedance matrix for the power system shown in Figure 9.25 is given by

$$Z_{bus} = j\begin{bmatrix} 0.150 & 0.075 & 0.140 & 0.135 \\ 0.075 & 0.1875 & 0.090 & 0.0975 \\ 0.140 & 0.090 & 0.2533 & 0.210 \\ 0.135 & 0.0975 & 0.210 & 0.2475 \end{bmatrix}$$

A three-phase fault occurs at bus4 through a fault impedance of $Z_f = j0.0025$ per unit. Using the bus impedance matrix calculate the fault current, bus voltages and line currents during fault. Check your result using the **Zbuild** and **symfault** programs.

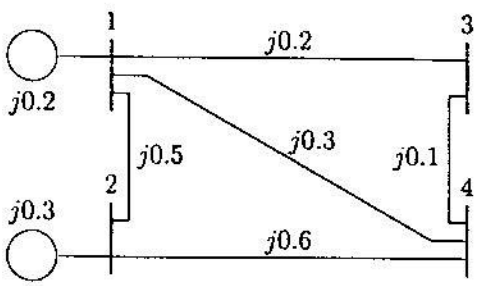


FIGURE 9.25
One-line diagram for Problem 9.12.

9.13. Repeat Example 9.7 for a bolted three-phase fault at bus 9.

9.14. Repeat Example 9.8 for a bolted three-phase fault at bus 9.

9.15. Repeat Example 9.9 for a bolted three-phase fault at bus 9.

9.16. The 6-bus power system network of an electric utility company is shown in Figure 9.26. The line and transformer data containing the series resistance and reactance in per unit, and one-half of the total capacitance in per unit susceptance on a 100-MVA base, is tabulated below.

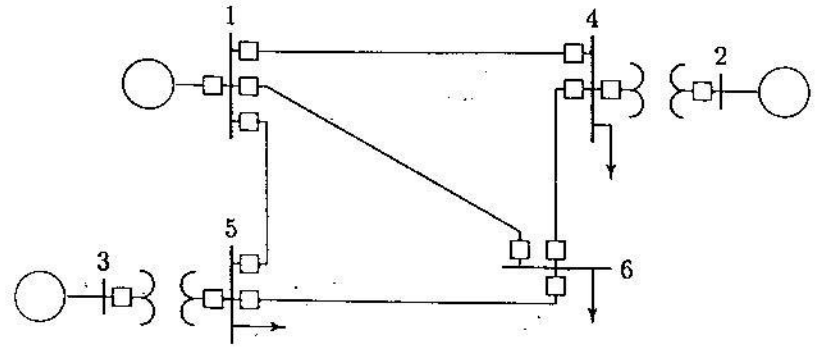


FIGURE 9.26
One-line diagram for Problem 9.16.

LINE AND TRANSFORMER DATA				
Bus No.	Bus No.	R, PU	X, PU	$\frac{1}{2}$ B, PU
1	4	0.035	0.225	0.0065
1	5	0.025	0.105	0.0045
1	6	0.040	0.215	0.0055
2	4	0.000	0.035	0.0000
3	5	0.000	0.042	0.0000
4	6	0.028	0.125	0.0035
5	6	0.026	0.175	0.0300

The transient impedance of the generators on a 100-MVA base are given below.

GEN. TRANSIENT IMPEDANCE, PU		
Gen. No.	R_a	X'_d
1	0	0.20
2	0	0.15
3	0	0.25

Neglecting the shunt capacitors and the loads, use **Zbus = zbuild(zdata)** function to obtain the bus impedance matrix. Assuming all the prefault bus voltages are equal to $1\angle 0°$, use **symfault(zdata, Zbus)** function to compute the fault current, bus voltages, and line currents for a bolted fault at bus 6. When using **Zbus = zbuild(zdata)** function, the generator reactances must be included in the **zdata** array with bus zero as the reference bus.

9.17. In Problem 9.16 consider the shunt capacitors and neglect the loads. use **zbuildpi(linedata, gendata, yload)** function to obtain the bus impedance matrix. Assuming all the prefault bus voltages are equal to $1\angle 0°$, use **symfault(linedata, Zbus)** function to compute the fault current, bus voltages, and line currents for a bolted fault at bus 6.

9.18. Repeat the symmetrical three-phase short circuit analysis for Problem 9.16 considering the prefault bus voltages and the effect of load currents. The load data is as follows.

LOAD DATA		
Bus	Load	
No.	MW	Mvar
1	0	0
2	0	0
3	0	0
4	100	70
5	90	30
6	160	110

Voltage magnitude, generation schedule, and the reactive power limits for the regulated buses are tabulated below. Bus 1, whose voltage is specified as $V_1 = 1.06\angle 0°$, is taken as the slack bus.

GENERATION DATA				
Bus	Voltage	Generation,	Mvar Limits	
No.	Mag.	MW	Min.	Max.
1	1.060			
2	1.040	150.0	0.0	140.0
3	1.030	100.0	0.0	90.0

Use anyone of the power flow programs to obtain the prefault bus voltages and the load admittance. The power flow program returns the prefault bus voltage array **V** and the bus load admittance array **yload**.

CHAPTER 10

SYMMETRICAL COMPONENTS AND UNBALANCED FAULT

10.1 INTRODUCTION

Different types of unbalanced faults are the *single line-to-ground fault*, *line-to-line fault*, and *double line-to-ground fault*.

The fault study that was presented in Chapter 9 has considered only three-phase balanced faults, which lends itself to a simple per phase approach. Various methods have been devised for the solution of unbalanced faults. However, since the one-line diagram simplifies the solution of the balanced three-phase problems, the method of symmetrical components that resolves the solution of unbalanced circuit into a solution of a number of balanced circuits is used. In this chapter, the symmetrical components method is discussed. It is then applied to the unbalanced faults, which allows once again the treatment of the problem on a simple per phase basis. Two functions are developed for the symmetrical components transformations. These are **abc2sc**, which provides transformation from phase quantities to symmetrical components, and **sc2abc** for the inverse transformation. In addition, these functions produce plots of unbalanced phasors and their symmetrical components. Finally, unbalanced faults are computed using the concept of symmetrical components. Three functions named **lgfault(zdata0, zbus0, zdata1, zbus1, zdata2, zbus2, V)**, **llfault(zdata1, zbus1, zdata2, zbus2, V)**, and **dlgfault(zdata0, zbus0, zdata1, zbus1, zdata2, zbus2, V)** are developed for the line-to-ground, line-to-line, and the double line-to-ground fault studies.

10.2 FUNDAMENTALS OF SYMMETRICAL COMPONENTS

Symmetrical components allow unbalanced phase quantities such as currents and voltages to be replaced by three separate balanced symmetrical components.

In three-phase system the phase sequence is defined as the order in which they pass through a positive maximum. Consider the phasor representation of a three-phase balanced current shown in Figure 10.1(a).

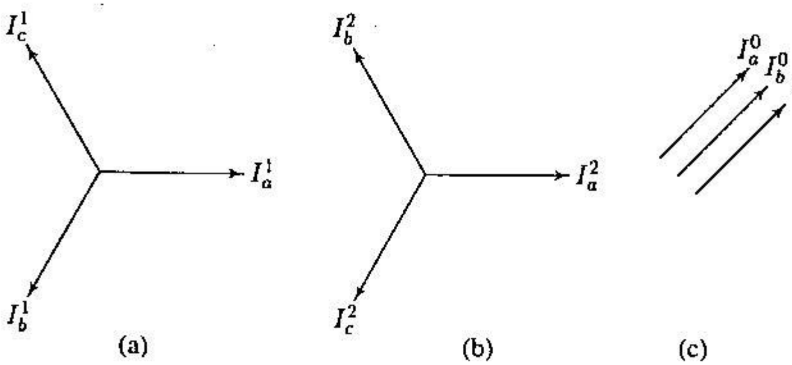


FIGURE 10.1
Representation of symmetrical components.

By convention, the direction of rotation of the phasors is taken to be counterclockwise. The three phasors are written as

$$\begin{aligned} I_a^1 &= I_a^1\angle 0^\circ &&= I_a^1 \\ I_b^1 &= I_a^1\angle 240^\circ &&= a^2 I_a^1 \\ I_c^1 &= I_a^1\angle 120^\circ &&= aI_a^1 \end{aligned} \tag{10.1}$$

where we have defined an operator a that causes a counterclockwise rotation of 120°, such that

$$\begin{aligned} a &= 1\angle 120^\circ = -0.5 + j0.866 \\ a^2 &= 1\angle 240^\circ = -0.5 - j0.866 \\ a^3 &= 1\angle 360^\circ = 1 + j0 \end{aligned} \tag{10.2}$$

From above, it is clear that

$$1 + a + a^2 = 0 \tag{10.3}$$

The order of the phasors is *abc*. This is designated the *positive phase sequence*. When the order is *acb* as in Figure 10.1(b), it is designated the *negative phase sequence*. The negative phase sequence quantities are represented as

$$\begin{aligned} I_a^2 &= I_a^2\angle 0^\circ &&= I_a^2 \\ I_b^2 &= I_a^2\angle 120^\circ &&= aI_a^2 \\ I_c^2 &= I_a^2\angle 240^\circ &&= a^2 I_a^2 \end{aligned} \tag{10.4}$$

When analyzing certain types of unbalanced faults, it will be found that a third set of balanced phasors must be introduced. These phasors, known as the *zero phase sequence*, are found to be in phase with each other. Zero phase sequence currents, as in Figure 10.1(c), would be designated

$$I_a^0 = I_b^0 = I_c^0 \tag{10.5}$$

The superscripts 1, 2, and 0 are being used to represent positive, negative, and zero-sequence quantities, respectively. In some texts the notation 0, +, – is used instead of 0, 1, 2. The symmetrical components method was introduced by Dr. C. L. Fortescue in 1918. Based on his theory, three-phase unbalanced phasors of a three-phase system can be resolved into three balanced systems of phasors as follows.

1. Positive-sequence components consisting of a set of balanced three-phase components with a phase sequence *abc*.
2. Negative-sequence components consisting of a set of balanced three-phase components with a phase sequence *acb*.
3. Zero-sequence components consisting of three single-phase components, all equal in magnitude but with the same phase angles.

Consider the three-phase unbalanced currents I_a, I_b, and I_c shown in Figure 10.2 (page 405). We are seeking to find the three symmetrical components of the current such that

$$\begin{aligned} I_a &= I_a^0 + I_a^1 + I_a^2 \\ I_b &= I_b^0 + I_b^1 + I_b^2 \\ I_c &= I_c^0 + I_c^1 + I_c^2 \end{aligned} \tag{10.6}$$

According to the definition of the symmetrical components as given by (10.1), (10.4), and (10.5), we can rewrite (10.6) all in terms of phase a components.

$$\begin{aligned} I_a &= I_a^0 + I_a^1 + I_a^2 \\ I_b &= I_a^0 + a^2 I_a^1 + aI_a^2 \\ I_c &= I_a^0 + aI_a^1 + a^2 I_a^2 \end{aligned} \tag{10.7}$$

or

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} \tag{10.8}$$

We rewrite the above equation in matrix notation as

$$\mathbf{I}^{abc} = \mathbf{A}\ \mathbf{I}_a^{012} \tag{10.9}$$

where $\mathbf{A}$ is known as *symmetrical components transformation matrix* (SCTM) which transforms phasor currents $\mathbf{I}^{abc}$ into component currents $\mathbf{I}_a^{012}$, and is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \tag{10.10}$$

Solving (10.9) for the symmetrical components of currents, we have

$$\mathbf{I}_a^{012} = \mathbf{A}^{-1}\ \mathbf{I}^{abc} \tag{10.11}$$

The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \tag{10.12}$$

From (10.10) and (10.12), we conclude that

$$\mathbf{A}^{-1} = \frac{1}{3}\mathbf{A}^* \tag{10.13}$$

Substituting for $\mathbf{A}^{-1}$ in (10.11), we have

$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} \tag{10.14}$$

or in component form, the symmetrical components are

$$\begin{aligned} I_a^0 &= \tfrac{1}{3}(I_a + I_b + I_c) \\ I_a^1 &= \tfrac{1}{3}(I_a + aI_b + a^2 I_c) \\ I_a^2 &= \tfrac{1}{3}(I_a + a^2 I_b + aI_c) \end{aligned} \tag{10.15}$$

From (10.15), we note that the zero-sequence component of current is equal to one-third of the sum of the phase currents. Therefore, when the phase currents sum to zero, e.g., in a three-phase system with ungrounded neutral, the zero-sequence current cannot exist. If the neutral of the power system is grounded, zero-sequence current flows between the neutral and the ground.

Similar expressions exist for voltages. Thus the unbalanced phase voltages in terms of the symmetrical components voltages are

$$\begin{aligned} V_a &= V_a^0 + V_a^1 + V_a^2 \\ V_b &= V_a^0 + a^2 V_a^1 + aV_a^2 \\ V_c &= V_a^0 + aV_a^1 + a^2 V_a^2 \end{aligned} \tag{10.16}$$

or in matrix notation

$$\mathbf{V}^{abc} = \mathbf{A}\ \mathbf{V}_a^{012} \tag{10.17}$$

The symmetrical components in terms of the unbalanced voltages are

$$\begin{aligned} V_a^0 &= \tfrac{1}{3}(V_a + V_b + V_c) \\ V_a^1 &= \tfrac{1}{3}(V_a + aV_b + a^2 V_c) \\ V_a^2 &= \tfrac{1}{3}(V_a + a^2 V_b + aV_c) \end{aligned} \tag{10.18}$$

or in matrix notation

$$\mathbf{V}_a^{012} = \mathbf{A}^{-1}\ \mathbf{V}^{abc} \tag{10.19}$$

The apparent power may also be expressed in terms of the symmetrical components. The three-phase complex power is

$$S_{(3\phi)} = \mathbf{V}^{abc^T} \mathbf{I}^{abc^*} \tag{10.20}$$

Substituting (10.9) and (10.17) in (10.20), we obtain

$$\begin{aligned} S_{(3\phi)} &= \left(\mathbf{A}\mathbf{V}_a^{012}\right)^T \left(\mathbf{A}\mathbf{I}_a^{012}\right)^* \\ &= \mathbf{V}_a^{012^T} \mathbf{A}^T \mathbf{A}^* \mathbf{I}_a^{012^*} \end{aligned} \tag{10.21}$$

Since $\mathbf{A}^T = \mathbf{A}$, then from (10.13), $\mathbf{A}^T\mathbf{A}^* = 3$, and the complex power becomes

$$\begin{aligned} S_{(3\phi)} &= 3\left(\mathbf{V}^{012^T}\mathbf{I}^{012^*}\right) \\ &= 3V_a^0 I_a^{0^*} + 3V_a^1 I_a^{1^*} + 3V_a^2 I_a^{2^*} \end{aligned} \tag{10.22}$$

Equation (10.22) shows that the total unbalanced power can be obtained from the sum of the symmetrical component powers. Often the subscript a of the symmetrical components are omitted, e.g., I^0, I^1, and I^2 are understood to refer to phase a.

Transformation from phase quantities to symmetrical components in *MATLAB* is very easy. Once the symmetrical components transformation matrix **A** is defined, its inverse is found using the *MATLAB* function **inv**. However, for quick calculations and graphical demonstration, the following functions are developed for symmetrical components analysis.

sctm The symmetrical components transformation matrix **A** is defined in this script file. Typing **sctm** defines **A**.

phasor(F) This function makes plots of phasors. The variable F may be expressed in an $n \times 1$ array in rectangular complex form or as an $n \times 2$ matrix. In the latter case, the first column is the phasor magnitude and the second column is its phase angle in degree.

F_{012} = **abc2sc**(F_{abc}) This function returns the symmetrical components of a set of unbalanced phasors in rectangular form. F_{abc} may be expressed in a 3×1 array in rectangular complex form or as a 3×2 matrix. In the latter case, the first column is the phasor magnitude and the second column is its phase angle in degree for a, b, and c phases. In addition, the function produces a plot of the unbalanced phasors and its symmetrical components.

F_{abc} = **sc2abc**(F_{012}) This function returns the unbalanced phasor in rectangular form when symmetrical components are specified. F_{012} may be expressed in a 3×1 array in rectangular complex form or as a 3×2 matrix. In the latter case, the first column is the phasor magnitude and the second column is its phase angle in degree for the zero-, positive-, and negative-sequence components, respectively. In addition, the function produces a plot of the unbalanced phasors and its symmetrical components.

Z_{012} = **zabc2sc**(Z_{abc}) This function transforms the phase impedance matrix to the sequence impedance matrix, given by (10.30).

F_p = **rec2pol**(F_r) This function converts the rectangular phasor F_r into polar form F_p.

F_r = **pol2rec**(F_p) This function converts the polar phasor F_p into rectangular form F_r.

Example 10.1

Obtain the symmetrical components of a set of unbalanced currents $I_a = 1.6\angle 25°$, $I_b = 1.0\angle 180°$, and $I_c = 0.9\angle 132°$.

The commands

```
Iabc = [1.6    25
        1.0   180
        0.9   132];
I012 = abc2sc(Iabc);  % Symmetrical components of phase a
I012p= rec2pol(I012)          % Rectangular to polar form
```

result in

```
I012P =
        0.4512    96.4529
        0.9435    -0.0550
        0.6024    22.3157
```

and the plots of the phasors are shown in Figure 10.2.

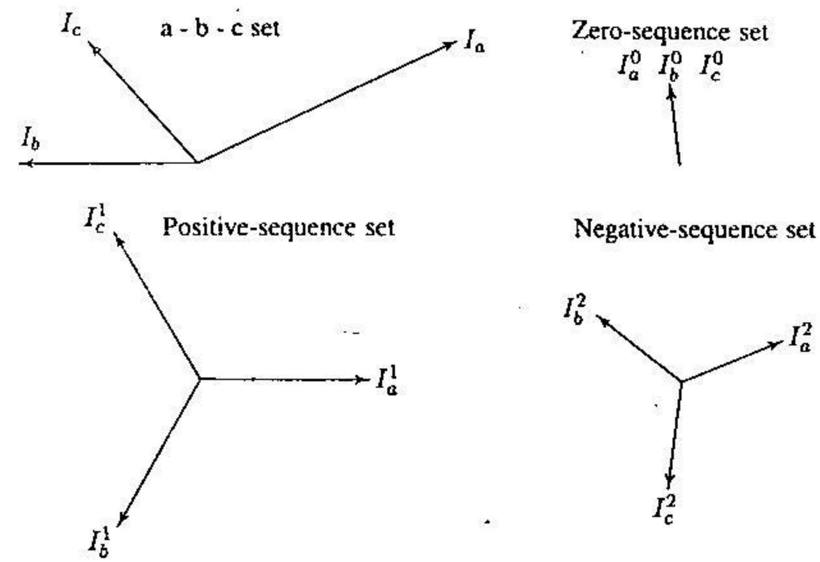


FIGURE 10.2
Resolution of unbalanced phasors into symmetrical components.

Example 10.2

The symmetrical components of a set of unbalanced three-phase voltages are $V_a^0 = 0.6\angle 90°$, $V_a^1 = 1.0\angle 30°$, and $V_a^2 = 0.8\angle -30°$. Obtain the original unbalanced phasors.

The commands

```
V012 = [0.6    90
        1.0    30
        0.8   -30];
Vabc = sc2abc(V012);%Unbalanced phasor to symmetrical comp.
Vabcp= rec2pol(Vabc)              % Rectangular to polar form
```

result in

```
  Vabcp =
          1.7088     24.1825
          0.400      90.0000
          1.7088    155.8175
```

and the plots of the phasors are shown in Figure 10.3.

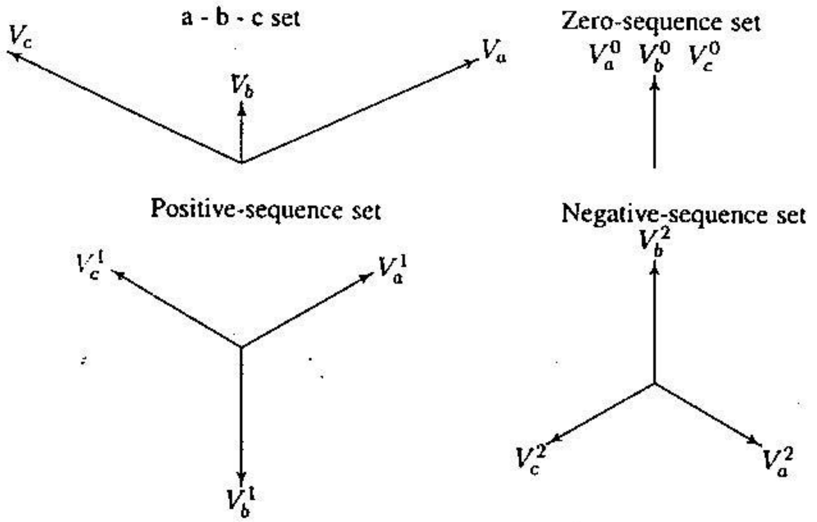


FIGURE 10.3
Transformation of the symmetrical components into phasor components.

10.3 SEQUENCE IMPEDANCES

This is the impedance of an equipment or component to the current of different sequences. The impedance offered to the flow of positive-sequence currents is known as the *positive-sequence impedance* and is denoted by Z^1. The impedance offered to the flow of negative-sequence currents is known as the *negative-sequence impedance*, shown by Z^2. When zero-sequence currents flow, the impedance is

called the *zero-sequence impedance*, shown by Z^0. The sequence impedances of transmission lines, generators, and transformers are considered briefly here.

10.3.1 SEQUENCE IMPEDANCES OF Y-CONNECTED LOADS

A three-phase balanced load with self and mutual elements is shown in Figure 10.4. The load neutral is grounded through an impedance Z_n.

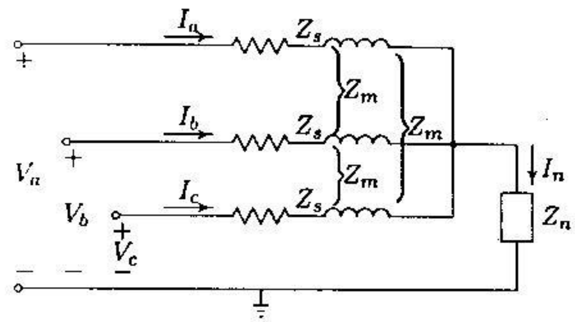


FIGURE 10.4
Balanced Y-connected load.

The line-to-ground voltages are

$$\begin{aligned} V_a &= Z_s I_a + Z_m I_b + Z_m I_c + Z_n I_n \\ V_b &= Z_m I_a + Z_s I_b + Z_m I_c + Z_n I_n \\ V_c &= Z_m I_a + Z_m I_b + Z_s I_c + Z_n I_n \end{aligned} \tag{10.23}$$

From Kirchhoff's current law, we have

$$I_n = I_a + I_b + I_c \tag{10.24}$$

Substituting for I_n from (10.24) into (10.23) and rewriting this equation in matrix form, yields

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} Z_s + Z_n & Z_m + Z_n & Zm + Z_n \\ Z_m + Z_n & Z_s + Z_n & Zm + Z_n \\ Z_m + Z_n & Z_m + Z_n & Zs + Z_n \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} \tag{10.25}$$

or in compact form

$$\mathbf{V}^{abc} = \mathbf{Z}^{abc}\mathbf{I}^{abc} \tag{10.26}$$

where

$$\mathbf{Z}^{abc} = \begin{bmatrix} Z_s + Z_n & Z_m + Z_n & Z_m + Z_n \\ Z_m + Z_n & Z_s + Z_n & Z_m + Z_n \\ Z_m + Z_n & Z_m + Z_n & Z_s + Z_n \end{bmatrix} \tag{10.27}$$

Writing $\mathbf{V}^{abc}$ and $\mathbf{I}^{abc}$ in terms of their symmetrical components, we get

$$\mathbf{A}\mathbf{V}_a^{012} = \mathbf{Z}^{abc}\mathbf{A}\mathbf{I}_a^{012} \tag{10.28}$$

Multiplying (10.28) by $\mathbf{A}^{-1}$, we get

$$\begin{aligned} \mathbf{V}_a^{012} &= \mathbf{A}^{-1}\mathbf{Z}^{abc}\mathbf{A}\mathbf{I}_a^{012} \\ &= \mathbf{Z}^{012}\mathbf{I}_a^{012} \end{aligned} \tag{10.29}$$

where

$$\mathbf{Z}^{012} = \mathbf{A}^{-1}\mathbf{Z}^{abc}\mathbf{A} \tag{10.30}$$

Substituting for $\mathbf{Z}^{abc}$, $\mathbf{A}$, and $\mathbf{A}^{-1}$ from (10.27), (10.10), and (10.12), we have

$$\mathbf{Z}^{012} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} Z_s+Z_n & Z_m+Z_n & Z_m+Z_n \\ Z_m+Z_n & Z_s+Z_n & Z_m+Z_n \\ Z_m+Z_n & Z_m+Z_n & Z_s+Z_n \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \tag{10.31}$$

Performing the above multiplications, we get

$$\mathbf{Z}^{012} = \begin{bmatrix} Z_s + 3Z_n + 2Z_m & 0 & 0 \\ 0 & Z_s - Z_m & 0 \\ 0 & 0 & Z_s - Z_m \end{bmatrix} \tag{10.32}$$

When there is no mutual coupling, we set $Z_m = 0$, and the impedance matrix becomes

$$\mathbf{Z}^{012} = \begin{bmatrix} Z_s + 3Z_n & 0 & 0 \\ 0 & Z_s & 0 \\ 0 & 0 & Z_s \end{bmatrix} \tag{10.33}$$

The impedance matrix has nonzero elements appearing only on the principal diagonal, and it is a diagonal matrix. Therefore, for a balanced load, the three sequences are independent. That is, currents of each phase sequence will produce voltage drops of the same phase sequence only. This is a very important property, as it permits the analysis of each sequence network on a per phase basis.

10.3.2 SEQUENCE IMPEDANCES OF TRANSMISSION LINES

Transmission line parameters were derived in Chapter 4. For static devices such as transmission lines, the phase sequence has no effect on the impedance, because the voltages and currents encounter the same geometry of the line, irrespective of the sequence. Thus, positive- and negative-sequence impedances are equal, i.e., $Z^1 = Z^2$.

In deriving the line parameters, the effect of ground and shielding conductors were neglected. Zero-sequence currents are in phase and flow through the a,b,c conductors to return through the grounded neutral. The ground or any shielding wire are effectively in the path of zero sequence. Thus, Z^0, which includes the effect of the return path through the ground, is generally different from Z^1 and Z^2. The determination of the zero sequence impedance with the presence of earth neutral wires is quite involved and the interested reader is referred to the Carson's formula [14]. To get an idea of the order of Z^0 we will consider the following simplified configuration. Consider 1-m length of a three-phase line with equilaterally spaced conductors as shown in Figure 10.5. The phase conductors carry zero-sequence (single-phase) currents with return paths through a grounded neutral. The ground surface is approximated to an equivalent fictitious conductor located at the average distance D_n from each of the three phases. Since conductor n carries the return current in opposite direction, we have

$$I_a^0 + I_b^0 + I_c^0 + I_n = 0 \tag{10.34}$$

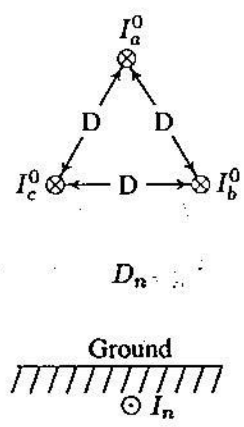


FIGURE 10.5
Zero-sequence current flow with earth return.

Since $I_a^0 = I_b^0 = I_c^0$, we have

$$I_n = -3I_a^0 \tag{10.35}$$

Utilizing the relation for the flux linkages of a conductor in a group expressed by (4.29), the total flux linkage of phase a conductor is

$$\lambda_{a0} = 2 \times 10^{-7}\left(I_a^0 \ln\frac{1}{r'} + I_b^0 \ln\frac{1}{D} + I_c^0 \ln\frac{1}{D} + I_n \ln\frac{1}{D_n}\right) \tag{10.36}$$

Substituting for I_b^0, I_c^0, and I_n in terms of I_a^0, we get

$$\lambda_{a0} = 2 \times 10^{-7} I_a^0\left(\ln\frac{1}{r'} + \ln\frac{1}{D} + \ln\frac{1}{D} - 3\ln\frac{1}{D_n}\right)$$

$$= 2 \times 10^{-7} I_a^0 \ln\frac{D_n^3}{r'D^2} \quad \text{Wb/m} \tag{10.37}$$

Since $L_0 = \lambda_{a0}/I_a^0$, the zero sequence inductance per phase in mH per kilometer length is

$$L_0 = 0.2\ln\frac{D_n^3}{r'D^2}$$

$$= 0.2\ln\frac{DD_n^3}{r'D^3}$$

$$= 0.2\ln\frac{D}{r'} + 3\left(0.2\ln\frac{D_n}{D}\right) \quad \text{mH/Km} \tag{10.38}$$

The first term above is the same as the positive-sequence inductance given by (4.33). Thus the zero sequence reactance can be expressed as

$$X^0 = X^1 + 3X_n \tag{10.39}$$

where

$$X_n = 2\pi f\left(0.2\ln\frac{D_n}{D}\right) \quad \text{m}\Omega\text{/km} \tag{10.40}$$

The zero-sequence impedance of the transmission line is more than three times larger than the positive- or negative-sequence impedance.

10.3.3 SEQUENCE IMPEDANCES OF SYNCHRONOUS MACHINE

The inductances of a synchronous machine depend upon the phase order of the sequence current relative to the direction of rotation of the rotor. The positive-sequence generator impedance is the value found when positive-sequence current flows from the action of an imposed positive-sequence set of voltages. We have seen that the generator positive-sequence reactance varies, and in Section 9.2 one of the reactances X_d'', X_d', or X_d was used for the balanced three-phase fault studies.

When negative-sequence currents are impressed in the stator, the net flux in the air gap rotates at opposite direction to that of the rotor. That is, the net flux rotates at twice synchronous speed relative to the rotor. Since the field voltage is associated with the positive-sequence variables, the field winding has no influence. Consequently, only the damper winding produces an effect in the quadrature axis. Hence, there is no distinction between the transient and subtransient reactances in the quadrature axis as there is in the direct axis. The negative-sequence reactance is close to the positive-sequence subtransient reactance, i.e.,

$$X^2 \simeq X_d'' \tag{10.41}$$

Zero-sequence impedance is the impedance offered by the machine to the flow of the zero-sequence current. We recall that a set of zero sequence currents are all identical. Therefore, if the spatial distribution of mmf is assumed sinusoidal, the resultant air-gap flux would be zero, and there is no reactance due to armature reaction. The machine offers a very small reactance due to the leakage flux. Therefore, the zero-sequence reactance is approximated to the leakage reactance, i.e.,

$$X^0 \simeq X_\ell \tag{10.42}$$

10.3.4 SEQUENCE IMPEDANCES OF TRANSFORMER

In Chapter 3 we obtained the per phase equivalent circuit for a three-phase transformer. In power transformers, the core losses and the magnetization current are on the order of 1 percent of the rated value; therefore, the magnetizing branch is neglected. The transformer is modeled with the equivalent series leakage impedance. Since the transformer is a static device, the leakage impedance will not change if the phase sequence is changed. Therefore, the positive- and negative-sequence impedances are the same. Also, if the transformer permits zero-sequence current flow at all, the phase impedance to zero-sequence is equal to the leakage impedance, and we have

$$Z^0 = Z^1 = Z^2 = Z_\ell \tag{10.43}$$

From Section 3.9.1, we recall that in a Y-Δ, or a Δ-Y transformer, the positive-sequence line voltage on HV side leads the corresponding line voltage on the

LV side by 30°. For the negative-sequence voltage the corresponding phase shift is $-30°$. The equivalent circuit for the zero-sequence impedance depends on the winding connections and also upon whether or not the neutrals are grounded. Figure 10.6 shows some of the more common transformer configurations and their zero-sequence equivalent circuits. We recall that in a transformer, when the core reluctance is neglected, there is an exact mmf balance between the primary and secondary. This means that current can flow in the primary only if there is a current in the secondary. Based on this observation we can check the validity of the zero-sequence circuits by applying a set of zero-sequence voltage to the primary and calculating the resulting currents.

(a) Y-Y connections with both neutrals grounded – We know that the zero sequence current equals the sum of phase currents. Since both neutrals are grounded, there is a path for the zero sequence current to flow in the primary and secondary, and the transformer exhibits the equivalent leakage impedance per phase as shown in Figure 10.6(a).

(b) Y-Y connection with the primary neutral grounded – The primary neutral is grounded, but since the secondary neutral is isolated, the secondary phase current must sum up to zero. This means that the zero-sequence current in the secondary is zero. Consequently, the zero sequence current in the primary is zero, reflecting infinite impedance or an open circuit as shown in Figure 10.6(b).

(c) Y-Δ with grounded neutral – In this configuration the primary currents can flow because there is zero-sequence circulating current in the Δ-connected secondary and a ground return path for the Y-connected primary. Note that no zero-sequence current can leave the Δ terminals, thus there is an isolation between the primary and secondary sides as shown in Figure 10.6(c).

(d) Y-Δ connection with isolated neutral – In this configuration, because the neutral is isolated, zero sequence current cannot flow and the equivalent circuit reflects an infinite impedance or an open as shown in Figure 10.6(d).

(e) Δ-Δ connection – In this configuration zero-sequence currents circulate in the Δ-connected windings, but no currents can leave the Δ terminals, and the equivalent circuit is as shown in Figure 10.6(e).

Notice that the neutral impedance plays an important part in the equivalent circuit. When the neutral is grounded through an impedance Z_n, because $I_n = 3I_0$, in the equivalent circuit the neutral impedance appears as $3Z_n$ in the path of I_0.

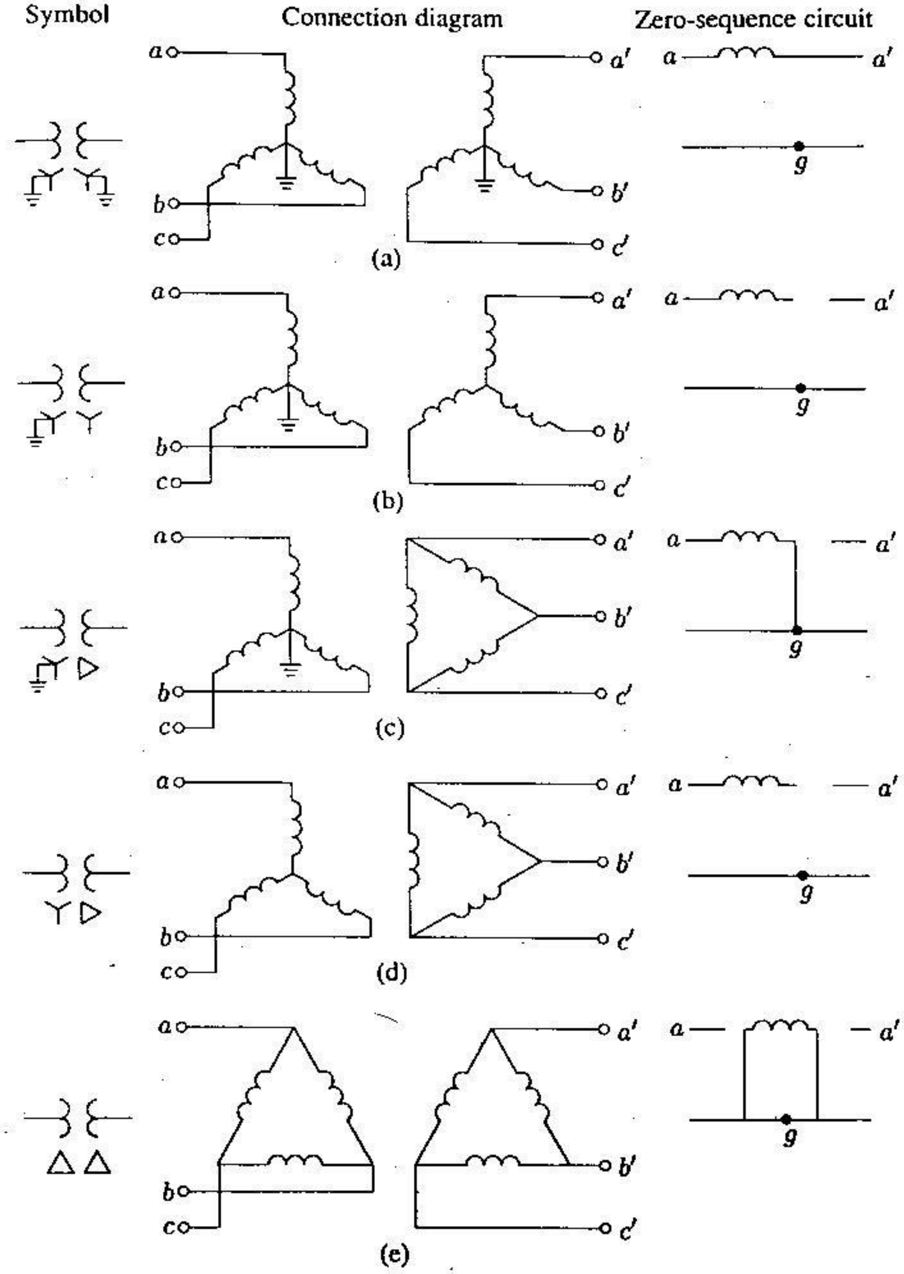


FIGURE 10.6
Transformer zero-sequence equivalent circuits.

Example 10.3

A balanced three-phase voltage of 100-V line-to-neutral is applied to a balanced Y-connected load with ungrounded neutral as shown in Figure 10.7. The three-phase load consists of three mutually-coupled reactances. Each phase has a series reactance of $Z_s = j12\ \Omega$, and the mutual coupling between phases is $Z_m = j4\ \Omega$.

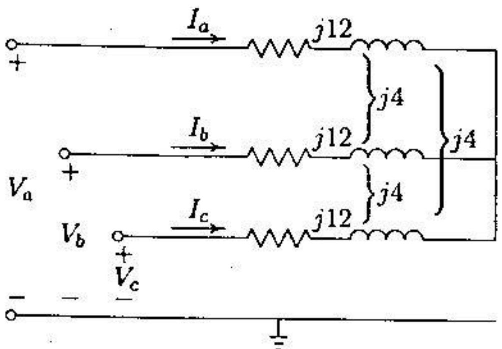


FIGURE 10.7
Circuit for Example 10.3.

(a) Determine the line currents by mesh analysis without using symmetrical components.
(b) Determine the line currents using symmetrical components.

(a) Applying KVL to the two independent mesh equations yields

$$Z_sI_a + Z_mI_b - Z_sI_b - Z_mI_a = V_a - V_b = |V_L|\angle\pi/6$$
$$Z_sI_b + Z_mI_c - Z_sI_c - Z_mI_b = V_b - V_c = |V_L|\angle-\pi/2$$

Also from KCL, we have

$$I_a + I_b + I_c = 0$$

Writing above equations in matrix form, results in

$$\begin{bmatrix} (Z_s - Z_m) & -(Z_s - Z_m) & 0 \\ 0 & (Z_s - Z_m) & -(Z_s - Z_m) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} |V_L|\angle\pi/6 \\ |V_L|\angle-\pi/2 \\ 0 \end{bmatrix}$$

or in compact form

$$\mathbf{Z}_{mesh}\mathbf{I}^{abc} = \mathbf{V}_{mesh}$$

Solving the above equations results in the line currents

$$\mathbf{I}^{abc} = \mathbf{Z}_{mesh}{}^{-1}\mathbf{V}_{mesh}$$

The following commands

```
% (a) Solution by mesh analysis
Zs=j*12; Zm=j*4; Va = 100; VL=Va*sqrt(3);
Z= [(Zs-Zm)  -(Zs-Zm)       0
      0       (Zs-Zm)  -(Zs-Zm)
      1          1         1   ];
V=[VL*cos(pi/6)+j*VL*sin(pi/6)
   VL*cos(-pi/2)+j*VL*sin(-pi/2)
         0                      ];
Y=inv(Z)
Iabc=Y*V;                    % Line currents (Rectangular form)
Iabcp=[abs(Iabc), angle(Iabc)*180/pi] % Line currents (Polar)
```

result in

```
Iabcp =
        12.5    -90.0
        12.5    150.0
        12.5     30.0
```

(b) Using the symmetrical components method, we have

$$\mathbf{V}^{012} = \mathbf{Z}^{012}\mathbf{I}^{012}$$

where

$$\mathbf{V}^{012} = \begin{bmatrix} 0 \\ V_a \\ 0 \end{bmatrix}$$

and from (10.32)

$$\mathbf{Z}^{012} = \begin{bmatrix} Z_s + 2Z_m & 0 & 0 \\ 0 & Z_s - Z_m & 0 \\ 0 & 0 & Z_s - Z_m \end{bmatrix}$$

for the sequence components of currents, we get

$$\mathbf{I}^{012} = [\mathbf{Z}^{012}]^{-1}\mathbf{V}^{012}$$

We write the following commands

```
% (b) Solution by symmetrical components method
Z012=[Zs+2*Zm   0      0          % Symmetrical components matrix
        0     Zs-Zm    0
        0       0    Zs-Zm];
V012=[0; Va ; 0];     %Symmetrical components of phase voltages
I012=inv(Z012)*V012; %Symmetrical components of line  currents
a=cos(2*pi/3)+j*sin(2*pi/3);
A=[ 1  1  1; 1 a^2  a; 1 a a^2];          % Transformation matrix
Iabc=A*I012;                      % Line currents (Rectangular form)
Iabcp=[abs(Iabc), angle(Iabc)*180/pi]  % Line currents (Polar)
```

which result in

```
Iabcp =
        12.5     -90.0
        12.5     150.0
        12.5      30.0
```

This is the same result as in part (a).

Example 10.4

A three-phase unbalanced source with the following phase-to-neutral voltages

$$\mathbf{V}^{abc} = \begin{bmatrix} 200 & \angle 25^\circ \\ 100 & \angle -155^\circ \\ 80 & \angle 100^\circ \end{bmatrix}$$

is applied to the circuit in Figure 10.4 (page 407). The load series impedance per phase is $Z_s = 8 + j24$ and the mutual impedance between phases is $Z_m = j4$. The load and source neutrals are solidly grounded. Determine

(a) The load sequence impedance matrix $\mathbf{Z}^{012} = \mathbf{A}^{-1}\mathbf{Z}^{abc}\mathbf{A}$.
(b) The symmetrical components of voltage.
(c) The symmetrical components of current.
(d) The load phase currents.
(e) The complex power delivered to the load in terms of symmetrical components, $S_{3\phi} = 3(V_a^0 I_a^{0*} + V_a^1 I_a^{1*} + V_a^2 I_a^{2*})$.
(f) The complex power delivered to the load by summing up the power in each phase, $S_{3\phi} = V_a I_a^* + V_b I_b^* + V_c I_c^*$.

We write the following commands


```
Vabc = [200     25
         100  -155
          80   100];
Zabc = [8+j*24      j*4      j*4
           j*4  8+j*24       j*4
          j*4       j*4  8+j*24];
Z012 = zabc2sc(Zabc)      % Symmetrical components of impedance
V012 = abc2sc(Vabc);        % Symmetrical components of voltage
V012p= rec2pol(V012)                % Rectangular to polar form
I012 = inv(Z012)*V012;      % Symmetrical components of current
I012p= rec2pol(I012)                % Rectangular to polar form
Iabc = sc2abc(I012);                           % Phase currents
Iabcp= rec2pol(Iabc)                % Rectangular to polar form
S3ph =3*(V012.')*conj(I012)%Power using symmetrical components
Vabcr = Vabc(:, 1).*(cos(pi/180*Vabc(:, 2)) +...
j*sin(pi/180*Vabc(:, 2)));
S3ph=(Vabcr.')*conj(Iabc)
                      % Power using phase currents and voltages
```

The result is

```
Z012 =
        8.00 + 32.00i   0.00 +  0.00i   0.00 +  0.00i
        0.00 +  0.00i   8.00 + 20.00i   0.00 +  0.00i
        0.00 -  0.00i   0.00 -  0.00i   8.00 + 20.00i

V012p =
         47.7739     57.6268
        112.7841     -0.0331
         61.6231     45.8825

I012p =
          1.4484    -18.3369
          5.2359    -68.2317
          2.8608    -22.3161

Iabcp =
          8.7507    -47.0439
          5.2292    143.2451
          3.0280     39.0675

S3ph =
        9.0471e+002+ 2.3373e+003i

S3ph =
        9.0471e+002+ 2.3373e+003i
```

10.4 SEQUENCE NETWORKS OF A LOADED GENERATOR

Figure 10.8 represents a three-phase synchronous generator with neutral grounded through an impedance Z_n. The generator is supplying a three-phase balanced load.

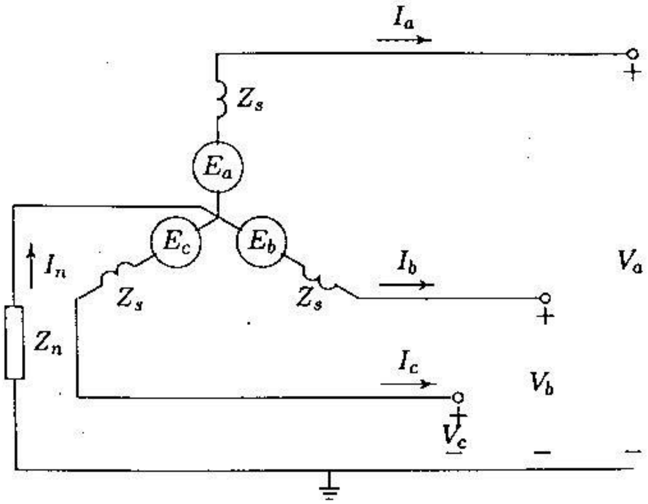


FIGURE 10.8
Three-phase balanced source and impedance.

The synchronous machine generates balanced three-phase internal voltages and is represented as a positive-sequence set of phasors

$$\mathbf{E}^{abc} = \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} E_a \tag{10.44}$$

The machine is supplying a three-phase balanced load. Applying Kirchhoff's voltage law to each phase we obtain

$$\begin{aligned} V_a &= E_a - Z_sI_a - Z_nI_n \\ V_b &= E_b - Z_sI_b - Z_nI_n \\ V_c &= E_c - Z_cI_c - Z_nI_n \end{aligned} \tag{10.45}$$

Substituting for $I_n = I_a + I_b + I_c$, and writing (10.45) in matrix form, we get

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix} - \begin{bmatrix} Z_s + Z_n & Z_n & Z_n \\ Z_n & Z_s + Z_n & Z_n \\ Z_n & Z_n & Z_s + Z_n \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} \tag{10.46}$$

or in compact form, we have

$$\mathbf{V}^{abc} = \mathbf{E}^{abc} - \mathbf{Z}^{abc}\mathbf{I}^{abc} \tag{10.47}$$

where $\mathbf{V}^{abc}$ is the phase terminal voltage vector and $\mathbf{I}^{abc}$ is the phase current vector. Transforming the terminal voltages and current phasors into their symmetrical components results in

$$\mathbf{A}\mathbf{V}_a^{012} = \mathbf{A}\mathbf{E}_a^{012} - \mathbf{Z}^{abc}\mathbf{A}\mathbf{I}_a^{012} \tag{10.48}$$

Multiplying (10.48) by $\mathbf{A}^{-1}$, we get

$$\begin{aligned} \mathbf{V}_a^{012} &= \mathbf{E}_a^{012} - \mathbf{A}^{-1}\mathbf{Z}^{abc}\mathbf{A}\mathbf{I}_a^{012} \\ &= \mathbf{E}_a^{012} - \mathbf{Z}^{012}\mathbf{I}_a^{012} \end{aligned} \tag{10.49}$$

where

$$\mathbf{Z}^{012} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} Z_s + Z_n & Z_n & Z_n \\ Z_n & Z_s + Z_n & Z_n \\ Z_n & Z_n & Z_s + Z_n \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \tag{10.50}$$

Performing the above multiplications, we get

$$\mathbf{Z}^{012} = \begin{bmatrix} Z_s + 3Z_n & 0 & 0 \\ 0 & Z_s & 0 \\ 0 & 0 & Z_s \end{bmatrix} = \begin{bmatrix} Z^0 & 0 & 0 \\ 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \end{bmatrix} \tag{10.51}$$

Since the generated emf is balanced, there is only positive-sequence voltage, i.e.,

$$\mathbf{E}_a^{012} = \begin{bmatrix} 0 \\ E_a \\ 0 \end{bmatrix} \tag{10.52}$$

Substituting for $\mathbf{E}_a^{012}$ and $\mathbf{Z}^{012}$ in (10.49), we get

$$\begin{bmatrix} V_a^0 \\ V_a^1 \\ V_a^2 \end{bmatrix} = \begin{bmatrix} 0 \\ E_a \\ 0 \end{bmatrix} - \begin{bmatrix} Z^0 & 0 & 0 \\ 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \end{bmatrix} \begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} \tag{10.53}$$

Since the above equation is very important, we write it in component form, and we get

$$\begin{aligned} V_a^0 &= 0 - Z^0I_a^0 \\ V_a^1 &= E_a - Z^1I_a^1 \\ V_a^2 &= 0 - Z^2I_a^2 \end{aligned} \tag{10.54}$$

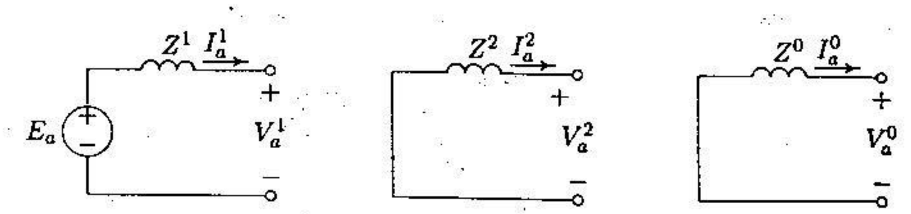


FIGURE 10.9
Sequence networks: (a) Positive-sequence; (b) negative-sequence; (c) zero-sequence.

The three equations given by (10.54) can be represented by the three equivalent sequence networks shown in Figure 10.9.

We make the following important observations.

- The three sequences are independent.
- The positive-sequence network is the same as the one-line diagram used in studying balanced three-phase currents and voltages.
- Only the positive-sequence network has a voltage source. Therefore, the positive-sequence current causes only positive-sequence voltage drops.
- There is no voltage source in the negative- or zero-sequence networks.
- Negative- and zero-sequence currents cause negative- and zero-sequence voltage drops only.
- The neutral of the system is the reference for positive-and negative-sequence networks, but ground is the reference for the zero-sequence networks. Therefore, the zero-sequence current can flow only if the circuit from the system neutrals to ground is complete.
- The grounding impedance is reflected in the zero sequence network as $3Z_n$.
- The three-sequence systems can be solved separately on a per phase basis. The phase currents and voltages can then be determined by superposing their symmetrical components of current and voltage respectively.

We are now ready with mathematical tools to analyze various types of unbalanced faults. First, the fault current is obtained using Thévenin's method and algebraic manipulation of sequence networks. The analysis will then be extended to find the bus voltages and fault current during fault, for different types of faults using the bus impedance matrix.

10.5 SINGLE LINE-TO-GROUND FAULT

Figure 10.10 illustrates a three-phase generator with neutral grounded through impedance Z_n.

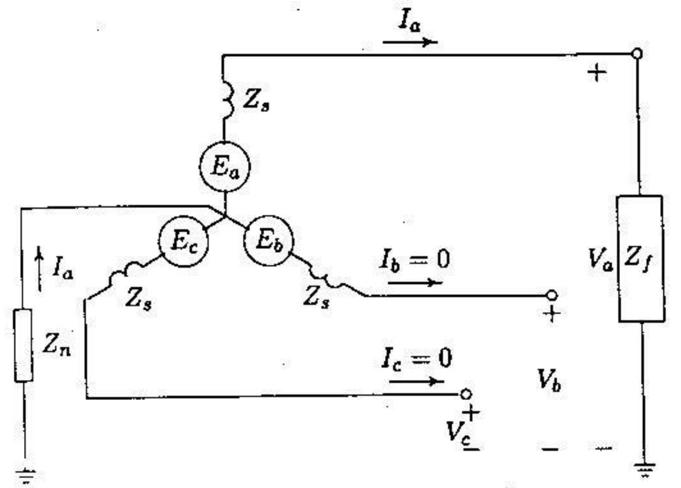


FIGURE 10.10
Line-to-ground fault on phase a.

Suppose a line-to-ground fault occurs on phase a through impedance Z_f. Assuming the generator is initially on no-load, the boundary conditions at the fault point are

$$V_a = Z_f I_a \tag{10.55}$$

$$I_b = I_c = 0 \tag{10.56}$$

Substituting for $I_b = I_c = 0$, the symmetrical components of currents from (10.14) are

$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} I_a \\ 0 \\ 0 \end{bmatrix} \tag{10.57}$$

From the above equation, we find that

$$I_a^0 = I_a^1 = I_a^2 = \frac{1}{3} I_a \tag{10.58}$$

Phase a voltage in terms of symmetrical components is

$$V_a = V_a^0 + V_a^1 + V_a^2 \tag{10.59}$$

Substituting for V_a^0, V_a^1, and V_a^2 from (10.54) and noting $I_a^0 = I_a^1 = I_a^2$, we get

$$V_a = E_a - (Z^1 + Z^2 + Z^0)I_a^0 \tag{10.60}$$

where $Z^0 = Z_s + 3Z_n$. Substituting for V_a from (10.55), and noting $I_a = 3I_a^0$, we get

$$3Z_f I_a^0 = E_a - (Z^1 + Z^2 + Z^0)I_a^0 \tag{10.61}$$

or

$$I_a^0 = \frac{E_a}{Z^1 + Z^2 + Z^0 + 3Z_f} \tag{10.62}$$

The fault current is

$$I_a = 3I_a^0 = \frac{3E_a}{Z^1 + Z^2 + Z^0 + 3Z_f} \tag{10.63}$$

Substituting for the symmetrical components of currents in (10.54), the symmetrical components of voltage and phase voltages at the point of fault are obtained.

Equations (10.58) and (10.62) can be represented by connecting the sequence networks in series as shown in the equivalent circuit of Figure 10.11. Thus, for line-to-ground faults, the Thévenin impedance to the point of fault is obtained for each sequence network, and the three sequence networks are placed in series. In many practical applications, the positive- and negative-sequence impedances are found to be equal. If the generator neutral is solidly grounded, $Z_n = 0$ and for bolted faults $Z_f = 0$.

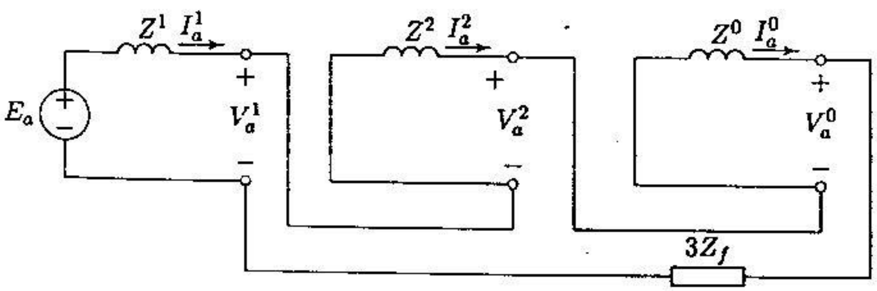


FIGURE 10.11
Sequence network connection for line-to-ground fault.

10.6 LINE-TO-LINE FAULT

Figure 10.12 shows a three-phase generator with a fault through an impedance Z_f between phases b and c. Assuming the generator is initially on no-load, the boundary conditions at the fault point are

$$V_b - V_c = Z_f I_b \tag{10.64}$$

$$I_b + I_c = 0 \tag{10.65}$$

$$I_a = 0 \tag{10.66}$$

Substituting for $I_a = 0$, and $I_c = -I_b$, the symmetrical components of currents from (10.14) are

$$\begin{bmatrix} I_a^0 \\ I_a^1 \\ I_a^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 0 \\ I_b \\ -I_b \end{bmatrix} \tag{10.67}$$

From the above equation, we find that

$$I_a^0 = 0 \tag{10.68}$$

$$I_a^1 = \frac{1}{3}(a - a^2)I_b \tag{10.69}$$

$$I_a^2 = \frac{1}{3}(a^2 - a)I_b \tag{10.70}$$

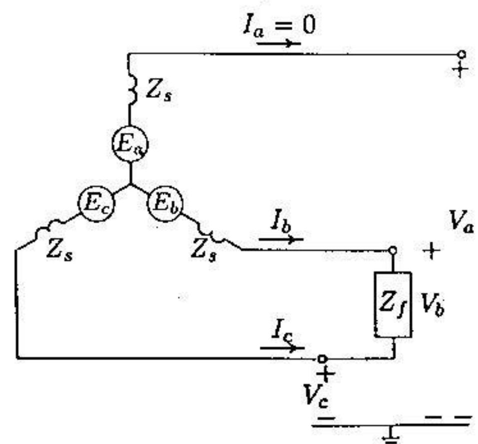


FIGURE 10.12
Line-to-line fault between phase b and c.

Also, from (10.69) and (10.70), we note that

$$I_a^1 = -I_a^2 \tag{10.71}$$

From (10.16), we have

$$\begin{aligned} V_b - V_c &= (a^2 - a)(V_a^1 - V_a^2) \\ &= Z_f I_b \end{aligned} \tag{10.72}$$

Substituting for V_a^1 and V_a^2 from (10.54) and noting $I_a^2 = -I_a^1$, we get

$$(a^2 - a)[E_a - (Z^1 + Z^2)I_a^1] = Z_f I_b \tag{10.73}$$

Substituting for I_b from (10.69), we get

$$E_a - (Z^1 + Z^2)I_a^1 = Z_f \frac{3I_a^1}{(a - a^2)(a^2 - a)} \tag{10.74}$$

Since $(a - a^2)(a^2 - a) = 3$, solving for I_a^1 results in

$$I_a^1 = \frac{E_a}{Z^1 + Z^2 + Z_f} \tag{10.75}$$

The phase currents are

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ I_a^1 \\ -I_a^1 \end{bmatrix} \tag{10.76}$$

The fault current is

$$I_b = -I_c = (a^2 - a)I_a^1 \tag{10.77}$$

or

$$I_b = -j\sqrt{3}\, I_a^1 \tag{10.78}$$

Substituting for the symmetrical components of currents in (10.54), the symmetrical components of voltage and phase voltages at the point of fault are obtained.

Equations (10.71) and (10.75) can be represented by connecting the positive- and negative-sequence networks in opposition as shown in the equivalent circuit of Figure 10.13. In many practical applications, the positive- and negative-sequence impedances are found to be equal. For a bolted fault, $Z_f = 0$.

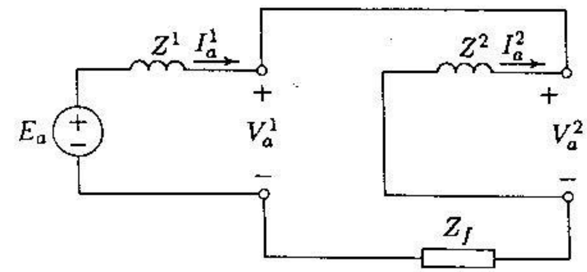


FIGURE 10.13
Sequence network connection for line-to-line fault.

10.7 DOUBLE LINE-TO-GROUND FAULT

Figure 10.14 shows a three-phase generator with a fault on phases b and c through an impedance Z_f to ground. Assuming the generator is initially on no-load, the boundary conditions at the fault point are

$$V_b = V_c = Z_f(I_b + I_c) \tag{10.79}$$
$$I_a = I_a^0 + I_a^1 + I_a^2 = 0 \tag{10.80}$$

From (10.16), the phase voltages V_b and V_c are

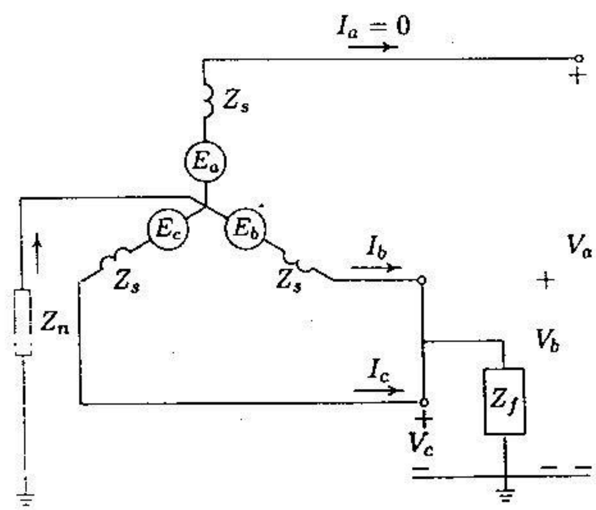


FIGURE 10.14
Double line-to-ground fault.

$$V_b = V_a^0 + a^2V_a^1 + aV_a^2 \quad (10.81)$$
$$V_c = V_a^0 + aV_a^1 + a^2V_a^2 \quad (10.82)$$

Since $V_b = V_c$, from above we note that

$$V_a^1 = V_a^2 \quad (10.83)$$

Substituting for the symmetrical components of currents in (10.79), we get

$$\begin{aligned} V_b &= Z_f(I_a^0 + a^2I_a^1 + aI_a^2 + I_a^0 + aI_a^1 + a^2I_a^2) \\ &= Z_f(2I_a^0 - I_a^1 - I_a^2) \\ &= 3Z_fI_a^0 \end{aligned} \quad (10.84)$$

Substituting for V_b from (10.84) and for V_a^2 from (10.83) into (10.81), we have

$$\begin{aligned} 3Z_fI_a^0 &= V_a^0 + (a^2 + a)V_a^1 \\ &= V_a^0 - V_a^1 \end{aligned} \quad (10.85)$$

Substituting for the symmetrical components of voltage from (10.54) into (10.85) and solving for I_a^0, we get

$$I_a^0 = -\frac{E_a - Z^1I_a^1}{Z^0 + 3Z_f} \quad (10.86)$$

Also, substituting for the symmetrical components of voltage in (10.83), we obtain

$$I_a^2 = -\frac{E_a - Z^1I_a^1}{Z^2} \quad (10.87)$$

Substituting for I_a^0 and I_a^2 into (10.80) and solving for I_a^1, we get

$$I_a^1 = \frac{E_a}{Z^1 + \frac{Z^2(Z^0+3Z_f)}{Z^2+Z^0+3Z_f}} \quad (10.88)$$

Equations (10.86)–(10.88) can be represented by connecting the positive-sequence impedance in series with the parallel combination of the negative-sequence and zero-sequence networks as shown in the equivalent circuit of Figure 10.15. The value of I_a^1 found from (10.88) is substituted in (10.86) and (10.87), and I_a^0 and I_a^2 are found. The phase currents are then found from (10.8). Finally, the fault current is obtained from

$$I_f = I_b + I_c = 3I_a^0 \quad (10.89)$$

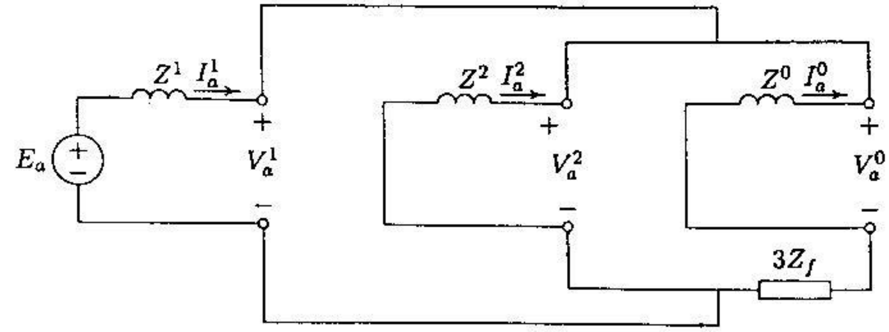


FIGURE 10.15
Sequence network connection for double line-to-ground fault.

Example 10.5

The one-line diagram of a simple power system is shown in Figure 10.16. The neutral of each generator is grounded through a current-limiting reactor of 0.25/3 per unit on a 100-MVA base. The system data expressed in per unit on a common 100-MVA base is tabulated below. The generators are running on no-load at their rated voltage and rated frequency with their emfs in phase.

Determine the fault current for the following faults.

(a) A balanced three-phase fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

(b) A single line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.10$ per unit.

(c) A line-to-line fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

(d) A double line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

Item	Base MVA	Voltage Rating	X^1	X^2	X^0
G_1	100	20 kV	0.15	0.15	0.05
G_2	100	20 kV	0.15	0.15	0.05
T_1	100	20/220 kV	0.10	0.10	0.10
T_2	100	20/220 kV	0.10	0.10	0.10
L_{12}	100	220 kV	0.125	0.125	0.30
L_{13}	100	220 kV	0.15	0.15	0.35
L_{23}	100	220 kV	0.25	0.25	0.7125

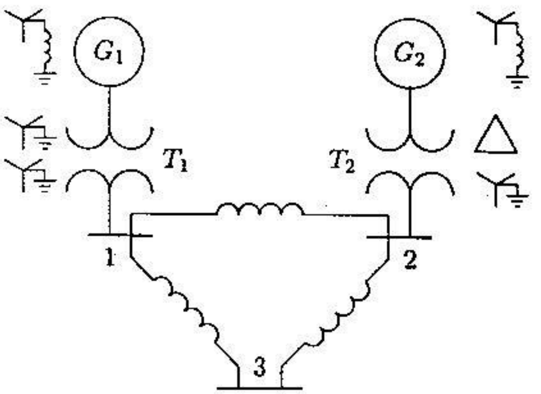


FIGURE 10.16
The one-line diagram for Example 10.5.

The positive-sequence impedance network is shown in Figure 10.17.

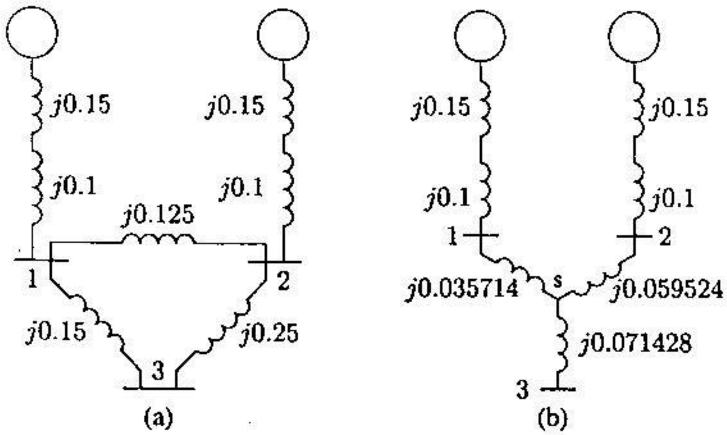


FIGURE 10.17
Positive-sequence impedance diagram for Example 10.5.

To find Thévenin impedance viewed from the faulted bus (bus 3), we convert the delta formed by buses 123 to an equivalent Y as shown in Figure 10.17(b).

$$Z_{1s} = \frac{(j0.125)(j0.15)}{j0.525} = j0.0357143$$

$$Z_{2s} = \frac{(j0.125)(j0.25)}{j0.525} = j0.0595238$$

$$Z_{3s} = \frac{(j0.15)(j0.25)}{j0.525} = j0.0714286$$

Combining the parallel branches, the positive-sequence Thévenin impedance is

$$Z_{33}^1 = \frac{(j0.2857143)(j0.3095238)}{j0.5952381} + j0.0714286$$
$$= j0.1485714 + j0.0714286 = j0.22$$

This is shown in Figure 10.18(a).

(a) Positive-sequence network (b) Negative-sequence network

FIGURE 10.18
Reduction of the positive-sequence Thévenin equivalent network.

Since the negative-sequence impedance of each element is the same as the positive-sequence impedance, we have

$$Z_{33}^2 = Z_{33}^1 = j0.22$$

and the negative-sequence network is as shown in Figure 10.18(b). The equivalent circuit for the zero-sequence network is constructed according to the transformer winding connections of Figure 10.6 and is shown in Figure 10.19.

To find Thévenin impedance viewed from the faulted bus (bus 3), we convert the delta formed by buses 123 to an equivalent Y as shown in Figure 10.19(b).

$$Z_{1s} = \frac{(j0.30)(j0.35)}{j1.3625} = j0.0770642$$

$$Z_{2s} = \frac{(j0.30)(j0.7125)}{j1.3625} = j0.1568807$$

$$Z_{3s} = \frac{(j0.35)(j0.7125)}{j1.3625} = j0.1830257$$

Combining the parallel branches, the zero-sequence Thévenin impedance is

$$Z_{33}^0 = \frac{(j0.4770642)(j0.2568807)}{j0.7339449} + j0.1830275$$
$$= j0.1669725 + j0.1830275 = j0.35$$

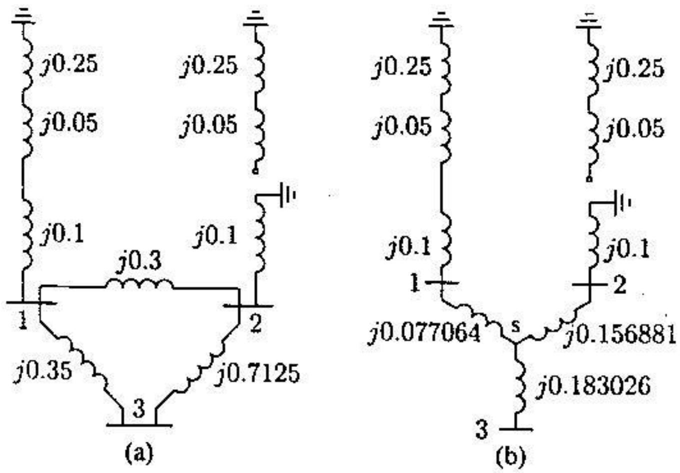


FIGURE 10.19
Zero-sequence impedance diagram for Example 10.5.

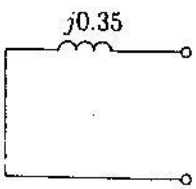


FIGURE 10.20
Zero-sequence network for Example 10.5.

The zero-sequence impedance diagram is shown in Figure 10.20.

(a) Balanced three-phase fault at bus 3.

Assuming the no-load generated emfs are equal to 1.0 per unit, the fault current is

$$I_3^a(F) = \frac{V_{3(0)}^a}{Z_{33}^1 + Z_f} = \frac{1.0}{j0.22 + j0.1} = -j3.125 \quad \text{pu}$$
$$= 820.1\angle{-90^\circ} \quad \text{A}$$

(b) Single line-to-ground fault at bus 3.

From (10.62), the sequence components of the fault current are

$$I_3^0 = I_3^1 = I_3^2 = \frac{V_3^a(0)}{Z_{33}^1 + Z_{33}^2 + Z_{33}^0 + 3Z_f}$$
$$= \frac{1.0}{j0.22 + j0.22 + j0.35 + 3(j0.1)}$$
$$= -j0.9174 \quad \text{pu}$$

The fault current is

$$\begin{bmatrix} I_3^a \\ I_3^b \\ I_3^c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} I_3^0 \\ I_3^0 \\ I_3^0 \end{bmatrix} = \begin{bmatrix} 3I_3^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -j2.7523 \\ 0 \\ 0 \end{bmatrix} \quad \text{pu}$$

(c) Line-to line fault at bus 3.

The zero-sequence component of current is zero, i.e.,

$$I_3^0 = 0$$

From (10.75), the positive- and negative-sequence components of the fault current are

$$I_3^1 = -I_3^2 = \frac{V_{3(0)}^a}{Z_{33}^1 + Z_{33}^2 + Z_f} = \frac{1}{j0.22 + j0.22 + j0.1} = -j1.8519 \quad \text{pu}$$

The fault current is

$$\begin{bmatrix} I_3^a \\ I_3^b \\ I_3^c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ -j1.8519 \\ j1.8519 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.2075 \\ 3.2075 \end{bmatrix}$$

(d) Double line-to line-fault at bus 3.

From (10.88), the positive-sequence component of the fault current is

$$I_3^1 = \frac{V_{3(0)}^a}{Z_{33}^1 + \frac{Z_{33}^2(Z_{33}^0 + 3Z_f)}{Z_{33}^2 + Z_{33}^0 + 3Z_f}} = \frac{1}{j0.22 + \frac{j0.22(j0.35 + j0.3)}{j0.22 + j0.35 + j.3}} = -j2.6017 \quad \text{pu}$$

The negative-sequence component of current from (10.87) is

$$I_3^2 = -\frac{V_{3(0)}^a - Z_{33}^1 I_3^1}{Z_{33}^2} = -\frac{1 - (j0.22)(-j2.6017)}{j0.22} = j1.9438 \quad \text{pu}$$

The zero-sequence component of current from (10.86) is

$$I_3^0 = -\frac{V_{3(0)}^a - Z_{33}^1 I_3^1}{Z_{33}^0 + 3Z_f} = -\frac{1 - (j0.22)(-j2.6017)}{j0.35 + j0.3} = j0.6579 \quad \text{pu}$$

and the phase currents are

$$\begin{bmatrix} I_3^a \\ I_3^b \\ I_3^c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} j0.6579 \\ -j2.6017 \\ j1.9438 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.058\angle 165.93^\circ \\ 4.058\angle 14.07^\circ \end{bmatrix}$$

The fault current is

$$I_3(F) = I_3^b + I_3^c = 1.9732\angle 90^\circ$$

10.8 UNBALANCED FAULT ANALYSIS USING BUS IMPEDANCE MATRIX

We have seen that when the network is balanced, the symmetrical components impedances are diagonal, so that it is possible to calculate $\mathbf{Z}_{bus}$ separately for zero-, positive-, and negative-sequence networks. Also, we have observed that for a fault at bus k, the diagonal element in the k axis of the bus impedance matrix $\mathbf{Z}_{bus}$ is the Thévenin impedance to the point of fault. In order to obtain a solution for the unbalanced faults, the bus impedance matrix for each sequence network is obtained separately, then the sequence impedances $Z^0{}_{kk}$, $Z^1{}_{kk}$, and $Z^2{}_{kk}$ are connected together as described in Figures 10.11, 10.13, and 10.15. The fault formulas for various unbalanced faults is summarized below. In writing the symmetrical components of voltage and currents, the subscript a is left out and the symmetrical components are understood to refer to phase a.

10.8.1 SINGLE LINE-TO-GROUND FAULT USING $\mathbf{Z}_{bus}$

Consider a fault between phase a and ground through an impedance Z_f at bus k as shown in Figure 10.21. The line-to-ground fault requires that positive-, negative-, and zero-sequence networks for phase a be placed in series in order to compute the zero-sequence fault current as given by (10.62). Thus, in general, for a fault at bus k, the symmetrical components of fault current is

$$I_k^0 = I_k^1 = I_k^2 = \frac{V_k(0)}{Z_{kk}^1 + Z_{kk}^2 + Z_{kk}^0 + 3Z_f} \tag{10.90}$$

where Z_{kk}^1, Z_{kk}^2, and Z_{kk}^0 are the diagonal elements in the k axis of the corresponding bus impedance matrix and $V_k(0)$ is the prefault voltage at bus k. The fault phase current is

$$I_k^{abc} = \mathbf{A} I_k^{012} \tag{10.91}$$

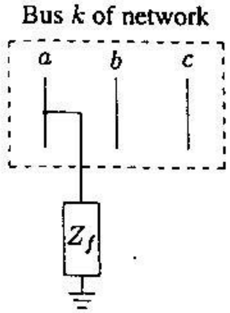


FIGURE 10.21
Line-to-ground fault at bus k.

10.8.2 LINE-TO-LINE FAULT USING $\mathbf{Z}_{bus}$

Consider a fault between phases b and c through an impedance Z_f at bus k as shown in Figure 10.22.

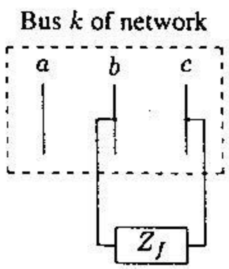


FIGURE 10.22
Line-to-line fault at bus k.

The phase a sequence network of Figure 10.13 is applicable here, where the positive- and negative-sequence networks are placed in opposition. The symmetrical components of the fault current as given from (10.68), (10.71), and (10.75) are

$$I_k^0 = 0 \tag{10.92}$$

$$I_k^1 = -I_k^2 = \frac{V_k(0)}{Z_{kk}^1 + Z_{kk}^2 + Z_f} \tag{10.93}$$

where Z_{kk}^1, and Z_{kk}^2 are the diagonal elements in the k axis of the corresponding bus impedance matrix. The fault phase current is then obtained from (10.91).

10.8.3 DOUBLE LINE-TO-GROUND FAULT USING $\mathbf{Z}_{bus}$

Consider a fault between phases b and c through an impedance Z_f to ground at bus k as shown in Figure 10.23.

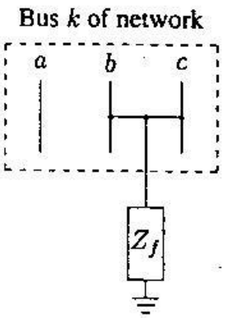


FIGURE 10.23
Double line-to-ground fault at bus k.

The phase a sequence network of Figure 10.15 is applicable here, where the positive-sequence impedance is placed in series with the parallel combination of the negative- and zero-sequence networks. The symmetrical components of the fault current as given from (10.86)–(10.88) are

$$I_k^1 = \frac{V_k(0)}{Z_{kk}^1 + \frac{Z_{kk}^2(Z_{kk}^0+3Z_f)}{Z_{kk}^2+Z_{kk}^0+3Z_f}} \tag{10.94}$$

$$I_k^2 = -\frac{V_k(0) - Z_{kk}^1 I_k^1}{Z_{kk}^2} \tag{10.95}$$

$$I_k^0 = -\frac{V_k(0) - Z_{kk}^1 I_k^1}{Z_{kk}^0 + 3Z_f} \tag{10.96}$$

where Z_{kk}^1, and Z_{kk}^2, and Z_{kk}^0 are the diagonal elements in the k axis of the corresponding bus impedance matrix. The phase currents are obtained from (10.91), and the fault current is

$$I_k(F) = I_k^b + I_k^c \tag{10.97}$$

10.8.4 BUS VOLTAGES AND LINE CURRENTS DURING FAULT

Using the sequence components of the fault current given by the formulas in (10.54), the symmetrical components of the ith bus voltages during fault are obtained

$$V_i^0(F) = 0 - Z_{ik}^0 I_k^0$$

$$V_i^1(F) = V_i^1(0) - Z_{ik}^1 I_k^1 \tag{10.98}$$
$$V_i^2(F) = 0 - Z_{ik}^2 I_k^2$$

where $V_i^1(0) = V_i(0)$ is the prefault phase voltage at bus i. The phase voltages during fault are

$$V_i^{abc} = \mathbf{A} V_i^{012} \tag{10.99}$$

The symmetrical components of fault current in line i to j is given by

$$I_{ij}^0 = \frac{V_i^0(F) - V_j^0(F)}{z_{ij}^0}$$
$$I_{ij}^1 = \frac{V_i^1(F) - V_j^1(F)}{z_{ij}^1} \tag{10.100}$$
$$I_{ij}^2 = \frac{V_i^2(F) - V_j^2(F)}{z_{ij}^2}$$

where z_{ij}^0, z_{ij}^1, and z_{ij}^2, are the zero-, positive-, and negative-sequence components of the actual line impedance between buses i and j. Having obtained the symmetrical components of line current, the phase fault current in line i to j is

$$I_{ij}^{abc} = \mathbf{A} I_{ij}^{012} \tag{10.101}$$

Example 10.6

Solve Example 10.5 using the bus impedance matrix. In addition, for each type of fault determine the bus voltages and line currents during fault.

Using the function **Zbus = zbuild(zdata)**, $\mathbf{Z}_{bus}^1$ and $\mathbf{Z}_{bus}^0$ are found for the positive-sequence network of Figure 10.17 and the zero-sequence network of Figure 10.19. The positive-sequence bus impedance matrix is

$$\mathbf{Z}_{bus}^1 = \begin{bmatrix} j0.1450 & j0.1050 & j0.1300 \\ j0.1050 & j0.1450 & j0.1200 \\ j0.1300 & j0.1200 & j0.2200 \end{bmatrix}$$

and the zero-sequence bus impedance matrix is

$$\mathbf{Z}_{bus}^0 = \begin{bmatrix} j0.1820 & j0.0545 & j0.1400 \\ j0.0545 & j0.0864 & j0.0650 \\ j0.1400 & j0.0650 & j0.3500 \end{bmatrix}$$

Since positive- and negative-sequence reactances for the system in Example 10.5 are identical, $\mathbf{Z}_{bus}^1 = \mathbf{Z}_{bus}^2$.

(a) Balanced three-phase fault at bus 3 through a fault impedance $Z_f = j0.1$.

The symmetrical components of fault current is given by

$$I_3^{012}(F) = \begin{bmatrix} 0 \\ \frac{1}{Z_{33}^1 + Z_f} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{j0.22 + j0.1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -j3.125 \\ 0 \end{bmatrix}$$

The fault current is

$$I_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ -j3.125 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.125\angle-90^\circ \\ 3.125\angle150^\circ \\ 3.125\angle30^\circ \end{bmatrix}$$

For balanced fault we only have the positive-sequence component of voltage. Thus, from (10.98), bus voltages during fault for phase a are

$$\begin{aligned} V_1(F) &= 1 - Z_{13}^1 I_3(F) = 1 - j0.13(-j3.125) = 0.59375 \\ V_2(F) &= 1 - Z_{23}^1 I_3(F) = 1 - j0.12(-j3.125) = 0.62500 \\ V_3(F) &= 1 - Z_{33}^1 I_3(F) = 1 - j0.22(-j3.125) = 0.31250 \end{aligned}$$

Fault currents in lines for phase a are

$$\begin{aligned} I_{21}(F) &= \frac{V_2(F) - V_1(F)}{z_{12}^1} = \frac{0.62500 - 0.59375}{j0.125} = 0.2500\angle-90^\circ \\ I_{13}(F) &= \frac{V_1(F) - V_3(F)}{z_{13}^1} = \frac{0.59375 - 0.31250}{j0.15} = 0.1875\angle-90^\circ \\ I_{23}(F) &= \frac{V_2(F) - V_3(F)}{z_{23}^1} = \frac{0.62500 - 0.31250}{j0.25} = 0.125\angle-90^\circ \end{aligned}$$

(b) Single line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.1$.

From (10.90), the symmetrical components of fault current is given by

$$\begin{aligned} I_3^0(F) = I_3^1(F) = I_3^2(F) &= \frac{1.0}{Z_{33}^1 + Z_{33}^2 + Z_{33}^0 + 3Z_f} \\ &= \frac{1.0}{j0.22 + j0.22 + j0.35 + j3(0.1)} = -j0.9174 \end{aligned}$$

The fault current is

$$I_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} -j0.9174 \\ -j0.9174 \\ -j0.9174 \end{bmatrix} = \begin{bmatrix} 2.7523\angle-90^\circ \\ 0\angle0^\circ \\ 0\angle0^\circ \end{bmatrix}$$

From (10.98), the symmetrical components of bus voltages during fault are

$$V_1^{012}(F) = \begin{bmatrix} 0 - Z_{13}^0 I_3^0 \\ V_1^1(0) - Z_{13}^1 I_3^1 \\ 0 - Z_{13}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.140(-j0.9174) \\ 1 - j0.130(-j0.9174) \\ 0 - j0.130(-j0.9174) \end{bmatrix} = \begin{bmatrix} -0.1284 \\ 0.8807 \\ -0.1193 \end{bmatrix}$$

$$V_2^{012}(F) = \begin{bmatrix} 0 - Z_{23}^0 I_3^0 \\ V_2^1(0) - Z_{23}^1 I_3^1 \\ 0 - Z_{23}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.065(-j0.9174) \\ 1 - j0.120(-j0.9174) \\ 0 - j0.120(-j0.9174) \end{bmatrix} = \begin{bmatrix} -0.0596 \\ 0.8899 \\ -0.1101 \end{bmatrix}$$

$$V_3^{012}(F) = \begin{bmatrix} 0 - Z_{33}^0 I_3^0 \\ V_3^1(0) - Z_{33}^1 I_3^1 \\ 0 - Z_{33}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.350(-j0.9174) \\ 1 - j0.220(-j0.9174) \\ 0 - j0.220(-j0.9174) \end{bmatrix} = \begin{bmatrix} -0.3211 \\ 0.7982 \\ -0.2018 \end{bmatrix}$$

Bus voltages during fault are

$$V_1^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} -0.1284 \\ 0.8807 \\ -0.1193 \end{bmatrix} = \begin{bmatrix} 0.633\angle0^\circ \\ 1.0046\angle-120.45^\circ \\ 1.0046\angle+120.45^\circ \end{bmatrix}$$

$$V_2^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} -0.0596 \\ 0.8899 \\ -0.1101 \end{bmatrix} = \begin{bmatrix} 0.7207\angle0^\circ \\ 0.9757\angle-117.43^\circ \\ 0.9757\angle+117.43^\circ \end{bmatrix}$$

$$V_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} -0.3211 \\ 0.7982 \\ -0.2018 \end{bmatrix} = \begin{bmatrix} 0.2752\angle0^\circ \\ 1.0647\angle-125.56^\circ \\ 1.0647\angle+125.56^\circ \end{bmatrix}$$

The symmetrical components of fault currents in lines for phase a are

$$I_{21}^{012} = \begin{bmatrix} \frac{V_2^0(F) - V_1^0(F)}{z_{12}^0} \\ \frac{V_2^1(F) - V_1^1(F)}{z_{12}^1} \\ \frac{V_2^2(F) - V_1^2(F)}{z_{12}^2} \end{bmatrix} = \begin{bmatrix} \frac{-0.0596-(-0.1284)}{j0.3} \\ \frac{0.8899-0.8807)}{j0.125} \\ \frac{-0.1101-(-0.1193)}{j0.125} \end{bmatrix} = \begin{bmatrix} 0.2294\angle-90^\circ \\ 0.0734\angle-90^\circ \\ 0.0734\angle-90^\circ \end{bmatrix}$$

$$I_{13}^{012} = \begin{bmatrix} \frac{V_1^0(F) - V_3^0(F)}{z_{13}^0} \\ \frac{V_1^1(F) - V_3^1(F)}{z_{13}^1} \\ \frac{V_1^2(F) - V_3^2(F)}{z_{13}^2} \end{bmatrix} = \begin{bmatrix} \frac{-0.1284-(-0.3211)}{j0.35} \\ \frac{0.8807-0.7982)}{j0.15} \\ \frac{-0.1193-(-0.2018)}{j0.15} \end{bmatrix} = \begin{bmatrix} 0.5505\angle-90^\circ \\ 0.5505\angle-90^\circ \\ 0.5505\angle-90^\circ \end{bmatrix}$$

$$I_{23}^{012} = \begin{bmatrix} \frac{V_2^0(F)-V_3^0(F)}{z_{23}^0} \\ \frac{V_2^1(F)-V_3^1(F)}{z_{23}^1} \\ \frac{V_2^2(F)-V_3^2(F)}{z_{23}^2} \end{bmatrix} = \begin{bmatrix} \frac{-0.0596-(-0.3211)}{j0.7125} \\ \frac{0.8899-0.7982)}{j0.25} \\ \frac{-0.1101-(-0.2018)}{j0.25} \end{bmatrix} = \begin{bmatrix} 0.3670\angle-90^\circ \\ 0.3670\angle-90^\circ \\ 0.3670\angle-90^\circ \end{bmatrix}$$

The line fault currents are

$$I_{21}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.2294\angle-90^\circ \\ 0.0734\angle-90^\circ \\ 0.0734\angle-90^\circ \end{bmatrix} = \begin{bmatrix} 0.3761\angle-90^\circ \\ 0.1560\angle-90^\circ \\ 0.1560\angle-90^\circ \end{bmatrix}$$

$$I_{13}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.5505\angle-90^\circ \\ 0.5505\angle-90^\circ \\ 0.5505\angle-90^\circ \end{bmatrix} = \begin{bmatrix} 1.6514\angle-90^\circ \\ 0 \\ 0 \end{bmatrix}$$

$$I_{23}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.3670\angle-90^\circ \\ 0.3670\angle-90^\circ \\ 0.3670\angle-90^\circ \end{bmatrix} = \begin{bmatrix} 1.1009\angle-90^\circ \\ 0 \\ 0 \end{bmatrix}$$

(c) Line-to-line fault at bus 3 through a fault impedance $Z_f = j0.1$.

From (10.92) and (10.93), the symmetrical components of fault current are

$$I_3^0 = 0$$

$$I_3^1 = -I_3^2 = \frac{V_3(0)}{Z_{33}^1 + Z_{33}^2 + Z_f} = \frac{1}{j0.22 + j0.22 + j0.1} = -j1.8519$$

The fault current is

$$I_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ -j1.8519 \\ j1.8519 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.2075 \\ 3.2075 \end{bmatrix}$$

From (10.98), the symmetrical components of bus voltages during fault are

$$V_1^{012}(F) = \begin{bmatrix} 0 \\ V_1^1(0) - Z_{13}^1 I_3^1 \\ 0 - Z_{13}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - j0.130(-j1.8519) \\ 0 - j0.130(j1.8519) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7593 \\ 0.2407 \end{bmatrix}$$

$$V_2^{012}(F) = \begin{bmatrix} 0 \\ V_2^1(0) - Z_{23}^1 I_3^1 \\ 0 - Z_{23}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - j0.120(-j1.8519 \\ 0 - j0.120(j1.8519) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7778 \\ 0.2222 \end{bmatrix}$$

$$V_3^{012}(F) = \begin{bmatrix} 0 \\ V_3^1(0) - Z_{33}^1 I_3^1 \\ 0 - Z_{33}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - j0.220(-j1.8519) \\ 0 - j0.220(j1.8519) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5926 \\ 0.4074 \end{bmatrix}$$

Bus voltages during fault are

$$V_1^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.7593 \\ 0.2407 \end{bmatrix} = \begin{bmatrix} 1\angle 0^\circ \\ 0.672\angle-138.07^\circ \\ 0.672\angle+138.07^\circ \end{bmatrix}$$

$$V_2^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.7778 \\ 0.2222 \end{bmatrix} = \begin{bmatrix} 1\angle 0^\circ \\ 0.6939\angle-136.10^\circ \\ 0.6939\angle+136.10^\circ \end{bmatrix}$$

$$V_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5926 \\ 0.4074 \end{bmatrix} = \begin{bmatrix} 1\angle 0^\circ \\ 0.5251\angle-162.21^\circ \\ 0.5251\angle+162.21^\circ \end{bmatrix}$$

The symmetrical components of fault currents in lines for phase a are

$$I_{21}^{012} = \begin{bmatrix} 0 \\ \frac{V_2^1(F)-V_1^1(F)}{z_{12}^1} \\ \frac{V_2^2(F)-V_1^2(F)}{z_{12}^2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{0.7778-0.7593)}{j0.125} \\ \frac{0.2222-0.2407}{j0.125} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.148\angle-90^\circ \\ 0.148\angle+90^\circ \end{bmatrix}$$

$$I_{13}^{012} = \begin{bmatrix} 0 \\ \frac{V_1^1(F)-V_3^1(F)}{z_{13}^1} \\ \frac{V_1^2(F)-V_3^2(F)}{z_{13}^2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{0.7593-0.5926}{j0.15} \\ \frac{0.2407-0.4074}{j0.15} \end{bmatrix} = \begin{bmatrix} 0 \\ 1.1111\angle-90^\circ \\ 1.1111\angle+90^\circ \end{bmatrix}$$

$$I_{23}^{012} = \begin{bmatrix} 0 \\ \frac{V_2^1(F)-V_3^1(F)}{z_{23}^1} \\ \frac{V_2^2(F)-V_3^2(F)}{z_{23}^2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{0.7778-0.5926}{j0.25} \\ \frac{0.2222-0.4074}{j0.25} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7407\angle-90^\circ \\ 0.7407\angle+90^\circ \end{bmatrix}$$

The line fault currents are

$$I_{21}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.148\angle-90^\circ \\ 0.148 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2566 \\ 0.2566 \end{bmatrix}$$

$$I_{13}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.1111\angle-90^\circ \\ 1.1111\angle+90^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ -1.9245 \\ 1.9245 \end{bmatrix}$$

$$I_{23}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.7407\angle-90^\circ \\ 0.7407\angle+90^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ -1.283 \\ 1.283 \end{bmatrix}$$

(d) Double line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.1$.

From (10.94)–(10.96), the symmetrical components of fault current is given by

$$I_3^1 = \frac{V_3(0)}{Z_{33}^1 + \frac{Z_{33}^2(Z_{33}^0+3Z_f)}{Z_{33}^2+Z_{33}^0+3Z_f}} = -\frac{1}{j0.22 + \frac{j0.22(j0.35+j0.3)}{j0.22+j0.35+j0.3}} = -j2.6017$$

$$I_3^2 = -\frac{V_3(0) - Z_{33}^1 I_{33}^1}{Z_{33}^2} - \frac{1 - j0.22(-j2.6017)}{j0.22} = j1.9438$$

$$I_3^0 = -\frac{V_3(0) - Z_{33}^1 I_{33}^1}{Z_{33}^0 + 3Z_f} = -\frac{1 - j0.22(-j2.6017)}{j0.35 + j0.3} = j0.6579$$

The phase currents at the faulted bus are

$$I_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} j0.6579 \\ -j2.6017 \\ j1.9438 \end{bmatrix} = \begin{bmatrix} 0 \\ 4.0583\angle165.93^\circ \\ 4.0583\angle14.07^\circ \end{bmatrix}$$

and the total fault current is

$$I_3^b + I_3^c = 4.0583\angle165.93^\circ - 4.0583\angle14.07^\circ = 1.9732\angle90^\circ$$

From (10.98), the symmetrical components of bus voltages during fault are

$$V_1^{012}(F) = \begin{bmatrix} 0 - Z_{13}^0 I_3^0 \\ V_1^1(0) - Z_{13}^1 I_3^1 \\ 0 - Z_{13}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.140(j0.6579) \\ 1 - j0.130(-j2.6017) \\ 0 - j0.130(j1.9438) \end{bmatrix} = \begin{bmatrix} 0.0921 \\ 0.6618 \\ 0.2527 \end{bmatrix}$$

$$V_2^{012}(F) = \begin{bmatrix} 0 - Z_{23}^0 I_3^0 \\ V_2^1(0) - Z_{23}^1 I_3^1 \\ 0 - Z_{23}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.065(0.6579) \\ 1 - j0.120(-j2.6017 \\ 0 - j0.120(j1.9438) \end{bmatrix} = \begin{bmatrix} 0.0428 \\ 0.6878 \\ 0.2333 \end{bmatrix}$$

$$V_3^{012}(F) = \begin{bmatrix} 0 - Z_{33}^0 I_3^0 \\ V_3^1(0) - Z_{33}^1 I_3^1 \\ 0 - Z_{33}^2 I_3^2 \end{bmatrix} = \begin{bmatrix} 0 - j0.350(0.6579) \\ 1 - j0.220(-j2.6017) \\ 0 - j0.220(j1.9438) \end{bmatrix} = \begin{bmatrix} 0.2303 \\ 0.4276 \\ 0.4276 \end{bmatrix}$$

Bus voltages during fault are

$$V_1^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.0921 \\ 0.6618 \\ 0.2527 \end{bmatrix} = \begin{bmatrix} 1.0066\angle0^\circ \\ 0.5088\angle-135.86^\circ \\ 0.5088\angle+135.86^\circ \end{bmatrix}$$

$$V_2^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.0428 \\ 0.6878 \\ 0.2333 \end{bmatrix} = \begin{bmatrix} 0.9638\angle0^\circ \\ 0.5740\angle-136.70^\circ \\ 0.5740\angle+136.70^\circ \end{bmatrix}$$

$$V_3^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.2303 \\ 0.4276 \\ 0.4276 \end{bmatrix} = \begin{bmatrix} 1.0855\angle0^\circ \\ 0.1974\angle180^\circ \\ 0.1974\angle+180^\circ \end{bmatrix}$$

The symmetrical components of fault currents in lines for phase a are

$$I_{12}^{012} = \begin{bmatrix} \frac{V_1^0(F)-V_2^0(F)}{z_{12}^0} \\ \frac{V_1^1(F)-V_2^1(F)}{z_{12}^1} \\ \frac{V_1^2(F)-V_2^2(F)}{z_{12}^2} \end{bmatrix} = \begin{bmatrix} \frac{0.0921-0.0428}{j0.3} \\ \frac{0.6618-0.687)}{j0.125} \\ \frac{0.2527-0.2333}{j0.125} \end{bmatrix} = \begin{bmatrix} 0.1645\angle-90^\circ \\ 0.2081\angle+90^\circ \\ 0.1555\angle-90^\circ \end{bmatrix}$$

$$I_{13}^{012} = \begin{bmatrix} \frac{V_1^0(F)-V_3^0(F)}{z_{13}^0} \\ \frac{V_1^1(F)-V_3^1(F)}{z_{13}^1} \\ \frac{V_1^2(F)-V_3^2(F)}{z_{13}^2} \end{bmatrix} = \begin{bmatrix} \frac{0.0921-0.2303}{j0.35} \\ \frac{0.6618-0.4276}{j0.15} \\ \frac{0.2527-0.4276}{j0.15} \end{bmatrix} = \begin{bmatrix} 0.3947\angle+90^\circ \\ 1.5610\angle-90^\circ \\ 1.1663\angle+90^\circ \end{bmatrix}$$

$$I_{23}^{012} = \begin{bmatrix} \frac{V_2^0(F)-V_3^0(F)}{z_{23}^0} \\ \frac{V_2^1(F)-V_3^1(F)}{z_{23}^1} \\ \frac{V_2^2(F)-V_3^2(F)}{z_{23}^2} \end{bmatrix} = \begin{bmatrix} \frac{0.0428-0.2303}{j0.7125} \\ \frac{0.6878-0.4276}{j0.25} \\ \frac{0.2333-0.4276}{j0.25} \end{bmatrix} = \begin{bmatrix} 0.2632\angle+90^\circ \\ 1.0407\angle-90^\circ \\ 0.7775\angle+90^\circ \end{bmatrix}$$

The line fault currents are

$$I_{12}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.1645\angle-90^\circ \\ 0.2081\angle+90^\circ \\ 0.1555\angle-90^\circ \end{bmatrix} = \begin{bmatrix} 0.1118\angle-90^\circ \\ 0.3682\angle-31.21 \\ 0.3682\angle-148.79^\circ \end{bmatrix}$$

$$I_{13}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.3947\angle+90^\circ \\ 1.5610\angle-90^\circ \\ 1.1663\angle+90^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ 2.435\angle165.93^\circ \\ 2.435\angle14.07^\circ \end{bmatrix}$$

$$I_{23}^{abc}(F) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0.2632\angle+90^\circ \\ 1.0407\angle-90^\circ \\ 0.7775\angle+90^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ 1.6233\angle165.93^\circ \\ 1.6233\angle14.07^\circ \end{bmatrix}$$

10.9 UNBALANCED FAULT PROGRAMS

Three functions are developed for the unbalanced fault analysis. These functions are are **lgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2, V)**, **llfault(zdata1, Zbus1,zdata2, Zbus2, V)**, and **dlgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2, V)**. **lgfault** is designed for the single line-to-ground fault analysis, **llfault** for the line-to-line fault analysis, and **dlgfault** for the double line-to-ground fault analysis of a power system network. **lgfault** and **dlgfault** require the positive-, negative-, and zero-sequence bus impedance matrices **Zbus0**, **Zbus1**, and **Zbus2**, and **llfault** requires the positive- and negative-sequence bus impedance matrices **Zbus1**, and **Zbus2**. The last argument **V** is optional. If it is not included, the program sets all the prefault bus voltages to 1.0 per unit. If the variable **V** is included, the prefault bus voltages must be specified by the array **V** containing bus numbers and the complex bus voltage. The voltage vector **V** is automatically generated following the execution of any of the power flow programs.

The bus impedance matrices may be obtained from **Zbus0 = zbuild(zdata0)**, and **Zbus1 = zbuild(zdata1)**. The argument **zdata1** contains the positive-sequence network impedances. **zdata0** contains the zero-sequence network impedances. Arguments **zdata0**, **zdata1** and **zdata2** are an $e \times 4$ matrices containing the impedance data of an e-element network. Columns 1 and 2 are the element bus numbers and columns 3 and 4 contain the element resistance and reactance, respectively, in per unit. Bus number 0 to generator buses contain generator impedances. These may be the subtransient, transient, or synchronous reactances. Also, any other shunt impedances such as capacitors and load impedances to ground (bus 0) may be included in this matrix.

The negative-sequence network has the same topology as the positive-sequence network. The line and transformer negative-sequence impedances are the same as the positive-sequence impedances, however, the generator negative-sequence reactances are different from the positive-sequence values. In the fault analysis of large power system usually the negative-sequence network impedances are assumed to be identical to the positive-sequence impedances. The zero-sequence network topology is different from the positive-sequence network. The zero-sequence network must be constructed according to the transformer winding connections of Figure 10.6. All transformer connections except Y-Y with both neutral grounded result in isolation between the primary and secondary in the zero-sequence network. For these connections the corresponding resistance and reactance columns in the zero-sequence data must be filled with **inf**. For grounded Y-Δ connections, additional entries must be included to represent the transformer impedance from bus 0 to the grounded Y-side. In case the neutral is grounded through an impedance Z_n, an impedance of $3Z_n$ must be added to the transformer reactance. The reader is reminded of the 30° phase shift in a Y-Δ or Δ-Y transformer. According to the ASA convention, the positive-sequence voltage is advanced by 30° when stepping up from the low-voltage side to the high-voltage side. Similarly, the negative-sequence voltage is retarded by 30° when stepping up from low-voltage to the high-voltage side. The phase shifts due to Δ-Y transformers have no effect on the bus voltages and line currents in that part of the system where the fault occurs. However, on the other side of the Δ-Y transformers, the sequence voltages, and currents must be shifted in phase before transforming to the phase quantities. The unbalanced fault programs presently ignores the 30° phase shift in the Δ-Y transformers.

The other function for the formation of the bus impedance matrix is **Zbus = zbuildpi(linedata, gendata, yload)**, which is compatible with the power flow programs. The first argument **linedata** is consistent with the data required for the power flow solution. Columns 1 and 2 are the line bus numbers. Columns 3 through 5 contain the line resistance, reactance, and one-half of the total line charging susceptance in per unit on the specified MVA base. The last column is for the transformer tap setting; for lines, 1 must be entered in this column. The generator reactances are not included in the **linedata** for the power flow program and must be specified separately as required by the **gendata** in the second argument. **gendata** is an $e_g \times 4$ matrix, where each row contains bus 0, generator bus number, resistance and reactance. The last argument **yload** is optional. This is a two-column matrix containing bus number and the complex load admittance. This data is provided by any of the power flow programs **lfgauss**, **lfnewton** or **decouple**. **yload** is automatically generated following the execution of the above power flow programs.

The program prompts the user to enter the faulted bus number and the fault impedance **Zf**. The program obtains the total fault current, bus voltages and line currents during the fault. The use of the above functions are demonstrated in the following examples.

Example 10.7

Use the **lgfault**, **llfault**, and **dlgfault** functions to compute the fault current, bus voltages and line currents in the circuit given in Example 10.5 for the following fault.

(a) A balanced three-phase fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

(b) A single-line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

(c) A line-to-line fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

(d) A double line-to-ground fault at bus 3 through a fault impedance $Z_f = j0.1$ per unit.

In this example all shunt capacitances and loads are neglected and all the prefault bus voltages are assumed to be unity. The positive-sequence impedance diagram in Figure 10.17 is described by the variable **zdata1** and the zero-sequence impedance diagram in Figure 10.19 is described by the variable **zdata0**. The negative-sequence data is assumed to be the same as the positive-sequence data. We use the following commands.

```
zdata1 = [0   1   0    0.25
          0   2   0    0.25
          1   2   0    0.125
          1   3   0    0.15
          2   3   0    0.25];

zdata0 = [0   1   0    0.40
          0   2   0    0.10
          1   2   0    0.30
          1   3   0    0.35
          2   3   0    0.7125];

zdata2 = zdata1;
Zbus1 = zbuild(zdata1)
Zbus0 = zbuild(zdata0)
Zbus2 = Zbus1;
symfault(zdata1, Zbus1)
lgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2)
llfault(zdata1, Zbus1, zdata2, Zbus2)
dlgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2)
```

The result is

```
Three-phase balanced fault analysis
Enter Faulted Bus No. -> 3
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = j*0.1
Balanced three-phase fault at bus No. 3
Total fault current =   3.1250 per unit

Bus Voltages during fault in per unit
     Bus      Voltage        Angle
     No.      Magnitude      Degree
      1        0.5938        0.0000
      2        0.6250        0.0000
      3        0.3125        0.0000

Line currents for fault at bus No. 3
```



```
     From       To      Current      Angle
     Bus        Bus     Magnitude    Degree
      G          1       1.6250     -90.0000
      1          3       1.8750     -90.0000
      G          2       1.5000     -90.0000
      2          1       0.2500     -90.0000
      2          3       1.2500     -90.0000
      3          F       3.1250     -90.0000

Another fault location?
Enter 'y' or 'n' within single quote -> 'n'

Line-to-ground fault analysis
Enter Faulted Bus No. -> 3
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = j*0.1
Single line to-ground fault at bus No. 3
Total fault current =    2.7523 per unit

Bus Voltages during the fault in per unit
     Bus     -------Voltage Magnitude-------
     No.     Phase a      Phase b      Phase c
      1      0.6330       1.0046       1.0046
      2      0.7202       0.9757       0.9757
      3      0.2752       1.0647       1.0647

Line currents for fault at bus No. 3
     From       To      -----Line Current Magnitude----
     Bus        Bus     Phase a     Phase b     Phase c
      1          3      1.6514      0.0000      0.0000
      2          1      0.3761      0.1560      0.1560
      2          3      1.1009      0.0000      0.0000
      3          F      2.7523      0.0000      0.0000

Another fault location?
Enter 'y' or 'n' within single quote -> 'n'

Line-to-line fault analysis
Enter Faulted Bus No. -> 3
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = j*0.1
Line-to-line fault at bus No. 3
Total fault current =    3.2075 per unit

Bus Voltages during the fault in per unit
```

```
Bus     -------Voltage Magnitude-------
No.     Phase a     Phase b     Phase c
1       1.0000      0.6720      0.6720
2       1.0000      0.6939      0.6939
3       1.0000      0.5251      0.5251

Line currents for fault at bus No. 3
     From     To      -----Line Current Magnitude----
     Bus      Bus     Phase a     Phase b     Phase c
      1        3      0.0000      1.9245      1.9245
      2        1      0.0000      0.2566      0.2566
      2        3      0.0000      1.2830      1.2830
      3        F      0.0000      3.2075      3.2075

Another fault location?
Enter 'y' or 'n' within single quote -> 'n'

Double line-to-ground fault analysis
Enter Faulted Bus No. -> 3
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = j*0.1
Double line-to-ground fault at bus No. 3
Total fault current =     1.9737 per unit

Bus Voltages during the fault in per unit
     Bus     -------Voltage Magnitude-------
     No.     Phase a     Phase b     Phase c
     1       1.0066      0.5088      0.5088
     2       0.9638      0.5740      0.5740
     3       1.0855      0.1974      0.1974

Line currents for fault at bus No. 3
     From     To      -----Line Current Magnitude----
     Bus      Bus     Phase a     Phase b     Phase c
      1        3      0.0000      2.4350      2.4350
      2        1      0.1118      0.3682      0.3682
      2        3      0.0000      1.6233      1.6233
      3        F      0.0000      4.0583      4.0583

Another fault location?
Enter 'y' or 'n' within single quote -> 'n'
```

Example 10.8

The 11-bus power system network of an electric utility company is shown in Figure 10.24. The positive- and zero-sequence reactances of the lines and transformers in per unit on a 100-MVA base is tabulated below. The transformer connections are shown in Figure 10.24. The Δ-Y transformer between buses 11 and 7 is grounded through a reactor of reactance 0.08 per unit. The generators positive-, and zero-sequence reactances including the reactance of grounding neutrals on a 100-MVA base is also tabulated below. Resistances, shunt reactances, and loads are neglected, and all negative-sequence reactances are assumed equal to the positive-sequence reactances. Use **zbuild** function to obtain the positive- and zero-sequence bus impedance matrices. Assuming all the prefault bus voltages are equal to $1\angle 0°$, use **lgfault**, **llfault**, and **dlgfault** to compute the fault current, bus voltages, and line currents for the following unbalanced faults.

(a) A bolted single line-to-ground fault at bus 8.
(b) A bolted line-to-line fault at bus 8.
(c) A bolted double line-to-ground fault at bus 8.

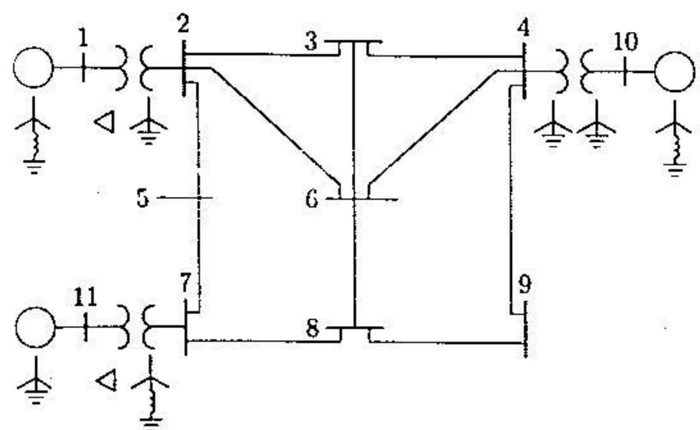


FIGURE 10.24
One-line diagram for Example 10.8.

GENERATOR TRANSIENT IMPEDANCE, PU			
Gen. No.	X^1	X^0	X_n
1	0.20	0.06	0.05
10	0.15	0.04	0.05
11	0.25	0.08	0.00

LINE AND TRANSFORMER DATA			
Bus No.	Bus No.	X^1, PU	X^0, PU
1	2	0.06	0.06
2	3	0.30	0.60
2	5	0.15	0.30
2	6	0.45	0.90
3	4	0.40	0.80
3	6	0.40	0.80
4	6	0.60	1.00
4	9	0.70	1.10
4	10	0.08	0.08
5	7	0.43	0.80
6	8	0.48	0.95
7	8	0.35	0.70
7	11	0.10	0.10
8	9	0.48	0.90

The equivalent circuit for the zero-sequence network is constructed according to the transformer winding connections of Figure 10.6 and is shown in Figure 10.25.

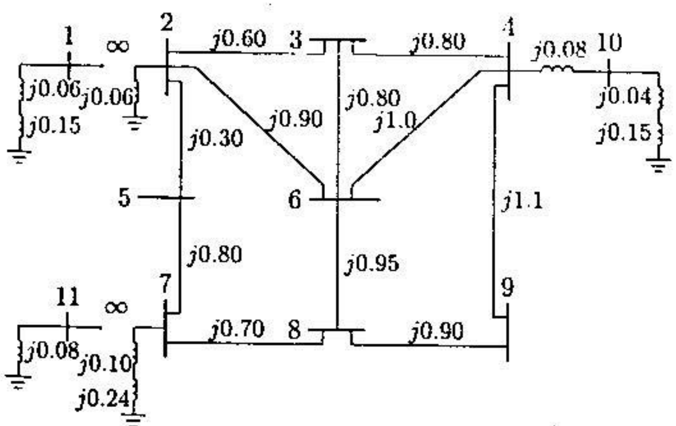


FIGURE 10.25
Zero-sequence network for Example 10.8.

When using **zbuild** function, the generator reactances must be included in the impedance data with bus zero as the reference bus.

The Δ-Y transformers result in isolation between the primary and secondary in the zero-sequence network. For these connections **inf** is entered in the corresponding resistance and reactance columns in the zero-sequence data. For grounded Y-Δconnections, additional entries are included to represent the transformer impedance from bus 0 to the grounded Y-side. The generators and transformers neutral reactor are included in the zero-sequence circuit each with a reactance of $3X_n$.

The positive- and zero-sequence impedance data and the required commands are as follows.

```
zdata1 = [ 0    1    0.00    0.20
           0   10    0.00    0.15
           0   11    0.00    0.25
           1    2    0.00    0.06
           2    3    0.00    0.30
           2    5    0.00    0.15
           2    6    0.00    0.45
           3    4    0.00    0.40
           3    6    0.00    0.40
           4    6    0.00    0.60
           4    9    0.00    0.70
           4   10    0.00    0.08
           5    7    0.00    0.43
           6    8    0.00    0.48
           7    8    0.00    0.35
           7   11    0.00    0.10
           8    9    0.00    0.48];

zdata0 = [ 0    1    0.00    0.06+3*0.05
           0   10    0.00    0.04+3*0.05
           0   11    0.00    0.08
           0    2    0.00    0.06
           0    7    0.00    0.10+3*.08
           1    2     inf     inf
           2    3    0.00    0.60
           2    5    0.00    0.30
           2    6    0.00    0.90
           3    4    0.00    0.80
           3    6    0.00    0.80
           4    6    0.00    1.00
           4    9    0.00    1.10
           4   10    0.00    0.08
           5    7    0.00    0.80
           6    8    0.00    0.95
           7    8    0.00    0.70
           7   11     inf     inf
           8    9    0.00    0.90];
```

```
zdata2=zdata1;
Zbus0 = zbuild(zdata0)
Zbus1 = zbuild(zdata1)
Zbus2 = Zbus1;
lgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2)
llfault(zdata1, Zbus1,zdata2, Zbus2)
dlgfault(zdata0, Zbus0, zdata1, Zbus1, zdata2, Zbus2)
```

The result is

```
Line-to-ground fault analysis
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Single line to-ground fault at bus No. 8
Total fault current =    2.8135 per unit

Bus Voltages during the fault in per unit
Bus    -------Voltage Magnitude-------
No.    Phase a    Phase b    Phase c
1      0.8907     0.9738     0.9738
2      0.8377     0.9756     0.9756
3      0.7451     0.9954     0.9954
4      0.7731     1.0063     1.0063
5      0.7824     0.9823     0.9823
6      0.5936     1.0123     1.0123
7      0.6295     0.9995     0.9995
8      0.0000     1.0898     1.0898
9      0.3299     1.0453     1.0453
10     0.8612     0.9995     0.9995
11     0.8231     0.9588     0.9588

Line currents for fault at bus No. 8
From    To      -----Line Current Magnitude----
Bus     Bus     Phase a    Phase b    Phase c
 1       2      0.5464     0.2732     0.2732
 2       3      0.2113     0.0407     0.0407
 2       6      0.3966     0.0207     0.0207
 3       6      0.2877     0.0073     0.0073
 4       3      0.0764     0.0479     0.0479
 4       6      0.2540     0.0255     0.0255
 4       9      0.5311     0.0023     0.0023
 5       2      0.2753     0.0023     0.0023
 6       8      0.9383     0.0121     0.0121
 7       5      0.2753     0.0023     0.0023
```



```
 7       8      1.3441     0.0098     0.0098
 8       F      2.8135     0.0000     0.0000
 9       8      0.5311     0.0023     0.0023
10       4      0.8615     0.0711     0.0711
11       7      0.7075     0.3538     0.3538

Another fault location?
Enter 'y' or 'n' within single quote -> 'n'

Line-to-line fault analysis
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Line-to-line fault at bus No. 8
Total fault current =    2.9060 per unit

Bus Voltages during the fault in per unit
Bus    -------Voltage Magnitude-------
No.    Phase a    Phase b    Phase c
1      1.0000     0.8576     0.8576
2      1.0000     0.8168     0.8168
3      1.0000     0.7757     0.7757
4      1.0000     0.8157     0.8157
5      1.0000     0.7838     0.7838
6      1.0000     0.6871     0.6871
7      1.0000     0.6947     0.6947
8      1.0000     0.5000     0.5000
9      1.0000     0.5646     0.5646
10     1.0000     0.8778     0.8778
11     1.0000     0.7749     0.7749

Line currents for fault at bus No. 8
From    To      -----Line Current Magnitude----
Bus     Bus     Phase a    Phase b    Phase c
 1       2      0.0000     0.8465     0.8465
 2       3      0.0000     0.1762     0.1762
 2       5      0.0000     0.2820     0.2820
 2       6      0.0000     0.3883     0.3883
 3       6      0.0000     0.3047     0.3047
 4       3      0.0000     0.1285     0.1285
 4       6      0.0000     0.2887     0.2887
 4       9      0.0000     0.5461     0.5461
 5       7      0.0000     0.2820     0.2820
 6       8      0.0000     0.9817     0.9817
 7       8      0.0000     1.3782     1.3782
```

8	F	0.0000	2.9060	2.9060
9	8	0.0000	0.5461	0.5461
10	4	0.0000	0.9633	0.9633
11	7	0.0000	1.0962	1.0962

```
Another fault location?
Enter 'y' or 'n' within single quote -> 'n'

Double line-to-ground fault analysis
Enter Faulted Bus No. -> 8
Enter Fault Impedance Zf = R + j*X in
complex form (for bolted fault enter 0). Zf = 0
Double line-to-ground fault at bus No. 8
Total fault current =     2.4222 per unit

Bus Voltages during the fault in per unit
```

Bus No.	Phase a	Phase b	Phase c
	-------Voltage Magnitude-------		
1	0.9530	0.8441	0.8441
2	0.9562	0.7884	0.7884
3	0.9919	0.7122	0.7122
4	1.0107	0.7569	0.7569
5	0.9686	0.7365	0.7365
6	1.0208	0.5666	0.5666
7	0.9992	0.5907	0.5907
8	1.1391	0.0000	0.0000
9	1.0736	0.3151	0.3151
10	0.9991	0.8455	0.8455
11	0.9239	0.7509	0.7509

Line currents for fault at bus No. 8

From Bus	To Bus	Phase a	Phase b	Phase c
		-----Line Current Magnitude----		
1	2	0.2352	0.8546	0.8546
2	3	0.0350	0.2069	0.2069
2	5	0.0020	0.3063	0.3063
2	6	0.0178	0.4278	0.4278
3	6	0.0063	0.3277	0.3277
4	3	0.0413	0.1290	0.1290
4	6	0.0220	0.3050	0.3050
4	9	0.0020	0.5924	0.5924
5	7	0.0020	0.3063	0.3063
6	8	0.0104	1.0596	1.0596
7	8	0.0084	1.4963	1.4963
8	F	0.0000	3.1483	3.1483

9	8	0.0020	0.5924	0.5924
10	4	0.0612	1.0217	1.0217
11	7	0.3046	1.1067	1.1067

```
Another fault location?
Enter 'y' or 'n' within single quote -> 'n'
```

PROBLEMS

10.1. Obtain the symmetrical components for the set of unbalanced voltages $V_a = 300\angle-120°$, $V_b = 200\angle 90°$, and $V_c = 100\angle-30°$.

10.2. The symmetrical components of a set of unbalanced three-phase currents are $I_a^0 = 3\angle-30°$, $I_a^1 = 5\angle 90°$, and $I_a^2 = 4\angle 30°$. Obtain the original unbalanced phasors.

10.3. The operator a is defined as $a = 1\angle 120°$; show that

(a) $\frac{(1+a)}{(1+a^2)} = 1\angle 120°$

(b) $\frac{(1-a)^2}{(1+a)^2} = 3\angle-180°$

(c) $(a - a^2)(a^2 - a) = 3\angle 0°$

(d) $V_{an}^1 = \frac{1}{\sqrt{3}} V_{bc}^1 \angle 90°$

(e) $V_{an}^2 = \frac{1}{\sqrt{3}} V_{bc}^2 \angle-90°$

10.4. The line-to-line voltages in an unbalanced three-phase supply are $V_{ab} = 1000\angle 0°$, $V_{bc} = 866.0254\angle-150°$, and $V_{ca} = 500\angle 120°$. Determine the symmetrical components for line and phase voltages, then find the phase voltages V_{an}, V_{bn}, and V_{cn}.

10.5. In the three-phase system shown in Figure 10.26, phase a is on no load and phases b and c are short-circuited to ground.

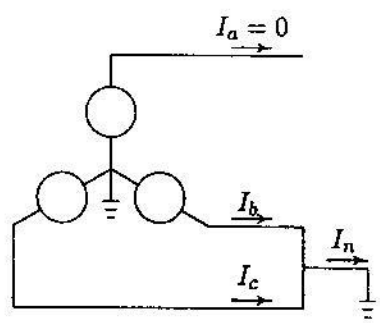


FIGURE 10.26
Circuit for Problem 10.5.

The following currents are given:
$I_b = 91.65\angle 160.9^\circ$
$I_n = 60.00\angle 90^\circ$
Find the symmetrical components of current I_a^0, I_a^1, and I_a^2.

10.6. A balanced three-phase voltage of 360-V line-to-neutral is applied to a balanced Y-connected load with ungrounded neutral, as shown in Figure 10.27. The three-phase load consists of three mutually-coupled reactances. Each phase has a series reactance of $Z_s = j24\ \Omega$, and the mutual coupling between phases is $Z_m = j6\ \Omega$.
(a) Determine the line currents by mesh analysis without using symmetrical components.
(b) Determine the line currents using symmetrical components.

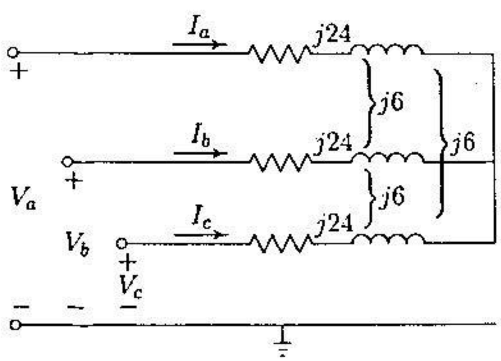


FIGURE 10.27
Circuit for Problem 10.6.

10.7. A three-phase unbalanced source with the following phase-to-neutral voltages

$$\mathbf{V}^{abc} = \begin{bmatrix} 300 & \angle -120^\circ \\ 200 & \angle 90^\circ \\ 100 & \angle -30^\circ \end{bmatrix}$$

is applied to the circuit in Figure 10.28. The load series impedance per phase is $Z_s = 10 + j40$ and the mutual impedance between phases is $Z_m = j5$. The load and source neutrals are solidly grounded. Determine
(a) The load sequence impedance matrix, $\mathbf{Z}^{012} = \mathbf{A}^{-1}\mathbf{Z}^{abc}\mathbf{A}$.
(b) The symmetrical components of voltage.
(c) The symmetrical components of current.
(d) The load phase currents.

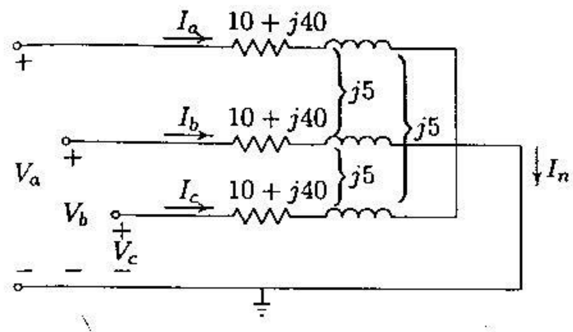


FIGURE 10.28
Circuit for Problem 10.7.

(e) The complex power delivered to the load in terms of symmetrical components, $S_{3\phi} = 3(V_a^0 I_a^{0*} + V_a^1 I_a^{1*} + V_a^2 I_a^{2*})$.
(f) The complex power delivered to the load by summing up the power in each phase, $S_{3\phi} = V_a I_a^* + V_b I_b^* + V_c I_c^*$.

10.8. The line-to-line voltages in an unbalanced three-phase supply are $V_{ab} = 600\angle 36.87^\circ$, $V_{bc} = 800\angle 126.87^\circ$, and $V_{ca} = 1000\angle -90^\circ$. A Y-connected load with a resistance of 37 Ω per phase is connected to the supply. Determine
(a) The symmetrical components of voltage.
(b) The phase voltages.
(c) The line currents.

10.9. A generator having a solidly grounded neutral and rated 50-MVA, 30-kV has positive-, negative-, and zero-sequence reactances of 25, 15, and 5 percent, respectively. What reactance must be placed in the generator neutral to limit the fault current for a bolted line-to-ground fault to that for a bolted three-phase fault?

10.10. What reactance must be placed in the neutral of the generator of Problem 9 to limit the magnitude of the fault current for a bolted double line-to-ground fault to that for a bolted three-phase fault?

10.11. Three 15-MVA, 30-kV synchronous generators A, B, and C are connected via three reactors to a common bus bar, as shown in Figure 10.29. The neutrals of generators A and B are solidly grounded, and the neutral of generator C is grounded through a reactor of 2.0 Ω. The generator data and the reactance of the reactors are tabulated below. A line-to-ground fault occurs on phase a of the common bus bar. Neglect prefault currents and assume generators are operating at their rated voltage. Determine the fault current in phase a.

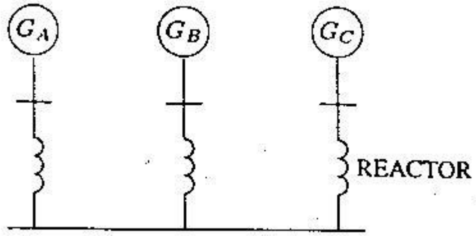


FIGURE 10.29
Circuit for Problem 10.11.

Item	X^1	X^2	X^0
G_A	0.25 pu	0.155 pu	0.056 pu
G_B	0.20 pu	0.155 pu	0.056 pu
G_C	0.20 pu	0.155 pu	0.060 pu
Reactor	6.0 Ω	6.0 Ω	6.0 Ω

10.12. Repeat Problem 10.11 for a bolted line-to-line fault between phases b and c.

10.13. Repeat Problem 10.11 for a bolted double line-to-ground fault on phases b and c.

10.14. The zero-, positive-, and negative-sequence bus impedance matrices for a three-bus power system are

$$\mathbf{Z}_{bus}^0 = j\begin{bmatrix} 0.20 & 0.05 & 0.12 \\ 0.05 & 0.10 & 0.08 \\ 0.12 & 0.08 & 0.30 \end{bmatrix} \quad \text{pu}$$

$$\mathbf{Z}_{bus}^1 = \mathbf{Z}_{bus}^2 = j\begin{bmatrix} 0.16 & 0.10 & 0.15 \\ 0.10 & 0.20 & 0.12 \\ 0.15 & 0.12 & 0.25 \end{bmatrix} \quad \text{pu}$$

Determine the per unit fault current and the bus voltages during fault for
(a) A bolted three-phase fault at bus 2.
(b) A bolted single line-to-ground fault at bus 2.
(c) A bolted line-to-line fault at bus 2.
(d) A bolted double line-to-ground fault at bus 2.

10.15. The reactance data for the power system shown in Figure 10.30 in per unit on a common base is as follows:

Item	X^1	X^2	X^0
G_1	0.10	0.10	0.05
G_2	0.10	0.10	0.05
T_1	0.25	0.25	0.25
T_2	0.25	0.25	0.25
Line 1–2	0.30	0.30	0.50

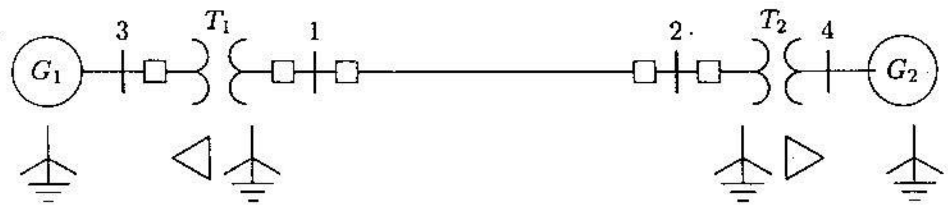


FIGURE 10.30
Circuit for Problem 10.15.

Obtain the Thévenin sequence impedances for the fault at bus 1 and compute the fault current in per unit for the following faults:

(a) A bolted three-phase fault at bus 1.
(b) A bolted single line-to-ground fault at bus 1.
(c) A bolted line-to-line fault at bus 1.
(d) A bolted double line-to-ground fault at bus 1.

10.16. For Problem 10.15, obtain the bus impedance matrices for the sequence networks. A bolted single line-to-ground fault occurs at bus 1. Find the fault current, the three-phase bus voltages during fault, and the line currents in each phase. Check your results using the **zbuild** and **lgfault** programs.

10.17. Repeat Problem 10.16 for a bolted line-to-line fault. Check your results using the **zbuild** and **llfault** programs.

10.18. Repeat Problem 10.16 for a bolted double line-to-ground fault. Check your results using the **zbuild** and **dlgfault** programs.

10.19. The positive-sequence reactances for the power system shown in Figure 10.31 are in per unit on a common MVA base. Resistances are neglected and the negative-sequence impedances are assumed to be the same as the positive-sequence impedances. A bolted line-to-line fault occurs between phases b and c at bus 2. Before the fault occurrence, all bus voltages are 1.0 per unit. Obtain the positive-sequence bus impedance matrix. Find the fault current, the three-phase bus voltages during fault, and the line currents in each phase. Check your results using the **zbuild** and **llfault** programs.

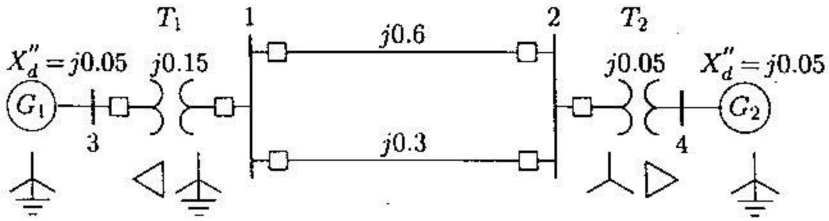


FIGURE 10.31
Circuit for Problem 10.19.

10.20. Use the **lgfault**, **llfault**, and **dlgfault** functions to compute the fault current, bus voltages, and line currents in the circuit given in Example 10.8 for the following unbalanced fault.

(a) A bolted single line-to-ground fault at bus 9.
(b) A bolted line-to-line fault at bus 9.
(c) A bolted double line-to-ground fault at bus 9.

All shunt capacitances and loads are neglected and the negative-sequence data is assumed to be the same as the positive-sequence data. All the prefault bus voltages are assumed to be unity.

10.21. The six-bus power system network of an electric utility company is shown in Figure 10.32. The positive- and zero-sequence reactances of the lines and transformers in per unit on a 100-MVA base is tabulated below.

LINE AND TRANSFORMER DATA			
Bus No.	Bus No.	X^1, PU	X^0, PU
1	4	0.225	0.400
1	5	0.105	0.200
1	6	0.215	0.390
2	4	0.035	0.035
3	5	0.042	0.042
4	6	0.125	0.250
5	6	0.175	0.350

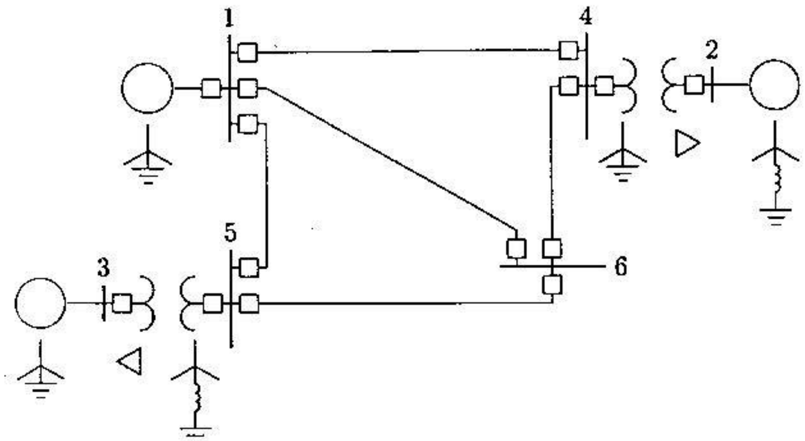


FIGURE 10.32
One-line diagram for Problem 10.32.

The transformer connections are shown in Figure 10.32. The Δ-Y transformer between buses 3 and 5 is grounded through a reactor of reactance 0.10 per unit. The generator's positive- and zero-sequence reactances including the reactance of grounding neutrals on a 100-MVA base is tabulated below.

GENERATOR TRANSIENT IMPEDANCE, PU			
Gen. No.	X^1	X^0	X_n
1	0.20	0.06	0.00
2	0.15	0.04	0.05
3	0.25	0.08	0.00

Resistances, shunt reactances, and loads are neglected, and all negative-sequence reactances are assumed equal to the positive-sequence reactances. Use **zbuild** function to obtain the positive- and zero-sequence bus impedance matrices. Assume all the prefault bus voltages are equal to $1\angle 0°$, use **lgfault**, **llfault**, and **dlgfault** to compute the fault current, bus voltages, and line currents for the following unbalanced faults.

(a) A bolted single line-to-ground fault at bus 6.
(b) A bolted line-to-line fault at bus 6.
(c) A bolted double line-to-ground fault at bus 6.

CHAPTER 11

STABILITY

11.1 INTRODUCTION

The tendency of a power system to develop restoring forces equal to or greater than the disturbing forces to maintain the state of equilibrium is known as *stability*. If the forces tending to hold machines in synchronism with one another are sufficient to overcome the disturbing forces, the system is said to remain stable (to stay in synchronism).

The stability problem is concerned with the behavior of the synchronous machines after a disturbance. For convenience of analysis, stability problems are generally divided into two major categories — *steady-state stability* and *transient stability*. Steady-state stability refers to the ability of the power system to regain synchronism after small and slow disturbances, such as gradual power changes. An extension of the steady-state stability is known as the *dynamic stability*. The dynamic stability is concerned with small disturbances lasting for a long time with the inclusion of automatic control devices. Transient stability studies deal with the effects of large, sudden disturbances such as the occurrence of a fault, the sudden outage of a line or the sudden application or removal of loads. Transient stability studies are needed to ensure that the system can withstand the transient condition following a major disturbance. Frequently, such studies are conducted when new generating and transmitting facilities are planned. The studies are helpful in determining such things as the nature of the relaying system needed, critical clearing time of circuit breakers, voltage level of, and transfer capability between systems.

11.2 SWING EQUATION

Under normal operating conditions, the relative position of the rotor axis and the resultant magnetic field axis is fixed. The angle between the two is known as the *power angle* or *torque angle*. During any disturbance, rotor will decelerate or accelerate with respect to the synchronously rotating air gap mmf, and a relative motion begins. The equation describing this relative motion is known as the *swing equation*. If, after this oscillatory period, the rotor locks back into synchronous speed, the generator will maintain its stability. If the disturbance does not involve any net change in power, the rotor returns to its original position. If the disturbance is created by a change in generation, load, or in network conditions, the rotor comes to a new operating power angle relative to the synchronously revolving field.

In order to understand the significance of the power angle we refer to the combined phasor/vector diagram of a two-pole cylindrical rotor generator illustrated in Figure 3.2. From this figure we see that the power angle δ_r is the angle between the rotor mmf F_r and the resultant air gap mmf F_{sr}, both rotating at synchronous speed. It is also the angle between the no-load generated emf E and the resultant stator voltage E_{sr}. If the generator armature resistance and leakage flux are neglected, the angle between E and the terminal voltage V, denoted by δ, is considered as the power angle.

Consider a synchronous generator developing an electromagnetic torque T_e and running at the synchronous speed ω_{sm}. If T_m is the driving mechanical torque, then under steady-state operation with losses neglected we have

$$T_m = T_e \tag{11.1}$$

A departure from steady state due to a disturbance results in an accelerating ($T_m > T_e$) or decelerating ($T_m < T_e$) torque T_a on the rotor.

$$T_a = T_m - T_e \tag{11.2}$$

If J is the combined moment of inertia of the prime mover and generator, neglecting frictional and damping torques, from law's of rotation we have

$$J\frac{d^2\theta_m}{dt^2} = T_a = T_m - T_e \tag{11.3}$$

where θ_m is the angular displacement of the rotor with respect to the stationary reference axis on the stator. Since we are interested in the rotor speed relative to synchronous speed, the angular reference is chosen relative to a synchronously rotating reference frame moving with constant angular velocity ω_{sm}, that is

$$\theta_m = \omega_{sm}t + \delta_m \tag{11.4}$$

where δ_m is the rotor position before disturbance at time $t = 0$, measured from the synchronously rotating reference frame. Derivative of (11.4) gives the rotor angular velocity

$$\omega_m = \frac{d\theta_m}{dt} = \omega_{ms} + \frac{d\delta_m}{dt} \tag{11.5}$$

and the rotor acceleration is

$$\frac{d^2\theta_m}{dt^2} = \frac{d^2\delta_m}{dt^2} \tag{11.6}$$

Substituting (11.6) into (11.3), we have

$$J\frac{d^2\delta_m}{dt^2} = T_m - T_e \tag{11.7}$$

Multiplying (11.7) by ω_m, results in

$$J\omega_m\frac{d^2\delta_m}{dt^2} = \omega_m T_m - \omega_m T_e \tag{11.8}$$

Since angular velocity times torque is equal to the power, we write the above equation in terms of power

$$J\omega_m\frac{d^2\delta_m}{dt^2} = P_m - P_e \tag{11.9}$$

The quantity $J\omega_m$ is called the inertia constant and is denoted by M. It is related to kinetic energy of the rotating masses, W_k.

$$W_k = \frac{1}{2}J\omega_m^2 = \frac{1}{2}M\omega_m \tag{11.10}$$

or

$$M = \frac{2W_k}{\omega_m} \tag{11.11}$$

Although M is called inertia constant, it is not really constant when the rotor speed deviates from the synchronous speed. However, since ω_m does not change by a large amount before stability is lost, M is evaluated at the synchronous speed and is considered to remain constant, i.e.,

$$M = \frac{2W_k}{\omega_{sm}} \tag{11.12}$$

The swing equation in terms of the inertia constant becomes

$$M\frac{d^2\delta_m}{dt^2} = P_m - P_e \tag{11.13}$$

It is more convenient to write the swing equation in terms of the electrical power angle δ. If p is the number of poles of a synchronous generator, the electrical power angle δ is related to the mechanical power angle δ_m by

$$\delta = \frac{p}{2}\delta_m \tag{11.14}$$

also,

$$\omega = \frac{p}{2}\omega_m \tag{11.15}$$

Swing equation in terms of electrical power angle is

$$\frac{2}{p}M\frac{d^2\delta}{dt^2} = P_m - P_e \tag{11.16}$$

Since power system analysis is done in per unit system, the swing equation is usually expressed in per unit. Dividing (11.16) by the base power S_B, and substituting for M from (11.12) results in

$$\frac{2}{p}\frac{2W_K}{\omega_{sm}S_B}\frac{d^2\delta}{dt^2} = \frac{P_m}{S_B} - \frac{P_e}{S_B} \tag{11.17}$$

We now define the important quantity known as the H *constant* or *per unit inertia constant*.

$$H = \frac{\text{kinetic energy in MJ at rated speed}}{\text{machine rating in MVA}} = \frac{W_K}{S_B} \tag{11.18}$$

The unit of H is seconds. The value of H ranges from 1 to 10 seconds, depending on the size and type of machine. Substituting in (11.17), we get

$$\frac{2}{p}\frac{2H}{\omega_{sm}}\frac{d^2\delta}{dt^2} = P_{m(pu)} - P_{e(pu)} \tag{11.19}$$

where $P_{m(pu)}$ and $P_{e(pu)}$ are the per unit mechanical power and electrical power, respectively. The electrical angular velocity is related to the mechanical angular velocity by $\omega_{sm} = (2/p)\omega_s$. (11.19) in terms of electrical angular velocity is

$$\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = P_{m\,(pu)} - P_{e\,(pu)} \tag{11.20}$$

The above equation is often expressed in terms of frequency f_0, and to simplify the notation, the subscript pu is omitted and the powers are understood to be in per unit.

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_e \tag{11.21}$$

where δ is in electrical radian. If δ is expressed in electrical degrees, the swing equation becomes

$$\frac{H}{180 f_0}\frac{d^2\delta}{dt^2} = P_m - P_e \tag{11.22}$$

11.3 SYNCHRONOUS MACHINE MODELS FOR STABILITY STUDIES

The representation of a synchronous machine during transient conditions was discussed in Chapter 8. In Section 8.6 the cylindrical rotor machine was modeled with a constant voltage source behind proper reactances, which may be X_d'', X_d', or X_d. The simplest model for stability analysis is the classical model, where saliency is ignored, and the machine is represented by a constant voltage E' behind the direct axis transient reactance X_d'.

Consider a generator connected to a major substation of a very large system through a transmission line as shown in Figure 11.1.

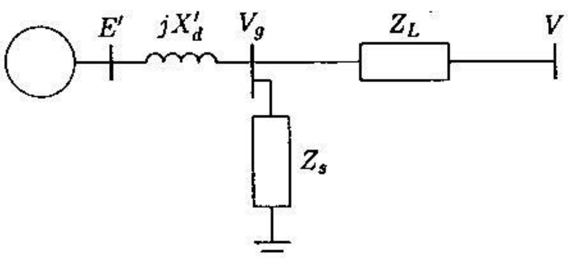


FIGURE 11.1
One machine connected to an infinite bus.

The substation bus voltage and frequency is assumed to remain constant. This is commonly referred to as an *infinite bus*, since its characteristics do not change regardless of the power supplied or consumed by any device connected to it. The generator is represented by a constant voltage behind the direct axis transient reactance X_d'. The node representing the generator terminal voltage V_g can be eliminated

by converting the Y-connected impedances to an equivalent Δ with admittances given by

$$\begin{aligned} y_{10} &= \frac{Z_L}{jX_d'Z_s + jX_d'Z_L + Z_LZ_s} \\ y_{20} &= \frac{jX_d'}{jX_d'Z_s + jX_d'Z_L + Z_LZ_s} \\ y_{12} &= \frac{Z_s}{jX_d'Z_s + jX_d'Z_L + Z_LZ_s} \end{aligned} \tag{11.23}$$

The equivalent circuit with internal voltage represented by node 1 and the infinite bus by node 2 is shown in Figure 11.2. Writing the nodal equations, we have

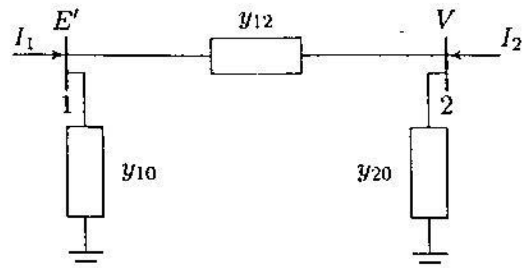


FIGURE 11.2
Equivalent circuit of one machine connected to an infinite bus.

$$\begin{aligned} I_1 &= (y_{10} + y_{12})E' - y_{12}V \\ I_2 &= -y_{12}E' + (y_{20} + y_{12})V \end{aligned} \tag{11.24}$$

The above equations can be written in terms of the bus admittance matrix as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}\begin{bmatrix} E' \\ V \end{bmatrix} \tag{11.25}$$

The diagonal elements of the bus admittance matrix are $Y_{11} = y_{10} + y_{12}$, and $Y_{22} = y_{20} + y_{12}$. The off-diagonal elements are $Y_{12} = Y_{21} = -y_{12}$. Expressing the voltages and admittances in polar form, the real power at node 1 is given by

$$\begin{aligned} P_e &= \Re[E'I_1^*] \\ &= \Re[|E'|\angle\delta(|Y_{11}|\angle-\theta_{11}|E'|\angle-\delta + |Y_{12}|\angle-\theta_{12}|V|\angle 0)] \end{aligned}$$

or

$$P_e = |E'|^2|Y_{11}|\cos\theta_{11} + |E'||V||Y_{12}|\cos(\delta - \theta_{12}) \tag{11.26}$$

The power flow equation given by (6.25) when applied to the above two-bus power system results in the same expression as (11.26). In most systems, Z_L and Z_s are predominantly inductive. If all resistances are neglected, $\theta_{11} = \theta_{12} = 90°$, $Y_{12} = B_{12} = 1/X_{12}$, and we obtain a simplified expression for power

$$P_e = |E'||V||B_{12}|\cos(\delta - 90°)$$

or

$$P_e = \frac{|E'||V|}{X_{12}}\sin\delta \tag{11.27}$$

This is the simplest form of the power flow equation and is basic to an understanding of all stability problems. The relation shows that the power transmitted depends upon the transfer reactance and the angle between the two voltages. The curve P_e versus δ is known as the *power angle curve* and is shown in Figure 11.3.

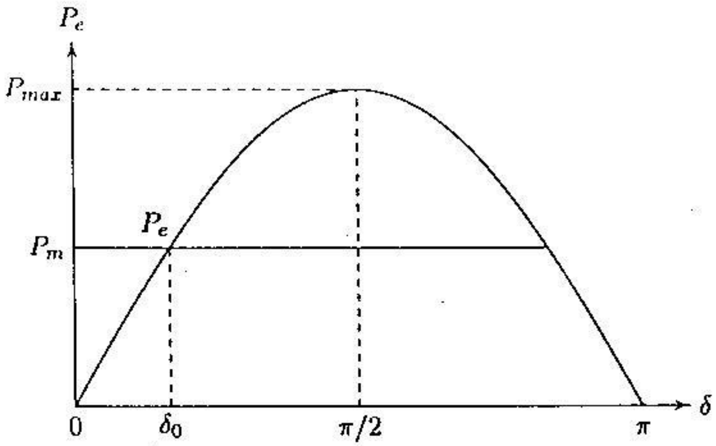


FIGURE 11.3
Power-angle curve.

The gradual increase of the generator power output is possible until the maximum electrical power is transferred. This maximum power is referred to as the ***steady-state stability limit***, and occurs at an angular displacement of 90°.

$$P_{max} = \frac{|E'||V|}{X_{12}} \tag{11.28}$$

If an attempt were made to advance δ further by further increasing the shaft input, the electrical power output will decrease from the P_{max} point. The machine will accelerate, causing loss of synchronism with the infinite bus bar. The electric power equation in terms of P_{max} is

$$P_e = P_{max}\sin\delta \tag{11.29}$$

When a generator is suddenly short-circuited, the current during the transient period is limited by its transient reactance X'_d. Thus, for transient stability problems, with the saliency neglected, the machine is represented by the voltage E' behind the reactance X'_d. If V_g is the generator terminal voltage and I_a is the prefault steady state generator current, E' is computed from

$$E' = V_g + jX'_d I_a \tag{11.30}$$

Since the field winding has a small resistance, the field flux linkages will tend to remain constant during the initial disturbance, and thus the voltage E' is assumed constant. The transient power-angle curve has the same general form as the steady-state curve; however, it attains larger peak compared to the steady-state peak value.

11.3.1 SYNCHRONOUS MACHINE MODEL INCLUDING SALIENCY

In Section 3.4 we developed the two-axis model of a synchronous machine under steady state conditions taking into account the effect of saliency. The phasor diagram of the salient-pole machine under steady state conditions, with armature resistance neglected, was presented in Figure 3.8. This phasor diagram is represented in Figure 11.4.

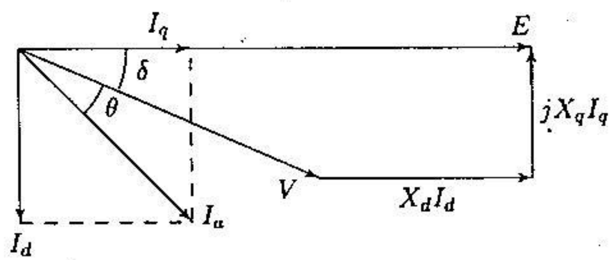


FIGURE 11.4
Phasor diagram during transient period.

The power-angle equation was given by (3.32). This equation presented in per unit is

$$P_e = \frac{|E||V|}{X_d}\sin\delta + |V|^2\frac{X_d - X_q}{2X_dX_q}\sin 2\delta \tag{11.31}$$

and

$$|E| = |V|\cos\delta + X_d I_d$$

or

$$|E| = |V|\cos\delta + X_d|I_a|\sin(\delta+\theta) \tag{11.32}$$

where E is the no-load generated emf in per unit and V is the generator terminal voltage in per unit. X_d and X_q are direct and quadrature axis reactances of the synchronous machine. For a derivation of the above formula, refer to Section 3.4. For a given power delivered at a given terminal voltage, we must compute E. In order to do that, we must first compute δ as follows:

$$\begin{aligned}|V|\sin\delta &= X_q I_q \\ &= X_q|I_a|\cos(\delta+\theta) \\ &= X_q|I_a|(\cos\delta\cos\theta - \sin\delta\sin\theta)\end{aligned}$$

From the above relation, δ is found to be

$$\delta = \tan^{-1}\frac{X_q|I_a|\cos\theta}{|V| + X_q|I_a|\sin\theta} \tag{11.33}$$

Substituting for δ from (11.33) into (11.32) will result in the voltage E.

A logical extension of the model would be to include the effect of transient saliency. Since the machine circuits are largely inductive, the flux linkages tend to remain constant during the early part of the transient period. During the transient period, the direct axis transient reactance is X_d'. Since the field is on the direct axis X_q', the quadrature axis transient reactance remains the same as X_q. The phasor diagram under transient condition is shown in Figure 11.5. Following the procedure

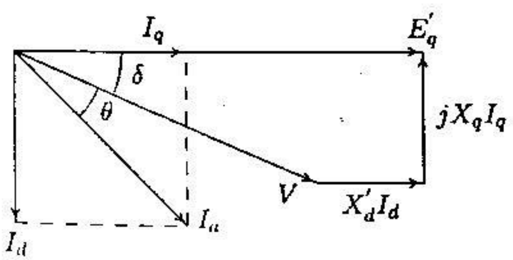


FIGURE 11.5
Phasor diagram during transient period.

in Section 3.4, the transient power-angle equation expressed in per unit becomes

$$P_e = \frac{|E_q'||V|}{X_d'}\sin\delta + |V|^2\frac{X_d' - X_q}{2X_d'X_q}\sin 2\delta \tag{11.34}$$

This equation represents the approximate behavior of the synchronous machine during the early part of transient period. We now have to determine E_q'. From the phasor diagram in Figure 11.5, we have

$$|E_q'| = |V|\cos\delta + X_d' I_d$$

or

$$|E_q'| = |V|\cos\delta + X_d'|I_a|\sin(\delta+\theta)$$

From (11.32), we find

$$|I_a|\sin(\delta+\theta) = \frac{|E| - |V|\cos\delta}{X_d}$$

and substitute it in the above equation to get

$$|E_q'| = \frac{X_d'|E| + (X_d - X_d')|V|\cos\delta}{X_d} \tag{11.35}$$

The prefault excitation voltage and power angles are computed from (11.32) and (11.33).

In this section we presented two simple models for cylindrical rotor and salient rotor synchronous machines. The choice of model for a given situation must, in general, depend upon the type of study being conducted as well as the data available. Although these models are useful for many stability studies, it is not adequate for many situations. More accurate models must include the effects of the various rotor circuits.

Example 11.1

Consider a synchronous machine characterized by the following parameters:

$$X_d = 1.0 \quad X_q = 0.6 \quad X_d' = 0.3 \text{ per unit}$$

and negligible armature resistance. The machine is connected directly to an infinite bus of voltage 1.0 per unit. The generator is delivering a real power of 0.5 per unit at 0.8 power factor lagging. Determine the voltage behind transient reactance and the transient power-angle equation for the following cases.

(a) Neglecting the saliency effect
(b) Including the effect of saliency

$$\theta = \cos^{-1} 0.8 = 36.87^\circ$$
$$S = \frac{0.5}{0.8}\angle 36.87^\circ = 0.625\angle 36.87^\circ \text{ pu}$$

The prefault steady state current is

$$I_a = \frac{S^*}{V^*} = 0.625\angle -36.87^\circ \text{ pu}$$

(a) With saliency neglected, the voltage behind transient reactance is

$$E' = V + jX'_d I_a = 1.0 + (j0.3)(0.625\angle -36.87^\circ) = 1.1226\angle 7.679^\circ \text{ pu}$$

The transient power-angle curve is given by

$$P_e = \frac{|E'||V|}{X'_d}\sin\delta = \frac{(1.1226)(1)}{0.3}\sin\delta$$

or

$$P_e = 3.7419 \sin\delta$$

(b) When the saliency effect is considered, the initial steady state power angle given by (11.33) is

$$\delta = \tan^{-1}\frac{X_q|I_a|\cos\theta}{|V| + X_q|I_a|\sin\theta} = \tan^{-1}\frac{(0.6)(0.625)(0.8)}{1.0 + (0.6)(.625)(.6)} = 13.7608^\circ$$

The steady state excitation voltage E, given by (11.32), is

$$|E| = |V|\cos\delta + X_d|I_a|\sin(\delta + \theta)$$
$$= (1.0)\cos(13.7608^\circ) + (1.0)(0.625)\sin(13.7608^\circ + 36.87^\circ) = 1.4545 \text{ pu}$$

The transient voltage E'_q given by (11.35) is

$$|E'_q| = \frac{X'_d|E| + (X_d - X'_d)|V|\cos\delta}{X_d}$$
$$= \frac{(0.3)(1.4545) + (1.0 - 0.3)(1.0)(\cos 13.7608^\circ)}{1.0} = 1.1162 \text{ pu}$$

and from (11.34) the transient power-angle equation is

$$P_e = \frac{(1.1162)(1)}{0.3}\sin\delta + \frac{(1.0)^2(0.3 - 0.6)}{2(0.3)(0.6)}\sin 2\delta$$

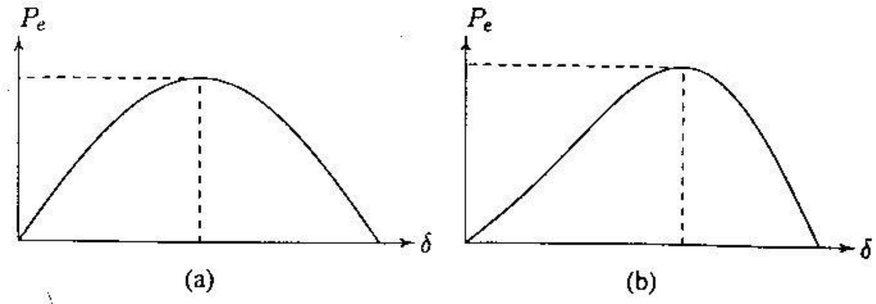


FIGURE 11.6
Transient power-angle curve for Example 11.1.

or

$$P_e = 3.7208 \sin\delta - 0.8333 \sin 2\delta$$

Using *MATLAB*, the power-angle equations obtained in (a) and (b) are plotted as shown in Figure 11.6. We use the function **[Pmax, k] = max(P)** to find the maximum power in case (b). The maximum power is found to be 4.032, occurring at angle $\delta(k) = 110.01^\circ$.

The coefficient of $\sin 2\delta$ is relatively small, and since $X'_d < X_q$, it is negative. Thus, the $\sin 2\delta$ term has the property of subtracting from the $\sin\delta$ term in the region $0^\circ < \delta < 90^\circ$, but adding to it in the region $90^\circ < \delta < 180^\circ$. During sudden impact, when δ swings from its initial value to the maximum value for marginal stability, the overall effect of the $\sin 2\delta$ term has the tendency to average out to zero. For this reason, the $\sin 2\delta$ term is often ignored in the approximate power-angle equation.

11.4 STEADY-STATE STABILITY — SMALL DISTURBANCES

The steady-state stability refers to the ability of the power system to remain in synchronism when subjected to small disturbances. It is convenient to assume that the disturbances causing the changes disappear. The motion of the system is free, and stability is assured if the system returns to its original state. Such a behavior can be determined in a linear system by examining the characteristic equation of the system. It is assumed that the automatic controls, such as voltage regulator and governor, are not active. The actions of governor and excitation system and control devices are discussed in Chapter 12 when dealing with dynamic stability.

To illustrate the steady-state stability problem, we consider the dynamic behavior of a one-machine system connected an infinite bus bar as shown in Figure 11.1. Substituting for the electrical power from (11.29) into the swing equation given in (11.21) results in

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_{max}\sin\delta \tag{11.36}$$

The swing equation is a nonlinear function of the power angle. However, for small disturbances, the swing equation may be linearized with little loss of accuracy as follows. Consider a small deviation $\Delta\delta$ in power angle from the initial operating point δ_0, i.e.,

$$\delta = \delta_0 + \Delta\delta \tag{11.37}$$

Substituting in (11.36), we get

$$\frac{H}{\pi f_0}\frac{d^2(\delta_0+\Delta\delta)}{dt^2} = P_m - P_{max}\sin(\delta_0+\Delta\delta)$$

or

$$\frac{H}{\pi f_0}\frac{d^2\delta_0}{dt^2} + \frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} = P_m - P_{max}(\sin\delta_0\cos\Delta\delta + \cos\delta_0\sin\Delta\delta)$$

Since $\Delta\delta$ is small, $\cos\Delta\delta \cong 1$ and $\sin\Delta\delta \cong \Delta\delta$, and we have

$$\frac{H}{\pi f_0}\frac{d^2\delta_0}{dt^2} + \frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} = P_m - P_{max}\sin\delta_0 - P_{max}\cos\delta_0\,\Delta\delta$$

Since at the initial operating state

$$\frac{H}{\pi f_0}\frac{d^2\delta_0}{dt^2} = P_m - P_{max}\sin\delta_0$$

The above equation reduces to linearized equation in terms of incremental changes in power angle, i.e.,

$$\frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} + P_{max}\cos\delta_0\,\Delta\delta = 0 \tag{11.38}$$

The quantity $P_{max}\cos\delta_0$ in (11.38) is the slope of the power-angle curve at δ_0. It is known as the *synchronizing coefficient*, denoted by P_s. This coefficient plays an important part in determining the system stability, and is given by

$$P_s = \left.\frac{dP}{d\delta}\right|_{\delta_0} = P_{max}\cos\delta_0 \tag{11.39}$$

Substituting in (11.38), we have

$$\frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} + P_s\Delta\delta = 0 \tag{11.40}$$

The solution of the above second-order differential equation depends on the roots of the characteristic equation given by

$$s^2 = -\frac{\pi f_0}{H}P_s \tag{11.41}$$

When P_s is negative, we have one root in the right-half s-plane, and the response is exponentially increasing and stability is lost. When P_s is positive, we have two roots on the j-ω axis, and the motion is oscillatory and undamped. The system is marginally stable with a natural frequency of oscillation given by

$$\omega_n = \sqrt{\frac{\pi f_0}{H}P_s} \tag{11.42}$$

It can be seen from Figure 11.3 that the range where P_s (i.e., the slope $dP/d\delta$) is positive lies between 0 and 90° with a maximum value at no-load ($\delta_0 = 0$).

As long as there is a difference in angular velocity between the rotor and the resultant rotating air gap field, induction motor action will take place between them, and a torque will be set up on the rotor tending to minimize the difference between the two angular velocities. This is called the *damping torque*. The damping power is approximately proportional to the speed deviation.

$$P_d = D\frac{d\delta}{dt} \tag{11.43}$$

The damping coefficient D may be determined either from design data or by test. Additional damping torques are caused by the speed/torque characteristic of the prime mover and the load dynamic, which are not considered here. When the synchronizing power coefficient P_s is positive, because of the damping power, oscillations will damp out eventually, and the operation at the equilibrium angle will be restored. No loss of synchronism occurs and the system is stable.

If damping is accounted for, the linearized swing equation becomes

$$\frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} + P_s\,\Delta\delta = 0 \tag{11.44}$$

or

$$\frac{d^2\Delta\delta}{dt^2} + \frac{\pi f_0}{H}D\frac{d\Delta\delta}{dt} + \frac{\pi f_0}{H}P_s\,\Delta\delta = 0 \tag{11.45}$$

or in terms of the standard second-order differential equation, we have

$$\frac{d^2\Delta\delta}{dt^2} + 2\zeta\omega_n \frac{d\Delta\delta}{dt} + \omega_n^2\,\Delta\delta = 0 \tag{11.46}$$

where ω_n, the natural frequency of oscillation is given by (11.42), and ζ is defined as the dimensionless damping ratio, given by

$$\zeta = \frac{D}{2}\sqrt{\frac{\pi f_0}{HP_s}} \tag{11.47}$$

The characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{11.48}$$

For normal operating conditions, $\zeta = D/2\sqrt{\frac{\pi f_0}{HP_s}} < 1$, and roots of the characteristic equation are complex

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n + j\omega_d \end{aligned} \tag{11.49}$$

where ω_d is the damped frequency of oscillation given by

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{11.50}$$

It is clear that for positive damping, roots of the characteristic equation have negative real part if synchronizing power coefficient P_s is positive. The response is bounded and the system is stable.

We now write (11.46) in state variable form. This makes it possible to extend the analysis to multimachine systems. Let

$$\begin{aligned} x_1 &= \Delta\delta \quad \text{and} \quad x_2 = \Delta\omega = \dot{\Delta\delta} \quad \text{then} \\ \dot{x}_1 &= x_2 \quad \text{and} \quad \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2 \end{aligned}$$

Writing the above equations in matrix, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11.51}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{11.52}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \tag{11.53}$$

This is the unforced state variable equation or the *homogeneous* state equation. If the state variables x_1 and x_2 are the desired response, we define the output vector $\mathbf{y}(t)$ as

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11.54}$$

or

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{11.55}$$

Taking the Laplace transform, we have

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{11.56}$$

where

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ \omega_n^2 & s + 2\zeta\omega_n \end{bmatrix} \tag{11.57}$$

Substituting for $(s\mathbf{I} - \mathbf{A})^{-1}$, we have

$$\mathbf{X}(s) = \frac{\begin{bmatrix} s + 2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix} \mathbf{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

When the rotor is suddenly perturbed by a small angle $\Delta\delta_0$, $x_1(0) = \Delta\delta_0$ and $x_2(0) = \Delta\omega_0 = 0$, and we obtain

$$\Delta\delta(s) = \frac{(s + 2\zeta\omega_n)\Delta\delta_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and

$$\Delta\omega(s) = -\frac{\omega_n^2\,\Delta\delta_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Taking inverse Laplace transforms results in the *zero-input* response

$$\Delta\delta = \frac{\Delta\delta_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta) \tag{11.58}$$

and

$$\Delta\omega = -\frac{\omega_n \Delta\delta_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t \tag{11.59}$$

where ω_d is the damped frequency of oscillation, and θ is given by

$$\theta = \cos^{-1}\zeta \tag{11.60}$$

The motion of rotor relative to the synchronously revolving field is

$$\delta = \delta_0 + \frac{\Delta\delta_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta) \tag{11.61}$$

and the rotor angular frequency is

$$\omega = \omega_0 - \frac{\omega_n \Delta\delta_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t \tag{11.62}$$

The response time constant is

$$\tau = \frac{1}{\zeta\omega_n} = \frac{2H}{\pi f_0 D} \tag{11.63}$$

and the response settles in approximately four time constants, and the settling time is

$$t_s \cong 4\tau \tag{11.64}$$

From (11.42) and (11.47), we note that as inertia constant H increases, the natural frequency and the damping ratio decreases, resulting in a longer settling time. An increase in the synchronizing power coefficient P_s results in an increase in the natural frequency and a decrease in the damping ratio.

Example 11.2

A 60-Hz synchronous generator having inertia constant H = 9.94 MJ/MVA and a transient reactance $X_d' = 0.3$ per unit is connected to an infinite bus through a purely reactive circuit as shown in Figure 11.7. Reactances are marked on the diagram on a common system base. The generator is delivering real power of 0.6 per unit, 0.8 power factor lagging to the infinite bus at a voltage of $V = 1$ per unit.

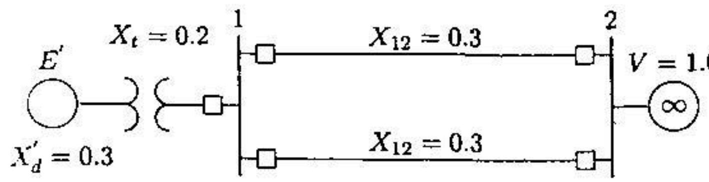


FIGURE 11.7
One-line diagram for Example 11.2.

Assume the per unit damping power coefficient is $D = 0.138$. Consider a small disturbance of $\Delta\delta = 10° = 0.1745$ radian. For example, the breakers open and then quickly close. Obtain equations describing the motion of the rotor angle and the generator frequency.

The transfer reactance between the generated voltage and the infinite bus is

$$X = 0.3 + 0.2 + \frac{0.3}{2} = 0.65$$

The per unit apparent power is

$$S = \frac{0.6}{0.8}\angle\cos^{-1}0.8 = 0.75\angle 36.87°$$

The current is

$$I = \frac{S^*}{V^*} = \frac{0.75\angle -36.87°}{1.0\angle 0°} = 0.75\angle -36.87°$$

The excitation voltage is

$$E' = V + jXI = 1.0\angle 0° + (j0.65)(0.75\angle -36.87°) = 1.35\angle 16.79°$$

Thus, the initial operating power angle is 16.79° = 0.2931 radian. The synchronizing power coefficient given by (11.39) is

$$P_s = P_{max}\cos\delta_0 = \frac{(1.35)(1)}{0.65}\cos 16.79° = 1.9884$$

The undamped angular frequency of oscillation and damping ratio are

$$\omega_n = \sqrt{\frac{\pi f_0}{H} P_s} = \sqrt{\frac{(\pi)(60)}{9.94} 1.9884} = 6.1405 \text{ rad/sec}$$

$$\zeta = \frac{D}{2}\sqrt{\frac{\pi f_0}{H P_s}} = \frac{0.138}{2}\sqrt{\frac{(\pi)(60)}{(9.94)(1.9884)}} = 0.2131$$

The linearized force-free equation which determines the mode of oscillation given by (11.46) with δ in radian is

$$\frac{d^2\Delta\delta}{dt^2} + 2.62\frac{d\Delta\delta}{dt} + 37.7\Delta\delta = 0$$

From (11.50), the damped angular frequency of oscillation is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 6.1405\sqrt{1-(0.2131)^2} = 6.0 \text{ rad/sec}$$

corresponding to a damped oscillation frequency of

$$f_d = \frac{6.0}{2\pi} = 0.9549 \text{ Hz}$$

From (11.61) and (11.62), the motion of rotor relative to the synchronously revolving field in electrical degrees and the frequency excursion in Hz are given by equations

$$\delta = 16.79^\circ + 10.234e^{-1.3t}\sin(6.0t + 77.6966^\circ)$$
$$f = 60 - 0.1746e^{-1.3t}\sin 6.0t$$

The above equations are written in *MATLAB* commands as follows

```
E=1.35; V=1.0; H=9.94; X=0.65; Pm=0.6; D= 0.138; f0 = 60;
Pmax = E*V/X, d0 = asin(Pm/Pmax)                  % Max. power
Ps = Pmax*cos(d0)          % Synchronizing power coefficient
wn = sqrt(pi*60/H*Ps)% Undamped frequency of  oscillation
z = D/2*sqrt(pi*60/(H*Ps))                   % Damping ratio
wd=wn*sqrt(1-z^2), fd=wd/(2*pi) %Damped frequency oscill.
tau = 1/(z*wn)                               % Time constant
th = acos(z)                               % Phase angle theta
Dd0 = 10*pi/180;                    % Initial angle in radian
t = 0:.01:3;
Dd = Dd0/sqrt(1-z^2)*exp(-z*wn*t).*sin(wd*t + th);
d = (d0+Dd)*180/pi;                    % Power angle in degree
Dw = -wn*Dd0/sqrt(1-z^2)*exp(-z*wn*t).*sin(wd*t);
f = f0 + Dw/(2*pi);                          % Frequency in Hz
subplot(2,1,1), plot(t, d), grid
xlabel('t  sec'), ylabel('Delta    degree')
subplot(2,1,2), plot(t,f), grid
xlabel('t  sec'), ylabel('Frequency    Hz')
subplot(111)
```

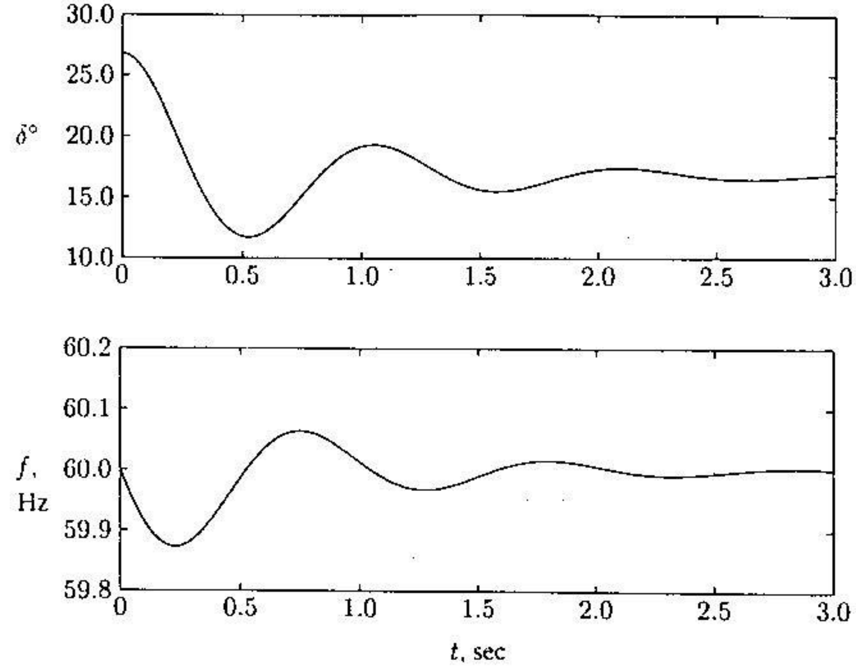



FIGURE 11.8
Natural responses of the rotor angle and frequency for machine of Example 11.2.

The result is shown in Figure 11.8.

The response shows that a small disturbance will be followed by a relatively slowly damped oscillation, or swing, of the rotor, before steady state operation at synchronous speed is resumed. In the case of a steam turbine generator, oscillations subside in a matter of two to three seconds. In the above example, the response settles in about $t_s \simeq 4\tau = 4(1/1.3) \simeq 3.1$ seconds. We also observe that the oscillations are fairly low in frequency, in the order of 0.955 Hz.

The formulation of the one-machine system with all control devices inactive resulted in a second-order differential equation or a two-dimensional state equation. Later on, when the analysis is extended to a multimachine system, an n-dimensional state variables equation is obtained. *MATLAB* Control Toolbox provides a function named **initial** for simulating continuous-time linear systems due to an initial condition on the states. Given the system

$$\dot{\mathbf{x}}(t) = \mathbf{AX}(t) + \mathbf{Bu}(t) \qquad (11.65)$$
$$\mathbf{y} = \mathbf{Cx}(t) + \mathbf{Du}(t)$$

$[\mathbf{y}, \mathbf{x}] = \mathbf{initial}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{x_0}, \mathbf{t})$ returns the output and state responses of the system to the initial condition $\mathbf{x_0}$. The matrices $\mathbf{y}$ and $\mathbf{x}$ contain the output and state response of the system at the regularly spaced time vector t.

From (11.52)–(11.54), the zero-input state equation for Example 11.2 is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -37.705 & -2.617 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The initial state variables are $\Delta\delta_0 = 10° = 0.1745$ radian, and $\Delta\dot{\delta}_0 = 0$. The following *MATLAB* commands are used to obtain the zero-input response for Example 11.2.

```
A = [0  1; -37.705  -2.617];
B = [0; 0];                             % Column B zero-input
C=[1 0; 0 1];%Unity matrix defining output y as x1 and x2
D = [0; 0];
Dx0 = [0.1745; 0];                        % Initial conditions
[y, x] = initial(A, B, C, D, Dx0, t);
Dd = x(:, 1); Dw = x(:, 2);   % State variables x1 and x2
d = (d0 + Dd)*180/pi;                  % Power angle in degree
f = f0 + Dw/(2*pi);                          % Frequency in Hz
subplot(2,1,1), plot(t, d), grid
xlabel('t  sec'), ylabel('Delta    Degree')
subplot(2,1,2), plot(t, f), grid
xlabel('t  sec'), ylabel('Frequency    Hz'),subplot(111)
```

The simulation results are exactly the same as the graphs shown in Figure 11.8.

Although it is convenient to assume that the disturbances causing the changes disappear, we will now investigate the system response to small power impacts. Assume the power input is increased by a small amount ΔP. Then the linearized swing equation becomes

$$\frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} + P_s\,\Delta\delta = \Delta P \tag{11.66}$$

or

$$\frac{d^2\Delta\delta}{dt^2} + \frac{\pi f_0}{H}D\frac{d\Delta\delta}{dt} + \frac{\pi f_0}{H}P_s\,\Delta\delta = \frac{\pi f_0}{H}\Delta P \tag{11.67}$$

or in terms of the standard second-order differential equation, we have

$$\frac{d^2\Delta\delta}{dt^2} + 2\zeta\omega_n\frac{d\Delta\delta}{dt} + \omega_n^2\,\Delta\delta = \Delta u \tag{11.68}$$

where

$$\Delta u = \frac{\pi f_0}{H}\Delta P \tag{11.69}$$

and ω_n and ζ are given by (11.42) and (11.47), respectively. Transforming to the state variable form, we have

$$x_1 = \Delta\delta \quad \text{and} \quad x_2 = \Delta\omega = \Delta\dot{\delta} \quad \text{then}$$
$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

Writing the above equations in matrix, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta u \tag{11.70}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\Delta u(t) \tag{11.71}$$

This is the forced state variable equation or the *zero-state* equation, and with x_1 and x_2 the desired response, the output vector $\mathbf{y}(t)$ is given by (11.55). Taking the Laplace transform of the state equation (11.71) with zero initial states results in

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\,\Delta U(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\Delta U(s) \tag{11.72}$$

where

$$\Delta U(s) = \frac{\Delta u}{s}$$

Substituting for $(s\mathbf{I} - \mathbf{A})^{-1}$, we have

$$\mathbf{X}(s) = \frac{\begin{bmatrix} s + 2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{\Delta u}{s}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

or

$$\Delta\delta(s) = \frac{\Delta u}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

and

$$\Delta\omega(s) = \frac{\Delta u}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

Taking inverse Laplace transforms results in the step response

$$\Delta\delta = \frac{\Delta u}{\omega_n^2}[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta)] \qquad (11.73)$$

where $\theta = \cos^{-1}\zeta$ and

$$\Delta\omega = \frac{\Delta u}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t \qquad (11.74)$$

Substituting for Δu from (11.69), the motion of rotor relative to the synchronously revolving field in electrical radian becomes

$$\delta = \delta_0 + \frac{\pi f_0 \Delta P}{H\omega_n^2}[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta)] \qquad (11.75)$$

and the rotor angular frequency in radian per second is

$$\omega = \omega_0 + \frac{\pi f_0 \Delta P}{H\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_d t \qquad (11.76)$$

Example 11.3

The generator of Example 11.2 is operating in the steady state at $\delta_0 = 16.79°$ when the input power is increased by a small amount $\Delta P = 0.2$ per unit. The generator excitation and the infinite bus bar voltage are the same as before, i.e., $E' = 1.35$ per unit and $V = 1.0$ per unit.
(a) Using (11.75) and (11.76), obtain the step response for the rotor angle and the generator frequency.
(b) Obtain the response using the *MATLAB* step function.
(c) Obtain a *SIMULINK* block diagram representation of the state-space model and simulate to obtain the response.

(a) Substituting for H, δ_0, ζ, and ω_n evaluated in Example 11.2 in (11.75), and expressing the power angle in degree, we get

$$\delta = 16.79° + \frac{(180)(60)(0.2)}{(9.94)(6.1405)^2}[1 - \frac{1}{\sqrt{1-(0.2131)^2}} e^{-1.3t} \sin(6t + 77.6966°)]$$

or

$$\delta = 16.79° + 5.7631[1 - 1.0235e^{-1.3t} \sin(6t + 77.6966°)]$$

Also, substituting the values in (11.76) and expressing the frequency in Hz, we get

$$f = 60 + \frac{(60)(0.2)}{2(9.94)(6.1405)\sqrt{1-(0.2131)^2}} e^{-1.3t} \sin 6t$$

or

$$f = 60 + 0.10e^{-1.3t} \sin 6t$$

The above functions are plotted over a range of 0 to 3 seconds and the result is shown in Figure 11.9.

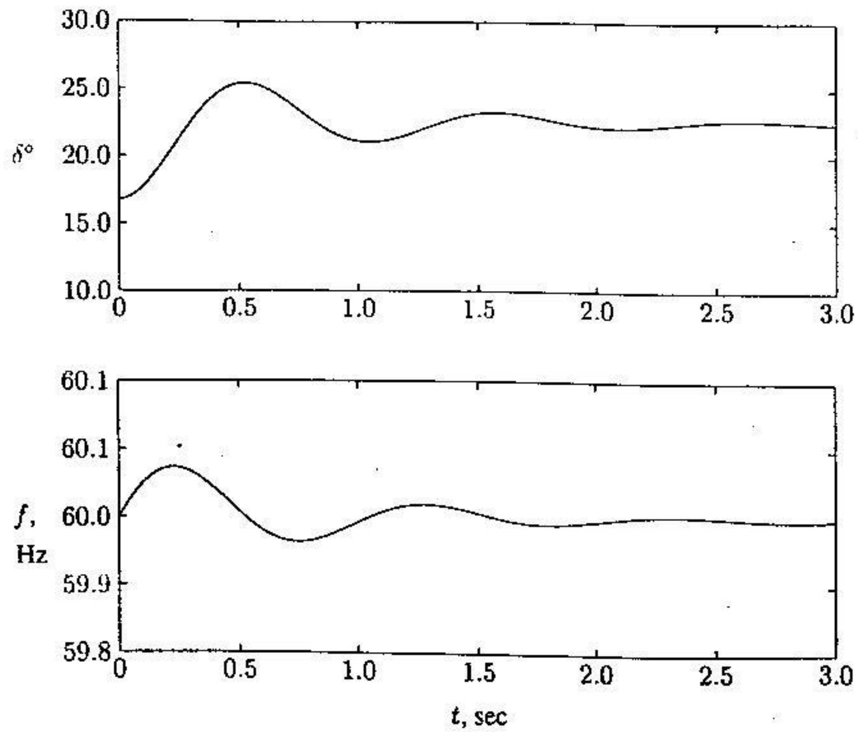


FIGURE 11.9
Step responses of the rotor angle and frequency for machine of Example 11.3.

(b) The step response of the state equation can be obtained conveniently, using the *MATLAB* Control Toolbox **[y, x] = lsim(A, B, C, D, u, t)** function or **[y,x] = step(A, B, C, D, iu, t)** function. These functions are particularly useful when dealing with multimachine systems. $[\mathrm{y}, \mathrm{x}] = \mathrm{step}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathrm{iu}, \mathrm{t})$ returns the output and step responses of the system. The index **iu** specifies which input to be used for the step response. With only one input **iu** = 1. The matrices y and x contain the output and state response of the system at the regularly spaced time vector t.

From (11.70), the state equation for Example 11.3 is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -37.705 & -2.617 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

and

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From (11.69), $\Delta u = (60\pi/9.94)(0.2) = 3.79$. The following *MATLAB* commands are used to obtain the step response for Example 11.3.

```
A = [0  1; -37.705  -2.617];
Dp = 0.2; Du = 3.79;     % Small step change in power input
B = [0; 1]*Du;
C=[1 0; 0 1];%Unity matrix defining output y as x1 and x2
D = [0; 0];
[y, x] = step(A, B, C, D, 1, t);
Dd = x(:, 1); Dw = x(:, 2);   % State variables x1 and x2
d = (d0 + Dd)*180/pi;                 % Power angle in degree
f = f0 + Dw/(2*pi);                       % Frequency in Hz
subplot(2,1,1), plot(t, d), grid
xlabel('t  sec'), ylabel('Delta    degree')
subplot(2,1,2), plot(t, f), grid
xlabel('t  sec'), ylabel('Frequency    Hz'),subplot(111)
```

The simulation results are exactly the same as the analytical solution and the plots are shown in Figure 11.9.

(c) A *SIMULINK* model named **sim11ex3.mdl** is constructed as shown in Figure 11.10. The file is opened and is run in the *SIMULINK WINDOW*. The simulation results in the same response as shown in Figure 11.9.

The response shows that the oscillation subsides in approximately 3.1 seconds and a new steady state operating point is attained at $\delta = 22.5°$. For the linearized swing equation, the stability is entirely independent of the input, and for a positive damping coefficient the system is always stable as long as the synchronizing power coefficient is positive. Theoretically, power can be increased gradually

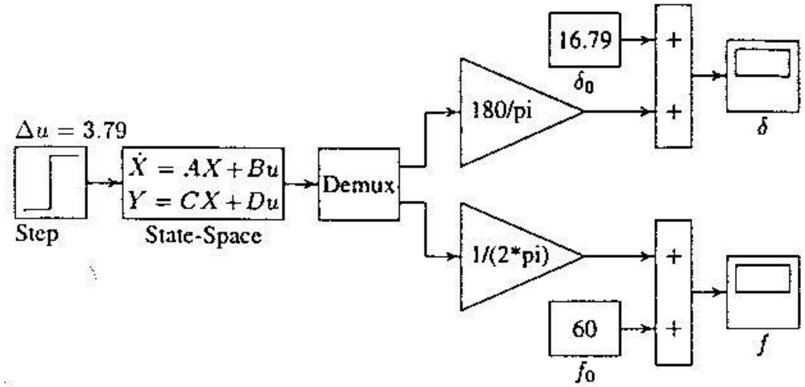


FIGURE 11.10
Simulation block diagram for Example 11.3.

up to the steady-state limit. It is important to note that the linearized equation is only valid for very small power impact and deviation from the operating state. Indeed, for a large sudden impact the nonlinear equation may result in unstable solution and stability is lost even if the impact is less than the steady-state power limit.

An important characteristic of the linear system is that the response is asymptotically stable if all roots of the characteristic equation have negative real part. The polynomial characteristic equation is obtained from the determinant of $(s\mathbf{I} - \mathbf{A})$ or eigenvalues of A. The eigenvalues provide very important results regarding the nature of response. The reciprocal of the real component of the eigenvalues gives the time constants, and the imaginary component gives the damped frequency of oscillations. Thus, the linear system expressed in the state variable form is asymptotically stable if and only if all of the eigenvalues of **A** lie in the left half of the complex plane. Therefore, to investigate the system stability of a multimachine system when subjected to small disturbances, all we need to do is to examine the eigenvalues of the **A** matrix. If the homogeneous state equation is written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{11.77}$$

we note that matrix **A** is the *Jacobian matrix* whose elements are partial derivatives of rows of $\mathbf{f}(\mathbf{x})$ with respect to state variables $x_1, x_2, \cdots, x_n$, evaluated at the equilibrium point. In *MATLAB* we can use the function $\mathrm{r} = \mathrm{eig}(\mathbf{A})$, which returns the eigenvalues of the **A** matrix. In Example 11.2, the **A** matrix was found to be

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -37.705 & -2.617 \end{bmatrix}$$

and the commands

```
A = [0  1; -37.705  -2.617];
r = eig(A)
```

result in

```
r =
    -1.3  +  6.00i
    -1.3  +  6.00i
```

The linearized model for small disturbances is very useful when the system is extended to include the governor action and the effect of automatic voltage regulators in a multimachine system. The linearized model allows the application of the linear control system analysis and compensation, which will be dealt with in Chapter 12.

11.5 TRANSIENT STABILITY — EQUAL-AREA CRITERION

The transient stability studies involve the determination of whether or not synchronism is maintained after the machine has been subjected to severe disturbance. This may be sudden application of load, loss of generation, loss of large load, or a fault on the system. In most disturbances, oscillations are of such magnitude that linearization is not permissible and the nonlinear swing equation must be solved.

A method known as the *equal-area criterion* can be used for a quick prediction of stability. This method is based on the graphical interpretation of the energy stored in the rotating mass as an aid to determine if the machine maintains its stability after a disturbance. The method is only applicable to a one-machine system connected to an infinite bus or a two-machine system. Because it provides physical insight to the dynamic behavior of the machine, application of the method to analysis of a single machine connected to a large system is considered here.

Consider a synchronous machine connected to an infinite bus. The swing equation with damping neglected as given by (11.21) is

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_e = P_a$$

where P_a is the accelerating power. From the above equation, we have

$$\frac{d^2\delta}{dt^2} = \frac{\pi f_0}{H}(P_m - P_e)$$

Multiplying both sides of the above equation by $2d\delta/dt$, we get

$$2\frac{d\delta}{dt}\frac{d^2\delta}{dt^2} = \frac{2\pi f_0}{H}(P_m - P_e)\frac{d\delta}{dt}$$

This may be written as

$$\frac{d}{dt}\left[\left(\frac{d\delta}{dt}\right)^2\right] = \frac{2\pi f_0}{H}(P_m - P_e)\frac{d\delta}{dt}$$

or

$$d\left[\left(\frac{d\delta}{dt}\right)^2\right] = \frac{2\pi f_0}{H}(P_m - P_e)d\delta$$

Integrating both sides,

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{2\pi f_0}{H}\int_{\delta_0}^{\delta}(P_m - P_e)d\delta$$

or

$$\frac{d\delta}{dt} = \sqrt{\frac{2\pi f_0}{H}\int_{\delta_0}^{\delta}(P_m - P_e)d\delta} \qquad (11.78)$$

Equation (11.78) gives the relative speed of the machine with respect to the synchronously revolving reference frame. For stability, this speed must become zero at some time after the disturbance. Therefore, from (11.78), we have for the stability criterion,

$$\int_{\delta_0}^{\delta}(P_m - P_e)d\delta = 0 \qquad (11.79)$$

Consider the machine operating at the equilibrium point δ_0, corresponding to the mechanical power input $P_{m0} = P_{e0}$ as shown in Figure 11.11. Consider a sudden step increase in input power represented by the horizontal line P_{m1}. Since $P_{m1} > P_{e0}$, the accelerating power on the rotor is positive and the power angle δ increases. The excess energy stored in the rotor during the initial acceleration is

$$\int_{\delta_0}^{\delta_1}(P_{m1} - P_e)d\delta = \text{area } abc = \text{area } A_1 \qquad (11.80)$$

With increase in δ, the electrical power increases, and when $\delta = \delta_1$, the electrical power matches the new input power P_{m1}. Even though the accelerating power is zero at this point, the rotor is running above synchronous speed; hence, δ and

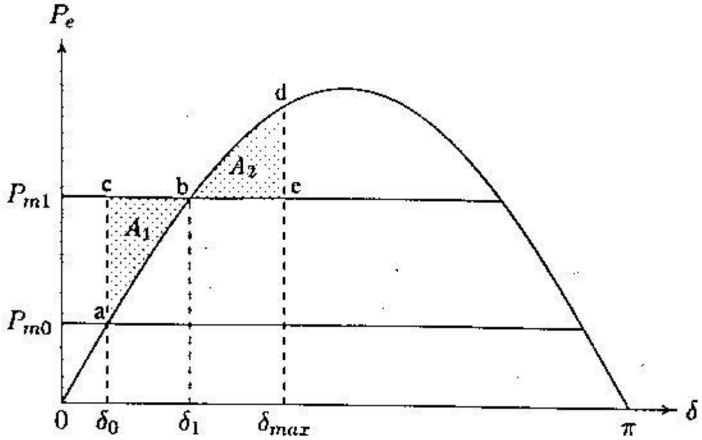


FIGURE 11.11
Equal-area criterion—sudden change of load.

electrical power P_e will continue to increase. Now $P_m < P_e$, causing the rotor to decelerate toward synchronous speed until $\delta = \delta_{max}$. According to (11.79), the rotor must swing past point b until an equal amount of energy is given up by the rotating masses. The energy given up by the rotor as it decelerates back to synchronous speed is

$$\int_{\delta_1}^{\delta_{max}} (P_{m1} - P_e) d\delta = \text{area } bde = \text{area } A_2 \qquad (11.81)$$

The result is that the rotor swings to point b and the angle δ_{max}, at which point

$$|\text{area } A_1| = |\text{area } A_2| \qquad (11.82)$$

This is known as the *equal-area criterion*. The rotor angle would then oscillate back and forth between δ_0 and δ_{max} at its natural frequency. The damping present in the machine will cause these oscillations to subside and the new steady state operation would be established at point b.

11.5.1 APPLICATION TO SUDDEN INCREASE IN POWER INPUT

The equal-area criterion is used to determine the maximum additional power P_m which can be applied for stability to be maintained. With a sudden change in the power input, the stability is maintained only if area A_2 at least equal to A_1 can be located above P_m. If area A_2 is less than area A_1, the accelerating momentum can never be overcome. The limit of stability occurs when δ_{max} is at the intersection of

line P_m and the power-angle curve for $90° < \delta < 180°$, as shown in Figure 11.12.

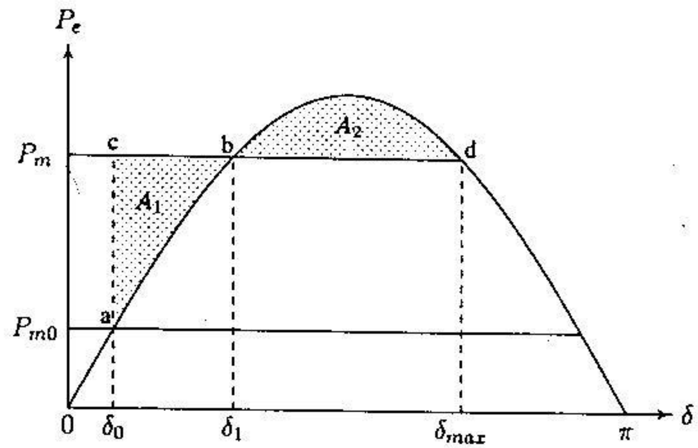


FIGURE 11.12
Equal-area criterion—maximum power limit.

Applying the equal-area criterion to Figure 11.12, we have

$$P_m(\delta_1 - \delta_0) - \int_{\delta_0}^{\delta_1} P_{max} \sin\delta \, d\delta = \int_{\delta_1}^{\delta_{max}} P_{max} \sin\delta \, d\delta - P_m(\delta_{max} - \delta_1)$$

Integrating the above expression yields

$$(\delta_{max} - \delta_0) P_m = P_{max}(\cos\delta_0 - \cos\delta_{max})$$

Substituting for P_m, from

$$P_m = P_{max} \sin\delta_{max}$$

into the above equation results in

$$(\delta_{max} - \delta_0) \sin\delta_{max} + \cos\delta_{max} = \cos\delta_0 \qquad (11.83)$$

The above nonlinear algebraic equation can be solved by an iterative technique for δ_{max}. Once δ_{max} is obtained, the maximum permissible power or the transient stability limit is found from

$$P_m = P_{max} \sin\delta_1 \qquad (11.84)$$

where

$$\delta_1 = \pi - \delta_{max} \qquad (11.85)$$

Equation (11.83) is a nonlinear function of angle δ_{max}, written as

$$f(\delta_{max}) = c \tag{11.86}$$

An iterative solution is obtained, using the Newton-Raphson method, described in Section 6.3. Starting with an initial estimate of $\pi/2 < \delta_{max}^{(k)} < \pi$, the Newton-Raphson algorithm gives

$$\Delta\delta_{max}^{(k)} = \frac{c - f(\delta_{max}^{(k)})}{\frac{df}{d\delta_{max}}\Big|_{\delta_{max}^{(k)}}} \tag{11.87}$$

where $df/d\delta_{max}$ is the derivative of (11.83) and is given by

$$\frac{df}{d\delta_{max}}\Big|_{\delta_{max}^{(k)}} = (\delta_{max}^{(k)} - \delta_0)\cos\delta_{max}^{(k)} \tag{11.88}$$

and

$$\delta_{max}^{(k+1)} = \delta_{max}^{(k)} + \Delta\delta_{max}^{(k)} \tag{11.89}$$

A solution is obtained when the difference between the absolute value of the successive iteration is less than a specified accuracy, i.e.,

$$|\delta_{max}^{(k+1)} - \delta_{max}^{(k)}| \le \epsilon \tag{11.90}$$

A function named **eacpower**(P_0, E, V, X) is developed for a one-machine system connected to an infinite bus. The function uses the above algorithm to find the sudden maximum permissible power that can be applied for critical stability. The function plots the power-angle curve and displays the shaded equal-areas. P_0, E, V, and X are the initial power, the transient internal voltage, the infinite bus bar voltage, and the transfer reactance, respectively, all in per unit. If **eacpower** is used without arguments, the user is prompted to enter the above quantities.

Example 11.4

The machine of Example 11.2 is delivering a real power of 0.6 per unit, at 0.8 power factor lagging to the infinite bus bar. The infinite bus bar voltage is 1.0 per unit. Determine

(a) The maximum power input that can be applied without loss of synchronism.
(b) Repeat (a) with zero initial power input. Assume the generator internal voltage remains constant at the value computed in (a).

In Example 11.2, the transfer reactance and the generator internal voltage were found to be $X = 0.65$ pu, and $E' = 1.35$ pu

(a) We use the following command:


```
P0 = 0.6; E = 1.35; V = 1.0; X = 0.65;
eacpower(P0, E, V, X)
```

which displays the graph shown in Figure 11.13 and results in

```
Initial power                          =   0.600 pu
Initial power angle                    =  16.791 degree
Sudden initial power                   =   1.084 pu
Total power for critical stability =   1.684 pu
Maximum angle swing                    = 125.840 pu
New operating angle                    =  54.160 degree
```

(b) The initial power input is set to zero, i.e., $P_o = 0$, and using **eacpower(Po, E, V, X)** displays the graph shown in Figure 11.14 with the following results:

```
Initial power                          =   0.00 pu
Initial power angle                    =   0.00 degree
Sudden initial power                   =   1.505 pu
Total power for critical stability =   1.505 pu
Maximum angle swing                    = 133.563 pu
New operating angle                    =  46.437 degree
```

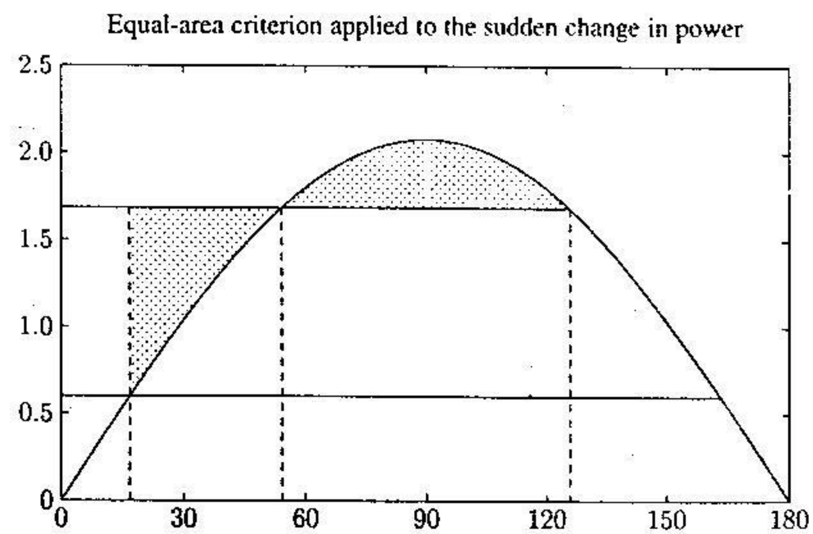


FIGURE 11.13
Maximum power limit by equal-area criterion for Example 11.4 (a).

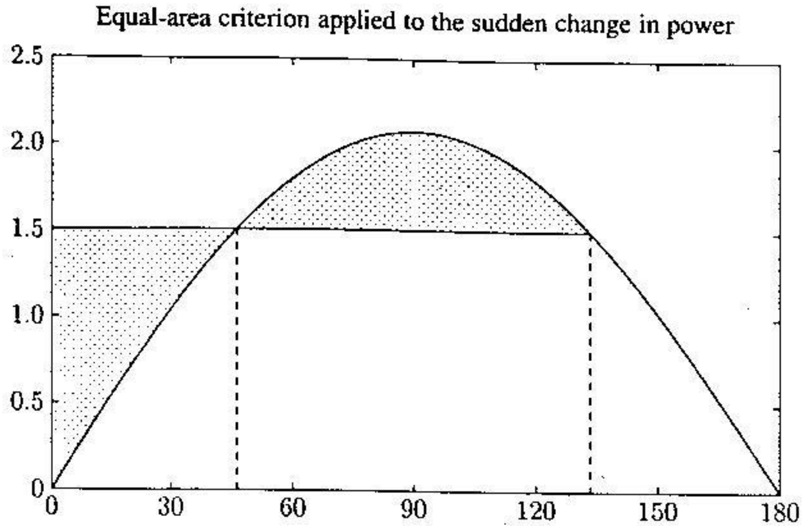


FIGURE 11.14
Maximum power limit by equal-area criterion for Example 11.4 (b).

11.6 APPLICATION TO THREE-PHASE FAULT

Consider Figure 11.15 where a generator is connected to an infinite bus bar through two parallel lines. Assume that the input power P_m is constant and the machine is operating steadily, delivering power to the system with a power angle δ_0 as shown in Figure 11.16. A temporary three-phase bolted fault occurs at the sending end of one of the line at bus 1.

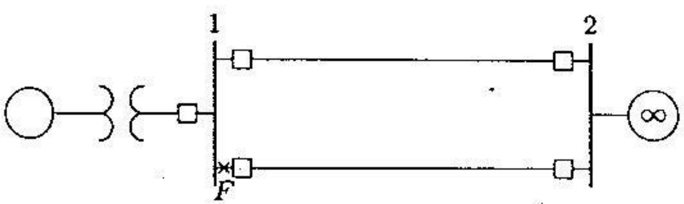


FIGURE 11.15
One-machine system connected to infinite bus, three-phase fault at F.

When the fault is at the sending end of the line, point F, no power is transmitted to the infinite bus. Since the resistances are neglected, the electrical power P_e is zero, and the power-angle curve corresponds to the horizontal axis. The machine accelerates with the total input power as the accelerating power, thereby increasing

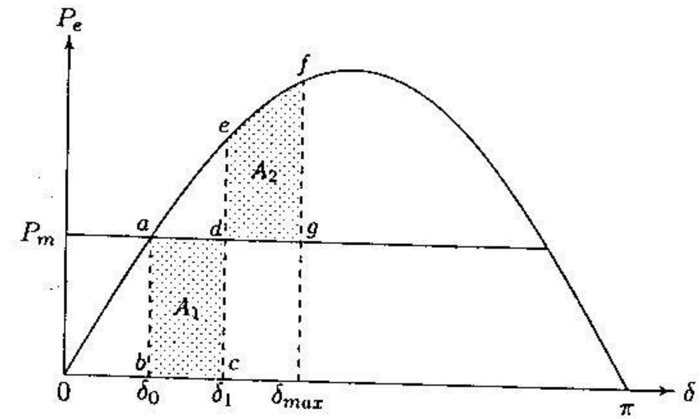


FIGURE 11.16
Equal-area criterion for a three-phase fault at the sending end.

its speed, storing added kinetic energy, and increasing the angle δ. When the fault is cleared, both lines are assumed to be intact. The fault is cleared at δ_1, which shifts the operation to the original power-angle curve at point e. The net power is now decelerating, and the previously stored kinetic energy will be reduced to zero at point f when the shaded area ($defg$), shown by A_2, equals the shaded area ($abcd$), shown by A_1. Since P_e is still greater than P_m, the rotor continues to decelerate and the path is retraced along the power-angle curve passing through points e and a. The rotor angle would then oscillate back and forth around δ_0 at its natural frequency. Because of the inherent damping, oscillation subsides and the operating point returns to the original power angle δ_0.

The critical clearing angle is reached when any further increase in δ_1 causes the area A_2, representing decelerating energy to become less than the area representing the accelerating energy. This occurs when δ_{max}, or point f, is at the intersection of line P_m and curve P_e, as shown in Figure 11.17. Applying equal-area criterion to Figure 11.17, we have

$$\int_{\delta_0}^{\delta_c} P_m \, d\delta = \int_{\delta_c}^{\delta_{max}} (P_{max} \sin\delta - P_m) d\delta$$

Integrating both sides, we have

$$P_m(\delta_c - \delta_0) = P_{max}(\cos\delta_c - \cos\delta_{max}) - P_m(\delta_{max} - \delta_c)$$

Solving for δ_c, we get

$$\cos\delta_c = \frac{P_m}{P_{max}}(\delta_{max} - \delta_0) + \cos\delta_{max} \tag{11.91}$$

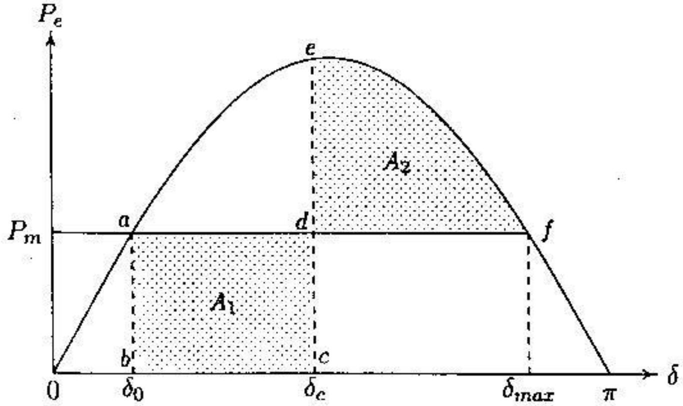


FIGURE 11.17
Equal-area criterion for critical clearing angle.

The application of equal-area criterion made it possible to find the critical clearing angle for the machine to remain stable. To find the critical clearing time, we still need to solve the nonlinear swing equation. For this particular case where the electrical power P_e during fault is zero, an analytical solution for critical clearing time can be obtained. The swing equation as given by (11.21), during fault with $P_e = 0$ becomes

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m$$

or

$$\frac{d^2\delta}{dt^2} = \frac{\pi f_0}{H}P_m$$

Integrating both sides

$$\frac{d\delta}{dt} = \frac{\pi f_0}{H}P_m\int_0^t dt = \frac{\pi f_0}{H}P_m t$$

Integrating again, we get

$$\delta = \frac{\pi f_0}{2H}P_m t^2 + \delta_0$$

Thus, if δ_c is the critical clearing angle, the corresponding critical clearing time is

$$t_c = \sqrt{\frac{2H(\delta_c - \delta_0)}{\pi f_0 P_m}} \tag{11.92}$$

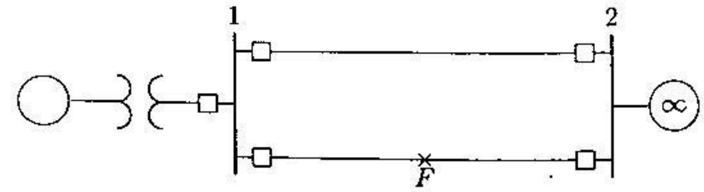


FIGURE 11.18
One-machine system connected to infinite bus, three-phase fault at F.

Now consider the fault location F at some distance away from the sending end as shown in Figure 11.18. Assume that the input power P_m is constant and the machine is operating steadily, delivering power to the system with a power angle δ_0 as shown in Figure 11.19. The power-angle curve corresponding to the prefault condition is given by curve A.

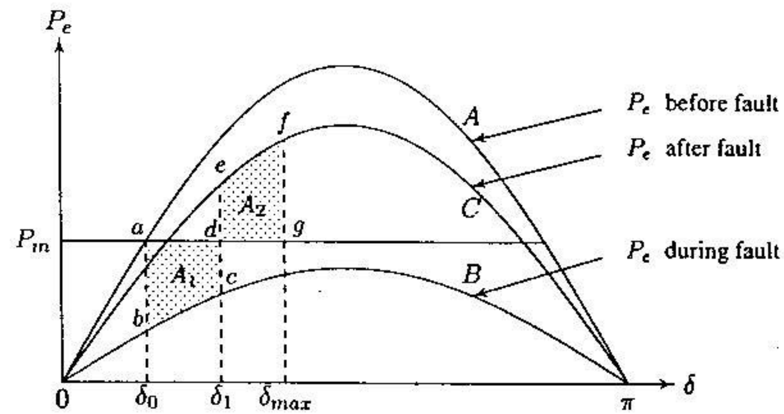


FIGURE 11.19
Equal-area criterion for a three-phase fault at the away from the sending end.

With fault location at F, away from the sending end, the equivalent transfer reactance between bus bars is increased, lowering the power transfer capability and the power-angle curve is represented by curve B. Finally, curve C represents the postfault power-angle curve, assuming the faulted line is removed. When the three-phase fault occurs, the operating point shifts immediately to point b on curve B. An excess of the mechanical input over its electrical output accelerates the rotor, thereby storing excess kinetic energy, and the angle δ increases. Assume the fault is cleared at δ_1 by isolating the faulted line. This suddenly shifts the operating point to e on curve C. The net power is now decelerating, and the previously stored kinetic energy will be reduced to zero at point f when the shaded area ($defg$) equals the shaded area ($abcd$). Since P_e is still greater than P_m, the rotor continues to decelerate, and the path is retraced along the power-angle curve passing through point e. The rotor angle will then oscillate back and forth around e at its natural frequency. The damping present in the machine will cause these oscillations to subside and a new steady state operation will be established at the intersection of P_m and curve C.

The critical clearing angle is reached when any further increase in δ_1 causes the area A_2, representing decelerating energy, to become less than the area representing the accelerating energy. This occurs when δ_{max}, or point f, is at the intersection of line P_m and curve C as shown in Figure 11.20.

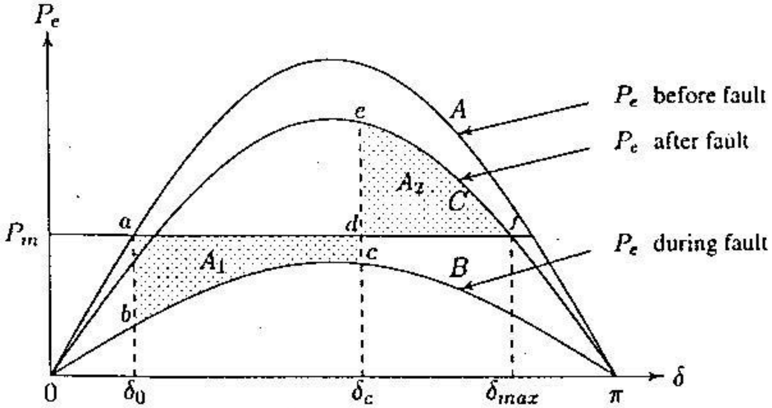


FIGURE 11.20
Equal-area criterion for critical clearing angle.

Applying equal-area criterion to Figure 11.20, we get

$$P_m(\delta_c - \delta_0) - \int_{\delta_0}^{\delta_c} P_{2\,max} \sin\delta \, d\delta = \int_{\delta_c}^{\delta_{max}} P_{3\,max} \sin\delta \, d\delta - P_m(\delta_{max} - \delta_c)$$

Integrating both sides, and solving for δ_c, we obtain

$$\cos\delta_c = \frac{P_m(\delta_{max} - \delta_0) + P_{3\,max}\cos\delta_{max} - P_{2\,max}\cos\delta_0}{P_{3\,max} - P_{2\,max}} \tag{11.93}$$

The application of equal-area criterion gives the critical clearing angle to maintain stability. However, because of the nonlinearity of the swing equation, an analytical solution for critical clearing time is not possible. In the next section we will discuss the numerical solution, which can readily be extended to large systems.

A function named **eacfault**(P_0, E, V, X_1, X_2, X_3) is developed for a one-machine system connected to an infinite bus. This function obtains the power-angle curve before fault, during fault, and after the fault clearance. The function uses equal-area criterion to find the critical clearing angle. For the case when power transfer during fault is zero, (11.92) is used to find the critical clearing time. The function plots the power-angle curve and displays the shaded equal-areas. P_0, E, and V are the initial power, the generator transient internal voltage, and the infinite bus bar voltage, all in per unit. X_1 is the transfer reactance before fault. X_2 is the transfer reactance during fault. If power transfer during fault is zero, **inf** must be used for X_2. Finally, X_3 is the postfault transfer reactance. If **eacfault** is used without arguments, the user is prompted to enter the above quantities.

Example 11.5

A 60-Hz synchronous generator having inertia constant H = 5 MJ/MVA and a direct axis transient reactance $X'_d = 0.3$ per unit is connected to an infinite bus through a purely reactive circuit as shown in Figure 11.21. Reactances are marked on the diagram on a common system base. The generator is delivering real power $P_e = 0.8$ per unit and $Q = 0.074$ per unit to the infinite bus at a voltage of $V = 1$ per unit.

(a) A temporary three-phase fault occurs at the sending end of the line at point F. When the fault is cleared, both lines are intact. Determine the critical clearing angle and the critical fault clearing time.

(b) A three-phase fault occurs at the middle of one of the lines, the fault is cleared, and the faulted line is isolated. Determine the critical clearing angle.

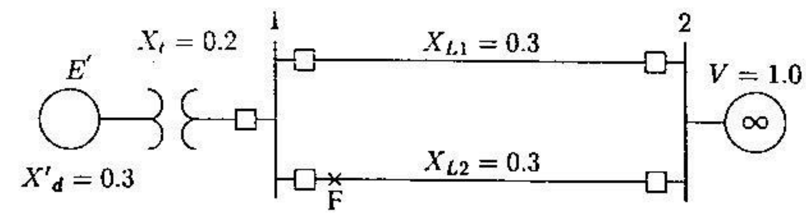


FIGURE 11.21
One-line diagram for Example 11.5.

The current flowing into the infinite bus is

$$I = \frac{S^*}{V^*} = \frac{0.8 - j0.074}{1.0\angle 0^\circ} = 0.8 - j0.074 \text{ pu}$$

The transfer reactance between internal voltage and the infinite bus before fault is

$$X_1 = 0.3 + 0.2 + \frac{0.3}{2} = 0.65$$

The transient internal voltage is

$$E' = V + jX_1 I = 1.0 + (j0.65)(0.8 - j0.074) = 1.17\angle 26.387^\circ \text{ pu}$$

(a) Since both lines are intact when the fault is cleared, the power-angle equation before and after the fault is

$$P_{max} \sin\delta = \frac{(1.17)(1.0)}{0.65} \sin\delta = 1.8 \sin\delta$$

The initial operating angle is given by

$$1.8 \sin\delta_0 = 0.8$$

or

$$\delta_0 = 26.388^\circ = 0.46055 \text{ rad}$$

and referring to Figure 11.17

$$\delta_{max} = 180^\circ - \delta_0 = 153.612^\circ = 2.681 \text{ rad}$$

Since the fault is at the beginning of the transmission line, the power transfer during fault is zero, and the critical clearing angle as given by (11.91) is

$$\cos\delta_c = \frac{0.8}{1.8}(2.681 - 0.46055) + \cos 153.61^\circ = 0.09106$$

Thus, the critical clearing angle is

$$\delta_c = \cos^{-1}(0.09106) = 84.775^\circ = 1.48 \text{ rad}$$

From (11.92), the critical clearing time is

$$t_c = \sqrt{\frac{2H(\delta_c - \delta_0)}{\pi f_0 P_m}} = \sqrt{\frac{(2)(5)(1.48 - 0.46055)}{(\pi)(60)(.8)}} = 0.26 \text{ second}$$

The use of function **eacfault**(P_m, E, V, X_1, X_2, X_3) to solve the above problem and to display power-angle plot with the shaded equal-areas is demonstrated below. We use the following commands

```
Pm = 0.8; E = 1.17; V = 1.0;
X1 = 0.65; X2 = inf; X3 = 0.65;
eacfault(Pm, E, V, X1, X2, X3)
```

The graph is displayed as shown in Figure 11.22 and the result is

```
Initial power angle      =  26.388
Maximum angle swing      = 153.612
Critical clearing angle  =  84.775
Critical clearing time   =   0.260 sec
```

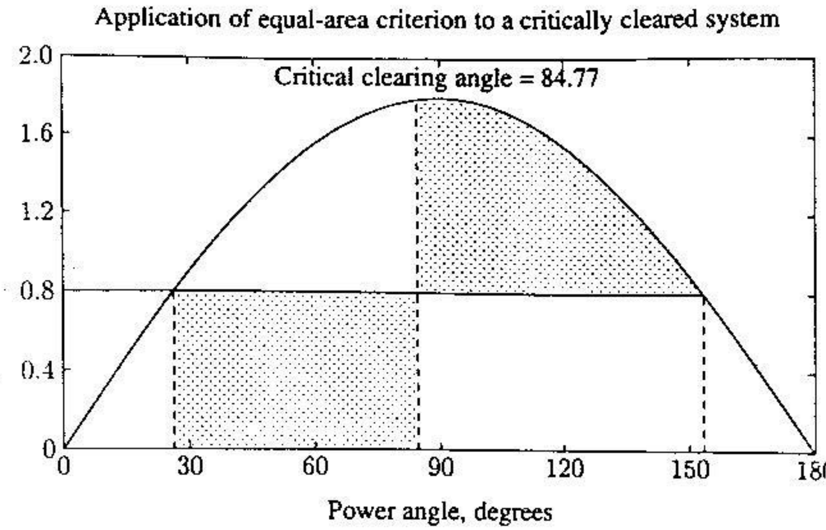



FIGURE 11.22
Equal-area criterion for Example 11.5 (a).

(b) The power-angle curve before the occurrence of the fault is the same as before, given by

$$P_{1\,max} = 1.8 \sin\delta$$

and the generator is operating at the initial power angle $\delta_0 = 26.4^\circ = 0.4605$ rad. The fault occurs at point F at the middle of one line, resulting in the circuit shown in Figure 11.23. The transfer reactance during fault may be found most readily by converting the Y-circuit ABF to an equivalent delta, eliminating junction C. The resulting circuit is shown in Figure 11.24.

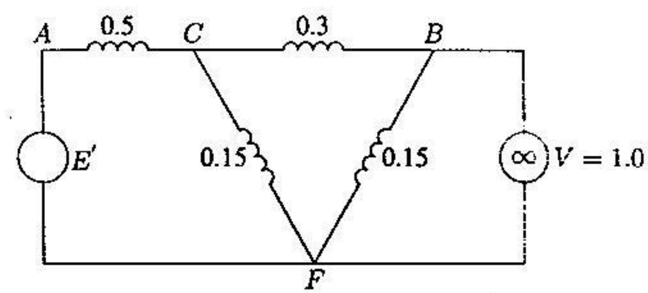


FIGURE 11.23
Equivalent circuit with three-phase fault at the middle of one line.

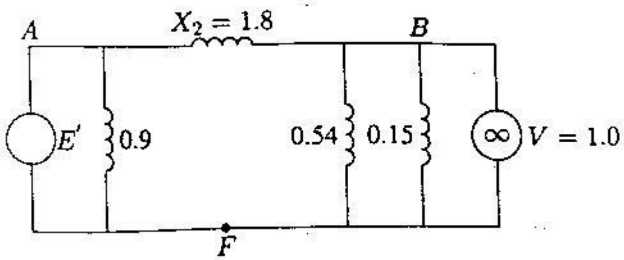


FIGURE 11.24
Equivalent circuit after Y-Δ transformation.

The equivalent reactance between generator and the infinite bus is

$$X_2 = \frac{(0.5)(0.3) + (0.5)(0.15) + (0.3)(0.15)}{0.15} = 1.8 \text{ pu}$$

Thus, the power-angle curve during fault is

$$P_{2max} \sin\delta = \frac{(1.17)(1.0)}{1.8} \sin\delta = 0.65 \sin\delta$$

When fault is cleared the faulted line is isolated. Therefore, the postfault transfer reactance is

$$X_3 = 0.3 + 0.2 + 0.3 = 0.8 \text{ pu}$$

and the power-angle curve is

$$P_{3max} \sin\delta = \frac{(1.17)(1.0)}{0.8} \sin\delta = 1.4625 \sin\delta$$

Referring to Figure 11.20

$$\delta_{max} = 180° - \sin^{-1}\left(\frac{0.8}{1.4625}\right) = 146.838° = 2.5628 \text{ rad}$$

Applying (11.93), the critical clearing angle is given by

$$\cos\delta_c = \frac{0.8(2.5628 - 0.46055) + 1.4625\cos 146.838° - 0.65\cos 26.388°}{1.4625 - 0.65} = -0.15356$$

Thus, the critical clearing angle is

$$\delta_c = \cos^{-1}(-0.15356) = 98.834°$$

Function eacfault(P_m, E, V, X_1, X_2, X_3) is used to solve part (b) and to display power-angle plot. We use the following commands

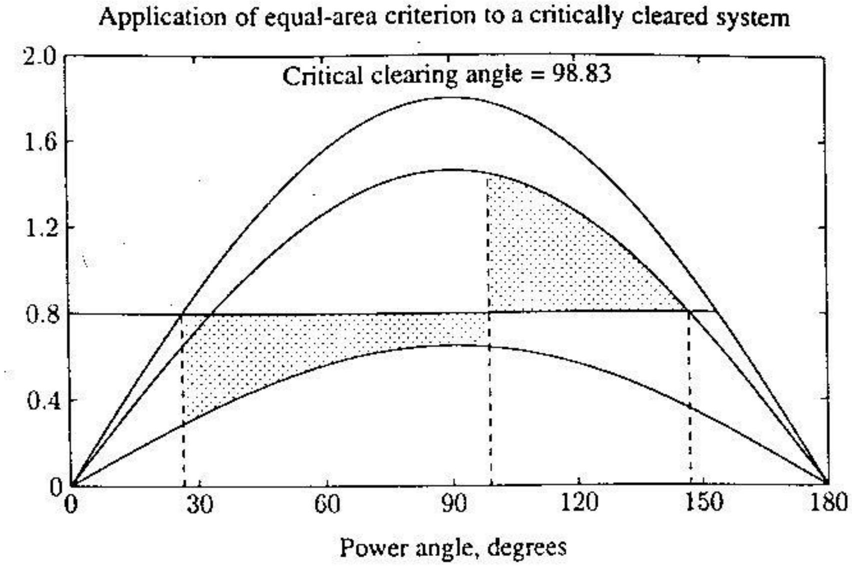


FIGURE 11.25
Equal-area criterion for Example 11.5 (b).

```
Pm = 0.8; E = 1.17; V = 1.0;
X1 = 0.65; X2 = 1.8; X3 = 0.8;
eacfault(Pm, E, V, X1, X2, X3)
```

The graph is displayed as shown in Figure 11.25 and the result is

```
Initial power angle      =  26.388
Maximum angle swing      = 146.838
Critical clearing angle  =  98.834
```

11.7 NUMERICAL SOLUTION OF NONLINEAR EQUATION

Numerical integration techniques can be applied to obtain approximate solutions of nonlinear differential equations. Many algorithms are available for numerical integration. Euler's method is the simplest and the least accurate of all numerical methods. It is presented here because of its simplicity. By studying this method, we will be able to grasp the basic ideas involved in numerical solutions of ODE and can more easily understand the more powerful methods such as the Runge-Kutta procedure.

Consider the first-order differential equation

$$\frac{dx}{dt} = f(t, x) \tag{11.94}$$

Euler's method is illustrated in Figure 11.26, where the curve shown represents the solution $x(t)$. If at t_0 the value of $x(t)$ denoted by x_0 is given, the curve can

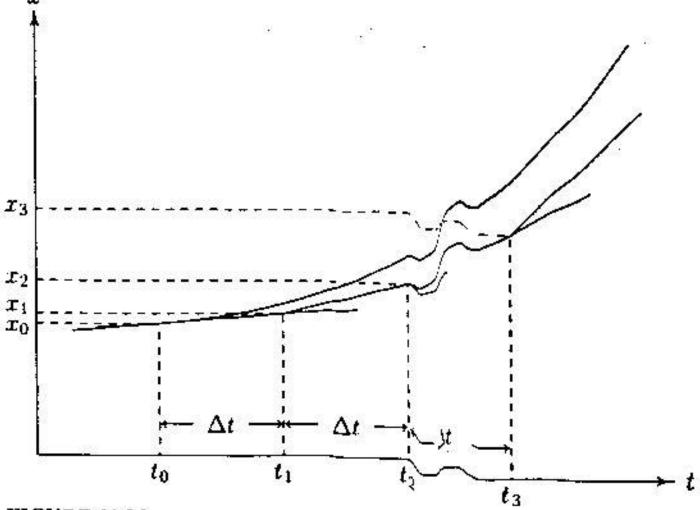


FIGURE 11.26
Graphical interpretation of Euler's method.

be approximated by the tangent evaluated at this point. For a small increment in t denoted by Δt, the increment in x is given by

$$\Delta x \approx \left.\frac{dx}{dt}\right|_{x_0} \Delta t$$

where $\left.\frac{dx}{dt}\right|_{x_0}$ is the slope of the curve at (t_0, x_0), which can be determined from (11.94). Thus, the value of x at $t_0 + \Delta t$ is

$$x_1 = x_0 + \Delta x = x_0 + \left.\frac{dx}{dt}\right|_{x_0} \Delta t$$

The above approximation is the Taylor series expansion of x around point (t_0, x_0), where higher-order terms have been discarded.

The subsequent values of x can be similarly determined. Hence, the computational algorithm is

$$x_{i+1} = x_i + \left.\frac{dx}{dt}\right|_{x_i} \Delta t \tag{11.95}$$

By applying the above algorithm successively, we can find approximate values of $x(t)$ at enough points from an initial state (t_0, x_0) to a final state (t_f, x_f). A graphical illustration is shown in Figure 11.26. Euler's method assumes that the slope is constant over the entire interval Δt causing the points to fall below the curve. An improvement can be obtained by calculating the slope at both the beginning and end of interval, and then averaging these slopes. This procedure is known as the modified Euler's method and is described as follows.

By using the derivative at the beginning of the step, the value at the end of the step ($t_1 = t_0 + \Delta t$) is predicted from

$$x_1^p = x_0 + \left.\frac{dx}{dt}\right|_{x_0} \Delta t$$

Using the predicted value of x_1^p, the derivative at the end of interval is determined by

$$\left.\frac{dx}{dt}\right|_{x_1^p} = f(t_1, x_1^p)$$

Then, the average value of the two derivatives is used to find the corrected value

$$x_1^c = x_0 + \left(\frac{\left.\frac{dx}{dt}\right|_{x_0} + \left.\frac{dx}{dt}\right|_{x_1^p}}{2}\right) \Delta t$$

Hence, the computational algorithm for the successive values is

$$x_{i+1}^c = x_i + \left(\frac{\left.\frac{dx}{dt}\right|_{x_i} + \left.\frac{dx}{dt}\right|_{x_{i+1}^p}}{2}\right) \Delta t \tag{11.96}$$

The problem with Euler's method is that there is numerical error introduced when discarding the higher-order terms in the series Taylor expansion. But by using a reasonably small value of Δt, we can decrease the error between successive points. If the step size is decreased too much, the number of steps increases and the computer round-off error increases directly with the number of operations. Thus, the step size must be selected small enough to obtain a reasonably accurate solution, but at the same time, large enough to avoid the numerical limitations of the computer.

The above technique can be applied to the solution of higher-order differential equations. An nth order differential equation can be expressed in terms of n first-order differential equations by the introduction of auxiliary variables. These variables are referred to as *state variables*, which may be physical quantities in a system. For example, given the second-order differential equation

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = c$$

and the initial conditions x_0 and $\left.\frac{dx}{dt}\right|_{x_0}$ at t_0, we introduce the following state variables.

$$x_1 = x$$
$$x_2 = \frac{dx}{dt}$$

Thus, the above second-order differential equation can be written as the two following simultaneous first-order differential equations.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{c}{a_2} - \frac{a_0}{a_2}x_1 - \frac{a_1}{a_2}x_2$$

There are many other more powerful techniques for the numerical solution of nonlinear equations. A popular technique is the Runge-Kutta method, which is based on formulas derived by using an approximation to replace the truncated Taylor series expansion. The interested reader should refer to textbooks on numerical techniques. *MATLAB* provides two powerful functions for the numerical solution of differential equations employing the Runge-Kutta-Fehlberg methods. These are **ode23** and **ode45**, based on the Fehlberg second- and third-order pair of formulas for medium accuracy and forth- and fifth-order pair for higher accuracy. The nth-order differential equation must be transformed into n first order differential equations and must be placed in an M-file that returns the state derivative of the equations. The formats for these functions are

```
[t, x] = ode23('xprime', tspan, x0)
[t, x] = ode45('xprime', tspan, x0)
```

where **tspan =[t0, tfinal]** is the time interval for the integration and **x0** is a column vector of initial conditions at time **t0**. **xprime** is the state derivative of the equations, defined in a file named **xprime.m**

11.8 NUMERICAL SOLUTION OF THE SWING EQUATION

To demonstrate the solution of the swing equation, consider Figure 11.18 where a generator is connected to an infinite bus bar through two parallel lines. Assume

that the input power P_m is constant. Under steady state operation $P_e = P_m$, and the initial power angle is given by

$$\delta_0 = \sin^{-1}\frac{P_m}{P_{1\,max}}$$

where

$$P_{1\,max} = \frac{|E'||V|}{X_1}$$

and X_1 is the transfer reactance before the fault. The rotor is running at synchronous speed, and the change in the angular velocity is zero, i.e.,

$$\Delta\omega_0 = 0$$

Now consider a three-phase fault at the middle of one line as shown in Figure 11.18. The equivalent transfer reactance between bus bars is increased, lowering the power transfer capability, and the amplitude of the power-angle equation becomes

$$P_{2\,max} = \frac{|E'||V|}{X_2}$$

where X_2 is the transfer reactance during fault. The swing equation given by (11.21) is

$$\frac{d^2\delta}{dt^2} = \frac{\pi f_0}{H}(P_m - P_{2\,max}\sin\delta) = \frac{\pi f_0}{H}P_a$$

The above swing equation is transformed into the state variable form as

$$\frac{d\delta}{dt} = \Delta\omega \qquad (11.97)$$
$$\frac{d\Delta\omega}{dt} = \frac{\pi f_0}{H}P_a$$

We now apply the modified Euler's method to the above equations. By using the derivatives at the beginning of the step, the value at the end of the step ($t_1 = t_0 + \Delta t$) is predicted from

$$\delta_{i+1}^p = \delta_i + \left.\frac{d\delta}{dt}\right|_{\Delta\omega_i}\Delta t$$
$$\Delta\omega_{i+1}^p = \Delta\omega_i + \left.\frac{d\Delta\omega}{dt}\right|_{\delta_i}\Delta t$$

Using the predicted value of δ_{i+1}^p, and $\Delta\omega_{i+1}^p$ the derivatives at the end of interval are determined by

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_{i+1}^p} = \Delta\omega_{i+1}^p$$
$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_{i+1}^p} = \left.\frac{\pi f_0}{H}P_a\right|_{\delta_{i+1}^p}$$

Then, the average value of the two derivatives is used to find the corrected value

$$\delta_{i+1}^c = \delta_i + \left(\frac{\left.\frac{d\delta}{dt}\right|_{\Delta\omega_i} + \left.\frac{d\delta}{dt}\right|_{\Delta\omega_{i+1}^p}}{2}\right)\Delta t$$

$$\Delta\omega_{i+1}^c = \Delta\omega_i + \left(\frac{\left.\frac{d\Delta\omega}{dt}\right|_{\delta_i} + \left.\frac{d\Delta\omega}{dt}\right|_{\delta_{i+1}^p}}{2}\right)\Delta t \qquad (11.98)$$

Based on the above algorithm, a function named **swingmeu**(P_m, E, V, X_1, X_2, X_3, H, f, t_c, t_f, Dt) is written for the transient stability analysis of a one-machine system. The function arguments are

- **P_m** Per unit mechanical power input, assumed to remain constant
- **E** Constant voltage back of transient reactance in per unit
- **V** Infinite bus bar voltage in per unit
- **X_1** Per unit reactance between buses E and V before fault
- **X_2** Per unit reactance between buses E and V during fault
- **X_3** Per unit reactance between buses E and V after fault clearance
- **H** Generator inertia constant in second, (MJ/MVA)
- **f** System nominal frequency
- **t_c** Fault clearing time
- **t_f** Final time for integration
- **Dt** Integration time interval, required for modified Euler

If **swingmeu** is used without the arguments, the user is prompted to enter the required data. In addition, based on the *MATLAB* automatic step size Runge-Kutta **ode23 and ode45** functions, two more functions are developed for the transient stability analysis of a one-machine system. These are **swingrk2**(P_m, E, V, X_1, X_2, X_3, H, f, t_c, t_f), based on **ode23**, and **swingrk4**(P_m, E, V, X_1, X_2, X_3, H, f, t_c, t_f), based on **ode45**. The function arguments are as defined above, except since these techniques use automatic step size, the argument **Dt** is not required. Again, if **swingrk2** and **swingrk4** are used without arguments, the user is prompted to enter the required data. All the functions above use a function named **cctime**(P_m, E, V, X_1, X_2, X_3, H, f), which obtains the critical clearing time of fault for critical stability.

Example 11.6

In the system of Example 11.5 a three-phase fault at the middle of one line is cleared by isolating the faulted circuit simultaneously at both ends.

(a) The fault is cleared in 0.3 second. Obtain the numerical solution of the swing equation for 1.0 second using the modified Euler method (function **swingmeu**) with a step size of $\Delta t = 0.01$ second. From the swing curve, determine the system stability.
(b) The **swingmeu** function automatically calls upon the **cctime** function and determines the critical clearing time. Repeat the simulation and obtain the swing plots for the critical clearing time, and when fault is cleared in 0.5 second.
(c) Obtain a *SIMULINK* block diagram model for the swing equation, and simulate for a fault clearing time of 0.3 and 0.5 second. Repeat the simulation until a critical clearing time is obtained.

(a) For the purpose of understanding the procedure, the computations are performed for one step. From Example 11.5, the power-angle curve before the occurrence of the fault is given by

$$P_{1\,max} = 1.8\sin\delta$$

and the generator is operating at the initial power angle

$$\delta_0 = 26.388° = 0.46055 \text{ rad}$$
$$\Delta\omega_0 = 0$$

The fault occurs at point F at the middle of one line, resulting in the circuit shown in Figure 11.23 (page 499). From the results obtained in Example 11.5, the accelerating power equation is

$$P_a = 0.8 - 0.65\sin\delta$$

Applying the modified Euler's method, the derivatives at the beginning of the step are

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_0} = 0$$
$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_0} = \frac{\pi(60)}{5}(0.8 - 0.65\sin 26.388°) = 19.2684 \ rad/sec^2$$

At the end of the first step ($t_1 = 0.01$), the predicted values are

$$\delta_1^p = 0.46055 + (0)(0.01) = 0.46055 \text{ rad} = 26.388°$$
$$\Delta\omega_1^p = 0 + (19.2684)(0.01) = 0.1927 \text{ rad/sec}$$

Using the predicted value of δ_1^p, and $\Delta\omega_1^p$, the derivatives at the end of interval are determined by

$$\left.\frac{d\delta}{dt}\right|_{\Delta\omega_1^p} = \Delta\omega_1^p = 0.1927 \ \text{rad/sec}$$

$$\left.\frac{d\Delta\omega}{dt}\right|_{\delta_1^p} = \frac{\pi(60)}{5}(0.8 - \sin 26.388°) = 19.2684 \ \text{rad/sec}^2$$

Then, the average value of the two derivatives is used to find the corrected value

$$\delta_1^c = 0.46055 + \frac{0 + 0.1927}{2}(0.01) = 0.4615 \ \text{rad}$$

$$\Delta\omega_1^c = 0.0 + \frac{19.2684 + 19.2684}{2}(0.01) = 0.1927 \ \text{rad/sec}$$

The process is continued for the successive steps, until at $t = 0.3$ second when the fault is cleared. From Example 11.5, the postfault accelerating power equation is

$$P_a = 0.8 - 1.4625 \sin\delta$$

The process is continued with the new accelerating equation until the specified final time $t_f = 1.0$ second. The complete computations are obtained using the **swingmeu** function as follows

```
Pm = 0.80;  E = 1.17;  V = 1.0;
X1 = 0.65; X2 = 1.80; X3 = 0.8;
H = 5; f = 60; tc = 0.3; tf = 1.0; Dt = 0.01;
swingmeu(Pm, E, V, X1, X2, X3, H, f, tc, tf, Dt)
```

The time interval and the corresponding power angle δ in degrees and the speed deviation $\Delta\omega$ in rad/sec are displayed in a tabular form. The swing plot is displayed as shown in Figure 11.27.

The swing curve shows that the power angle returns after a maximum swing indicating that with inclusion of system damping, the oscillations will subside and a new operating angle is attained. Hence, the system is found to be stable for this fault clearing time. The critical clearing time is determined by the program to be

```
Critical clearing time  = 0.4 second
Critical clearing angle = 98.83 degrees
```

(b) The above program is run for a clearing time of $t_c = 0.4$ second and $t_c = 0.5$ second with the results shown in Figure 11.28. The swing curve for $t_c = 0.4$ second corresponds to the critical clearing time. The swing curve for $t_c = 0.5$

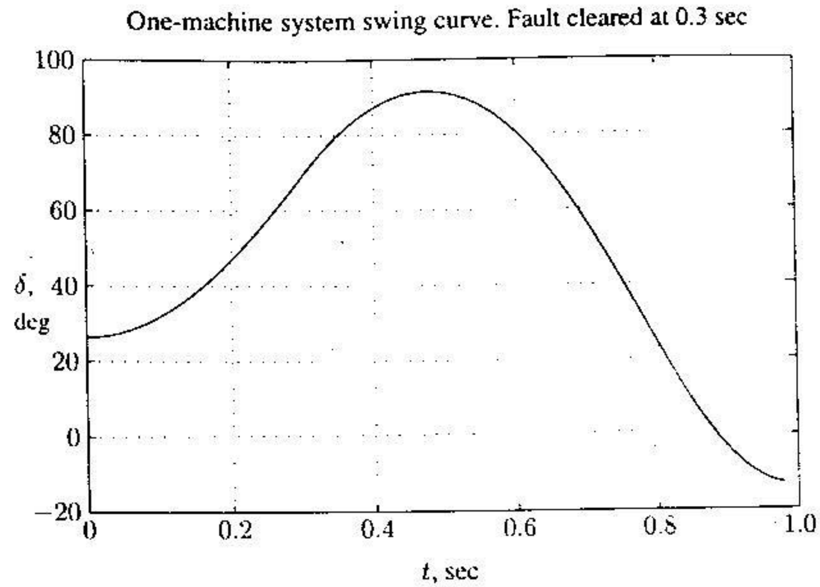


FIGURE 11.27
Swing curve for machine of Example 11.6. Fault cleared at 0.3 sec.

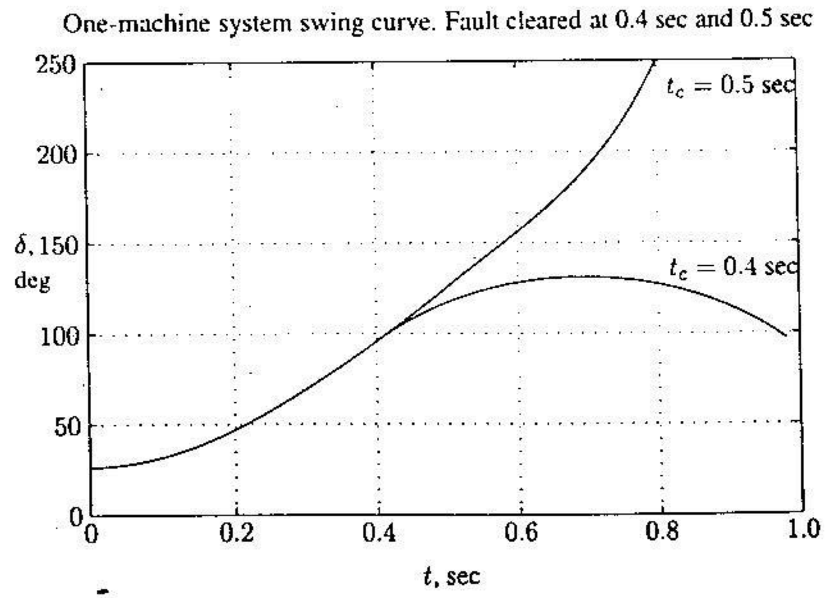


FIGURE 11.28
Swing curves for machine of Example 11.6, for fault clearance at 0.4 sec and 0.5 sec.

second shows that the power angle δ is increasing without limit. Hence, the system is unstable for this clearing time.

The swing curves for the machine in Problem 11.6 are obtained for fault clearing times of $t_c = 0.3$, $t_c = 0.4$, and $t_c = 0.5$, with **swingrk4** function, which uses the *MATLAB* **ode45** function. We use the following statements.

```
Pm = 0.80; E = 1.17; V = 1.0; H = 5.0; f = 60;
X1 = 0.65; X2 = 1.80; X3 = 0.8;
tc = 0.3; tf = 1;
swingrk4(Pm, E, V, X1, X2, X3, H, f, tc, tf)
tc = .5;
swingrk4(Pm, E, V, X1, X2, X3, H, f, tc, tf)
tc = .4;
swingrk4(Pm, E, V, X1, X2, X3, H, f, tc, tf)
```

The same numerical solutions are obtained and the swing curves are the same as the ones shown in Figures 11.27 and 11.28.

(c) Using the state-space representation of the swing equation, given in (11.97), a *SIMULINK* model named **sim11ex6.mdl** is constructed as shown in Figure 11.29.

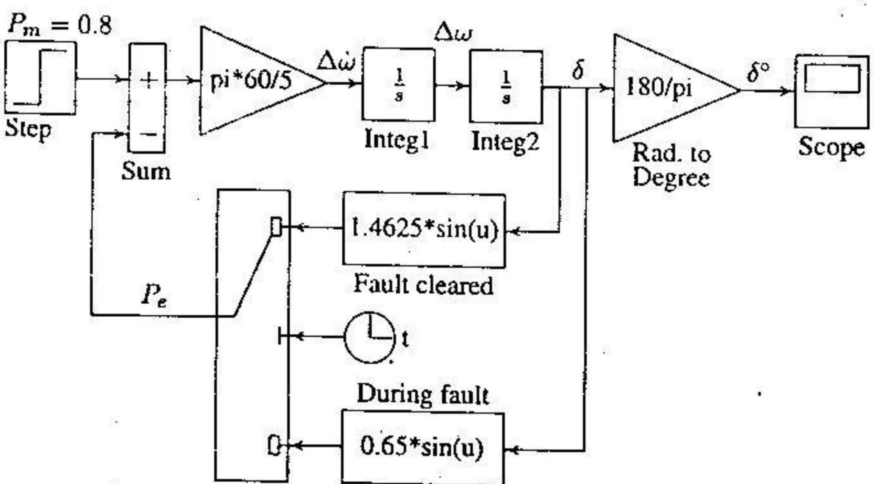


FIGURE 11.29
Simulation block diagram for Example 11.6.

The file is opened and is run in the *SIMULINK WINDOW*. Open the Switch Dialog Box and change the Switch Threshold setting for different values of fault clearing time. The simulation results in the same response as shown in Figure 11.27.

11.9 MULTIMACHINE SYSTEMS

Multimachine equations can be written similar to the one-machine system connected to the infinite bus. In order to reduce the complexity of the transient stability analysis, similar simplifying assumptions are made as follows.

1. Each synchronous machine is represented by a constant voltage source behind the direct axis transient reactance. This representation neglects the effect of saliency and assumes constant flux linkages.
2. The governor's actions are neglected and the input powers are assumed to remain constant during the entire period of simulation.
3. Using the prefault bus voltages, all loads are converted to equivalent admittances to ground and are assumed to remain constant.
4. Damping or asynchronous powers are ignored.
5. The mechanical rotor angle of each machine coincides with the angle of the voltage behind the machine reactance.
6. Machines belonging to the same station swing together and are said to be *coherent*. A group of coherent machines is represented by one equivalent machine.

The first step in the transient stability analysis is to solve the initial load flow and to determine the initial bus voltage magnitudes and phase angles. The machine currents prior to disturbance are calculated from

$$I_i = \frac{S_i^*}{V_i^*} = \frac{P_i - jQ_i}{V_i^*} \qquad i = 1, 2, \ldots, m \tag{11.99}$$

where m is the number of generators. V_i is the terminal voltage of the ith generator, P_i and Q_i are the generator real and reactive powers. All unknown values are determined from the initial power flow solution. The generator armature resistances are usually neglected and the voltages behind the transient reactances are then obtained.

$$E_i' = V_i + jX_d' I_i \tag{11.100}$$

Next, all loads are converted to equivalent admittances by using the relation

$$y_{i0} = \frac{S_i^*}{|V_i|^2} = \frac{P_i - jQ_i}{|V_i|^2} \tag{11.101}$$

To include voltages behind transient reactances, m buses are added to the n-bus power system network. The equivalent network with all loads converted to admittances is shown in Figure 11.30.

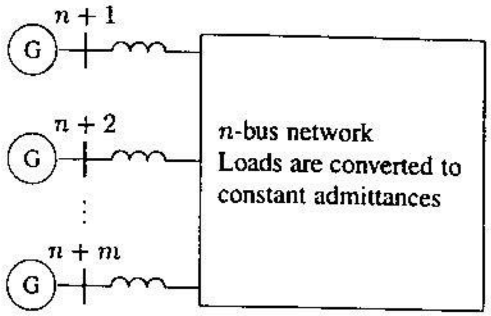


FIGURE 11.30
Power system representation for transient stability analysis.

Nodes $n+1, n+2, \ldots, n+m$ are the internal machine buses, i.e., the buses behind the transient reactances. The node voltage equation with node 0 as reference for this network, as given by (6.2), is

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \\ \hline I_{n+1} \\ \vdots \\ I_{n+m} \end{bmatrix} = \left[\begin{array}{ccc|ccc} Y_{11} & \cdots & Y_{1n} & Y_{1(n+1)} & \cdots & Y_{1(n+m)} \\ Y_{21} & \cdots & Y_{2n} & Y_{2(n+1)} & \cdots & Y_{2(n+m)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{nn} & Y_{n(n+1)} & \cdots & Y_{n(n+m)} \\ \hline Y_{(n+1)1} & \cdots & Y_{(n+1)n} & Y_{(n+1)(n+1)} & \cdots & Y_{(n+1)(n+m)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{(n+m)1} & \cdots & Y_{(n+m)n} & Y_{(n+m)(n+1)} & \cdots & Y_{(n+m)(n+m)} \end{array}\right] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ \hline E'_{n+1} \\ \vdots \\ E'_{n+m} \end{bmatrix} \tag{11.102}$$

or

$$\mathbf{I}_{bus} = \mathbf{Y}_{bus}\,\mathbf{V}_{bus} \tag{11.103}$$

where $\mathbf{I}_{bus}$ is the vector of the injected bus currents and $\mathbf{V}_{bus}$ is the vector of bus voltages measured from the reference node. The diagonal elements of the bus admittance matrix are the sum of admittances connected to it, and the off-diagonal elements are equal to the negative of the admittance between the nodes. This is similar to the Ifybus used in the power flow analysis. The difference is that additional nodes are added to include the machine voltages behind transient reactances. Also, diagonal elements are modified to include the load admittances.

To simplify the analysis, all nodes other than the generator internal nodes are eliminated using the Kron reduction formula. To eliminate the load buses, the bus admittance matrix in (11.102) is partitioned such that the n buses to be removed are represented in the upper n rows. Since no current enters or leaves the load buses, currents in the n rows are zero. The generator currents are denoted by the vector $\mathbf{I}_m$ and the generator and load voltages are represented by the vectors $\mathbf{E}'_m$ and $\mathbf{V}_n$, respectively. Then, Equation (11.102), in terms of submatrices, becomes

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{nn} & \mathbf{Y}_{nm} \\ \mathbf{Y}^t_{nm} & \mathbf{Y}_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{E}'_m \end{bmatrix} \tag{11.104}$$

The voltage vector $\mathbf{V}_n$ may be eliminated by substitution as follows.

$$0 = \mathbf{Y}_{nn}\mathbf{V}_n + \mathbf{Y}_{nm}\mathbf{E}'_m \tag{11.105}$$

$$\mathbf{I}_m = \mathbf{Y}^t_{nm}\mathbf{V}_n + \mathbf{Y}_{mm}\mathbf{E}'_m \tag{11.106}$$

from (11.105),

$$\mathbf{V}_n = -\mathbf{Y}_{nn}{}^{-1}\mathbf{Y}_{nm}\mathbf{E}'_m \tag{11.107}$$

Now substituting into (11.106), we have

$$\begin{aligned} \mathbf{I}_m &= [\mathbf{Y}_{mm} - \mathbf{Y}^t_{nm}\mathbf{Y}_{nn}{}^{-1}\mathbf{Y}_{nm}]\mathbf{E}'_m \\ &= \mathbf{Y}^{red}_{bus}\mathbf{E}'_m \end{aligned} \tag{11.108}$$

The reduced admittance matrix is

$$\mathbf{Y}^{red}_{bus} = \mathbf{Y}_{mm} - \mathbf{Y}^t_{nm}\mathbf{Y}_{nn}{}^{-1}\mathbf{Y}_{nm} \tag{11.109}$$

The reduced bus admittance matrix has the dimensions $(m \times m)$, where m is the number of generators.

The electrical power output of each machine can now be expressed in terms of the machine's internal voltages.

$$S^*_{ei} = E'^*_i I_i$$

or

$$P_{ei} = \Re[E'^*_i I_i] \tag{11.110}$$

where

$$I_i = \sum_{j=1}^{m} E'_j Y_{ij} \tag{11.111}$$

Expressing voltages and admittances in polar form, i.e., $E'_i = |E'_i|\angle\delta_i$ and $Y_{ij} = |Y_{ij}|\angle\theta_{ij}$, and substituting for I_i in (11.110), results in

$$P_{ei} = \sum_{j=1}^{m} |E'_i||E'_j||Y_{ij}|\cos(\theta_{ij} - \delta_i + \delta_j) \tag{11.112}$$

The above equation is the same as the power flow equation given by (6.52). Prior to disturbance, there is equilibrium between the mechanical power input and the electrical power output, and we have

$$P_{mi} = \sum_{j=1}^{m} |E'_i||E'_j||Y_{ij}|\cos(\theta_{ij} - \delta_i + \delta_j) \tag{11.113}$$

11.10 MULTIMACHINE TRANSIENT STABILITY

The classical transient stability study is based on the application of a three-phase fault. A solid three-phase fault at bus k in the network results in $V_k = 0$. This is simulated by removing the kth row and column from the prefault bus admittance matrix. The new bus admittance matrix is reduced by eliminating all nodes except the internal generator nodes. The generator excitation voltages during the fault and postfault modes are assumed to remain constant. The electrical power of the ith generator in terms of the new reduced bus admittance matrices are obtained from (11.112). The swing equation with damping neglected, as given by (11.21), for machine i becomes

$$\frac{H_i}{\pi f_0}\frac{d^2\delta_i}{dt^2} = P_{mi} - \sum_{j=1}^{m} |E'_i||E'_j||Y_{ij}|\cos(\theta_{ij} - \delta_i + \delta_j) \tag{11.114}$$

where Y_{ij} are the elements of the faulted reduced bus admittance matrix, and H_i is the inertia constant of machine i expressed on the common MVA base S_B. If H_{Gi} is the inertia constant of machine i expressed on the machine rated MVA S_{Gi}, then H_i is given by

$$H_i = \frac{S_{Gi}}{S_B}H_{Gi} \tag{11.115}$$

Showing the electrical power of the ith generator by P_e^f and transforming (11.114) into state variable model yields

$$\frac{d\delta_i}{dt} = \Delta\omega_i \qquad i = 1, \ldots, m \tag{11.116}$$

$$\frac{d\Delta\omega_i}{dt} = \frac{\pi f_0}{H_i}(P_m - P_e^f)$$

We have two state equations for each generator, with initial power angles δ_{0_i} and $\Delta\omega_{0_i} = 0$. The *MATLAB* function **ode23** is employed to solve the above $2m$ first-order differential equations. When the fault is cleared, which may involve the removal of the faulty line, the bus admittance matrix is recomputed to reflect the change in the network. Next, the postfault reduced bus admittance matrix is evaluated and the postfault electrical power of the ith generator shown by P_i^{pf} is readily determined from (11.112). Using the postfault power P_i^{pf}, the simulation is continued to determine the system stability, until the plots reveal a definite trend as to stability or instability. Usually the slack generator is selected as the reference, and the phase angle difference of all other generators with respect to the reference machine are plotted. Usually, the solution is carried out for two swings to show that the second swing is not greater than the first one. If the angle differences do not increase, the system is stable. If any of the angle differences increase indefinitely, the system is unstable.

Based on the above procedure, a program named **trstab** is developed for transient stability analysis of a multimachine network subjected to a balanced three-phase fault. The program **trstab** must be preceded by the power flow program. Any of the power flow programs **lfgauss**, **lfnewton**, or **decouple** can be used. In addition to the power flow data, generator data must be specified in a matrix named **gendata**. The first column contains the generator bus number terminal. Columns 2 and 3 contain resistance and transient reactance in per unit on the specified common MVA base, and the last column contain the machine inertia constant in seconds, expressed on the common MVA base. The program **trstab** automatically adds additional buses to include the generator impedances in the power flow line data. Also, the bus admittance matrix is modified to include the load admittances **yload**, returned by the power flow program. The program prompts the user to enter the faulted bus number, fault clearing time, and the line numbers of the removed faulty line. The program displays the prefault, faulted, and postfault reduced bus admittance matrices. The machine phase angles are tabulated and a plot of the swing curves is obtained. The program inquires for other fault clearing times and fault locations. The use of **trstab** program is demonstrated in the following example.

Example 11.7

The power system network of an electric utility company is shown in Figure 11.31. The load data and voltage magnitude, generation schedule, and the reactive power limits for the regulated buses are tabulated on the next page. Bus 1, whose voltage is specified as $V_1 = 1.06\angle 0°$, is taken as the slack bus. The line data containing the series resistance and reactance in per unit, and one-half of the total capacitance in per unit susceptance on a 100-MVA base is also tabulated as shown.

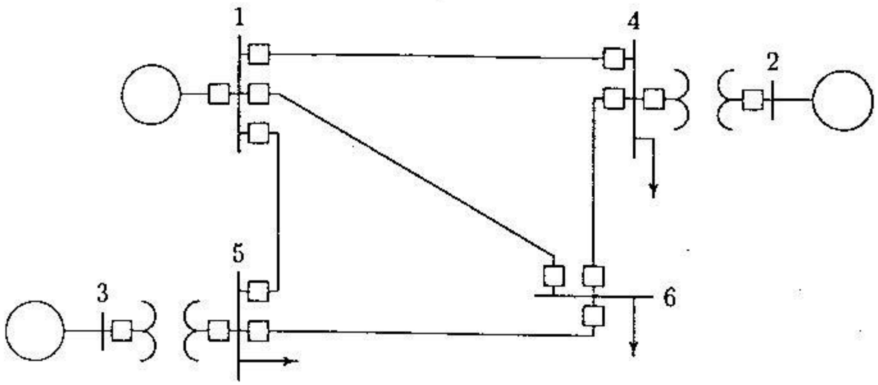


FIGURE 11.31
One-line diagram for Problem 11.7.

LOAD DATA		
Bus	Load	
No.	MW	Mvar
1	0	0
2	0	0
3	0	0
4	100	70
5	90	30
6	160	110

GENERATION SCHEDULE				
Bus	Voltage	Generation,	Mvar Limits	
No.	Mag.	MW	Min.	Max.
1	1.06			
2	1.04	150	0	140
3	1.03	100	0	90

LINE DATA				
Bus No.	Bus No.	R, PU	X, PU	$\frac{1}{2}B$, PU
1	4	0.035	0.225	0.0065
1	5	0.025	0.105	0.0045
1	6	0.040	0.215	0.0055
2	4	0.000	0.035	0.0000
3	5	0.000	0.042	0.0000
4	6	0.028	0.125	0.0035
5	6	0.026	0.175	0.0300

The generator's armature resistances and transient reactances in per unit, and the inertia constants in seconds expressed on a 100-MVA base are given below:

MACHINE DATA			
Gen.	R_a	X'_d	H
1	0	0.20	20
2	0	0.15	4
3	0	0.25	5

A three-phase fault occurs on line 5–6 near bus 6, and is cleared by the simultaneous opening of breakers at both ends of the line. Using the **trstab** program, perform a transient stability analysis. Determine the system stability for
(a) When the fault is cleared in 0.4 second
(b) When the fault is cleared in 0.5 second
(c) Repeat the simulation to determine the critical clearing time.

The required data and commands are as follows:

```
basemva = 100;  accuracy = 0.0001;  maxiter = 10;
%       Bus Bus  Voltage Angle --Load-- --Generator-- Injected
%        No code Mag. degree MW  Mvar  MW  Mvar Qmin Qmax Mvar
busdata=[1   1   1.06   0      0     0    0   0    0     0   0
         2   2   1.04   0      0     0  150   0    0   140   0
         3   2   1.03   0      0     0  100   0    0    90   0
         4   0   1.0    0    100    70    0   0    0     0   0
         5   0   1.0    0     90    30    0   0    0     0   0
         6   0   1.0    0    160   110    0   0    0     0   0];
% Line data
%          Bus  bus   R      X      1/2 B   1 for line code or
%          nl   nr    pu     pu     pu      tap setting value
linedata=[1   4   0.035  0.225  0.0065   1.0
          1   5   0.025  0.105  0.0045   1.0
          1   6   0.040  0.215  0.0055   1.0
          2   4   0.000  0.035  0.0000   1.0
          3   5   0.000  0.042  0.0000   1.0
          4   6   0.028  0.125  0.0035   1.0
          5   6   0.026  0.175  0.0300   1.0];
lfybus           % form the bus admittance matrix for power flow
lfnewton         % Power flow solution by Newton-Raphson method
busout           % Prints the power flow solution on the screen
% Generator data
%         Gen.  Ra    Xd'   H
gendata=[ 1     0    0.20  20
          2     0    0.15   4
          3     0    0.25   5];
trstab                       % Performs the stability analysis.
       % User is prompted to enter the clearing time of fault.
```

The power flow result is

```
              Power Flow Solution by Newton-Raphson Method
                 Maximum Power Mismatch = 1.80187e-007
                        No. of Iterations = 4

Bus  Voltage  Angle   -----Load-----  ---Generation--- Injected
No.  Mag.     degree   MW       Mvar    MW       Mvar    Mvar

1    1.060    0.000     0.000    0.00  105.287 107.335   0.00
2    1.040    1.470     0.000    0.00  150.000  99.771   0.00
3    1.030    0.800     0.000    0.00  100.000  35.670   0.00
4    1.008   -1.401   100.000   70.00    0.000   0.000   0.00
5    1.016   -1.499    90.000   30.00    0.000   0.000   0.00
6    0.941   -5.607   160.000  110.00    0.000   0.000   0.00

Total                 350.000  210.00  355.287 242.776   0.00
```

The **trstab** result is

```
Reduced prefault bus admittance matrix

 Ybf =
        0.3517 - 2.8875i   0.2542 + 1.1491i   0.1925 + 0.9856i
        0.2542 + 1.1491i   0.5435 - 2.8639i   0.1847 + 0.6904i
        0.1925 + 0.9856i   0.1847 + 0.6904i   0.2617 - 2.2835i

      G(i)    E'(i)     d0(i)     Pm(i)
       1     1.2781     8.9421    1.0529
       2     1.2035    11.8260    1.5000
       3     1.1427    13.0644    1.0000

Enter faulted bus No. -> 6

Reduced faulted bus admittance matrix

 Ypf =
        0.1913 - 3.5849i   0.0605 + 0.3644i   0.0523 + 0.4821i
        0.0605 + 0.3644i   0.3105 - 3.7467i   0.0173 + 0.1243i
        0.0523 + 0.4821i   0.0173 + 0.1243i   0.1427 - 2.6463i

Fault is cleared by opening a line.  The bus to bus numbers of
line to be removed must be entered within brackets, e.g. [5,7]
Enter the bus to bus Nos. of line to be removed -> [5, 6]
```



```
Reduced postfault bus admittance matrix

 Yaf =
        0.3392 - 2.8879i   0.2622 + 1.1127i   0.1637 + 1.0251i
        0.2622 + 1.1127i   0.6020 - 2.7813i   0.1267 + 0.5401i
        0.1637 + 1.0251i   0.1267 + 0.5401i   0.2859 - 2.0544i

Enter clearing time of fault in sec. tc = 0.4
Enter final simulation time in sec.  tf = 1.5
```

The phase angle differences of each machine with respect to the slack bus are printed in a tabular form on the screen, which is not presented here. The program also obtains a plot of the swing curves which is presented in Figure 11.32.

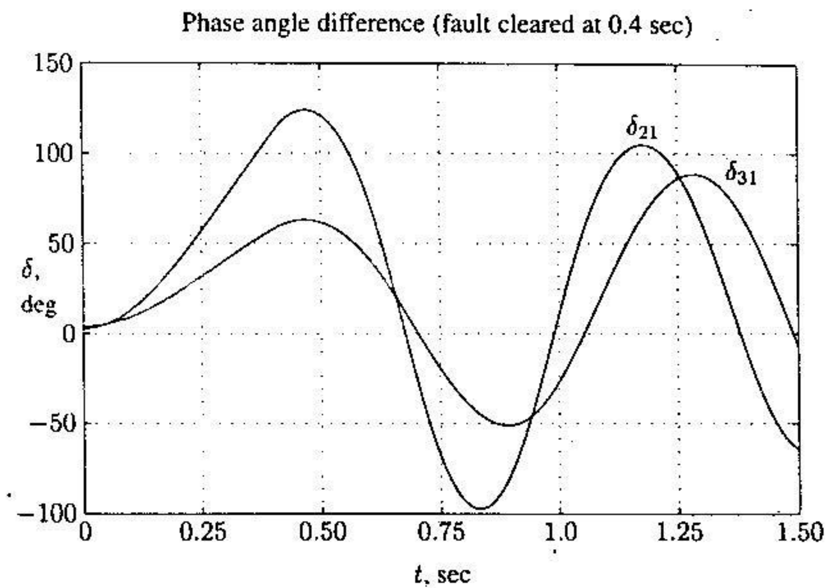


FIGURE 11.32
Plots of angle differences for machines 2 and 3 of Example 11.7.

Again the tabulated result is printed on the screen, and plots of the swing curves are obtained as shown in Figure 11.33. Figure 11.32 shows that the phase angle differences, after reaching a maximum of $\delta_{21} = 123.9°$ and $\delta_{31} = 62.95°$ will decrease, and the machines swing together. Hence, the system is found to be stable when fault is cleared in 0.4 second.

The program inquires for another fault clearing time, and the results continue as follows:

```
Another clearing time of fault?
Enter 'y' or 'n' within quotes -> 'y'
Enter clearing time of fault in sec. tc = 0.5
Enter final simulation time in sec.  tf = 1.5
```

The swing curves shown in Figure 11.33 show that machine 2 phase angle increases without limit. Thus, the system is unstable when fault is cleared in 0.5 second. The simulation is repeated for a clearing time of 0.45 second, which is found to be critically stable.

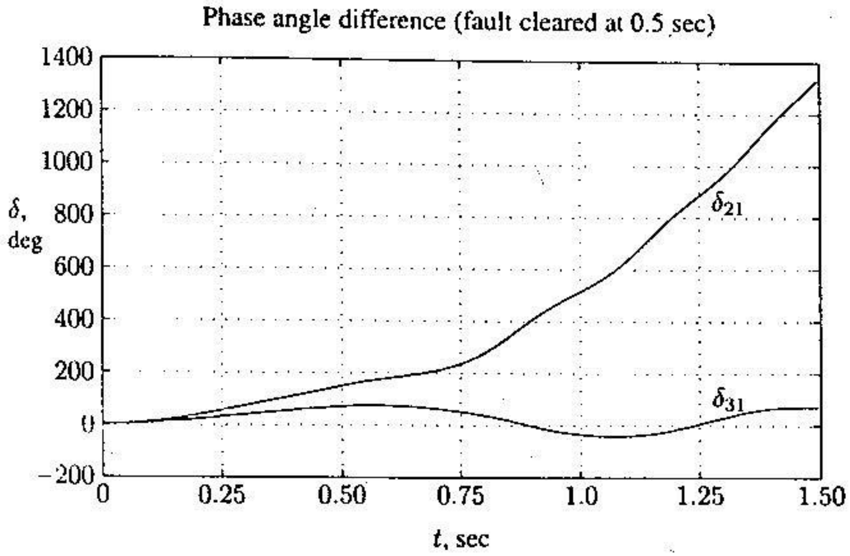


FIGURE 11.33
Plots of angle differences for machines 2 and 3 of Example 11.7.

PROBLEMS

11.1. A four-pole, 60-Hz synchronous generator has a rating of 200 MVA, 0.8 power factor lagging. The moment of inertia of the rotor is 45,100 kg·m^2. Determine M and H.

11.2. A two-pole, 60-Hz synchronous generator has a rating of 250 MVA, 0.8 power factor lagging. The kinetic energy of the machine at synchronous speed is 1080 MJ. The machine is running steadily at synchronous speed and delivering 60 MW to a load at a power angle of 8 electrical degrees. The load is suddenly removed. Determine the acceleration of the rotor. If the acceleration computed for the generator is constant for a period of 12 cycles, determine the value of the power angle and the rpm at the end of this time.

11.3. Determine the kinetic energy stored by a 250-MVA, 60-Hz, two-pole synchronous generator with an inertia constant H of 5.4 MJ/MVA. Assume the machine is running steadily at synchronous speed with a shaft input of 331,100 hp The electrical power developed suddenly changes from its normal value to a value of 200 MW. Determine the acceleration or deceleration of the rotor. If the acceleration computed for the generator is constant for a period of 9 cycles, determine the change in the power angle in that period and the rpm at the end of 9 cycles.

11.4. The swing equations of two interconnected synchronous machines are written as

$$\frac{H_1}{\pi f_0}\frac{d^2\delta_1}{dt^2} = P_{m1} - P_{e1}$$

$$\frac{H_2}{\pi f_0}\frac{d^2\delta_2}{dt^2} = P_{m2} - P_{e2}$$

Denote the relative power angle between the two machines by $\delta = \delta_1 - \delta_2$. Obtain a swing equation equivalent to that of a single machine in terms of δ, and show that

$$\frac{H}{\pi f_0}\frac{d^2\delta}{dt^2} = P_m - P_e$$

where

$$H = \frac{H_1 H_2}{H_1 + H_2}$$

$$P_m = \frac{H_2 P_{m1} - H_1 P_{m2}}{H_1 + H_2}$$

$$P_e = \frac{H_2 P_{e1} - H_1 P_{e2}}{H_1 + H_2}$$

11.5. Two synchronous generators represented by a constant voltage behind transient reactance are connected by a pure reactance $X = 0.3$ per unit, as shown in Figure 11.34. The generator inertia constants are $H_1 = 4.0$ MJ/MVA and $H_2 = 6$ MJ/MVA, and the transient reactances are $X_1' = 0.16$ and $X_2' = 0.20$ per unit. The system is operating in the steady state with $E_1' = 1.2$, $P_{m1} = 1.5$ and $E_2' = 1.1$, $P_{m2} = 1.0$ per unit. Denote the relative power

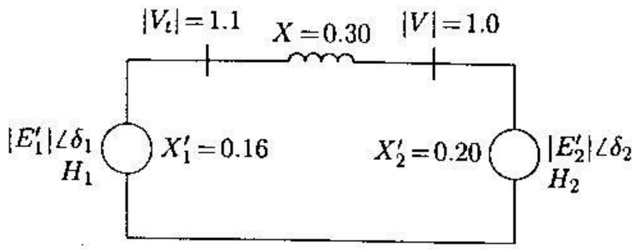


FIGURE 11.34
System of Problem 11.5.

angle between the two machines by $\delta = \delta_1 - \delta_2$. Referring to Problem 11.4, reduce the two-machine system to an equivalent one-machine against an infinite bus. Find the inertia constant of the equivalent machine, the mechanical input power, and the amplitude of its power angle curve, and obtain the equivalent swing equation in terms of δ.

11.6. A 60-Hz synchronous generator has a transient reactance of 0.2 per unit and an inertia constant of 5.66 MJ/MVA. The generator is connected to an infinite bus through a transformer and a double circuit transmission line, as shown in Figure 11.35. Resistances are neglected and reactances are expressed on a common MVA base and are marked on the diagram. The generator is delivering a real power of 0.77 per unit to bus bar 1. Voltage magnitude at bus 1 is 1.1. The infinite bus voltage $V = 1.0\angle 0°$ per unit. Determine the generator excitation voltage and obtain the swing equation as given by (11.36).

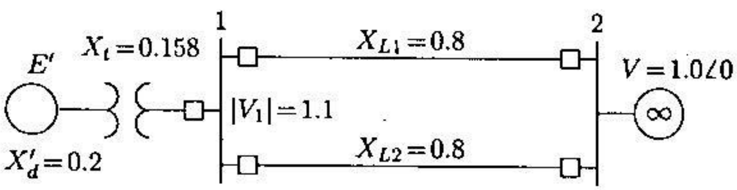


FIGURE 11.35
System of Problem 11.6.

11.7. A three-phase fault occurs on the system of Problem 11.6 at the sending end of the transmission lines. The fault occurs through an impedance of 0.082 per unit. Assume the generator excitation voltage remains constant at $E' = 1.25$ per unit. Obtain the swing equation during the fault.

11.8. The power-angle equation for a salient-pole generator is given by

$$P_e = P_{max}\sin\delta + P_K \sin 2\delta$$

Consider a small deviation in power angle from the initial operating point δ_0, i.e., $\delta = \delta + \delta_0$. Obtain an expression for the synchronizing power coefficient, similar to (11.39). Also, find the linearized swing equation in terms of $\Delta\delta$.

11.9. Consider the displacement x for a unit mass supported by a nonlinear spring as shown in Figure 11.36. The equation of motion is described by

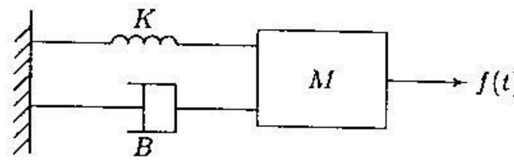


FIGURE 11.36
System of Problem 11.9.

$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + K\sin x(t) = f(t)$$

where M is the mass, B is the frictional coefficient and K is the spring constant. The system is at the steady state $x(0) = 0$ for $f(0) = 0$. A small perturbation $f(t) = f(0) + \Delta f(t)$ results in the displacement $x(t) = x(0) + \Delta x(t)$.

(a) Obtain a linearized expression for the motion of the system in terms of the system parameters, $\Delta x(t)$ and $\Delta f(t)$.
(b) For $M = 1.6$, $B = 9.6$, and $K = 40$, find the damping ratio ζ and the damped frequency of oscillation ω_d.

11.10. The machine in the power system of Problem 11.6 has a per unit damping coefficient of $D = 0.15$. The generator excitation voltage is $E' = 1.25$ per unit and the generator is delivering a real power of 0.77 per unit to the infinite bus at a voltage of $V = 1.0$ per unit. Write the linearized swing equation for this power system. Use (11.61) and (11.62) to find the equations describing the motion of the rotor angle and the generator frequency for a small disturbance of $\Delta\delta = 15°$. Use *MATLAB* to obtain the plots of rotor angle and frequency.

11.11. Write the linearized swing equation of Problem 11.10 in state variable form. Use $[y, x]$ = **initial**(A, B, C, D, x_0, t) and **plot** commands to obtain the zero-input response for the initial conditions $\Delta\delta = 15°$, and $\Delta\omega_n = 0$.

11.12. The generator of Problem 11.10 is operating in the steady state at $\delta_0 = 27.835°$ when the input power is increased by a small amount $\Delta P = 0.15$ per unit. The generator excitation and the infinite bus voltage are the same as before. Use (11.75) and (11.76) to find the equations describing the motion of the rotor angle and the generator frequency for a small disturbance of $\Delta P = 0.15$ per unit. Use *MATLAB* to obtain the plots of rotor angle and frequency.

11.13. Write the linearized swing equation of Problem 11.10 in state variable form. Use $[y, x] =$ **step**$(A, B, C, D, 1, t)$ and **plot** commands to obtain the zero-state response when the input power is increased by a small amount $\Delta P = 0.15$ per unit.

11.14. The machine of Problem 11.6 is delivering a real power input of 0.77 per unit to the infinite bus at a voltage of 1.0 per unit. The generator excitation voltage is $E' = 1.25$ per unit. Use **eacpower**(P_m, E, V, X) to find
(a) The maximum power input that can be added without loss of synchronism.
(b) Repeat (a) with zero initial power input. Assume the generator internal voltage remains constant at the value computed in (a).

11.15. The machine of Problem 11.6 is delivering a real power input of 0.77 per unit to the infinite bus at a voltage of 1.0 per unit. The generator excitation voltage is $E' = 1.25$ per unit.
(a) A temporary three-phase fault occurs at the sending end of one of the transmission lines. When the fault is cleared, both lines are intact. Using equal area criterion, determine the critical clearing angle and the critical fault clearing time. Use **eacfault**$(P_m, E, V, X_1, X_2, X_3)$ to check the result and to display the power-angle plot.
(b) A three-phase fault occurs at the middle of one of the lines, the fault is cleared, and the faulted line is isolated. Determine the critical clearing angle. Use **eacfault**$(P_m, E, V, X_1, X_2, X_3)$ to check the results and to display the power-angle plot.

11.16. The machine of Problem 11.6 is delivering a real power input of 0.77 per unit to the infinite bus at a voltage of 1.0 per unit. The generator excitation voltage is $E' = 1.25$ per unit. A three-phase fault at the middle of one line is cleared by isolating the faulted circuit simultaneously at both ends.
(a) The fault is cleared in 0.2 second. Obtain the numerical solution of the swing equation for 1.5 seconds. Select one of the functions **swingmeu, swingrk2,** or **swingrk4.**
(b) Repeat the simulation and obtain the swing plots when fault is cleared in 0.4 second, and for the critical clearing time.

11.17. Consider the power system network of Example 11.7 with the described operating condition. A three-phase fault occurs on line 1–5 near bus 5 and is cleared by the simultaneous opening of breakers at both ends of the line. Using the **trstab** program, perform a transient stability analysis. Determine the system stability for
(a) When the fault is cleared in 0.2 second
(b) When the fault is cleared in 0.4 second
(c) Repeat the simulation to determine the critical clearing time.

11.18. The power system network of an electric company is shown in Figure 11.37. The load data is as follows.

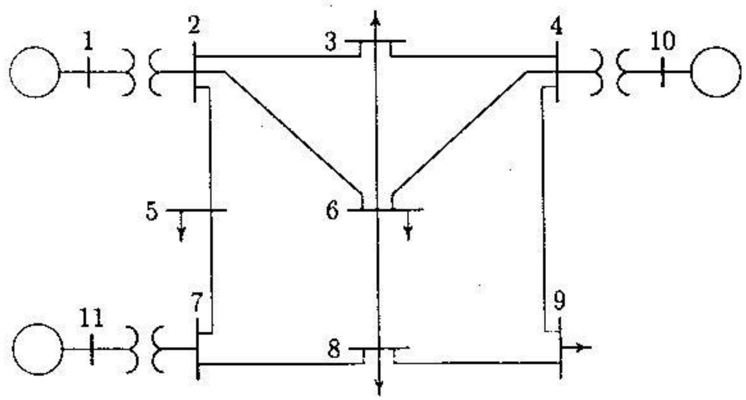


FIGURE 11.37
System of Problem 11.18.

LOAD DATA					
Bus	Load		Bus	Load	
No.	MW	Mvar	No.	MW	Mvar
1	0.0	0.0	7	0.0	0.0
2	0.0	0.0	8	110.0	90.0
3	150.0	120.0	9	80.0	50.0
4	0.0	0.0	10	0.0	0.0
5	120.0	60.0	11	0.0	0.0
6	140.0	90.0			

Voltage magnitude, generation schedule, and the reactive power limits for the regulated buses are tabulated below. Bus 1, whose voltage is specified as $V_1 = 1.04\angle 0°$, is taken as the slack bus.

GENERATION SCHEDULE				
Bus No.	Voltage Mag.	Generation, MW	Mvar Limits Min.	Max.
1	1.040			
10	1.035	200.0	0.0	180.0
11	1.030	160.0	0.0	120.0

The line data containing the per unit series impedance, and one-half of the shunt capacitive susceptance on a 100-MVA base is tabulated below.

LINE AND TRANSFORMER DATA				
Bus No.	Bus No.	R, PU	X, PU	$\frac{1}{2}B$, PU
1	2	0.000	0.006	0.000
2	3	0.008	0.030	0.004
2	5	0.004	0.015	0.002
2	6	0.012	0.045	0.005
3	4	0.010	0.040	0.005
3	6	0.004	0.040	0.005
4	6	0.015	0.060	0.008
4	9	0.018	0.070	0.009
4	10	0.000	0.008	0.000
5	7	0.005	0.043	0.003
6	8	0.006	0.048	0.000
7	8	0.006	0.035	0.004
7	11	0.000	0.010	0.000
8	9	0.005	0.048	0.000

The generator's armature resistance and transient reactances in per unit, and the inertia constants expressed on a 100-MVA base are given below.

MACHINE DATA			
Gen.	R_a	X'_d	H
1	0	0.20	12
10	0	0.15	10
11	0	0.25	9

A three-phase fault occurs on line 4–9, near bus 4, and is cleared by the simultaneous opening of breakers at both ends of the line. Using the **trstab** program, perform a transient stability analysis. Determine the stability for
(a) When the fault is cleared in 0.4 second
(b) When the fault is cleared in 0.8 second
(c) Repeat the simulation to determine the critical clearing time.

CHAPTER 12

POWER SYSTEM CONTROL

12.1 INTRODUCTION

So far, this text has concentrated on the problems of establishing a normal operating state and optimum scheduling of generation for a power system. This chapter deals with the control of active and reactive power in order to keep the system in the steady-state. In addition, simple models of the essential components used in the control systems are presented. The objective of the control strategy is to generate and deliver power in an interconnected system as economically and reliably as possible while maintaining the voltage and frequency within permissible limits.

Changes in real power affect mainly the system frequency, while reactive power is less sensitive to changes in frequency and is mainly dependent on changes in voltage magnitude. Thus, real and reactive powers are controlled separately. The *load frequency control* (LFC) loop controls the real power and frequency and the *automatic voltage regulator* (AVR) loop regulates the reactive power and voltage magnitude. Load frequency control (LFC) has gained in importance with the growth of interconnected systems and has made the operation of interconnected systems possible. Today, it is still the basis of many advanced concepts for the control of large systems.

The methods developed for control of individual generators, and eventually control of large interconnections, play a vital role in modern energy control centers. Modern energy control centers (ECC) are equipped with on-line computers performing all signal processing through the remote acquisition systems known as *supervisory control and data acquisition* (SCADA) systems. Only an introduction to power system control is presented here. This chapter utilizes some of the concepts of feedback control systems. Some students may not be fully versed in feedback theory. Therefore, a brief review of the fundamentals of linear control systems analysis and design is included in Appendix B. The use of *MATLAB CONTROL TOOLBOX* functions and some useful custom-made functions are also described in this appendix.

The role of *automatic generation control* (AGC) in power system operation, with reference to tie-line power control under normal operating conditions, is first analyzed. Typical responses to real power demand are illustrated using the latest simulation technique available by the *MATLAB SIMULINK* package. Finally, the requirement of reactive power and voltage regulation and the influence on stability of both speed and excitation controls, with use of suitable feedback signals, are examined.

12.2 BASIC GENERATOR CONTROL LOOPS

In an interconnected power system, load frequency control (LFC) and automatic voltage regulator (AVR) equipment are installed for each generator. Figure 12.1 represents the schematic diagram of the load frequency control (LFC) loop and the automatic voltage regulator (AVR) loop. The controllers are set for a particular operating condition and take care of small changes in load demand to maintain the frequency and voltage magnitude within the specified limits. Small changes in real power are mainly dependent on changes in rotor angle δ and, thus, the frequency. The reactive power is mainly dependent on the voltage magnitude (i.e., on the generator excitation). The excitation system time constant is much smaller than the prime mover time constant and its transient decay much faster and does not affect the LFC dynamic. Thus, the cross-coupling between the LFC loop and the AVR loop is negligible, and the load frequency and excitation voltage control are analyzed independently.

12.3 LOAD FREQUENCY CONTROL

The operation objectives of the LFC are to maintain reasonably uniform frequency, to divide the load between generators, and to control the tie-line interchange schedules. The change in frequency and tie-line real power are sensed, which is a measure of the change in rotor angle δ, i.e., the error $\Delta\delta$ to be corrected. The error signal, i.e., Δf and ΔP_{tie}, are amplified, mixed, and transformed into a real power command signal ΔP_V, which is sent to the prime mover to call for an increment in the torque.

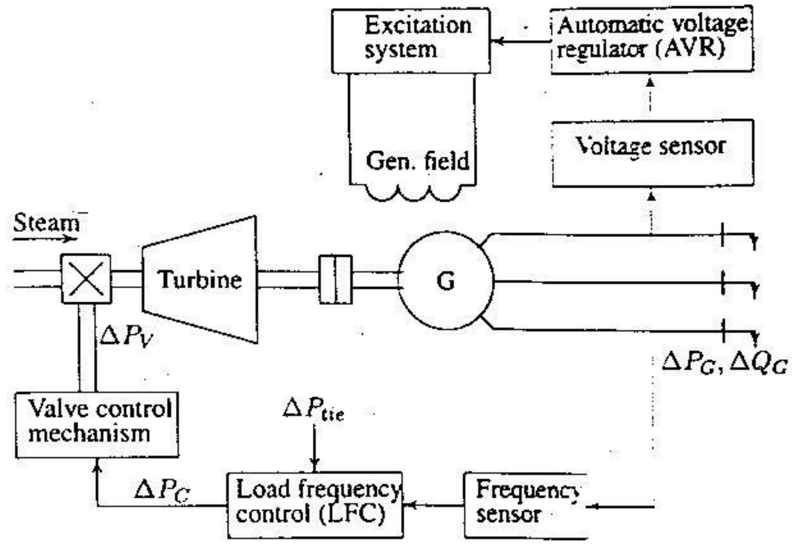


FIGURE 12.1
Schematic diagram of LFC and AVR of a synchronous generator

The prime mover, therefore, brings change in the generator output by an amount ΔP_g which will change the values of Δf and ΔP_{tie} within the specified tolerance.

The first step in the analysis and design of a control system is mathematical modeling of the system. The two most common methods are the transfer function method and the state variable approach. The state variable approach can be applied to portray linear as well as nonlinear systems. In order to use the transfer function and linear state equations, the system must first be linearized. Proper assumptions and approximations are made to linearize the mathematical equations describing the system, and a transfer function model is obtained for the following components.

12.3.1 GENERATOR MODEL

Applying the swing equation of a synchronous machine given by (11.21) to small perturbation, we have

$$\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} = \Delta P_m - \Delta P_e \tag{12.1}$$

or in terms of small deviation in speed

$$\frac{d\Delta \frac{\omega}{\omega_s}}{dt} = \frac{1}{2H}(\Delta P_m - \Delta P_e) \tag{12.2}$$

With speed expressed in per unit, without explicit per unit notation, we have

$$\frac{d\Delta\omega}{dt} = \frac{1}{2H}(\Delta P_m - \Delta P_e) \tag{12.3}$$

Taking Laplace transform of (12.3), we obtain

$$\Delta\Omega(s) = \frac{1}{2Hs}[\Delta P_m(s) - \Delta P_e(s)] \tag{12.4}$$

The above relation is shown in block diagram form in Figure 12.2.

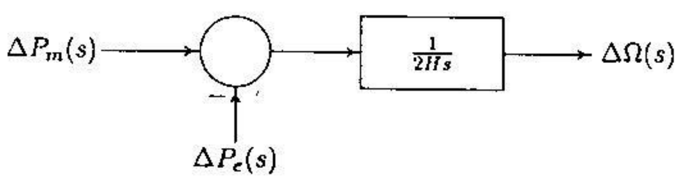


FIGURE 12.2
Generator block diagram.

12.3.2 LOAD MODEL

The load on a power system consists of a variety of electrical devices. For resistive loads, such as lighting and heating loads, the electrical power is independent of frequency. Motor loads are sensitive to changes in frequency. How sensitive it is to frequency depends on the composite of the speed-load characteristics of all the driven devices. The speed-load characteristic of a composite load is approximated by

$$\Delta P_e = \Delta P_L + D\Delta\omega \tag{12.5}$$

where ΔP_L is the nonfrequency-sensitive load change, and $D\Delta\omega$ is the frequency-sensitive load change. D is expressed as percent change in load divided by percent change in frequency. For example, if load is changed by 1.6 percent for a 1 percent change in frequency, then $D = 1.6$. Including the load model in the generator block diagram, results in the block diagram of Figure 12.3. Eliminating the simple feedback loop in Figure 12.3, results in the block diagram shown in Figure 12.4.

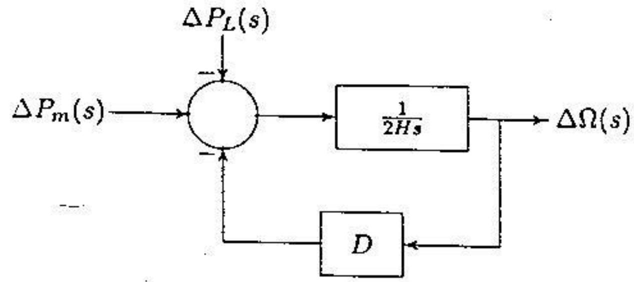


FIGURE 12.3
Generator and load block diagram.

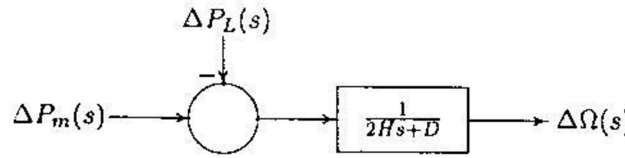


FIGURE 12.4
Generator and load block diagram.

12.3.3 PRIME MOVER MODEL

The source of mechanical power, commonly known as the *prime mover*, may be hydraulic turbines at waterfalls, steam turbines whose energy comes from the burning of coal, gas, nuclear fuel, and gas turbines. The model for the turbine relates changes in mechanical power output ΔP_m to changes in steam valve position ΔP_V. Different types of turbines vary widely in characteristics. The simplest prime mover model for the nonreheat steam turbine can be approximated with a single time constant τ_T, resulting in the following transfer function

$$G_T(s) = \frac{\Delta P_m(s)}{\Delta P_V(s)} = \frac{1}{1 + \tau_T s} \tag{12.6}$$

The block diagram for a simple turbine is shown in Figure 12.5.

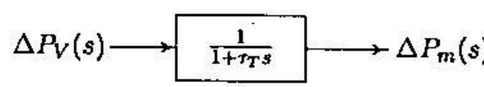


FIGURE 12.5
Block diagram for a simple nonreheat steam turbine.

The time constant τ_T is in the range of 0.2 to 2.0 seconds.

12.3.4 GOVERNOR MODEL

When the generator electrical load is suddenly increased, the electrical power exceeds the mechanical power input. This power deficiency is supplied by the kinetic energy stored in the rotating system. The reduction in kinetic energy causes the turbine speed and, consequently, the generator frequency to fall. The change in speed is sensed by the turbine governor which acts to adjust the turbine input valve to change the mechanical power output to bring the speed to a new steady-state. The earliest governors were the Watt governors which sense the speed by means of rotating *flyballs* and provides mechanical motion in response to speed changes. However, most modern governors use electronic means to sense speed changes. Figure 12.6 shows schematically the essential elements of a conventional Watt governor, which consists of the following major parts.

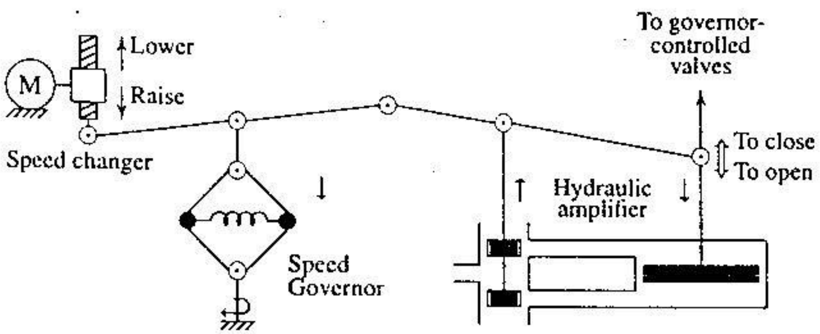


FIGURE 12.6
Speed governing system.

1. Speed Governor: The essential part are centrifugal flyballs driven directly or through gearing by the turbine shaft. The mechanism provides upward and downward vertical movements proportional to the change in speed.

2. Linkage Mechanism: These are links for transforming the flyballs movement to the turbine valve through a hydraulic amplifier and providing a feedback from the turbine valve movement.

3. Hydraulic Amplifier: Very large mechanical forces are needed to operate the steam valve. Therefore, the governor movements are transformed into high power forces via several stages of hydraulic amplifiers.

4. Speed Changer: The speed changer consists of a servomotor which can be operated manually or automatically for scheduling load at nominal frequency.

By adjusting this set point, a desired load dispatch can be scheduled at nominal frequency.

For stable operation, the governors are designed to permit the speed to drop as the load is increased. The steady-state characteristics of such a governor is shown in Figure 12.7.

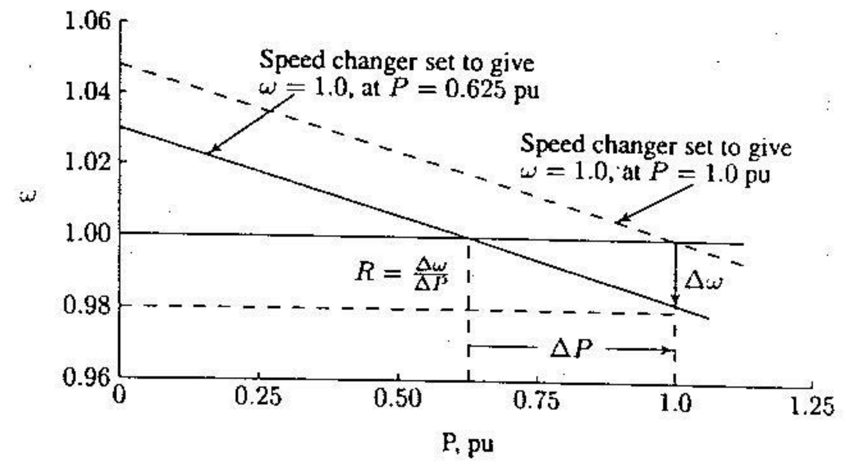


FIGURE 12.7
Governor steady-state speed characteristics.

The slope of the curve represents the speed regulation R. Governors typically have a speed regulation of 5–6 percent from zero to full load. The speed governor mechanism acts as a comparator whose output ΔP_g is the difference between the reference set power ΔP_{ref} and the power $\frac{1}{R}\Delta\omega$ as given from the governor speed characteristics, i.e.,

$$\Delta P_g = \Delta P_{ref} - \frac{1}{R}\Delta\omega \tag{12.7}$$

or in s-domain

$$\Delta P_g(s) = \Delta P_{ref}(s) - \frac{1}{R}\Delta\Omega(s) \tag{12.8}$$

The command ΔP_g is transformed through the hydraulic amplifier to the steam valve position command ΔP_V. Assuming a linear relationship and considering a simple time constant τ_g, we have the following s-domain relation

$$\Delta P_V(s) = \frac{1}{1+\tau_g}\Delta P_g(s) \tag{12.9}$$

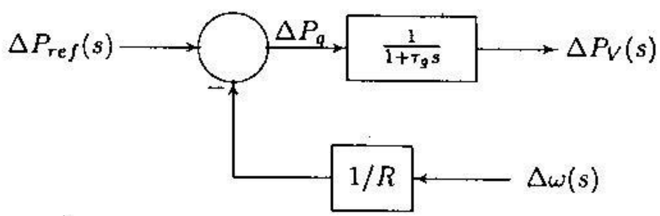


FIGURE 12.8
Block diagram representation of speed governing system for steam turbine.

Equations (12.8) and (12.9) are represented by the block diagram shown in Figure 12.8. Combining the block diagrams of Figures 12.4, 12.5, and 12.8 results in the complete block diagram of the load frequency control of an isolated power station shown in Figure 12.9. Redrawing the block diagram of Figure 12.9 with the load

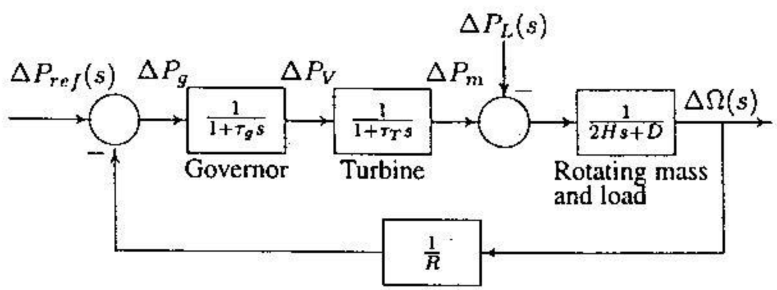


FIGURE 12.9
Load frequency control block diagram of an isolated power system.

change $-\Delta P_L(s)$ as the input and the frequency deviation $\Delta\Omega(s)$ as the output results in the block diagram shown in Figure 12.10. The open-loop transfer function of the block diagram in Figure 12.10 is

$$KG(s)H(s) = \frac{1}{R}\frac{1}{(2Hs+D)(1+\tau_g s)(1+\tau_T s)} \tag{12.10}$$

and the closed-loop transfer function relating the load change ΔP_L to the frequency deviation $\Delta\Omega$ is

$$\frac{\Delta\Omega(s)}{-\Delta P_L(s)} = \frac{(1+\tau_g s)(1+\tau_T s)}{(2Hs+D)(1+\tau_g s)(1+\tau_T s)+1/R} \tag{12.11}$$

or

$$\Delta\Omega(s) = -\Delta P_L(s)T(s) \tag{12.12}$$

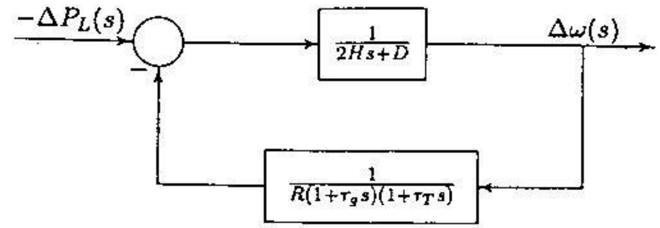


FIGURE 12.10
LFC block diagram with input $\Delta P_L(s)$ and output $\Delta\Omega(s)$.

The load change is a step input, i.e., $\Delta P_L(s) = \Delta P_L/s$. Utilizing the final value theorem, the steady-state value of $\Delta\omega$ is

$$\Delta\omega_{ss} = \lim_{s\to 0} s\Delta\Omega(s) = (-\Delta P_L)\frac{1}{D+1/R} \tag{12.13}$$

It is clear that for the case with no frequency-sensitive load (i.e., with $D = 0$), the steady-state deviation in frequency is determined by the governor speed regulation, and is

$$\Delta\omega_{ss} = (-\Delta P_L)R \tag{12.14}$$

When several generators with governor speed regulations $R_1, R_2, \ldots, R_n$ are connected to the system, the steady-state deviation in frequency is given by

$$\Delta\omega_{ss} = (-\Delta P_L)\frac{1}{D+1/R_1+1/R2+\cdots 1/R_n} \tag{12.15}$$

Example 12.1

An isolated power station has the following parameters

Turbine time constant $\tau_T = 0.5$ sec
Governor time constant $\tau_g = 0.2$ sec
Generator inertia constant $H = 5$ sec
Governor speed regulation = R per unit

The load varies by 0.8 percent for a 1 percent change in frequency, i.e., $D = 0.8$
(a) Use the Routh-Hurwitz array (Appendix B.2.1) to find the range of R for control system stability.
(b) Use *MATLAB* **rlocus** function to obtain the root locus plot.
(c) The governor speed regulation of Example 12.1 is set to $R = 0.05$ per unit. The turbine rated output is 250 MW at nominal frequency of 60 Hz. A sudden load change of 50 MW ($\Delta P_L = 0.2$ per unit) occurs.

(i) Find the steady-state frequency deviation in Hz.

(ii) Use *MATLAB* to obtain the time-domain performance specifications and the frequency deviation step response.

(d) Construct the *SIMULINK* block diagram (see Appendix A.17) and obtain the frequency deviation response for the condition in part (c).

Substituting the system parameters in the LFC block diagram of Figure 12.10 results in the block diagram shown in Figure 12.11. The open-loop transfer function

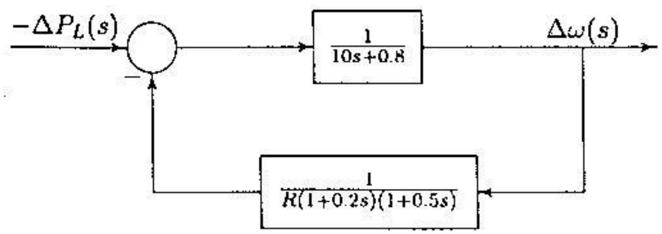


FIGURE 12.11
LFC block diagram for Example 12.1.

is

$$KG(s)H(s) = \frac{K}{(10s+0.8)(1+0.2s)(1+0.5s)} = \frac{K}{s^3+7.08s^2+10.56s+0.8}$$

where $K = \frac{1}{R}$

(a) The characteristic equation is given by

$$1 + KG(s)H(s) = 1 + \frac{K}{s^3+7.08s^2+10.56s+0.8} = 0$$

which results in the characteristic polynomial equation

$$s^3 + 7.08s^2 + 10.56s + 0.8 + K = 0$$

The Routh-Hurwitz array for this polynomial is then (see Appendix B.2.1)

$$\begin{array}{c|cc} s^3 & 1 & 10.56 \\ s^2 & 7.08 & 0.8+K \\ s^1 & \frac{73.965-K}{7.08} & 0 \\ s^0 & 0.8+K & 0 \end{array}$$

From the s^1 row, we see that for control system stability, K must be less than 73.965. Also from the s^0 row, K must be greater than -0.8. Thus, with positive values of K, for control system stability

$$K < 73.965$$

Since $R = \frac{1}{K}$, for control system stability, the governor speed regulation must be

$$R > \frac{1}{73.965} \quad \text{or} \quad R > 0.0135$$

For $K = 73.965$, the auxiliary equation from the s^2 row is

$$7.08s^2 + 74.765 = 0$$

or $s = \pm j3.25$. That is, for $R = 0.0135$, we have a pair of conjugate poles on the $j\omega$ axis, and the control system is marginally stable.

(b) To obtain the root-locus, we use the following commands.

```
num=1;
den = [1  7.08  10.56 .8];
figure(1), rlocus(num, den)
```

The result is shown in Figure 12.12.

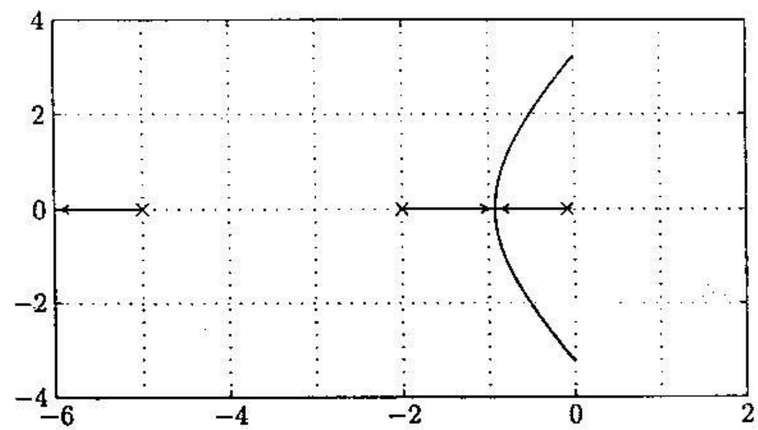


FIGURE 12.12
Root-locus plot for Example 12.1.

The loci intersect the $j\omega$ axis at $s = \pm j3.25$ for $K = 73.965$. Thus, the system is marginally stable for $R = \frac{1}{73.965} = 0.0135$.

(c) The closed-loop transfer function of the system shown in Figure 12.11 is

$$\frac{\Delta\Omega(s)}{-\Delta P_L(s)} = T(s) = \frac{(1+0.2s)(1+0.5s)}{(10s+0.8)(1+0.2s)(1+0.5s)+1/.05}$$
$$= \frac{0.1s^2+0.7s+1}{s^3+7.08s^2+10.56s+20.8}$$

(i) The steady-state frequency deviation due to a step input is

$$\Delta\omega_{ss} = \lim_{s\to 0} s\Delta\Omega(s) = \frac{1}{20.8}(-0.2) = -0.0096 \text{ pu}$$

Thus, the steady-state frequency deviation in hertz due to the sudden application of a 50-MW load is $\Delta f = (-0.0096)(60) = 0.576$ Hz.

(ii) To obtain the step response and the time-domain performance specifications, we use the following commands

```
PL = 0.2; numc = [0.1  0.7  1];
denc = [1   7.08  10.56  20.8];
t = 0:.02:10; c = -PL*step(num, den, t);
figure(2), plot(t, c), xlabel('t,  sec'), ylabel('pu')
title('Frequency deviation step response'), grid
  timespec(num, den)
```

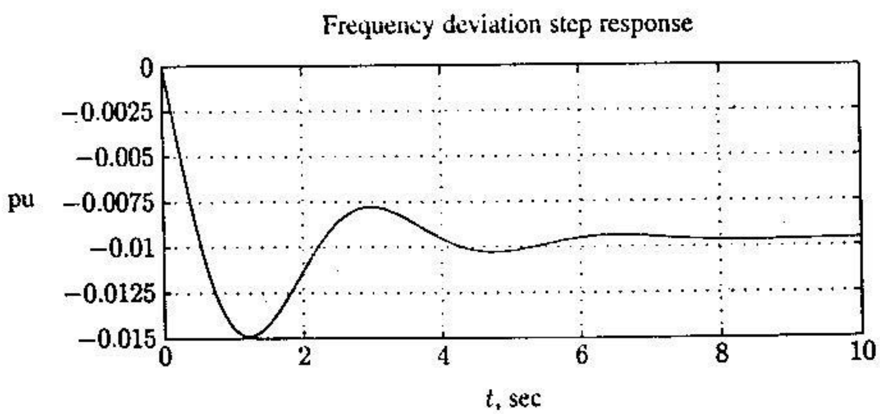


FIGURE 12.13
Frequency deviation step response for Example 12.1.

The frequency deviation step response is shown in Figure 12.13, and the time-domain performance specifications are

```
Peak time = 1.223     Percent overshoot = 54.80
Rise time = 0.418
Settling time = 6.8
```

(d) A *SIMULINK* model named **sim12ex1.mdl** is constructed as shown in Figure 12.14. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.13.

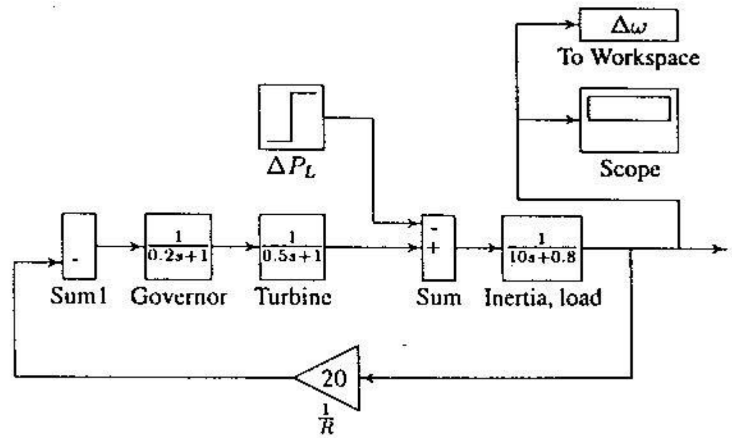


FIGURE 12.14
Simulation block diagram for Example 12.1.

Example 12.2

A single area consists of two generating units with the following characteristics.

Unit	Rating	Speed regulation R (pu on unit MVA base)
1	600 MVA	6%
2	500 MVA	4%

The units are operating in parallel, sharing 900 MW at the nominal frequency. Unit 1 supplies 500 MW and unit 2 supplies 400 MW at 60 Hz. The load is increased by 90 MW.

(a) Assume there is no frequency-dependent load, i.e., $D = 0$. Find the steady-state frequency deviation and the new generation on each unit.

(b) The load varies 1.5 percent for every 1 percent change in frequency, i.e., $D = 1.5$. Find the steady-state frequency deviation and the new generation on each unit.

First we express the governor speed regulation of each unit to a common MVA base. Select 1000 MVA for the apparent power base, then

$$R_1 = \frac{1000}{600}(0.06) = 0.1 \text{ pu}$$

$$R_2 = \frac{1000}{500}(0.05) = 0.08 \text{ pu}$$

The per unit load change is

$$\Delta P_L = \frac{90}{1000} = 0.09 \text{ pu}$$

(a) From (12.15) with $D = 0$, the per unit steady-state frequency deviation is

$$\Delta\omega_{ss} = \frac{-\Delta P_L}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{-0.09}{10 + 12.5} = -0.004 \text{ pu}$$

Thus, the steady-state frequency deviation in Hz is

$$\Delta f = (-0.004)(60) = -0.24 \text{ Hz}$$

and the new frequency is

$$f = f_0 + \Delta f = 60 - 0.24 = 59.76 \text{ Hz}$$

The change in generation for each unit is

$$\Delta P_1 = -\frac{\Delta\omega}{R_1} = -\frac{-0.004}{0.1} = 0.04 \text{ pu} = 40 \text{ MW}$$

$$\Delta P_2 = -\frac{\Delta\omega}{R_2} = -\frac{-0.004}{0.08} = 0.05 \text{ pu} = 50 \text{ MW}$$

Thus, unit 1 supplies 540 MW and unit 2 supplies 450 MW at the new operating frequency of 59.76 Hz.

MATLAB is used to plot the per unit speed characteristics of each governor as shown in Figure 12.15. As we can see from this figure, the initial generations are 0.5 and 0.40 per unit at the nominal frequency of 1.0 per unit. With the addition of 0.09 per unit power speed drops to 0.996 per unit. The new generations are 0.54

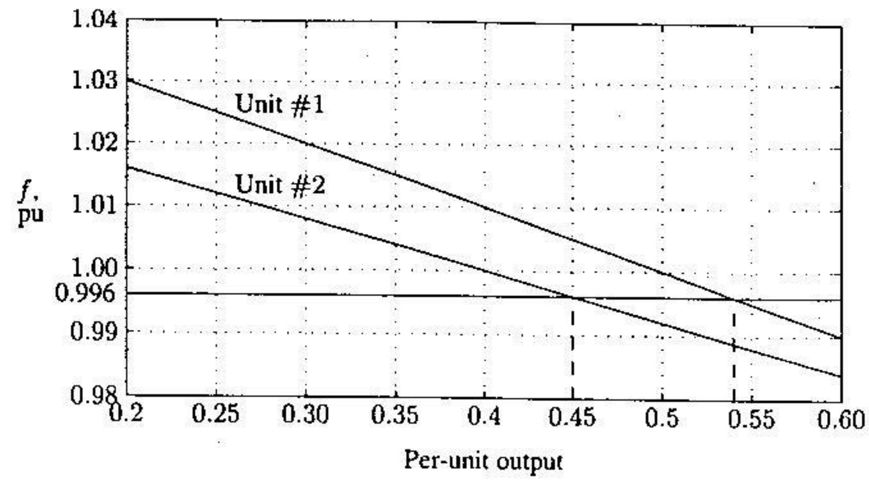


FIGURE 12.15
Load division between the two units of Example 12.2

and 0.45 per unit.

(b) For $D = 1.5$, the per unit steady-state frequency deviation is

$$\Delta\omega_{ss} = \frac{-\Delta P_L}{\frac{1}{R_1} + \frac{1}{R_2} + D} = \frac{-0.09}{10 + 12.5 + 1.5} = -0.00375 \text{ pu}$$

Thus, the steady-state frequency deviation in Hz is

$$\Delta f = (-0.00375)(60) = -0.225 \text{ Hz}$$

and the new frequency is

$$f = f_0 + \Delta f = 60 - 0.225 = 59.775 \text{ Hz}$$

The change in generation for each unit is

$$\Delta P_1 = -\frac{\Delta\omega}{R_1} = -\frac{-0.00375}{0.1} = 0.0375 \text{ pu} = 37.500 \text{ MW}$$

$$\Delta P_2 = -\frac{\Delta\omega}{R_2} = -\frac{-0.00375}{0.08} = 0.046875 \text{ pu} = 46.875 \text{ MW}$$

Thus, unit 1 supplies 537.5 MW and unit 2 supplies 446.875 MW at the new operating frequency of 59.775 Hz. The total change in generation is 84.375, which is 5.625 MW less than the 90 MW load change. This is because of the change in load due to frequency drop which is given by

$$\begin{aligned}\Delta\omega D &= (-0.00375)(1.5) = -0.005625 \text{ pu} \\ &= -5.625 \text{ MW}\end{aligned}$$

12.4 AUTOMATIC GENERATION CONTROL

If the load on the system is increased, the turbine speed drops before the governor can adjust the input of the steam to the new load. As the change in the value of speed diminishes, the error signal becomes smaller and the position of the governor flyballs gets closer to the point required to maintain a constant speed. However, the constant speed will not be the set point, and there will be an offset. One way to restore the speed or frequency to its nominal value is to add an integrator. The integral unit monitors the average error over a period of time and will overcome the offset. Because of its ability to return a system to its set point, integral action is also known as the *rest action*. Thus, as the system load changes continuously, the generation is adjusted automatically to restore the frequency to the nominal value. This scheme is known as the *automatic generation control* (AGC). In an interconnected system consisting of several pools, the role of the AGC is to divide the loads among system, stations, and generators so as to achieve maximum economy and correctly control the scheduled interchanges of tie-line power while maintaining a reasonably uniform frequency. Of course, we are implicitly assuming that the system is stable, so the steady-state is achievable. During large transient disturbances and emergencies, AGC is bypassed and other emergency controls are applied. In the following section, we consider the AGC in a single area system and in an interconnected power system.

12.4.1 AGC IN A SINGLE AREA SYSTEM

With the primary LFC loop, a change in the system load will result in a steady-state frequency deviation, depending on the governor speed regulation. In order to reduce the frequency deviation to zero, we must provide a reset action. The rest action can be achieved by introducing an integral controller to act on the load reference setting to change the speed set point. The integral controller increases the system type by 1 which forces the final frequency deviation to zero. The LFC system, with the addition of the secondary loop, is shown in Figure 12.16. The integral controller gain K_I must be adjusted for a satisfactory transient response. Combining the parallel branches results in the equivalent block diagram shown in Figure 12.17.

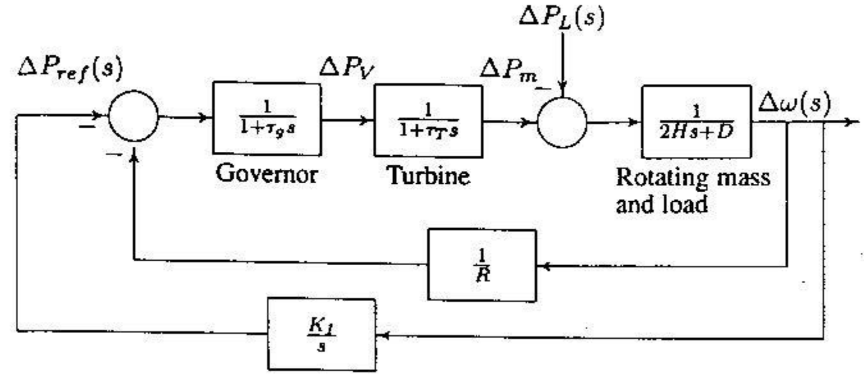


FIGURE 12.16
AGC for an isolated power system.

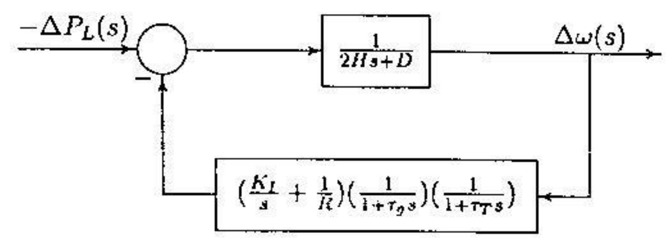


FIGURE 12.17
The equivalent block diagram of AGC for an isolated power system.

The closed-loop transfer function of the control system shown in Figure 12.17 with only $-\Delta P_L$ as input becomes

$$\frac{\Delta\Omega(s)}{-\Delta P_L(s)} = \frac{s(1+\tau_g s)(1+\tau_T s)}{s(2Hs+D)(1+\tau_g s)(1+\tau_T s) + K_I + s/R} \tag{12.16}$$

Example 12.3

The LFC system in Example 12.1 is equipped with the secondary integral control loop for automatic generation control.
(a) Use the *MATLAB* step function to obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.2$ per unit. Set the integral controller gain to $K_I = 7$.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation re-

sponse for the condition in part (a).

(a) Substituting for the system parameters in (12.16), with speed regulation adjusted to $R = 0.05$ per unit, results in the following closed-loop transfer function

$$T(s) = \frac{0.1s^3 + 0.7s^2 + s}{s^4 + 7.08s^3 + 10.56s^2 + 20.8s + 7}$$

To find the step response, we use the following commands

```
PL = 0.2;
KI = 7;
num = [0.1  0.7  1 0];
den = [1   7.08  10.56  20.8 KI];
t = 0:.02:12;
c = -PL*step(num, den, t);
plot(t, c), grid
xlabel('t, sec'), ylabel('pu')
title('Frequency deviation step response')
```

The step response is shown in Figure 12.18.

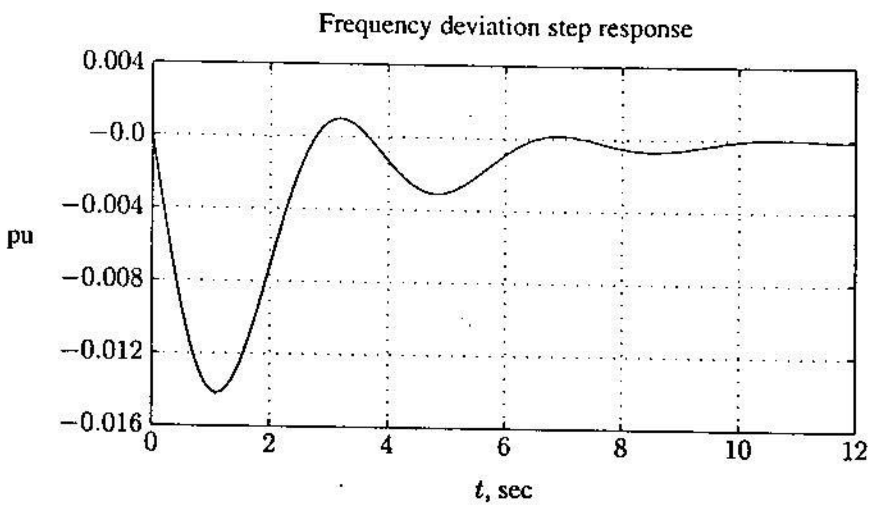


FIGURE 12.18
Frequency deviation step response for Example 12.3.

From the step response, we observe that the steady-state frequency deviation $\Delta\omega_{ss}$ is zero, and the frequency returns to its nominal value in approximately 10 seconds.

(b) A *SIMULINK* model named **sim12ex3.mdl** is constructed as shown in Figure 12.19. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.18.

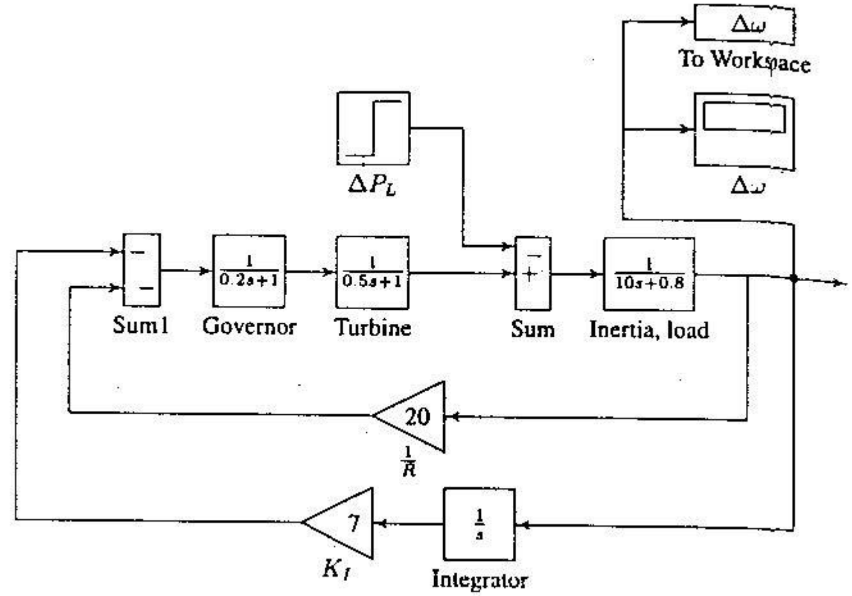


FIGURE 12.19
Simulation block diagram for Example 12.3.

12.4.2 AGC IN THE MULTIAREA SYSTEM

In many cases, a group of generators are closely coupled internally and swing in unison. Furthermore, the generator turbines tend to have the same response characteristics. Such a group of generators are said be *coherent*. Then it is possible to let the LFC loop represent the whole system, which is referred to as a *control area*. The AGC of a multiarea system can be realized by studying first the AGC for a two-area system. Consider two areas represented by an equivalent generating unit interconnected by a lossless tie line with reactance X_{tie}. Each area is represented by a voltage source behind an equivalent reactance as shown in Figure 12.20.

During normal operation, the real power transferred over the tie line is given by

$$P_{12} = \frac{|E_1||E_2|}{X_{12}} sin\delta_{12} \tag{12.17}$$

where $X_{12} = X_1 + X_{tie} + X_2$, and $\delta_{12} = \delta_1 - \delta_2$. Equation (12.17) can be linearized

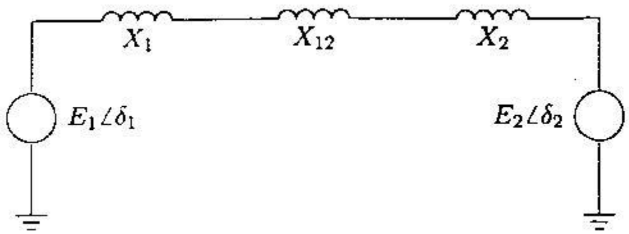


FIGURE 12.20
Equivalent network for a two-area power system.

for a small deviation in the tie-line flow ΔP_{12} from the nominal value, i.e.,

$$\begin{aligned}\Delta P_{12} &= \left.\frac{dP_{12}}{d\delta_{12}}\right|_{\delta_{12_0}} \Delta\delta_{12} \\ &= P_s\Delta\delta_{12}\end{aligned} \tag{12.18}$$

The quantity P_s is the slope of the power angle curve at the initial operating angle $\delta_{12_0} = \delta_{1_0} - \delta_{2_0}$. This was defined as the synchronizing power coefficient by (11.39) in Section 11.4. Thus we have

$$P_s = \left.\frac{dP_{12}}{d\delta_{12}}\right|_{\delta_{12_0}} = \frac{|E_1||E_2|}{X_{12}} \cos\Delta\delta_{12_0} \tag{12.19}$$

The tie-line power deviation then takes on the form

$$\Delta P_{12} = P_s(\Delta\delta_1 - \Delta\delta_2) \tag{12.20}$$

The tie-line power flow appears as a load increase in one area and a load decrease in the other area, depending on the direction of the flow. The direction of flow is dictated by the phase angle difference; if $\Delta\delta_1 > \Delta\delta_2$, the power flows from area 1 to area 2. A block diagram representation for the two-area system with LFC containing only the primary loop is shown in Figure 12.21.

Let us consider a load change ΔP_{L1} in area 1. In the steady-state, both areas will have the same steady-state frequency deviation, i.e.,

$$\Delta\omega = \Delta\omega_1 = \Delta\omega_2 \tag{12.21}$$

and

$$\begin{aligned}\Delta P_{m1} - \Delta P_{12} - \Delta P_{L1} &= \Delta\omega D_1 \\ \Delta P_{m2} + \Delta P_{12} &= \Delta\omega D_2\end{aligned} \tag{12.22}$$

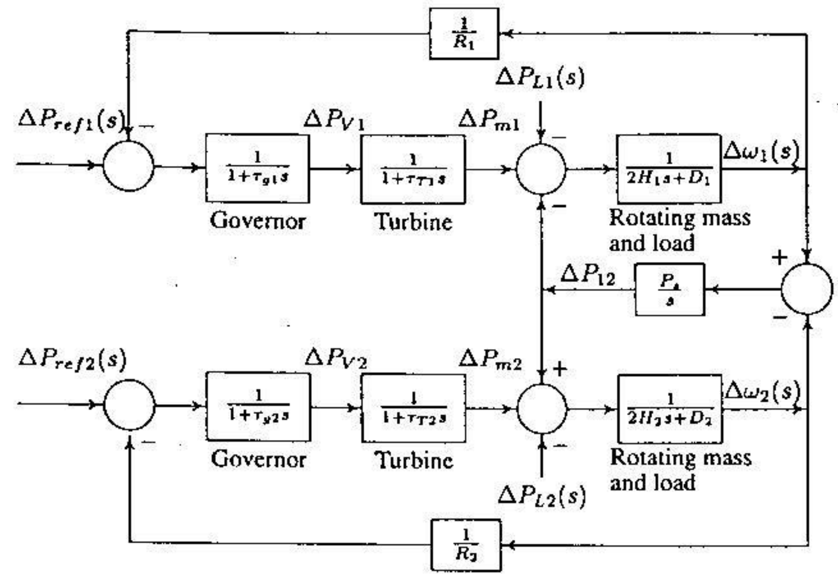


FIGURE 12.21
Two-area system with only primary LFC loop.

The change in mechanical power is determined by the governor speed characteristics, given by

$$\begin{aligned}\Delta P_{m1} &= \frac{-\Delta\omega}{R_1} \\ \Delta P_{m2} &= \frac{-\Delta\omega}{R_2}\end{aligned} \tag{12.23}$$

Substituting from (12.23) into (12.22), and solving for $\Delta\omega$, we have

$$\begin{aligned}\Delta\omega &= \frac{-\Delta P_{L1}}{(\frac{1}{R_1} + D_1) + (\frac{1}{R_2} + D_2)} \\ &= \frac{-\Delta P_{L1}}{B_1 + B2}\end{aligned} \tag{12.24}$$

where

$$\begin{aligned}B_1 &= \frac{1}{R_1} + D_1 \\ B_2 &= \frac{1}{R_2} + D_2\end{aligned} \tag{12.25}$$

B_1 and B_2 are known as the *frequency bias factors*. The change in the tie-line power is

$$\Delta P_{12} = -\frac{(\frac{1}{R_2} + D_2)\Delta P_{L1}}{(\frac{1}{R_1} + D_1)(\frac{1}{R_2} + D_2)} \tag{12.26}$$
$$= \frac{B_2}{B_1 + B_2}(-\Delta P_{L1})$$

Example 12.4

A two-area system connected by a tie line has the following parameters on a 1000-MVA common base

Area	1	2
Speed regulation	$R_1 = 0.05$	$R_2 = 0.0625$
Frequency-sens. load coeff.	$D_1 = 0.6$	$D_2 = 0.9$
Inertia constant	$H_1 = 5$	$H_2 = 4$
Base power	1000 MVA	1000 MVA
Governor time constant	$\tau_{g1} = 0.2$ sec	$\tau_{g2} = 0.3$ sec
Turbine time constant	$\tau_{T1} = 0.5$ sec	$\tau_{T2} = 0.6$ sec

The units are operating in parallel at the nominal frequency of 60 Hz. The synchronizing power coefficient is computed from the initial operating condition and is given to be $P_s = 2.0$ per unit. A load change of 187.5 MW occurs in area 1.
(a) Determine the new steady-state frequency and the change in the tie-line flow.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (a).

(a) The per unit load change in area 1 is

$$\Delta P_{L1} = \frac{187.5}{1000} = 0.1875 \text{ pu}$$

The per unit steady-state frequency deviation is

$$\Delta\omega_{ss} = \frac{-\Delta P_{L1}}{(\frac{1}{R_1} + D_1) + (\frac{1}{R_2} + D_2)} = \frac{-0.1875}{(20 + 0.6) + (16 + 0.9)} = -0.005 \text{ pu}$$

Thus, the steady-state frequency deviation in Hz is

$$\Delta f = (-0.005)(60) = -0.3 \text{ Hz}$$

and the new frequency is

$$f = f_0 + \Delta f = 60 - 0.3 = 59.7 \text{ Hz}$$

The change in mechanical power in each area is

$$\Delta P_{m1} = -\frac{\Delta\omega}{R_1} = -\frac{-0.005}{0.05} = 0.10 \text{ pu}$$
$$= 100 \text{ MW}$$

$$\Delta P_{m2} = -\frac{\Delta\omega}{R_2} = -\frac{-0.005}{0.0625} = 0.080 \text{ pu}$$
$$= 80 \text{ MW}$$

Thus, area 1 increases the generation by 100 MW and area 2 by 80 MW at the new operating frequency of 59.7 Hz. The total change in generation is 180 MW, which is 7.5 MW less than the 187.5 MW load change because of the change in the area loads due to frequency drop.

The change in the area 1 load is $\Delta\omega D_1 = (-0.005)(0.6) = -0.003$ per unit (−3.0 MW), and the change in the area 2 load is $\Delta\omega D_1 = (-0.005)(0.9) = -0.0045$ per unit (−4.5 MW). Thus, the change in the total area load is −7.5 MW. The tie-line power flow is

$$\Delta P_{12} = \Delta\omega\left(\frac{1}{R_2} + D2\right) = -0.005(16.9) = 0.0845 \text{ pu}$$
$$= -84.5 \text{ MW}$$

That is, 84.5 MW flows from area 2 to area 1. 80 MW comes from the increased generation in area 2, and 4.5 MW comes from the reduction in area 2 load due to frequency drop.

(b) A *SIMULINK* model named **sim12ex4.mdl** is constructed as shown in Figure 12.22. The file is opened and is run in the *SIMULINK* window. The simulation result is shown in Figure 12.23. The simulation diagram returns the vector DP, containing t, P_{m1}, P_{m2}, and P_{12}. A plot of the per unit power response is obtained in *MATLAB* as shown in Figure 12.24.

12.4.3 TIE-LINE BIAS CONTROL

In Example 12.4, where LFCs were equipped with only the primary control loop, a change of power in area 1 was met by the increase in generation in both areas associated with a change in the tie-line power, and a reduction in frequency. In the normal operating state, the power system is operated so that the demands of areas are satisfied at the nominal frequency. A simple control strategy for the normal mode is

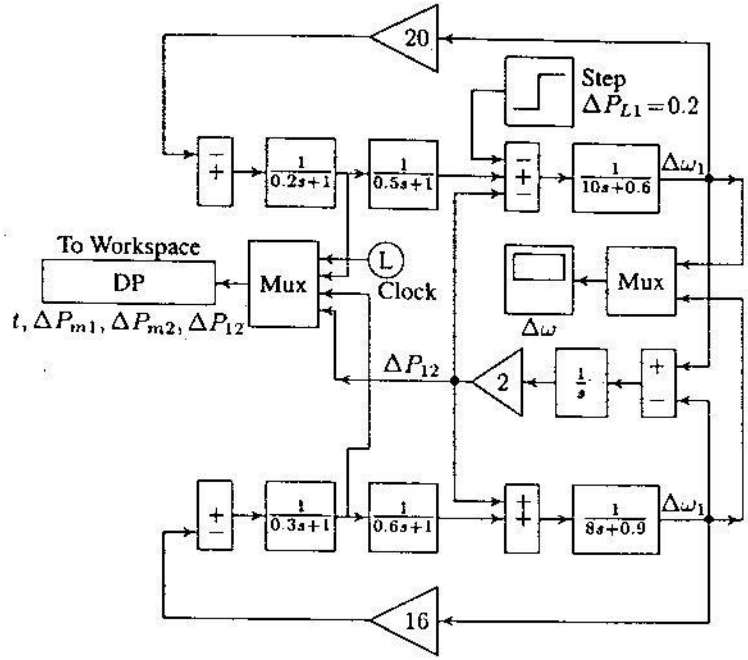


FIGURE 12.22
Simulation block diagram for Example 12.4.

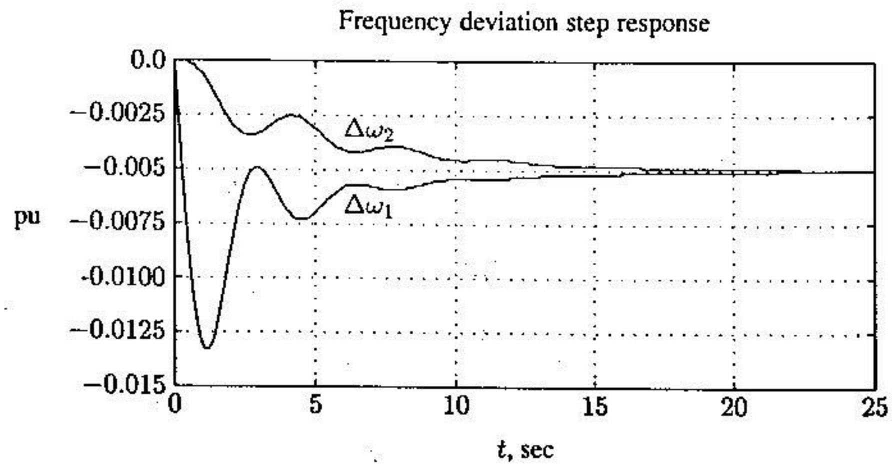


FIGURE 12.23
Frequency deviation step response for Example 12.4.

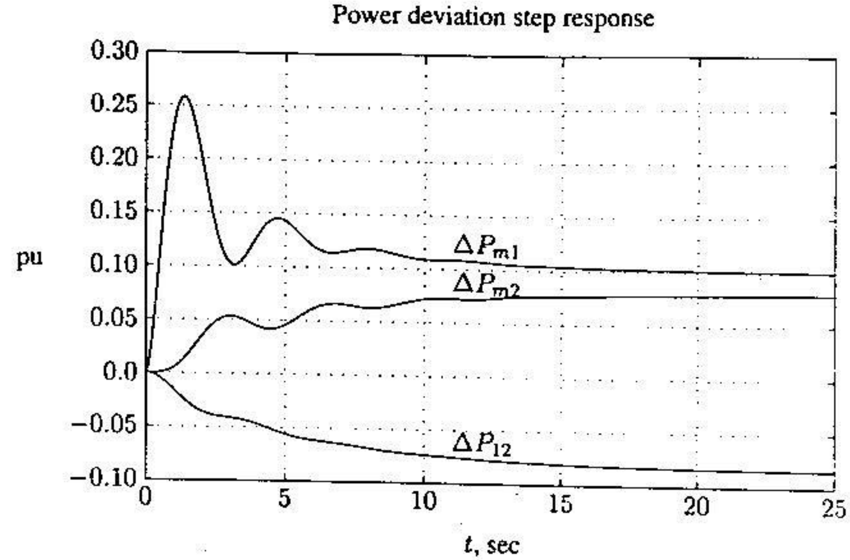


FIGURE 12.24
Power deviation step response for Example 12.4.

- Keep frequency approximately at the nominal value (60 Hz).
- Maintain the tie-line flow at about schedule.
- Each area should absorb its own load changes.

Conventional LFC is based upon tie-line bias control, where each area tends to reduce the area control error (ACE) to zero. The control error for each area consists of a linear combination of frequency and tie-line error.

$$\text{ACE}_i = \sum_{j=1}^{n} \Delta P_{ij} + K_i \, \Delta\omega \tag{12.27}$$

The area bias K_i determines the amount of interaction during a disturbance in the neighboring areas. An overall satisfactory performance is achieved when K_i is selected equal to the frequency bias factor of that area, i.e., $B_i = \frac{1}{R_i} + D_i$. Thus, the ACEs for a two-area system are

$$\begin{aligned}\text{ACE}_1 &= \Delta P_{12} + B_1 \, \Delta\omega_1 \\ \text{ACE}_2 &= \Delta P_{21} + B_2 \, \Delta\omega_2\end{aligned} \tag{12.28}$$

where ΔP_{12} and ΔP_{21} are departures from scheduled interchanges. ACEs are used as actuating signals to activate changes in the reference power set points, and when steady-state is reached, ΔP_{12} and $\Delta\omega$ will be zero. The integrator gain constant must be chosen small enough so as not to cause the area to go into a chase mode. The block diagram of a simple AGC for a two-area system is shown in Figure 12.25. We can easily extend the tie-line bias control to an n-area system.

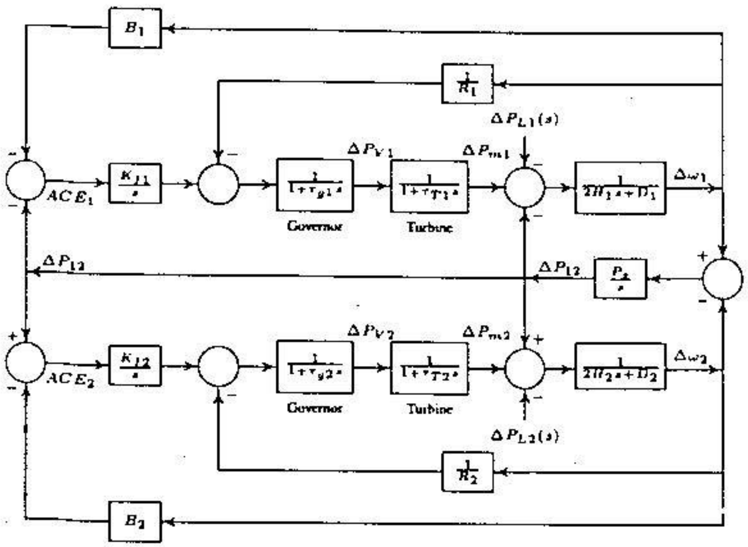


FIGURE 12.25
AGC block diagram for a two-area system.

Example 12.5

Construct the *SIMULINK* model for the two-area system of Example 12.4 with the inclusion of the ACEs, and obtain the frequency and power response for each area.

A *SIMULINK* model named **sim12ex5.mdl** is constructed as shown in Figure 12.26.

The file is opened and is run in the *SIMULINK* window. The integrator gain constants are adjusted for a satisfactory response. The simulation result for $K_{I1} = K_{I2} = 0.3$ is shown in Figure 12.27. The simulation diagram returns the vector ΔP, containing t, ΔP_{m1}, ΔP_{m2}, and ΔP_{12}. A plot of the per unit power response is obtained in *MATLAB* as shown in Figure 12.28. As we can see from Figure 12.27, the frequency deviation returns to zero with a settling time of approximately 20 seconds. Also, the tie-line power change reduces to zero, and the increase in area 1 load is met by the increase in generation ΔP_{m1}.

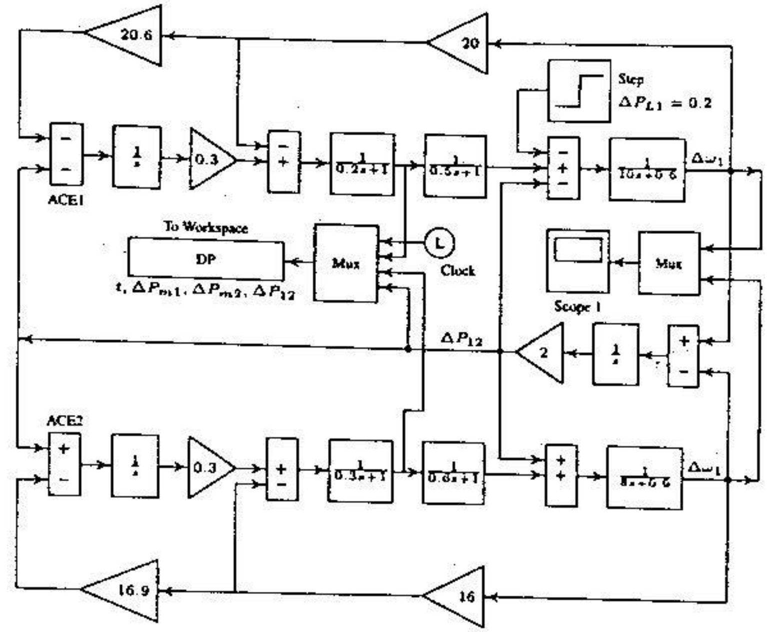


FIGURE 12.26
Simulation block diagram for Example 12.5.

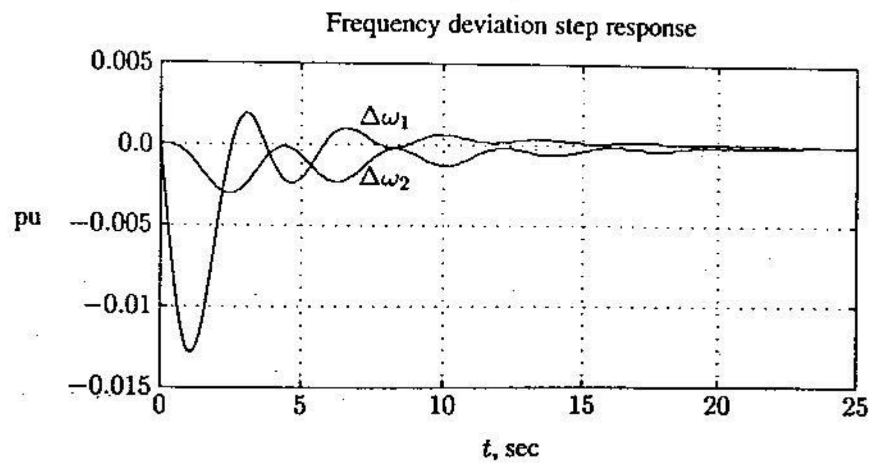


FIGURE 12.27
Frequency deviation step response for Example 12.5.

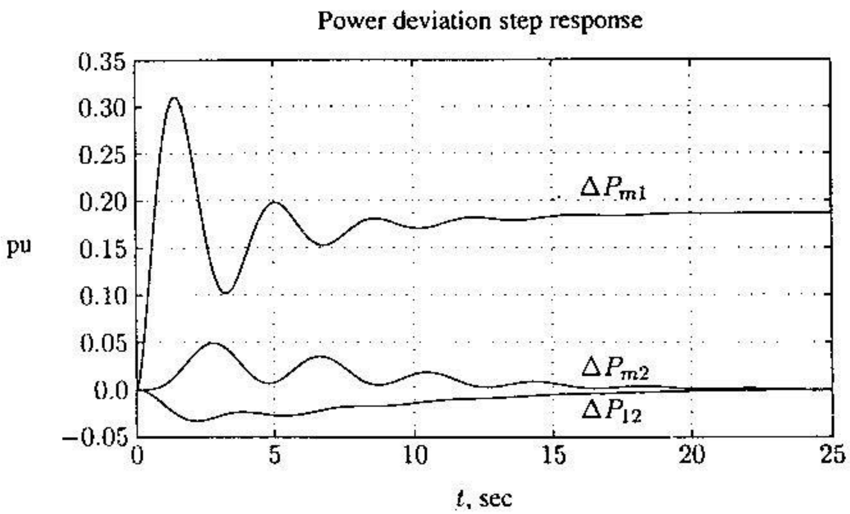


FIGURE 12.28
Power deviation step response for Example 12.5.

12.5 AGC WITH OPTIMAL DISPATCH OF GENERATION

The factors influencing power generation at minimum cost are operating efficiencies, fuel cost, and transmission losses. The optimal dispatch of generation was discussed in Chapter 7, and a program named **dispatch** was developed to find the optimal dispatch of generation for an interconnected power system.

The optimal dispatch of generation may be treated within the framework of LFC. In direct digital control systems, the digital computer is included in the control loop which scans the unit generation and tie-line flows. These settings are compared with the optimal settings derived from the solution of the optimal dispatch program, such as **dispatch** program developed in Chapter 7. If the actual settings are off from the optimal values, the computer generates the raise/lower pulses which are sent to the individual units. The allocation program will also take into account the tie-line power contracts between the areas.

With the development of modern control theory, several concepts are included in the AGC which go beyond the simple tie-line bias control. The fundamental approach is the use of more extended mathematical models. In retrospect, the AGC can be used to include the representation of the dynamics of the area, or even of the complete system.

Other concepts of the modern control theory are being employed, such as state estimation and optimal control with linear regulator utilizing constant feedback gains. In addition to the structures which aim at the control of deterministic

signals and disturbances, there are schemes which employ stochastic control concepts, e.g., minimization of some expected value of an integral quadratic error criterion. Usually, this results in the design of the Kalman filter, which is of value for the control of small random disturbances.

12.6 REACTIVE POWER AND VOLTAGE CONTROL

The generator excitation system maintains generator voltage and controls the reactive power flow. The generator excitation of older systems may be provided through slip rings and brushes by means of dc generators mounted on the same shaft as the rotor of the synchronous machine. However, modern excitation systems usually use ac generators with rotating rectifiers, and are known as *brushless excitation*.

As we have seen, a change in the real power demand affects essentially the frequency, whereas a change in the reactive power affects mainly the voltage magnitude. The interaction between voltage and frequency controls is generally weak enough to justify their analysis separately.

The sources of reactive power are generators, capacitors, and reactors. The generator reactive powers are controlled by field excitation. Other supplementary methods of improving the voltage profile on electric transmission systems are transformer load-tap changers, switched capacitors, step-voltage regulators, and static var control equipment. The primary means of generator reactive power control is the generator excitation control using *automatic voltage regulator* (AVR), which is discussed in this chapter. The role of an (AVR) is to hold the terminal voltage magnitude of a synchronous generator at a specified level. The schematic diagram of a simplified AVR is shown in Figure 12.29.

An increase in the reactive power load of the generator is accompanied by a drop in the terminal voltage magnitude. The voltage magnitude is sensed through a potential transformer on one phase. This voltage is rectified and compared to a dc set point signal. The amplified error signal controls the exciter field and increases the exciter terminal voltage. Thus, the generator field current is increased, which results in an increase in the generated emf. The reactive power generation is increased to a new equilibrium, raising the terminal voltage to the desired value. We will look briefly at the simplified models of the component involved in the AVR system.

12.6.1 AMPLIFIER MODEL

The excitation system amplifier may be a magnetic amplifier, rotating amplifier, or modern electronic amplifier. The amplifier is represented by a gain K_A and a time

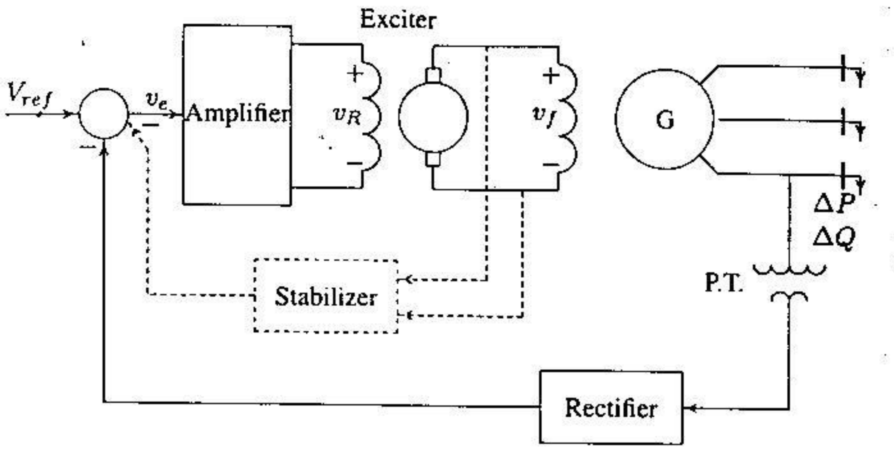


FIGURE 12.29
A typical arrangement of a simple AVR.

constant τ_A, and the transfer function is

$$\frac{V_R(s)}{V_e(s)} = \frac{K_A}{1 + \tau_A s} \tag{12.29}$$

Typical values of K_A are in the range of 10 to 400. The amplifier time constant is very small, in the range of 0.02 to 0.1 second, and often is neglected.

12.6.2 EXCITER MODEL

There is a variety of different excitation types. However, modern excitation systems uses ac power source through solid-state rectifiers such as SCR. The output voltage of the exciter is a nonlinear function of the field voltage because of the saturation effects in the magnetic circuit. Thus, there is no simple relationship between the terminal voltage and the field voltage of the exciter. Many models with various degrees of sophistication have been developed and are available in the IEEE recommendation publications. A reasonable model of a modern exciter is a linearized model, which takes into account the major time constant and ignores the saturation or other nonlinearities. In the simplest form, the transfer function of a modern exciter may be represented by a single time constant τ_E and a gain K_E, i.e.,

$$\frac{V_F(s)}{V_R(s)} = \frac{K_E}{1 + \tau_E s} \tag{12.30}$$

The time constant of modern exciters are very small.

12.6.3 GENERATOR MODEL

The synchronous machine generated emf is a function of the machine magnetization curve, and its terminal voltage is dependent on the generator load. In the linearized model, the transfer function relating the generator terminal voltage to its field voltage can be represented by a gain K_G and a time constant τ_G, and the transfer function is

$$\frac{V_t(s)}{V_F(s)} = \frac{K_G}{1 + \tau_G s} \tag{12.31}$$

These constants are load dependent, K_G may vary between 0.7 to 1, and τ_G between 1.0 and 2.0 seconds from full-load to no-load.

12.6.4 SENSOR MODEL

The voltage is sensed through a potential transformer and, in one form, it is rectified through a bridge rectifier. The sensor is modeled by a simple first order transfer function, given by

$$\frac{V_S(s)}{V_t(s)} = \frac{K_R}{1 + \tau_R s} \tag{12.32}$$

τ_R is very small, and we may assume a range of 0.01 to 0.06 second. Utilizing the above models results in the AVR block diagram shown in Figure 12.30.

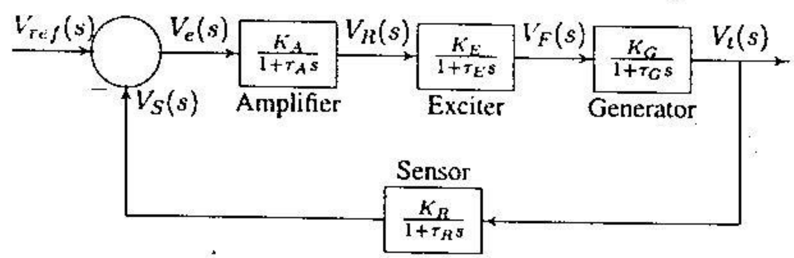


FIGURE 12.30
A simplified automatic voltage regulator block diagram.

The open-loop transfer function of the block diagram in Figure 12.30 is

$$KG(s)H(s) = \frac{K_A K_E K_G K_R}{(1 + \tau_A s)(1 + \tau_E s)(1 + \tau_G s)(1 + \tau_R s)} \tag{12.33}$$

and the closed-loop transfer function relating the generator terminal voltage $V_t(s)$ to the reference voltage $V_{ref}(s)$ is

$$\frac{V_t(s)}{V_{ref}(s)} = \frac{K_A K_E K_G K_R(1 + \tau_R s)}{(1 + \tau_A s)(1 + \tau_E s)(1 + \tau_G s)(1 + \tau_R s) + K_A K_E K_G K_R} \tag{12.34}$$

or

$$V_t(s) = T(s)V_{ref}(s) \tag{12.35}$$

For a step input $V_{ref}(s) = \frac{1}{s}$, using the final value theorem, the steady-state response is

$$V_{t_{ss}} = \lim_{s\to 0} sV_t(s) = \frac{K_A}{1+K_A} \tag{12.36}$$

Example 12.6

The AVR system of a generator has the following parameters

	Gain	Time constant
Amplifier	K_A	$\tau_A = 0.1$
Exciter	$K_E = 1$	$\tau_E = 0.4$
Generator	$K_G = 1$	$\tau_G = 1.0$
Sensor	$K_R = 1$	$\tau_R = 0.05$

(a) Use the Routh-Hurwitz array (Appendix B.2.1) to find the range of K_A for control system stability.
(b) Use *MATLAB* **rlocus** function to obtain the root locus plot.
(c) The amplifier gain is set to $K_A = 10$
  (i) Find the steady-state step response.
  (ii) Use *MATLAB* to obtain the step response and the time-domain performance specifications.
(d) Construct the *SIMULINK* block diagram and obtain the step response.

Substituting the system parameters in the AVR block diagram of Figure 12.30 results in the block diagram shown in Figure 12.31.

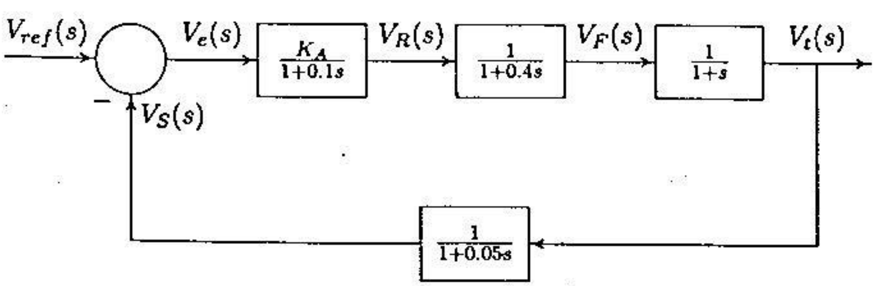


FIGURE 12.31
AVR block diagram for Example 12.6.

The open-loop transfer function of the AVR system shown in Figure 12.31 is

$$\begin{aligned} KG(s)H(s) &= \frac{K_A}{(1+0.1s)(1+0.4s)(1+s)(1+0.05s)} \\ &= \frac{500K_A}{(s+10)(s+2.5)(s+1)(s+20)} \\ &= \frac{500K_A}{s^4+33.5s^3+307.5s^2+775s+500} \end{aligned}$$

(a) The characteristic equation is given by

$$1 + KG(s)H(s) = 1 + \frac{500K_A}{s^4+33.5s^3+307.5s^2+775ss+500} = 0$$

which results in the characteristic polynomial equation

$$s^4 + 33.5s^3 + 307.5s^2 + 775s + 500 + 500K_A = 0$$

The Routh-Hurwitz array for this polynomial is then (see Appendix B.2.1)

s^4	1	307.5	$500 + 500K_A$
s^3	33.5	775	0
s^2	284.365	$500 + 500K_A$	0
s^1	$58.9K_A - 716.1$	0	0
s^0	$500 + 500K_A$		

From the s^1 row we see that, for control system stability, K_A must be less than 12.16, also from the s^0 row, K_A must be greater than -1. Thus, with positive values of K_A, for control system stability, the amplifier gain must be

$$K_A < 12.16$$

For $K = 12.16$, the auxiliary equation from the s^2 row is

$$284.365s^2 + 6580 = 0$$

or $s = \pm j4.81$. That is, for $K = 12.16$, we have a pair of conjugate poles on the $j\omega$ axis, and the control system is marginally stable.

(b) To obtain the root-locus plot for the range of K from 0 to 12.16, we use the following commands.

```
num=500;
den=[1  33.5  307.5  775  500];
figure(1), rlocus(num, den);
```

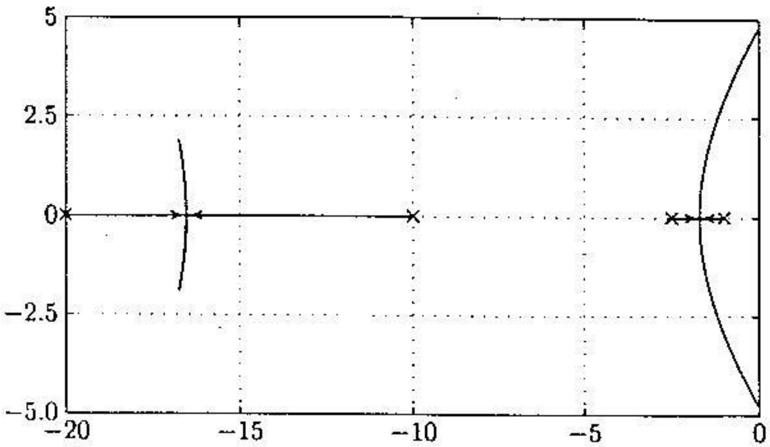


FIGURE 12.32
Root-locus plot for Example 12.6.

The result is shown in Figure 12.32. The loci intersect the $j\omega$ axis at $s = \pm j4.81$ for $K_A = 12.16$. Thus, the system is marginally stable for $K_A = 12.16$.
(c) The closed-loop transfer function of the system shown in Figure 12.31 is

$$\frac{V_t(s)}{V_{ref}(s)} = \frac{25K_A(s+20)}{s^4 + 33.5s^3 + 307.5s^2 + 775s + 500 + 500K_A}$$

(i) The steady-state response is

$$V_{t_{ss}} = \lim_{s \to 0} sV_t(s) = \frac{K_A}{1+K_A}$$

For the amplifier gain of $K_A = 10$, the steady-state response is

$$V_{t_{ss}} = \frac{10}{1+10} = 0.909$$

and the steady-state error is

$$V_{e_{ss}} = 1.0 - 0.909 = 0.091$$

In order to reduce the steady-state error, the amplifier gain must be increased, which results in an unstable control system.
(ii) To obtain the step response and the time-domain performance specifications, we use the following commands


```
KA= 10;
numc=KA*[25  500];
denc=[1  33.5  307.5  775  500+ 500*KA];
t=0:.05:20;
c=step(numc, denc, t);
figure(2), plot(t, c), grid
timespec(numc, denc)
```

The time-domain performance specifications are

```
Peak time = 0.791        Percent overshoot = 82.46
Rise time = 0.247
Settling time = 19.04
```

The terminal voltage step response is shown in Figure 12.33.

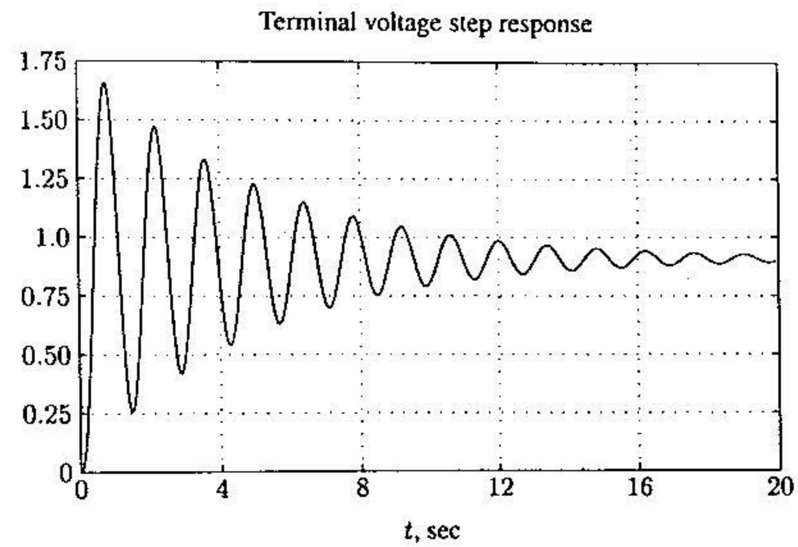


FIGURE 12.33
Terminal voltage step response for Example 12.6.

(d) A *SIMULINK* model named **sim12ex6.mdl** is constructed as shown in Figure 12.34. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.33. From the results, we see that for an amplifier gain $K_A = 10$, the response is highly oscillatory, with a very large overshoot and a long settling time. Furthermore, the steady-state error is over 9 percent. We cannot have a small steady-state error and a satisfactory transient response at the same time.

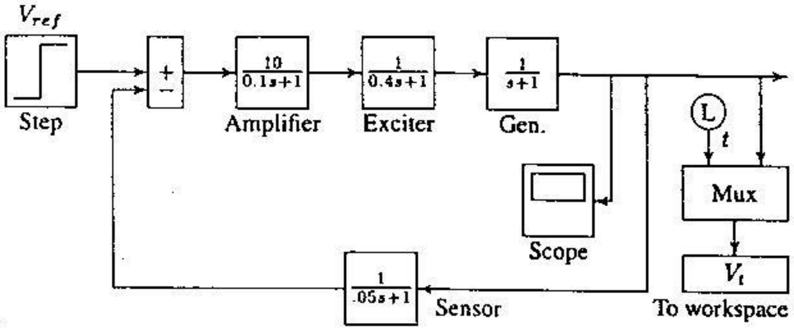


FIGURE 12.34
Simulation block diagram for Example 12.6.

12.6.5 EXCITATION SYSTEM STABILIZER — RATE FEEDBACK

As we have seen in Example 12.6, even for an small amplifier gain of $K_A = 10$, AVR step response is not satisfactory, and a value exceeding 12.16 results in an unbounded response. Thus, we must increase the relative stability by introducing a controller, which would add a zero to the AVR open-loop transfer function. One way to do this is to add a rate feedback to the control system as shown in Figure 12.35. By proper adjustment of K_F and τ_F, a satisfactory response can be obtained.

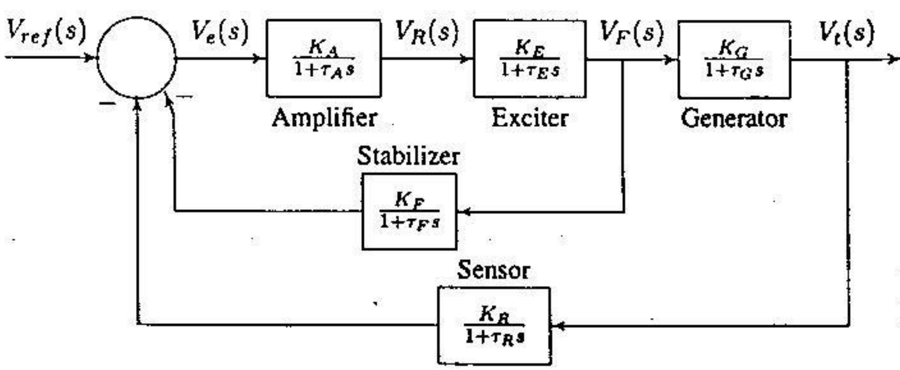


FIGURE 12.35
Block diagram of the compensated AVR system.

Example 12.7

A rate feedback stabilizer is added to the AVR system of Example 12.6. The stabilizer time constant is $\tau_F = 0.04$ second, and the derivative gain is adjusted to $K_F = 2$.
(a) Obtain the step response and the time-domain performance specifications.
(b) Construct the *SIMULINK* model and obtain the step response.

(a) Substituting for the parameters in the block diagram of Figure 12.35 and applying the Mason's gain formula, we obtain the closed-loop transfer function

$$\frac{V_t(s)}{V_{ref}(s)} = \frac{250(s^2 + 45s + 500)}{s^5 + 58.5s^4 + 13,645s^3 + 270,962.5s^2 + 274,875s + 137,500}$$

(i) The steady-state response is

$$V_{t_{ss}} = \lim_{s \to 0} sV_t(s) = \frac{(250)(500)}{137,500} = 0.909$$

To find the step response, we use the following commands

```
numc=250*[1 45 500];
denc=[1  58.5  13645  270962.5  274875 137500];
t=0:.05:10;
c=step(numc, denc, t); plot(t, c), grid
timespec(numc, denc)
```

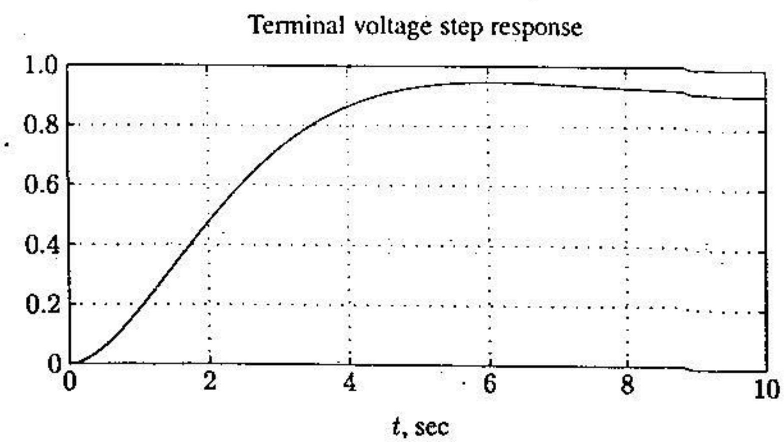


FIGURE 12.36
Terminal voltage step response for Example 12.7.

The step response is shown in Figure 12.36. The time-domain performance specifications are

```
Peak time = 6.08        Percent overshoot = 4.13
Rise time = 2.95
Settling time = 8.08
```

(b) A *SIMULINK* model named **sim12ex7.mdl** is constructed as shown in Figure 12.37.

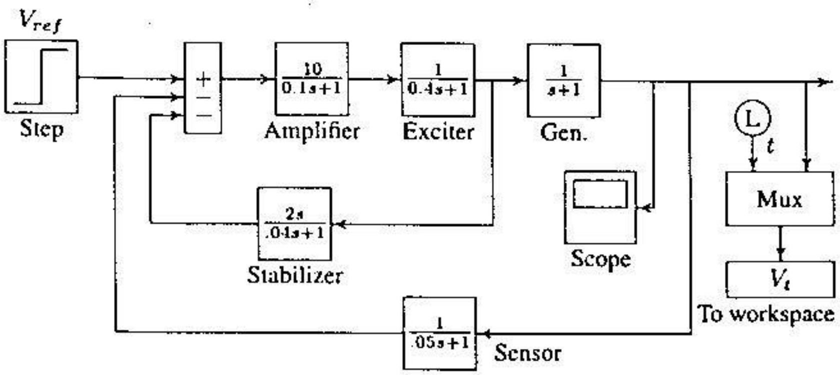


FIGURE 12.37
Simulation block diagram for Example 12.7.

The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.36. The results show a very satisfactory transient response with an overshoot of 4.13 percent and a settling time of approximately 8 seconds.

12.6.6 EXCITATION SYSTEM STABILIZER — PID CONTROLLER

One of the most common controllers available commercially is the *proportional integral derivative* (PID) controller. The PID controller is used to improve the dynamic response as well as to reduce or eliminate the steady-state error. The derivative controller adds a finite zero to the open-loop plant transfer function and improves the transient response. The integral controller adds a pole at origin and increases the system type by one and reduces the steady-state error due to a step function to zero. The PID controller transfer function is

$$G_C(s) = K_P + \frac{K_I}{s} + K_D s \tag{12.37}$$

The block diagram of an AVR compensated with a PID controller is shown in Figure 12.38.

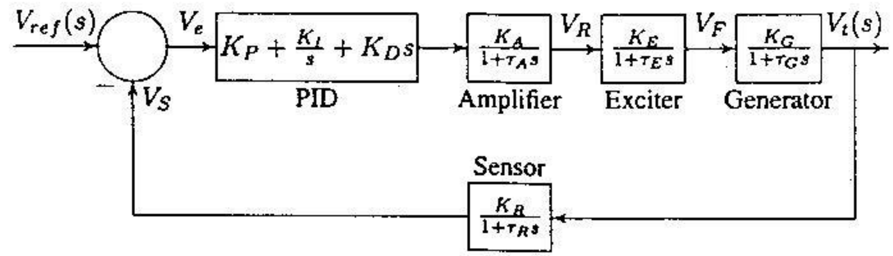


FIGURE 12.38
AVR system with PID controller.

Example 12.8

A PID controller is added in the forward path of the AVR system of Example 12.6 as shown in Figure 12.38. Construct the *SIMULINK* model. Set the proportional gain K_P to 1.0 and adjust K_I and K_D until a step response with a minimum overshoot and a very small settling time is obtained.

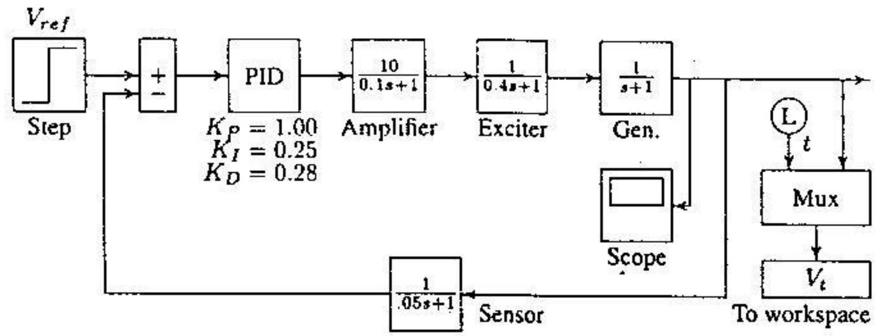


FIGURE 12.39
Simulation block diagram for Example 12.8.

A *SIMULINK* model named **sim12ex8.mdl** is constructed as shown in Figure 12.39. The file is opened and is run in the *SIMULINK* window. An integral gain of $K_I = 0.25$ and a derivative gain of $K_D = 0.28$ is found to be satisfactory. The response settles in about 1.4 seconds with a negligibly small overshoot. Note that the PID controller reduces the steady-state error to zero. The simulation result for the above settings is shown in Figure 12.40.

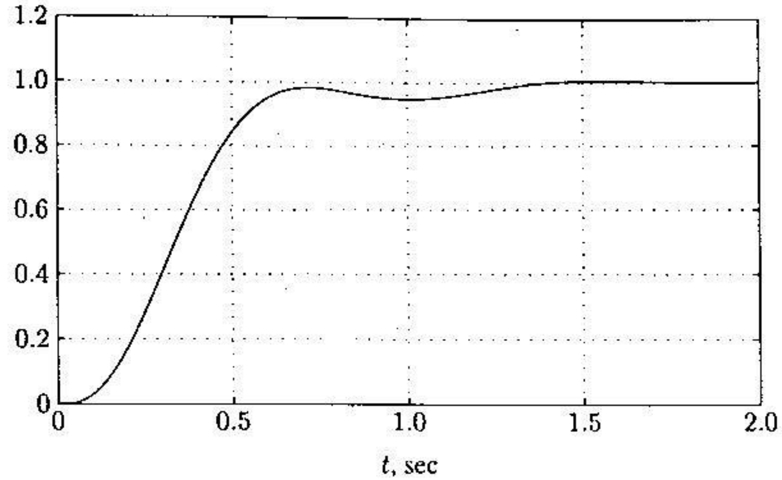


FIGURE 12.40
Terminal voltage step response for Example 12.8.

12.7 AGC INCLUDING EXCITATION SYSTEM

Since there is a weak coupling between the LFC and AVR systems, the frequency and voltage were controlled separately. We can study the coupling effect by extending the linearized AGC system to include the excitation system. In (12.17), we found that a small change in the real power is the product of the synchronizing power coefficient P_S and the change in the power angle $\Delta\delta$. If we include the small effect of voltage upon real power, we obtain the following linearized equation

$$\Delta P_e = P_s\,\Delta\delta + K_2 E' \tag{12.38}$$

where K_2 is the change in electrical power for a small change in the stator emf. Also, including the small effect of rotor angle upon the generator terminal voltage, we may write

$$\Delta V_t = K_5\,\Delta\delta + K_6 E' \tag{12.39}$$

where K_5 is the change in the terminal voltage for a small change in rotor angle at constant stator emf, and K_6 is the change in terminal voltage for a small change in the stator emf at constant rotor angle. Finally, modifying the generator field transfer function to include the effect of rotor angle, we may express the stator emf as

$$E' = \frac{K_G}{1+\tau_G}(V_f - K_4\,\Delta\delta) \tag{12.40}$$

The above constants depend upon the network parameters and the operating conditions. For the detailed derivation, see references 2 and 52. For a stable system, P_S is positive. Also, K_2, K_4, and K_6 are positive, but K_5 may be negative. Including (12.38)–(12.40) in the AGC system of Figure 12.16 and the AVR system of Figure 12.38, a linearized model for the combined LFC and AVR systems is obtained. A combined simulation block diagram is constructed in Example 12.9.

Example 12.9

An isolated power station has the following parameters

	Gain	Time constant
Turbine	$K_T = 1$	$\tau_T = 0.5$
Governor	$K_g = 1$	$\tau_g = 0.2$
Amplifier	$K_A = 10$	$\tau_A = 0.1$
Exciter	$K_E = 1$	$\tau_E = 0.4$
Generator	$K_G = 0.8$	$\tau_G = 1.4$
Sensor	$K_R = 1$	$\tau_R = 0.05$
Inertia	$H = 5$	
Regulation	$R = 0.05$	

The load varies by 0.8 percent for a 1 percent change in frequency, i.e., $D = 0.8$. Assume the synchronizing coefficient P_S is 1.5, and the voltage coefficient K_6 is 0.5. Also, the coupling constants are $K_2 = 0.2$, $K_4 = 1.4$, and $K_5 = -0.1$. Construct the combined *SIMULINK* block diagram and obtain the frequency deviation and terminal voltage responses for a load change of $\Delta P_{L1} = 0.2$ per unit.

A *SIMULINK* model named **sim12ex9.mdl** is constructed as shown in Figure 12.41. The file is opened and is run in the *SIMULINK* window. The integrator gain in the secondary LFC loop is set to a value of 6.0. The excitation PID controller is tuned for $K_P = 1$, $K_I = 0.25$, and $K_D = 0.3$. The speed deviation step response and the terminal voltage step response are shown in Figures 12.42 and 12.43. It is observed that when the coupling coefficients are set to zero, there is little change in the transient response. Thus, separate treatment of frequency and voltage control loops is justified.

12.8 INTRODUCTORY MODERN CONTROL APPLICATION

The classical design techniques used so far are based on the root-locus that utilize only the plant output for feedback with a dynamic controller. In this section, we employ modern control designs that require the use of all state variables to form a linear static controller.

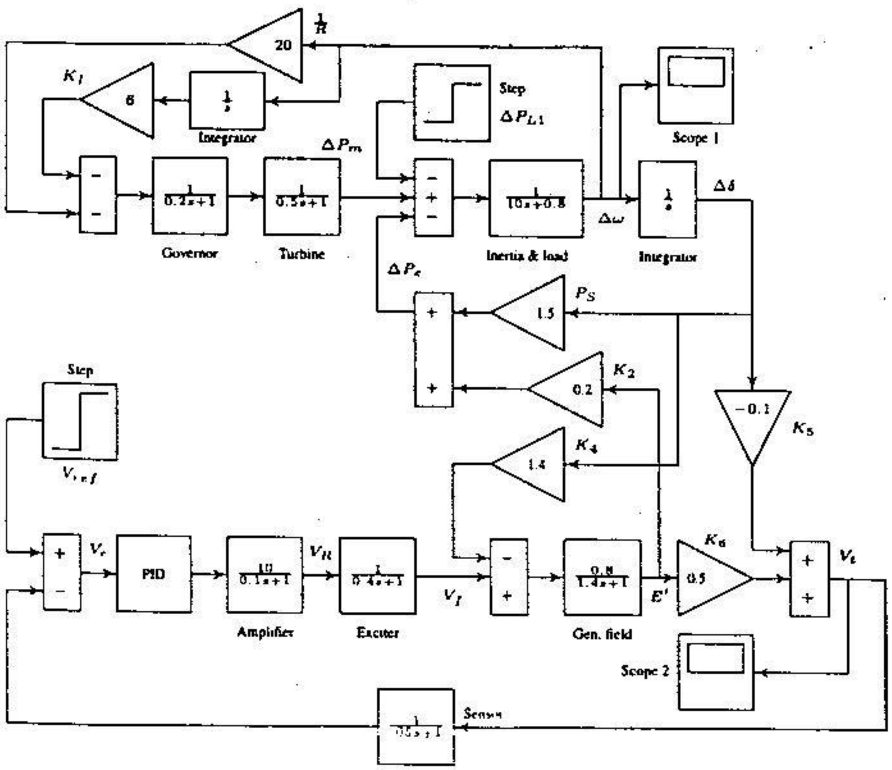


FIGURE 12.41
Simulation block diagram for Example 12.9.

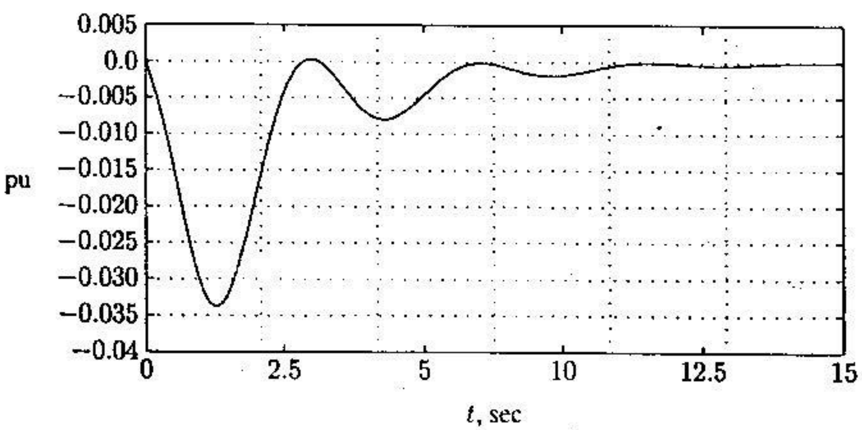


FIGURE 12.42
Frequency deviation step response for Example 12.8.

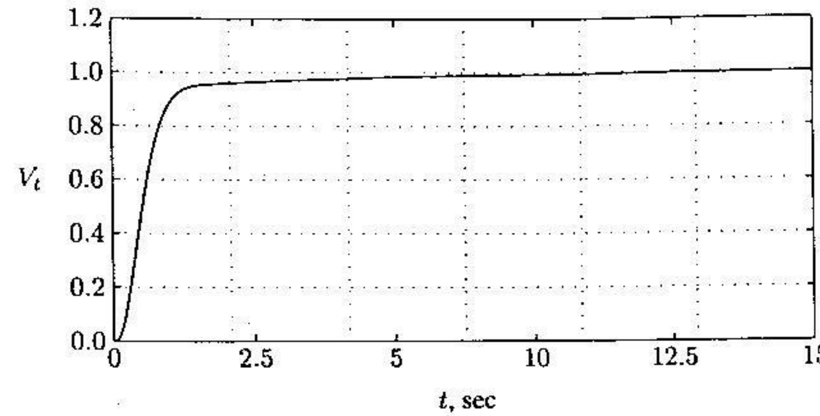


FIGURE 12.43
Terminal voltage step response for Example 12.8.

Modern control design is especially useful in multivariable systems. One approach in modern control systems accomplished by the use of state feedback is known as *pole-placement design*. The pole-placement design allows all roots of the system characteristic equation to be placed in desired locations. This results in a regulator with constant gain vector **K**.

The state-variable feedback concept requires that all states be accessible in a physical system. For systems in which all states are not available for feedback, a state estimator (observer) may be designed to implement the pole-placement design. The other approach to the design of regulator systems is the optimal control problem, where a specified mathematical performance criterion is minimized.

12.8.1 POLE-PLACEMENT DESIGN

The control is achieved by feeding back the state variables through a regulator with constant gains. Consider the control system presented in the state-variable form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{12.41}$$
$$y(t) = \mathbf{C}\mathbf{x}(\mathrm{t})$$

Now consider the block diagram of the system shown in Figure 12.44 with the following state feedback control

$$u(t) = -\mathbf{K}\mathbf{x}(t) \tag{12.42}$$

where **K** is a $1 \times n$ vector of constant feedback gains. The control system input $r(t)$ is assumed to be zero. The purpose of this system is to return all state variables to values of zero when the states have been perturbed.

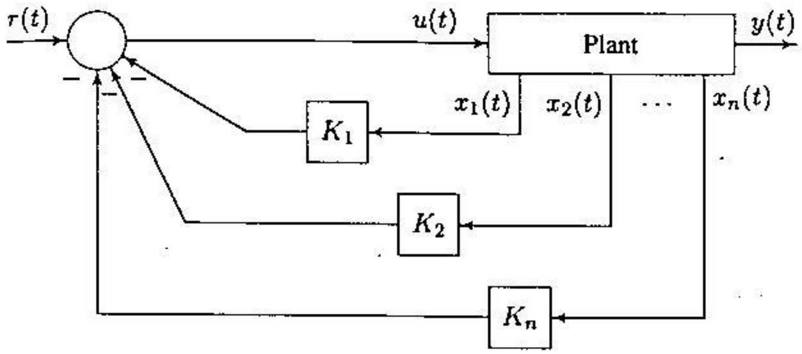


FIGURE 12.44
Control system design via pole placement.

Substituting (12.42) into (12.41), the compensated system state-variable representation becomes

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) = \mathbf{A_f}\mathbf{x}(t) \tag{12.43}$$

The compensated system characteristic equation is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0 \tag{12.44}$$

Assume the system is represented in the phase variable canonical form as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{12.45}$$

Substituting for $\mathbf{A}$ and $\mathbf{B}$ into (12.44), the compensated characteristic equation for the control system is found.

$$|s\mathbf{I}-\mathbf{A}+\mathbf{BK}| = s^n+(a_{n-1}+k_n)s^{n-1}+\cdots+(a_1+k_2)s+(a_0+k_1)=0 \tag{12.46}$$

For the specified closed-loop pole locations $-\lambda_1, \dots, -\lambda_n$, the desired characteristic equation is

$$\alpha_c(s) = (s+\lambda_1)\cdots(s+\lambda_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0 \tag{12.47}$$

The design objective is to find the gain matrix $\mathbf{K}$ such that the characteristic equation for the controlled system is identical to the desired characteristic equation.

Thus, the gain vector $\mathbf{K}$ is obtained by equating coefficients of equations (12.46) and (12.47), and for the ith coefficient we get

$$k_i = \alpha_{i-1} - a_{i-1} \tag{12.48}$$

If the state model is not in the phase-variable canonical form, we can use the transformation technique to transform the given state model to the phase-variable canonical form. The gain factor is obtained for this model and then transformed back to conform with the original model. This procedure results in the following formula, known as *Ackermann's formula*.

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{S}^{-1}\alpha_c(\mathbf{A}) \tag{12.49}$$

where the matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \tag{12.50}$$

and the notation $\alpha_c(\mathbf{A})$ is given by

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} \tag{12.51}$$

The function $[\mathbf{K}, \mathbf{A_f}] = \text{placepol}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{p})$ is developed for the pole-placement design. $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are system matrices and $\mathbf{p}$ is a row vector containing the desired closed-loop poles. This function returns the gain vector $\mathbf{K}$ and the closed-loop system matrix $\mathbf{A_f}$. Also, the *MATLAB Control System Toolbox* contains two functions for pole-placement design. Function $\mathbf{K} = \text{acker}(\mathbf{A}, \mathbf{B}, \mathbf{p})$ is for single input systems, and function $\mathbf{K} = \text{place}(\mathbf{A}, \mathbf{B}, \mathbf{p})$, which uses a more reliable algorithm, is for multiinput systems. The condition that must exist to place the closed-loop poles at the desired location is to be able to transform the given state model into phase-variable canonical form.

We demonstrate the use of pole-placement design by applying it to the LFC of an isolated power system considered before, which is represented again in Figure 12.45. The s-domain equations describing the block diagram shown in Figure 12.45 are

$$\begin{aligned} (1+\tau_g s)\Delta P_V(s) &= \Delta P_{ref} - \frac{1}{R}\Delta\Omega(s) \\ (1+\tau_T s)\Delta P_m(s) &= \Delta P_V \\ (2Hs + D)\Delta\Omega(s) &= \Delta P_m - \Delta P_L \end{aligned} \tag{12.52}$$

Solving for the first derivative term, we have

$$\begin{aligned} s\Delta P_V(s) &= -\frac{1}{\tau_g}\Delta P_V - \frac{1}{R\tau_g}\Delta\Omega(s) + \frac{1}{\tau_g}\Delta P_{ref}(s) \\ s\Delta P_m(s) &= \frac{1}{\tau_T}\Delta P_V - \frac{1}{\tau_T}\Delta P_m \\ s\Delta\Omega(s) &= \frac{1}{2H}\Delta P_m - \frac{D}{2H}\Delta\Omega(s) - \frac{1}{2H}\Delta P_L \end{aligned} \tag{12.53}$$

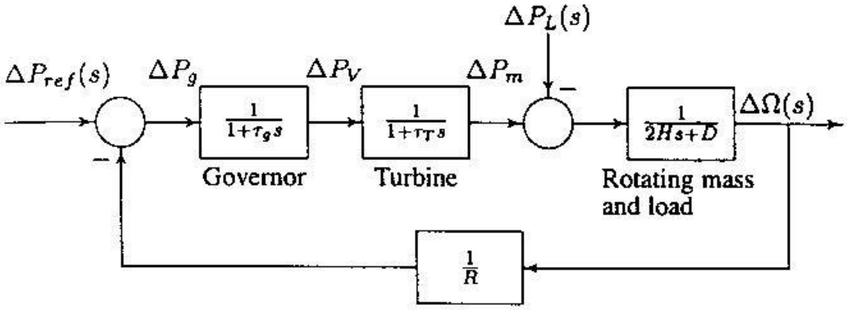


FIGURE 12.45
Load frequency control block diagram of an isolated power system.

Transforming into time-domain and expressing in matrix form the state equation becomes

$$\begin{bmatrix} \Delta\dot{P}_V \\ \Delta\dot{P}_m \\ \Delta\dot{\omega} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\tau_g} & 0 & \frac{-1}{R\tau_g} \\ \frac{1}{\tau_T} & \frac{-1}{\tau_T} & 0 \\ 0 & \frac{1}{2H} & \frac{-D}{2H} \end{bmatrix} \begin{bmatrix} \Delta P_V \\ \Delta P_m \\ \Delta\omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{2H} \end{bmatrix} \Delta P_L + \begin{bmatrix} \frac{1}{\tau_g} \\ 0 \\ 0 \end{bmatrix} \Delta P_{ref} \quad (12.54)$$

Example 12.10

Obtain the state variable representation of the LFC system of Example 12.1 with one input ΔP_L and perform the following analysis.
(a) Use the *MATLAB* **step** function to obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.2$ per unit.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (a).
(c) Use **placepole(A, B, C, p)** function to place the compensated closed-loop pole at $-2 \pm j6$ and -3. Obtain the frequency deviation step response of the compensated system.
(d) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (c).

Substituting the parameters of the system in Example 12.1 in the state equation (12.51) with $\Delta P_{ref} = 0$, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 0 & -100 \\ 2 & -2 & 0 \\ 0 & 0.1 & -0.08 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -0.1 \end{bmatrix} u$$

and the output equation is

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

where $y = \Delta\omega$ and

$$\mathbf{x} = \begin{bmatrix} \Delta P_V \\ \Delta P_m \\ \Delta\omega \end{bmatrix}$$

(a) We use the following commands

```
PL = 0.2;
A = [-5  0  -100; 2  -2  0; 0  0.1  -0.08];
B = [0;  0;  -0.1]; BPL = B*PL;
C = [0  0  1];  D = 0;
t=0:0.02:10;
[y, x] = step(A, BPL, C, D, 1, t);
figure(1), plot(t, y), grid
xlabel('t, sec'), ylabel('pu')
r =eig(A)
```

The frequency deviation step response result is shown in Figure 12.46, which is the same as the response obtained in Figure 12.13 using the transfer function method.

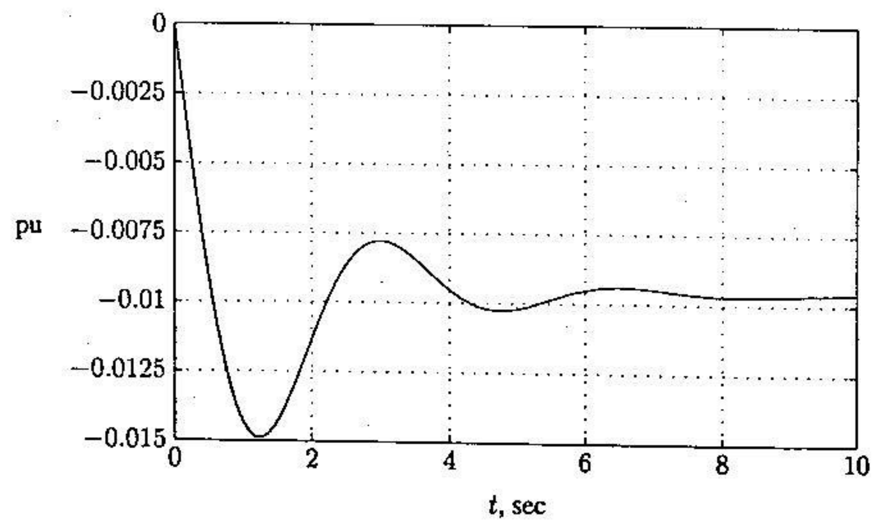


FIGURE 12.46
Uncompensated frequency deviation step response for Example 12.10.

The command **r = eig(A)** returns the roots of the characteristic equation, which are

```
r =
  -5.8863
  -0.5968 + 1.7825i
  -0.5968 - 1.7825i
```

(b) The *SIMULINK* state-space model can be used to obtain the response. A *SIMULINK* model named **sim12xxb.mdl** is constructed as shown in Figure 12.47. The state-space description dialog box is opened, and the **A**, **B**, **C**, and **D** constants are entered in the appropriate box in *MATLAB* matrix notation. The simulation

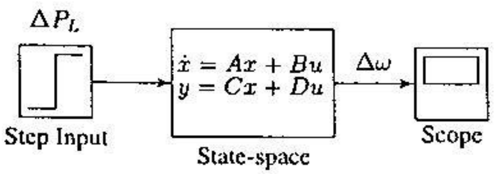


FIGURE 12.47
Simulation block diagram for Example 12.10 (b).

parameters are set to the appropriate values. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.46.

(c) We are seeking the feedback gain vector **K** to place the roots of the system characteristic equation at $-2 \pm j6$ and -3. The following commands are added to the previous file.

```
P=[-2.0+j*6   -2.0-j*6   -3];
[K, Af] = placepol(A, B, C, P);
t=0:0.02:4;
[y, x] = step(Af, BPL, C, D, 1, t);
figure(2), plot(t, y), grid
xlabel('t,  sec'), ylabel('pu')
```

The result is

```
Feedback gain vector K
  4.2   0.8   0.8
Uncompensated Plant transfer function:
Numerator      0   -0.10   -0.70   -1.0
Denominator    1    7.08   10.56   20.8
```



```
Compensated system closed-loop transfer function:
Numerator     0   -0.1    -0.7    -1
Denominator   1    7.0    52.0   120

Compensated system matrix A - B*K
   -5.00    0.00   -100.00
    2.00   -2.00      0.00
    0.42    0.18      0.00
```

and the frequency deviation step response is shown in Figure 12.48.

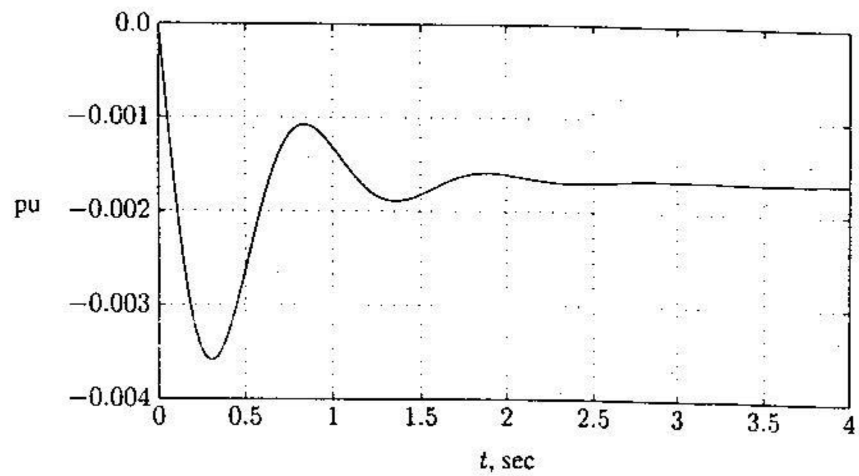


FIGURE 12.48
Compensated frequency deviation step response for Example 12.10.

Thus, the state feedback constants $K_1 = 4.2$, $K_2 = 0.8$, and $K_3 = 0.8$ result in the desired characteristic equation roots. The transient response is improved, and the response settles to a steady-state value of $\Delta_{ss} = -0.0017$ per unit in about 2.5 seconds.

(d) A *SIMULINK* model named **sim12xxd.mdl** is constructed as shown in Figure 12.49. In the state-space description dialog box, **C** is specified as an identity matrix of rank 3 to provide the three state variables as output. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.48.

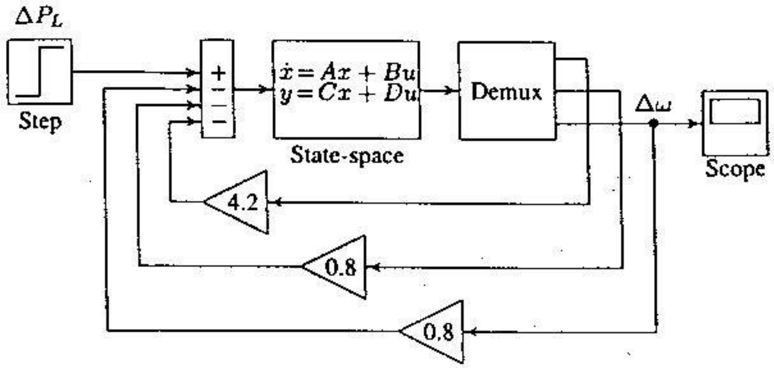


FIGURE 12.49
Simulation block diagram for Example 12.10 (d).

12.8.2 OPTIMAL CONTROL DESIGN

Optimal control is a branch of modern control theory that deals with designing controls for dynamic systems by minimizing a performance index that depends on the system variables. In this section, we will discuss the design of optimal controllers for linear systems with quadratic performance index, the so-called *linear quadratic regulator* (LQR) problem. The object of the optimal regulator design is to determine the optimal control law $\mathbf{u}^*(\mathbf{x}, t)$ which can transfer the system from its initial state to the final state such that a given performance index is minimized. The performance index is selected to give the best trade-off between performance and cost of control. The performance index that is widely used in optimal control design is known as the *quadratic performance index* and is based on minimum-error and minimum-energy criteria.

Consider the plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{12.55}$$

The problem is to find the vector $\mathbf{K}(t)$ of the control law

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{12.56}$$

which minimizes the value of a quadratic performance index $\mathbf{J}$ of the form

$$\mathbf{J} = \int_{t_0}^{t_f} (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})dt \tag{12.57}$$

subject to the dynamic plant equation in (12.55). In (12.57), $\mathbf{Q}$ is a positive semidefinite matrix, and $\mathbf{R}$ is a real symmetric matrix. $\mathbf{Q}$ is positive semidefinite, if all its

principal minors are nonnegative. The choice of the elements of $\mathbf{Q}$ and $\mathbf{R}$ allows the relative weighting of individual state variables and individual control inputs.

To obtain a formal solution, we can use the method of Lagrange multipliers. The constraint problem is solved by augmenting (12.55) into (12.57) using an n-vector of *Lagrange multipliers*, λ. The problem reduces to the minimization of the following unconstrained function.

$$\mathcal{L}(\mathbf{x}, \lambda, u, t) = [\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u}] + \lambda'[\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\mathbf{x}}] \tag{12.58}$$

The optimal values (denoted by the subscript $*$) are found by equating the partial derivatives to zero.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{A}\mathbf{X}^* + \mathbf{B}\mathbf{u}^* - \dot{\mathbf{x}}^* = 0 \quad \Rightarrow \quad \dot{\mathbf{x}}^* = \mathbf{A}\mathbf{X}^* + \mathbf{B}\mathbf{u}^* \tag{12.59}$$

$$\frac{\partial \mathcal{L}}{\partial u} = 2\mathbf{R}\mathbf{u}^* + \lambda'\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{u}^* = -\frac{1}{2}\mathbf{R}^{-1}\lambda'\mathbf{B} \tag{12.60}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\mathbf{x}'^*\mathbf{Q} + \dot{\lambda}' + \lambda'\mathbf{A} = 0 \quad \Rightarrow \quad \dot{\lambda} = -2\mathbf{Q}\mathbf{x}^* - \mathbf{A}'\lambda \tag{12.61}$$

Assume that there exists a symmetric, time-varying positive definite matrix $\mathbf{p}(t)$ satisfying

$$\lambda = 2\mathbf{p}(t)\mathbf{x}^* \tag{12.62}$$

Substituting (12.42) into (12.60) gives the optimal closed-loop control law

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\mathbf{p}(t)\mathbf{x}^* \tag{12.63}$$

Obtaining the derivative of (12.62), we have

$$\dot{\lambda} = 2(\dot{\mathbf{p}}\mathbf{x}^* + \mathbf{p}\dot{\mathbf{x}}^*) \tag{12.64}$$

Finally, equating (12.61) with (12.64), we obtain

$$\dot{\mathbf{p}}(t) = -\mathbf{p}(t)\mathbf{A} - \mathbf{A}'\mathbf{p}(t) - \mathbf{Q} + \mathbf{p}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{p}(t) \tag{12.65}$$

The above equation is referred to as the matrix *Riccati equation*. The boundary condition for (12.65) is $\mathbf{p}(t_f) = 0$. Therefore, (12.65) must be integrated backward in time. Since a numerical solution is performed forward in time, a dummy time variable $\tau = t_f - t$ is replaced for time t. Once the solution to (12.65) is obtained, the solution of the state equation (12.59) in conjunction with the optimum control equation (12.63) is obtained.

The function $[\tau, \mathbf{p}, \mathbf{K}, \mathbf{t}, \mathbf{x}]$= **riccati** is developed for the time-domain solution of the Riccati equation. The function returns the solution of the matrix Riccati equation, $\mathbf{p}(\tau)$, the optimal feedback gain vector $\mathbf{k}(\tau)$, and the initial state response $\mathbf{x}(t)$. In order to use this function, the user must declare the function $[\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{t_0}, \mathbf{t_f}, \mathbf{x_0}] = \mathbf{system}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{t_0}, \mathbf{t_f}, \mathbf{x_0})$ containing system matrices and the performance index matrices in an M-file named **system.m**.

The optimal controller gain is a time-varying state-variable feedback. Such feedback are inconvenient to implement, because they require the storage in computer memory of time-varying gains. An alternative control scheme is to replace the time-varying optimal gain $\mathbf{K}(t)$ by its constant steady-state value. In most practical applications, the use of the steady-state feedback gain is adequate. For linear time-invariant systems, since $\dot{\mathbf{p}} = 0$, when the process is of infinite duration, that is $t_f = \infty$, (12.65) reduces to the algebraic Riccati equation

$$\mathbf{pA} + \mathbf{A}'\mathbf{p} + \mathbf{Q} - \mathbf{pBR}^{-1}\mathbf{B}'\mathbf{p} = 0 \tag{12.66}$$

The *MATLAB Control System Toolbox* function **[k, p]=lqr2(A, B, Q, R)** can be used for the solution of the algebraic Riccati equation.

The LQR design procedure is in stark contrast to classical control design, where the gain matrix $\mathbf{K}$ is selected directly. To design the optimal LQR, the design engineer first selects the design parameter weight matrices $\mathbf{Q}$ and $\mathbf{R}$. Then, the feedback gain $\mathbf{K}$ is automatically given by matrix design equations and the closed-loop time responses are found by simulation. If these responses are unsuitable, new values of $\mathbf{Q}$ and $\mathbf{R}$ are selected and the design is repeated. This has the significant advantages of allowing all the control loops in a multiloop system to be closed simultaneously, while guaranteeing closed-loop stability.

Example 12.11

Design an LQR state feedback for the system described in Example 12.10.
(a) Find the optimal feedback gain vector to minimize the performance index

$$J = \int_0^\infty \left(20{x_1}^2 + 15{x_2}^2 + 5{x_3}^2 + 0.15u^2\right) dt$$

The admissible states and control values are unconstrained. Obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.2$ per unit.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (a).

For this system we have

$$\mathbf{A} = \begin{bmatrix} -5 & 0 & -100 \\ 2 & -2 & 0 \\ 0 & 0.1 & -0.08 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -0.1 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and $R = 0.15$.

(a) We use the following commands

```
PL=0.2;
A = [-5  0  -100; 2  -2  0; 0  0.1  -0.08];
B = [0;  0;  -0.1]; BPL=PL*B;
C = [0  0  1];  D = 0;
Q = [20 0  0; 0  10 0; 0  0  5];  R = .15;
[K, P] = lqr2(A, B, Q, R)
Af = A - B*K
t=0:0.02:1;
[y, x] = step(Af, BPL, C, D, 1, t);
plot(t, y), grid, xlabel('t,  sec'), ylabel('pu')
```

The result is

```
K =
    6.4128     1.1004 -112.6003
P =
    1.5388     0.3891    -9.6192
    0.3891     2.3721    -1.6506
   -9.6192    -1.6506   168.9004
Af =
   -5.0000          0 -100.0000
    2.0000    -2.0000          0
    0.6413     0.2100   -11.3400
```

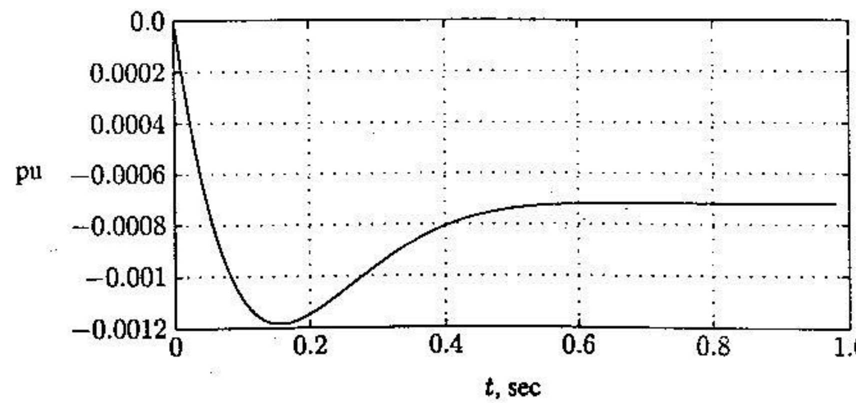


FIGURE 12.50
Frequency deviation step response for Example 12.11.

The frequency deviation step response is shown in Figure 12.50. The transient response settles to a steady-state value of $\Delta_{ss} = -0.0007$ per unit in about 0.6 second.

(b) A *SIMULINK* model named **sim12xx1.mdl** is constructed as shown in Figure 12.51. The state-space description dialog box is opened, and the **A**, **B**, **C**, and D constants are entered in the appropriate box in *MATLAB* matrix notation. Also, the LQR description dialog box is opened, and weighting matrix Q and weighting coefficient **R** are set to the given values. The simulation parameters are set to the

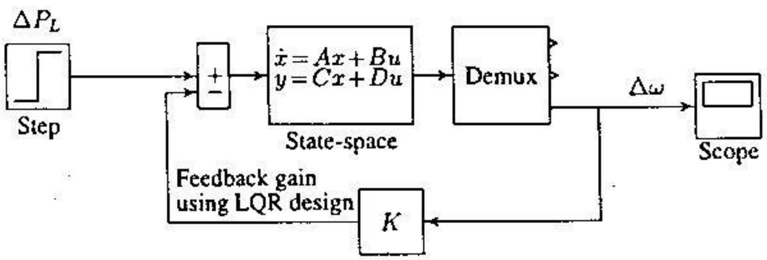


FIGURE 12.51
Simulation block diagram for Example 12.11.

appropriate values. The file is opened and is run in the *SIMULINK* window. The simulation results in the same response as shown in Figure 12.50.

PROBLEMS

12.1. A 250-MW, 60-Hz turbine generator set has a speed regulation of 5 percent based on its own rating. The generator frequency decreases from 60 Hz to a steady state value of 59.7Hz. Determine the increase in the turbine power output.

12.2. Two generating units rated for 250 MW and 400 MW have governor speed regulation of 6.0 and 6.4 percent, respectively, from no-load to full-load, respectively. They are operating in parallel and share a load of 500 MW. Assuming free governor action, determine the load shared by each unit.

12.3. A single area consists of two generating units, rated at 400 and 800 MVA, with speed regulation of 4 percent and 5 percent on their respective ratings. The units are operating in parallel, sharing 700 MW. Unit 1 supplies 200 MW and unit 2 supplies 500 MW at 1.0 per unit (60 Hz) frequency. The load is increased by 130 MW.
(a) Assume there is no frequency-dependent load, i.e., $D = 0$. Find the steady-state frequency deviation and the new generation on each unit.

(b) The load varies 0.804 percent for every 1 percent change in frequency, i.e., $D = 0.804$. Find the steady-state frequency deviation and the new generation on each unit.

12.4. An isolated power station has the LFC system as shown in Figure 12.9 with the following parameters

Turbine time constant $\tau_T = 0.5$ sec
Governor time constant $\tau_g = 0.25$ sec
Generator inertia constant $H = 8$ sec
Governor speed regulation = R per unit

The load varies by 1.6 percent for a 1 percent change in frequency, i.e., $D = 1.6$.
(a) Use the Routh-Hurwitz array (Appendix B.2.1) to find the range of R for control system stability.
(b) Use *MATLAB* **rlocus** function to obtain the root-locus plot.

12.5. The governor speed regulation of Problem 12.4 is set to $R = 0.04$ per unit. The turbine rated output is 200 MW at nominal frequency of 60 Hz. A sudden load change of 50 MW ($\Delta P_L = 0.25$ per unit) occurs.
(a) Find the steady-state frequency deviation in Hz.
(b) Obtain the closed-loop transfer function and use *MATLAB* to obtain the frequency deviation step response.
(c) Construct the *SIMULINK* block diagram and obtain the frequency deviation response.

12.6. The LFC system in Problem 12.5 is equipped with the secondary integral control loop for automatic generation control as shown in Figure 12.16.
(a) Use the *MATLAB* **step** function to obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.25$ per unit. Set the integral controller gain to $K_I = 9$.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (a).

12.7. The load changes of 200 MW and 150 MW occur simultaneously in areas 1 and 2 of the two-area system of Example 12.4. Modify the *SIMULINK* block diagram (sim12ex4.mdl), and obtain the frequency deviation and the power responses.

12.8. Modify the *SIMULINK* model for the two-area system of Example 12.5 with the tie-line bias control (sim12ex5.mdl) to include the load changes specified in Problem 12.7. Obtain the frequency and power response deviation for each area.

12.9. A generating unit has a simplified linearized AVR system as shown in Figure 12.52.
(a) Use the Routh-Hurwitz array (Appendix B.2.1) to find the range of K_A for control system stability.
(b) Use *MATLAB* **rlocus** function to obtain the root-locus plot.
(c) The amplifier gain is set to $K_A = 40$. Find the system closed-loop transfer function, and use *MATLAB* to obtain the step response.
(d) Construct the *SIMULINK* block diagram and obtain the step response.

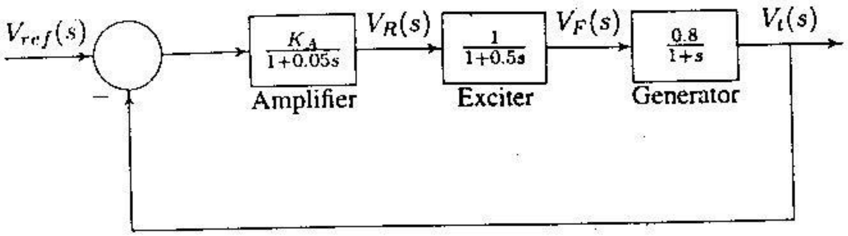


FIGURE 12.52
AVR system of Problem 12.9.

12.10. A rate feedback stabilizer is added to the AVR system of Problem 12.9 as shown in Figure 12.53. The stabilizer time constant is $\tau_F = 0.04$ second, and the derivative gain is adjusted to $K_F = 0.1$.
(a) Find the system closed-loop transfer function, and use *MATLAB* to obtain the step response.
(b) Construct the *SIMULINK* model, and obtain the step response.

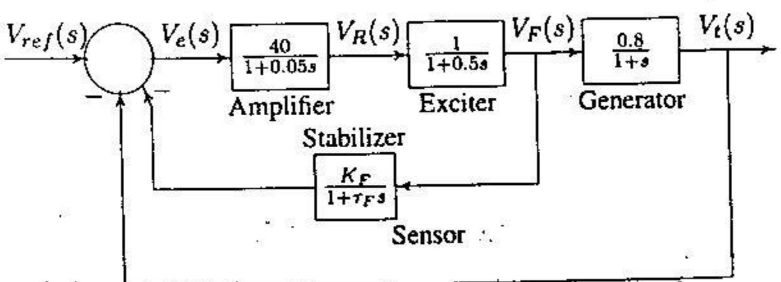


FIGURE 12.53
AVR system with rate feedback for Problem 12.10.

12.11. A PID controller is added in the forward path of the AVR system of Problem 12.9 as shown in Figure 12.54. Construct the *SIMULINK* model. Set the proportional gain K_P to 2.0, and adjust K_I and K_D until a step response

with a minimum overshoot and a very small settling time is obtained (suggested values $K_I = 0.2$, and $K_P = 0.25$).

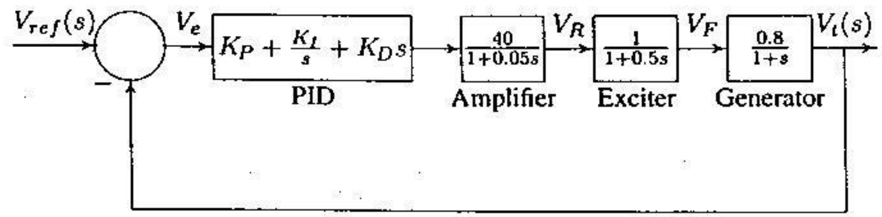


FIGURE 12.54
AVR system with PID controller for Problem 12.11.

12.12. Figure 12.55 shows an inverted pendulum of length L and mass m mounted on a cart of mass M, by means of a force u applied to the cart. This is a model of the attitude control of a space booster on takeoff. The differential equations describing the motion of the system is obtained by summing the forces on the pendulum, which result in the following nonlinear equations.

$$(M + m)\ddot{x} + mL\cos\theta\,\ddot{\theta} = mL\sin\theta\,\dot{\theta}^2 + u$$
$$mL\cos\theta\,\ddot{x} + mL^2\ddot{\theta} = mgL\sin\theta$$

(a) Linearize the above equations in the neighborhood of the zero initial

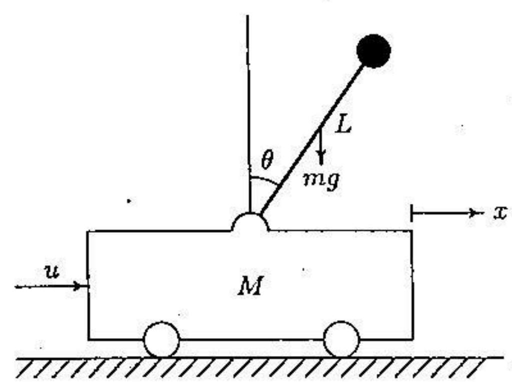


FIGURE 12.55
Inverted pendulum on a cart.

states. Hint: Substitute θ for $\sin\theta$, 1 for $\cos\theta$ and 0 for $\dot{\theta}^2$. With the state variables defined as $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$, and $x_4 = \dot{x}$, show that the linearized state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{ML}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

Assume $M = 4$ kg, $m = 0.2$ kg, $L = 0.5$ m, and $g = 9.81$ m/s^2.
(b) Use the *MATLAB* function **eig(A)** to find the roots of the system characteristic equation.
(c) Define C as the identity matrix of rank 4, i.e., C = eye(4) and D = zeros(4, 1). Use the *MATLAB* function $[\mathbf{y}, \mathbf{x}] = \text{initial}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X_0}, \mathbf{t})$ to simulate the system for 20 seconds in response to an initial condition offset of $\theta(0) = 0.1$ rad, and $x(0) = 0.1$ m (i.e., $x_0 = [0.1 \quad 0 \quad 0.1 \quad 0]$). Obtain a plot of θ and x, and comment on the stability of the system.
(d) You may have found that the inverted pendulum is unstable, that is, it will fall over unless a suitable control force via state feedback is used. The purpose is to design a control system such that for a small initial disturbance the pendulum can be brought back to the vertical position ($\theta = 0$), and the cart can be brought back to the reference position ($x = 0$). A simple method is to use a state feedback gain to place the compensated closed-loop poles in the left-half of the s-plane. Use the custom made function **[K, Af] = placepol(A, B, C, P)** and design a state feedback controller to place the compensated closed-loop poles at $-2 \pm j0.5$, -4, and -5. Simulate the system for 20 seconds in response to the same initial condition offset. Obtain a plot of θ, x, and $u = -Kx'$.

12.13. Construct the *SIMULINK* block diagram for the linearized model of the inverted pendulum described in Problem 12.12 (a) with the state feedback controller. Assume the state feedback gains are $K_1 = -170.367$, $K_2 = -38.054$, $K_3 = -17.3293$, and $K_4 = -24.1081$. Obtain the response for θ, x, and u for the initial condition offset of $\theta(0) = 0.1$ rad and $x(0) = 0.1$m (i.e., $x_0 = [0.1 \quad 0 \quad 0.1 \quad 0]$).

12.14. A classical problem in control systems is to find the optimal control law which will transfer the system from its initial state to the final state, such that a given performance index is minimized.
(a) Design an LQR state feedback for the linearized model of the inverted pendulum described in Problem 12.12, and find the optimal feedback gain vector to minimize the performance index

$$\mathbf{J} = \int_0^\infty \left(10{x_1}^2 + 10{x_2}^2 + 5{x_3}^2 + +5{x_4}^2 + 0.2u^2\right) dt$$

The admissible states and control values are unconstrained.
(b) Define C as the identity matrix of rank 4, i.e., C = eye(4) and D = zeros(4, 1). Use the *MATLAB* function **[K, P] = lqr2 (A, B, Q, R)** to design a state feedback controller in response to an initial condition offset of $\theta(0) = 0.1$ rad and $x(0) = 0.1$m (i.e., $x_0 = [0.1 \quad 0 \quad 0.1 \quad 0]$). Use the *MATLAB* function $[\mathbf{y}, \mathbf{x}] = \text{initial}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X_0}, \mathbf{t})$ to simulate the system for 20 seconds. Obtain a plot of θ, x, and the control law $u = -kx'$.

12.15. Construct the *SIMULINK* block diagram for the linearized model of the inverted pendulum described in Problem 12.12(a) using the *SIMULINK* LQR model. Obtain the response for θ, x, and u for the initial condition offset described in Problem 12.14.

12.16. Obtain the state variable representation of the LFC system of Problem 12.4 with one input ΔP_L, and perform the following analysis.
(a) Use the *MATLAB* **step** function to obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.2$ per unit.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (a).
(c) Use **placepol(A, B, C, p)** function to place the compensated closed-loop pole at $-4 \pm j6$ and -4. Obtain the frequency deviation step response of the compensated system.
(d) Construct the *SIMULINK* block diagram and obtain the frequency deviation response for the condition in part (c).

12.17. Design a LQR state feedback for the system described in Problem 12.16.
(a) Find the optimal feedback gain vector to minimize the performance index

$$\mathbf{J} = \int_0^\infty \left(40{x_1}^2 + 20{x_2}^2 + 10{x_3}^2 + 0.2u^2\right) dt$$

The admissible states and control values are unconstrained. Obtain the frequency deviation step response for a sudden load change of $\Delta P_L = 0.2$ per unit.
(b) Construct the *SIMULINK* block diagram and obtain the frequency deviation step response for the condition in part (a).

APPENDIX A

INTRODUCTION TO *MATLAB*

MATLAB, developed by Math Works Inc., is a software package for high performance numerical computation and visualization. The combination of analysis capabilities, flexibility, reliability, and powerful graphics makes *MATLAB* the premier software package for electrical engineers.

MATLAB provides an interactive environment with hundreds of reliable and accurate built-in mathematical functions. These functions provide solutions to a broad range of mathematical problems including matrix algebra, complex arithmetic, linear systems, differential equations, signal processing, optimization, nonlinear systems, and many other types of scientific computations. The most important feature of *MATLAB* is its programming capability, which is very easy to learn and to use, and which allows user-developed functions. It also allows access to Fortran algorithms and C codes by means of external interfaces. There are several optional toolboxes written for special applications such as signal processing, control systems design, system identification, statistics, neural networks, fuzzy logic, symbolic computations, and others. *MATLAB* has been enhanced by the very powerful *SIMULINK* program. *SIMULINK* is a graphical mouse-driven program for the simulation of dynamic systems. *SIMULINK* enables students to simulate linear, as well as nonlinear, systems easily and efficiently.

The following section describes the use of *MATLAB* and is designed to give a quick familiarization with some of the commands and capabilities of *MATLAB*. For a description of all other commands, *MATLAB* functions, and many other useful features, the reader is referred to the *MATLAB User's Guide*.

A.1 INSTALLING THE TEXT TOOLBOX

The diskette included with the book contains all the developed functions and chapter examples. The M-files reside in the directories labeled Folder 1 and Folder 2. The file names for chapter examples begin with the letters **chp**. For example, the M-file for Example 11.4 is **chp11ex4**. The appendix examples begin with **exa** and **exb** and a number. The disk contains a file called **setup.exe** that automates installation of all the files. Insert the diskette in the disk drive and use Windows to view its contents. For automatic installation double click on the **setup.exe** icon to start the installation. The installation program will prompt you for the location of the *MATLAB* directory and the name of the directory where you would like the files to be installed.

Alternatively, you can copy the M-files manually. To do this, create a subdirectory, such as **power**, where the *MATLAB* toolbox resides. Copy all the files from Folder 1 and Folder 2 to the subdirectory *matlab \toolbox\power*.

In the *MATLAB* 5 **Command Window** open the Path Browser by selecting **Set Path** from the **File** menu. From the **Path** menu choose **Add to Path**. Select the directory to add, choose **Add to back** option and press OK to add to the current *MATLAB* directory area. Save before exiting the Path Browser.

If you are running *MATLAB* 4 edit the **matlabrc.m** located in the subdirectory *matlab\bin*, where the search paths are specified. Describe the subdirectory just created by adding the statement

`';C:\matlab\toolbox\power',` ... at the end of this file.

A.2 RUNNING MATLAB

MATLAB supports almost every computational platform. *MATLAB* for *WINDOWS* is started by clicking on the *MATLAB* icon. The **Command window** is launched, and after some messages such as intro, demo, help help, info, and others, the prompt " ≫ " is displayed. The program is in an interactive command mode. Typing **who** or **whos** displays a list of variable names currently in memory. Also, the **dir** command lists all the files on the default directory. *MATLAB* has an on-line help facility, and its use is highly recommended. The command **help** provides a list of files, built-in functions and operators for which on-line help is available. The command

`help` function name

will give information on the specified function as to its purpose and use. The command

```
help help
```

will give information as to how to use the on-line help.

MATLAB has a demonstration program that shows many of its features. The command **demo** brings up a menu of the available demonstrations. This will provide a presentation of the most important *MATLAB* facilities. Follow the instructions on the screen – it is worth trying.

MATLAB 5.2 includes a Help Desk facility that provides access to on line help topics, documentation, getting started with *MATLAB*, online reference materials, *MATLAB* functions, real-time Workshop, and several toolboxes. The online documentation is available in HTML, via either Netscape Navigator Release 3.0 or Microsoft Internet Explorer 3.0. The command **helpdesk** launches the Help Desk, or you can use the **Help** menu to bring up the Help Desk.

If an expression with correct syntax is entered at the prompt in the Command window, it is processed immediately and the result is displayed on the screen. If an expression requires more than one line, the last character of the previous line must contain three dots "...". Characters following the percent sign are ignored. The (%) may be used anywhere in a program to add clarifying comments. This is especially helpful when creating a program. The command **clear** erases all variables in the Command window.

MATLAB is also capable of executing sequences of commands that are stored in files, known as script files or *M-files*. Clicking on **File**, **New M-file**, opens the **Edit window**. A program can be written and saved in ASCII format with a filename having extension .m in the directory where *MATLAB* runs. To run the program, click on the Command window and type the filename without the *.m* extension at the *MATLAB* command "≫". You can view the text Edit window simultaneously with the Command window. That is, you can use the two windows to edit and debug a script file repeatedly and run it in the Command window without ever quitting *MATLAB*.

In addition to the Command window and Edit window are the **Graphic windows** or **Figure windows** with black (default) background. The plots created by the graphic commands appear in these windows.

Another type of M-file is a *function file*. A function provides a convenient way to encapsulate some computation, which can then be used without worrying about its implementation. In contrast to the script file, a *function file* has a name following the word "function" at the beginning of the file. The filename must be the same as the "function" name. The first line of a function file must begin with the function statement having the following syntax

function [*output arguments*] = *function name* (*input arguments*)

The output argument(s) are variables returned. A function need not return a value. The input arguments are variables passed to the function. Variables generated in function files are local to the function. The use of **global** variables make defined variables common and accessible between the main script file and other function files. For example, the statement **global R S T** declares the variables R, S, and T to be global without the need for passing the variables through the input list. This statement goes before any executable statement in the script and function files that need to access the values of the global variables.

Normally, while an M-file is executing, the commands of the file are not displayed on the screen. The command **echo** allows M-files to be viewed as they execute. **echo off** turns off the echoing of all script files. Typing **what** lists M-files and Mat-files in the default directory.

MATLAB follows conventional Windows procedure. Information from the command screen can be printed by highlighting the desired text with the mouse and then choosing the **print Selected ...** from the **File** menu. If no text is highlighted the entire Command window is printed. Similarly, selecting **print** from the Figure window sends the selected graph to the printer. For a complete list and help on general purpose commands, type **help general**.

A.3 VARIABLES

Expressions typed without a variable name are evaluated by *MATLAB*, and the result is stored and displayed by a variable called **ans**. The result of an expression can be assigned to a variable name for further use. Variable names can have as many as 19 characters (including letters and numbers). However, the first character of a variable name must be a letter. *MATLAB* is case-sensitive. Lower and uppercase letters represent two different variables. The command **casesen** makes *MATLAB* insensitive to the case. Variables in script files are global. The expressions are composed of operators and any of the available functions. For example, if the following expression is typed

```
x = exp(-0.2696*.2)*sin(2*pi*0.2)/(0.01*sqrt(3)*log(18))
```

the result is displayed on the screen as

```
x =
    18.0001
```

and is assigned to x. If a variable name is not used, the result is assigned to the variable ans. For example, typing the expression

```
250/sin(pi/6)
```

results in

```
ans =
      500.0000
```

If the last character of a statement is a semicolon (;), the expression is executed, but the result is not displayed. However, the result is displayed upon entering the variable name. The command **disp** may be used to display a variable without printing its name. For example, **disp(x)** displays the value of the variable without printing its name. If **x** contains a text string, the string is displayed.

A.4 OUTPUT FORMAT

While all computations in *MATLAB* are done in double precision, the default format prints results with five significant digits. The format of the displayed output can be controlled by the following commands.

MATLAB Command	Display
format	Default. Same as **format short**
format short	Scaled fixed point format with 5 digits
format long	Scaled fixed point format with 15 digits
format short e	Floating point format with 5 digits
format long e	Floating point format with 15 digits
format short g	Best of fixed or floating point with 5 digits
format long g	Best of fixed or floating point with 15 digits
format hex	Hexadecimal format
format +	The symbols +, - and blank are printed for positive, negative, and zero elements
format bank	Fixed format for dollars and cents
format rat	Approximation by ratio of small integers
format compact	Suppress extra line feeds
format loose	Puts the extra line feeds back in

For more flexibility in the output format, the command **fprintf** displays the result with a desired format on the screen or to a specified filename. The general form of this command is the following.

```
fprintf{fstr, A,...)
```

writes the real elements of the variable or matrix `A,...` according to the specifications in the string argument of `fstr`. This string can contain format characters like *ANCI C* with certain exceptions and extensions. **fprintf** is "vectorized" for the case when A is nonscalar. The format string is recycled through the elements of A (columnwise) until all the elements are used up. It is then recycled in a similar manner through any additional matrix arguments. The characters used in the format string of the commands **fprintf** are listed in the table below.

Format codes		Control characters	
%e	scientific format, lower case e	\n	new line
%E	scientific format, upper case E	\r	beginning of the line
%f	decimal format	\b	back space
%s	string	\t	tab
%u	integer	\g	new page
%i	follows the type	''	apostrophe
%x	hexadecimal, lower case	\\	back slash
%X	hexadecimal, upper case	\a	bell

A simple example of the **fprintf** is

```
fprintf('Area = %7.3f Square meters \n', pi*4.5^2)
```

The results is

```
Area =  63.617 Square meters
```

The %7.3f prints a floating point number seven characters wide, with three digits after the decimal point. The sequence \n advances the output to the left margin on the next line.

The following command displays a formatted table of the natural logarithmic for numbers 10, 20, 40, 60, and 80

```
x = [10; 20; 40; 60; 80];
y = [x,  log(x)];
fprintf('\n Number   Natural log\n')
fprintf('%4i \t %8.3f\n',y')
```

The result is

```
Number   Natural log
  10        2.303
  20        2.996
  40        3.689
  60        4.094
  80        4.382
```

An M-file can prompt for input from the keyboard. The command **input** causes the computer to request data from the keyboard. For example, the command

```
R = input('Enter radius in meter ')
```

displays the text string

```
Enter radius in meter
```

and waits for a number to be entered. If a number, say 4.5 is entered, it is assigned to variable R and displayed as

```
R =
    4.5000
```

The command **keyboard** placed in an M-file will stop the execution of the file and permit the user to examine and change variables in the file. Pressing **cntrl-z** terminates the keyboard mode and returns to the invoking file. Another useful command is **diary** A:*filename*. This command creates a file on drive A, and all output displayed on the screen is sent to that file. **diary off** turns off the diary. The contents of this file can be edited and used for merging with a word processor file. Finally, the command **save** *filename* can be used to save the expressions on the screen to a file named *filename.mat*, and the statement **load** *filename* can be used to load the file *filename.mat*.

MATLAB has a useful collection of transcendental functions, such as exponential, logarithm, trigonometric, and hyperbolic functions. For a complete list and help on operators, type **help ops**, and for elementary math functions, type **help elfun**.

A.5 CHARACTER STRING

A sequence of characters in single quotes is called a *character string* or *text variable.*

```
c ='Good'
```

results in

```
c = Good
```

A text variable can be augmented with more text variables, for example,

```
cs = [c, ' luck']
```

produces

```
cs =
     Good luck
```


A.6 VECTOR OPERATIONS

An n vector is a row or a column array of n numbers. In *MATLAB*, elements enclosed by brackets and separated by semicolons generate a column vector.

For example, the statement

```
X = [ 2; -4; 8]
```

results in

```
X =
      2
     -4
      8
```

If elements are separated by blanks or commas, a row vector is produced. Elements may be any expression. The statement

```
R = [tan(pi/4)  sqrt(9) -5]
```

results in the output

```
R =
      1.0000    3.0000    -5.0000
```

The transpose of a column vector results in a row vector, and vice versa. For example

```
Y=R'
```

will produce

```
Y =
      1.0000
      3.0000
     -5.0000
```

MATLAB has two different types of arithmetic operations. Matrix arithmetic operations are defined by the rules of linear algebra. Array arithmetic operations are carried out element-by-element. The period character (.) distinguishes the array operations from the matrix operations. However, since the matrix and array operations are the same for addition and subtraction, the character pairs .+ and .- are not used.

Vectors of the same size can be added or subtracted, where addition is performed componentwise. However, for multiplication, specific rules must be followed in order to obtain the correct resulting values. The operation of multiplying a vector X with a scalar k (scalar multiplication) is performed componentwise. For example $P = 5 * R$ produces the output

```
P =
      5.0000    15.0000    -25.0000
```

The inner product or the *dot product* of two vectors X and Y denoted by $\langle X, Y\rangle$ is a scalar quantity defined by $\sum_{i=1}^{n} x_i y_i$. If X and Y are both column vectors defined above, the inner product is given by

```
S = X'*Y
```

and results in

```
S =
     -50
```

The operator (.* performs element-by-element operation. For example, for the previously defined vectors, X and Y, the statement

```
E = X.*Y
```

results in

```
E =
      2
    -12
    -40
```

The operator ./ performs element-by-element division. The two arrays must have the same size, unless one of them is a scalar. Array powers or element-by-element powers are denoted by (.^). The trigonometric functions, and other elementary mathematical functions such as **abs**, **sqrt**, **real**, and **log**, also operate element by element.

Various norms (measure of size) of a vector can be obtained. For example, the *Euclidean norm* is the square root of the inner product of the vector and itself. The command

```
N = norm(X)
```

produces the output

```
N =
    9.1652
```

The angle between two vectors X and Y is defined by $\cos\theta = \frac{\langle X,Y\rangle}{\|X\|\,\|Y\|}$. The statement


```
Theta = acos( X'*Y/(norm(X)*norm(Y)) )
```

results in the output

```
Theta =
        2.7444
```

where Theta is in radians.

The *zero vector*, also referred to as origin, is a vector with all components equal to zero. For example, to build a zero row vector of size 4, the following command

```
Z = zeros(1, 4)
```

results in

```
Z =
     0     0     0     0
```

The *one vector* is a vector with each component equal to one. To generate a one vector of size 4, use

```
I = ones(1, 4)
```

The result is

```
I =
     1     1     1     1
```

In *MATLAB*, the colon (:) can be used to generate a row vector. For example

```
x = 1:8
```

generates a row vector of integers from 1 to 8.

```
x =
      1   2   3   4   5   6   7   8
```

For increments other than unity, the following command

```
z = 0 : pi/3 : pi
```

results in

```
z =
      0000    1.0472    2.0944    3.1416
```

For negative increments

```
x = 5 : -1:1
```

results in

```
x =
      5    4    3    2    1
```

Alternatively, special vectors can be created, the command **linspace(x, y, n)** creates a vector with n elements that are spaced linearly between x and y. Similarly, the command **logspace(x, y, n)** creates a vector with n elements that are spaced in even logarithmic increments between 10^x and 10^y.

A.7 ELEMENTARY MATRIX OPERATIONS

In *MATLAB*, a matrix is created with a rectangular array of numbers surrounded by brackets. The elements in each row are separated by blanks or commas. A semicolon must be used to indicate the end of a row. Matrix elements can be any *MATLAB* expression. The statement

```
A = [ 6  1  2;  -1  8  3;  2  4  9]
```

results in the output

```
A =
       6    1    2
      -1    8    3
       2    4    9
```

If a semicolon is not used, each row must be entered in a separate line as shown below.

```
A = [ 6    1    2
     -1    8    3
      2    4    9]
```

The entire row or column of a matrix can be addressed by means of the symbol (:). For example

```
r3 =  A(3, :)
```

results in


```
r3 =
      2    4    9
```

Similarly, the statement $A(:,\ 2)$ addresses all elements of the second column in A.

Matrices of the same dimension can be added or subtracted. Two matrices, A and B, can be multiplied together to form the product AB if they are conformable. Two symbols are used for nonsingular matrix division. $A\backslash B$ is equivalent to $A^{-1}B$, and A/B is equivalent to AB^{-1}

Example A.1

For the matrix equation below, $AX = B$, determine the vector X.

$$\begin{bmatrix} 4 & -2 & -10 \\ 2 & 10 & -12 \\ -4 & -6 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 32 \\ -16 \end{bmatrix}$$

The following statements

```
A = [4  -2  -10;  2  10  -12;  -4  -6  16];
B = [-10;  32;  -16];
X = A\B
```

result in the output

```
X =
      2.0000
      4.0000
      1.0000
```

In addition to the built-in functions, numerous mathematical functions are available in the form of M-files. For the current list and their applications, see the *MATLAB User's Guide*.

Example A.2

Use the **inv** function to determine the inverse of matrix A in Example A.1 and then determine X. The following statements

```
A = [4  -2  -10;  2  10  -12;  -4  -6  16];
B = [-10;  32;  -16];
C = inv(A)
X = C*B
```

result in the output

```
C =
        2.2000    2.3000    3.1000
        0.4000    0.6000    0.7000
        0.7000    0.8000    1.1000
X =
        2.0000
        4.0000
        1.0000
```

Example A.3

Use the **lu** factorization function to express the matrix A of Example A.2 as the product of upper and lower triangular matrices, $A = LU$. Then find X from $X = U^{-1}L^{-1}B$. Typing

```
A = [ 4  -2  -10;  2  10  -12;  -4  -6  16 ]
B = [-10;  32  -16];
[L,U] = lu(A)
```

results in

```
L =
        1.0000         0         0
        0.5000    1.0000         0
       -1.0000   -0.7273    1.0000

U =
        4.0000   -2.0000  -10.0000
             0   11.0000   -7.0000
             0         0    0.9091
```

Now entering

```
X = inv(U)*inv(L)*B
```

results in

```
X =
        2.0000
        4.0000
        1.0000
```

Dimensioning is automatic in *MATLAB*. You can find the dimensions and rank of an existing matrix with the **size** and **rank** statements. For vectors, use the command **length**.

A.7.1 UTILITY MATRICES

There are many special utility matrices which are useful for matrix operations. A few examples are

eye(m, n)	Generates an m-by-n identity matrix.
zeros(m, n)	Generates an m-by-n matrix of zeros.
ones(m, n)	Generates an m-by-n matrix of ones.
diag(x)	Produces a diagonal matrix with the elements of **x** on the diagonal line.

For a complete list and help on elementary matrices and matrix manipulation, type **help elmat**. There are many other special built-in matrices. For a complete list and help on specialized matrices, type **help specmat**.

A.7.2 EIGENVALUES

If A is an n-by-n matrix, the n numbers λ that satisfy $Ax = \lambda x$ are the eigenvalues of A. They are found using **eig(A)**, which returns the eigenvalues in a column vector. Eigenvalues and eigenvectors can be obtained with a double assignment statement $[X, D] = \text{eig}(A)$. The diagonal elements of D are the eigenvalues and the columns of X are the corresponding eigenvectors such that $AX = XD$.

Example A.4

Find the eigenvalues and the eigenvectors of the matrix A given by

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

```
A = [ 0  1  -1;  -6  -11  6;  -6  -11  5];
[X,D] = eig(A)
```

The eigenvalues and the eigenvectors are obtained as follows

```
X =                                     D =
      -0.7071   0.2182  -0.0921              -1    0    0
       0.0000   0.4364  -0.5523               0   -2    0
      -0.7071   0.8729  -0.8285               0    0   -3
```

A.8 COMPLEX NUMBERS

All the *MATLAB* arithmetic operators are available for complex operations. The imaginary unit $\sqrt{-1}$ is predefined by two variables i and j. In a program, if other values are assigned to i and j, they must be redefined as imaginary units, or other characters can be defined for the imaginary unit.

```
j = sqrt(-1)     or i = sqrt(-1)
```

Once the complex unit has been defined, complex numbers can be generated.

Example A.5

Evaluate the following function $V = Zc \cosh g + \sinh g/Zc$, where $Zc = 200 + j300$ and $g = 0.02 + j1.5$

```
i  = sqrt(-1); Zc = 200 +  300*i; g  = 0.02 + 1.5*i;
v  = Zc *cosh(g) + sinh(g)/Zc
```

results in the output

```
v =
      8.1672 + 25.2172i
```

It is important to note that, when complex numbers are entered as matrix elements within brackets, we avoid any blank spaces. If spaces are provided around the complex number sign, it represents two separate numbers.

Example A.6

In the circuit shown in Figure A.1, determine the node voltages V_1 and V_2 and the power delivered by each source.

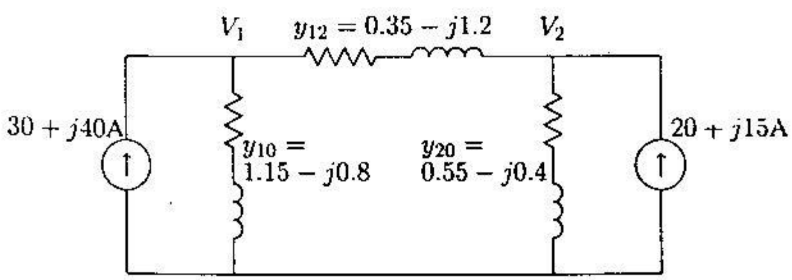


FIGURE A.1
Circuit for Example A.6.

Kirchhoff's current law results in the following matrix node equation.

$$\begin{bmatrix} 1.5 - j2.0 & -.35 + j1.2 \\ -.35 + j1.2 & 0.9 - j1.6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 30 + j40 \\ 20 + j15 \end{bmatrix}$$

and the complex power of each source is given by $S = VI^*$. The following program is written to yield solutions to V_1, V_2 and S using *MATLAB*.


```
j=sqrt(-1)                                          % Defining j
I=[30+j*40; 20+j*15]          % Column of node current phasors
Y=[1.5-j*2   -.35+j*1.2; -.35+j*1.2  .9-j*1.6]
                                    % Complex admittance matrix Y
disp('The solution is') V=inv(Y)*I  % Node voltage solution
S=V.*conj(I)                            % complex power at nodes
```

result in

```
The solution is
      V =

          3.5902 + 35.0928i
          6.0155 + 36.2212i

      S =

          1511.4 + 909.2i
           663.6 + 634.2i
```

The prime (′) transposes a real matrix; but for complex matrices, the symbol (.′) must be used to find the transpose.

A.9 POLYNOMIAL ROOTS AND CHARACTERISTIC POLYNOMIAL

If p is a row vector containing the coefficients of a polynomial, **roots(p)** returns a column vector whose elements are the roots of the polynomial. If r is a column vector containing the roots of a polynomial, **poly(r)** returns a row vector whose elements are the coefficients of the polynomial.

Example A.7

Find the roots of the following polynomial.

$$s^6 + 9s^5 + 31.25s^4 + 61.25s^3 + 67.75s^2 + 14.75s + 15$$

The polynomial coefficients are entered in a row vector in descending powers. The roots are found using **roots**.

```
p = [ 1  9  31.25  61.25  67.75  14.75  15 ]
r = roots(p)
```

The polynomial roots are obtained in column vector

```
r =
     -4.0000
     -3.0000
     -1.0000 + 2.0000i
     -1.0000 - 2.0000i
      0.0000 + 0.5000i
      0.0000 - 0.5000i
```

Example A.8

The roots of a polynomial are -1, -2, $-3 \pm j4$. Determine the polynomial equation.

Complex numbers may be entered using function i or j. The roots are then entered in a column vector. The polynomial equation is obtained using **poly** as follows

```
i = sqrt(-1)
r = [-1  -2  -3+4*i  -3-4*i ]
p = poly(r)
```

The coefficients of the polynomial equation are obtained in a row vector.

```
p =
     1   9   45   87   50
```

Therefore, the polynomial equation is

$$s^4 + 9s^3 + 45s^2 + 87s + 50 = 0$$

Example A.9

Determine the roots of the characteristic equation of the following matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The characteristic equation of the matrix is found by **poly**, and the roots of this equation are found by **roots**.

```
A = [ 0  1  -1;  -6  -11  6;  -6  -11  5];
p = poly(A)
r = roots(p)
```

The result is as follows


```
p =
      1.0000   6.0000   11.0000   6.0000
r =  -3.0000
     -2.0000
     -1.0000
```

The roots of the characteristic equation are the same as the eigenvalues of matrix A. Thus, in place of the **poly** and **roots** function, we may use

```
r = eig(A)
```

A.9.1 PRODUCT AND DIVISION OF POLYNOMIALS

The product of polynomials is the convolution of the coefficients. The division of polynomials is obtained by using the deconvolution command.

Example A.10

(a) Given $A = s^2 + 7s + 12$, and $B = s^2 + 9$, find $C = AB$.
(b) Given $Z = s^4 + 9s^3 + 37s^2 + 81s + 52$, and $Y = s^2 + 4s + 13$, find $X = \frac{Z}{Y}$.

The commands

```
A = [1   7   12]; B = [1   0   9];
C = conv(A, B)
Z = [1   9   37 81   52]; Y = [1   4   13];
[X, r] = deconv(Z, Y)
```

result in

```
C =
      1   7   21   63   108
X =
      1   5   4
r =
      0   0   0
```

A.9.2 POLYNOMIAL CURVE FITTING

In general, a polynomial fit to data in vector x and y is a function p of the form

$$p(x) = c_1 x^d + c_2 x^{d-1} + \cdots + c_n$$

The degree is d, and the number of coefficients is $n = d + 1$. Given a set of points in vectors x and y, **polyfit(x, y, d)** returns the coefficients of dth order polynomial in descending powers of x.

Example A.11

Find a polynomial of degree 3 to fit the following data

x	0	1	2	4	6	10
y	1	7	23	109	307	1231

```
x = [ 0  1  2  4  6  10];
y = [ 1  7  23  109  307  1231];
c = polyfit(x,y,3)
```

The coefficients of a third degree polynomial are found as follows

```
c =
      1.0000    2.0000    3.0000    1.0000
```

i.e., $y = x^3 + 2x^2 + 3x + 1$.

A.9.3 POLYNOMIAL EVALUATION

If c is a vector whose elements are the coefficients of a polynomial in descending powers, the **polyval(c, x)** is the value of the polynomial evaluated at **x**. For example, to evaluate the above polynomial at points 0, 1, 2, 3, and 4, use the commands

```
c = [1   2  3  1];
x = 0:1:4;
y = polyval(c, x)
```

which result in

```
y =
      7   23   55   109
```

A.9.4 PARTIAL-FRACTION EXPANSION

[r, p, k] = residue[b, a] finds the residues, poles, and direct terms of a partial fraction expansion of the ratio of two polynomials

$$\frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Vectors **b** and **a** specify the coefficients of the polynomials in descending powers of s. The residues are returned in column vector **r**, the pole locations in column vector **p**, and the direct terms in row vector **k**.

Example A.12

Determine the partial fraction expansion for

$$F(s) = \frac{2s^3 + 9s + 1}{s^3 + s^2 + 4s + 4}$$

```
      b = [ 2  0  9  1];
      a = [ 1  1  4  4];
[r,p,k] = residue(b,a)
```

The result is as follows

```
r =
      0.0000   -0.2500i
      0.0000   +0.2500i
     -2.0000
p =
      0.0000   +2.0000i
      0.0000   -2.0000i
     -1.0000
K =
      2.0000
```

Therefore the partial fraction expansion is

$$2 + \frac{-2}{s+1} + \frac{j0.25}{s+j2} + \frac{-j0.25}{s-j2} = 2 + \frac{-2}{s+1} + \frac{1}{s^2+4}$$

[b, a] = residue(r, p, K) converts the partial fraction expansion back to the polynomial $P(s)/Q(s)$.

For a complete list and help on matrix analysis, linear equations, eigenvalues, and matrix functions, type **help matfun**.

A.10 GRAPHICS

MATLAB can create high-resolution, publication-quality *2-D*, *3-D*, *linear*, *semilog*, *log*, *polar*, *bar chart* and *contour plots* on plotters, dot-matrix printers, and laser printers. Some of the *2-D* graph types are **plot**, **loglog**, **semilogx**, **semi -logy**, **polar**, and **bar**. The syntax for the above plots includes the following optional symbols and colors.

COLOR SPECIFICATION		*LINE STYLE-OPTION*	
Long name	*Short name*	*Style*	*Symbol*
black	k	solid	–
blue	b	dashed	- -
cyan	c	dotted	:
green	g	dash-dot	-.
magenta	m	point	.
red	r	circle	o
white	w	x-mark	x
yellow	y	plus	+
		star	*

You have three options for plotting multiple curves on the same graph. For example,

```
plot(x1, y1,'r', x2, y2, '+b', x3, y3, '--')
```

plots `(x1, y1)` with a solid red line, `(x2, y2)` with a blue + mark, and `(x3, y3)` with a dashed line. If **X** and **Y** are matrices of the same size, **plot(X, Y) will** plot the columns of **Y** versus the column of **X**.

Alternatively, the **hold** command can be used to place new plots on the previous graph. **hold on** holds the current plot and all axes properties; subsequent plot commands are added to the existing graph. **hold off** returns to the default mode whereby a new plot command replaces the previous plot. **hold**, by itself, toggles the hold state.

Another way for plotting multiple curves on the same graph is the use of the **line** command. For example, if a graph is generated by the command `plot(x1, y1)`, then the commands

```
line(x2, y2, '+b')
line(x3, y3, '--')
```

Add curve `(x2, y2)` with a blue + mark, and `(x2, y2)` with a dashed line to the existing graph generated by the previous plot command. Multiple figure windows can be created by the **figure** command. **figure**, by itself, opens a new figure window, and returns the next available figure number, known as the figure handle. **figure(h)** makes the figure with handle h the current figure for subsequent plotting commands. Plots may be annotated with title, $x - y$ labels and grid. The command **grid** adds a grid to the graph. The commands **title('*Graph title* ')** titles the plot, and **xlabel('*x-axis label* ')**, **ylabel('*y-axis label* ')** label the plot with the specified string argument. The command **text(*x-coordinate, y-coordinate*, '*text*')** can be used for placing text on the graph, where the coordinate values are taken from the current plot. For example, the statement


```
text(3.5, 1.5, 'Voltage')
```

will write Voltage at point (3.5, 1.5) in the current plot. Alternatively, you can use the **gtext**('*text*') command for interactive labeling. Using this command after a plot provides a crosshair in the Figure window and lets the user specify the location of the text by clicking the mouse at the desired location. Finally, the command **legend**(*string1, string2, string3*, ...) may be used to place a legend on the current plot using the specified strings as labels. This command has many optional arguments. For example, **legend(** *linetype1, string1, linetype2, string2, linetype3, string3*, ...) specifies the line types/color for each label at a suitable location. However, you can move the legend to a desired location with the mouse. **legend off** removes the legend from the current axes.

MATLAB provides automatic scaling. The command **axis([x min. x max. y min. y max.])** enforces the manual scaling. For example

```
axis([-10  40  -60  60])
```

produces an x-axis scale from -10 to 40 and a y-axis scale from -60 to 60. Typing **axis** again or **axis('auto')** resumes auto scaling. Also, the aspect ratio of the plot can be made equal to one with the command **axis('square')**. With a square aspect ratio, a line with slope 1 is at a true 45 degree angle. **axis('equal')** will make the x- and y-axis scaling factor and tic mark increments the same. For a complete list and help on general purpose graphic functions, and two- and three-dimensional graphics, see **help graphics, help plotxy**, and **help plotxyz**.

There are many other specialized commands for two-dimensional plotting. Among the most useful are the **semilogx** and **semilogy**, which produce a plot with an x-axis log scale and a y-axis log scale. An interesting graphic command is the **comet** plot. The command **comet(x, y)** plots the data in vectors **x** and **y** with a comet moving through the data points, and you can see the curve as it is being plotted. For a complete list and help on general purpose graphic functions and two-dimensional graphics, see **help graphics** and **help plotxy**.

A.11 GRAPHICS HARD COPY

The easiest way to obtain hard-copy printout is to make use of the Windows built-in facilities. In the Figure window, you can pull down the file menu and click on the **Print** command to send the current graph directly to the printer. You can also import a graph to your favorite word processor. To do this, select **Copy options** from the **Edit** pull-down menu, and check mark the **Invert background** option in the dialog box to invert the background. Then, use **Copy** command to copy the graph into the clipboard. Launch your word processor and use the **Paste** command to import the graph.

Some word processors may not provide the extensive support of the Windows graphics and the captured graph may be corrupted in color. To eliminate this problem use the command

```
system_dependent(14, 'on')
```

which sets the metafile rendering to the lowest common denominator. To set the metafile rendering to normal, use

```
system_dependent(14, 'off')
```

In addition *MATLAB* provides a function called **print** that can be used to produce high resolution graphic files. For example,

```
print -dhpgl [filename]
```

saves the graph under the specified *filename* with extension *hgl*. This file may be processed with an HPGL- compatible plotter. Similarly, the command

```
print -dilll [filename]
```

produces a graphic file compatible with the Adobe Illustrator'88. Another **print** option allows you to save and reload a figure. The command

```
print -dmfile [ filename ]
```

produces a MAT file and M-file to reproduce the figure again.

Example A.13

Create a linear X-Y plot for the following variables.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	10	10	16	24	30	38	52	68	82	96	123

For a small amount of data, you can type in data explicitly using brackets.

```
x = [ 0  0.5  1.0  1.5  2.0  2.5  3.0  3.5  4.0  4.5  5.0];
y = [10   10   16   24   30   38   52   68   82   96  123];
plot(x, y), grid
xlabel('x'), ylabel('y'), title('A simple plot example')
```


plot(x, y) produces a linear plot of y versus x on the screen, as shown in Figure A.2.

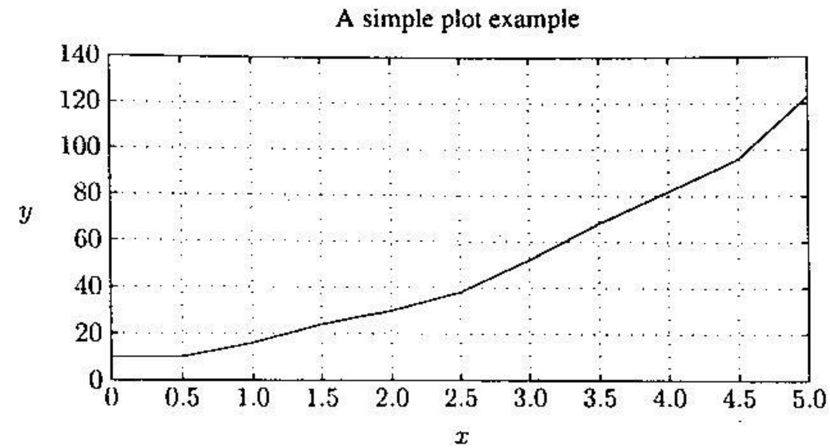


FIGURE A.2
Example of X-Y plot.

For large amounts of data, use the text editor to create a file with extension **m**. Typing the *filename* creates your data in the workspace.

Example A.14

Fit a polynomial of order 2 to the data in Example A.13. Plot the given data point with symbol x, and the fitted curve with a solid line. Place a boxed legend on the graph.

The command **p = polyfit(x, y, 2)** is used to find the coefficients of a polynomial of degree 2 that fits the data, and the command **yc = polyval(p, x)** is used to evaluate the polynomial at all values in **x**. We use the following command.

```
x = [ 0  0.5  1.0  1.5  2.0  2.5  3.0  3.5  4.0  4.5  5.0];
y = [10   10   16   24   30   38   52   68   82   96  123];
p = polyfit(x, y, 2)  % finds the coefficients of a polynomial
                                     % of degree 2 that fits the data
yc = polyval(p, x);%polynomial is evaluated at all points in x
plot(x, y,'x', x, yc)%plots data with  x and fitted polynomial
xlabel('x'), ylabel('y'), grid
title('Polynomial curve fitting')
legend('Actual data', 'Fitted polynomial')
```

The result is the array of coefficients of the polynomial of degree 2, and is

```
p =
    4.0232    2.0107    9.6783
```

Thus, the parabola $4.0x^2 + 2.0x + 9.68$ is found that fits the given data in the least-square sense. The plots are shown in Figure A.3.

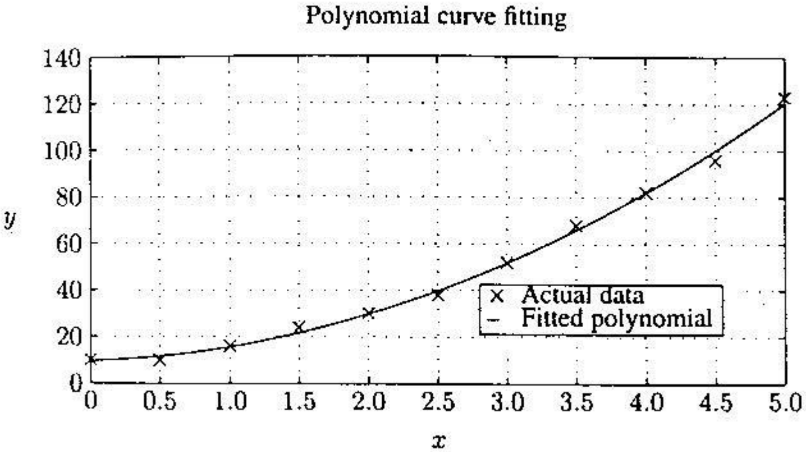


FIGURE A.3
Fitting a parabola to the data in Example A.13.

Example A.15

Plot function $y = 1 + e^{-2t}\sin(8t - \pi/2)$ from 0 to 3 seconds. Find the time corresponding to the peak value of the function and the peak value. The graph is to be labeled, titled, and have grid lines displayed.

Remember to use .* for the element-by-element multiplication of the two terms in the given equation. The command **[cp, k] = max(c)** returns the peak value and the index k corresponding to the peak time. We use the following commands.

```
t=0:.005:3; c = 1+ exp(-2*t).*sin(8*t - pi/2);
[cp, k] = max(c)  % cp is the maximum value of c at interval k
tp = t(k)                                    % tp is the peak time
plot(t, c), xlabel(' t - sec'), ylabel('c'), grid
title('Damped sine curve')
text(0.55,1.35,['cp =',num2str(cp)])%Text in quote & the value
    % of cp are printed on the graph at the specified location
text(0.55, 1.2, ['tp = ',num2str(tp)])
```

The result is


```
cp =
    1.4702
k =
    73
tp =
    0.3600
```

and the plot is shown in Figure A.4.

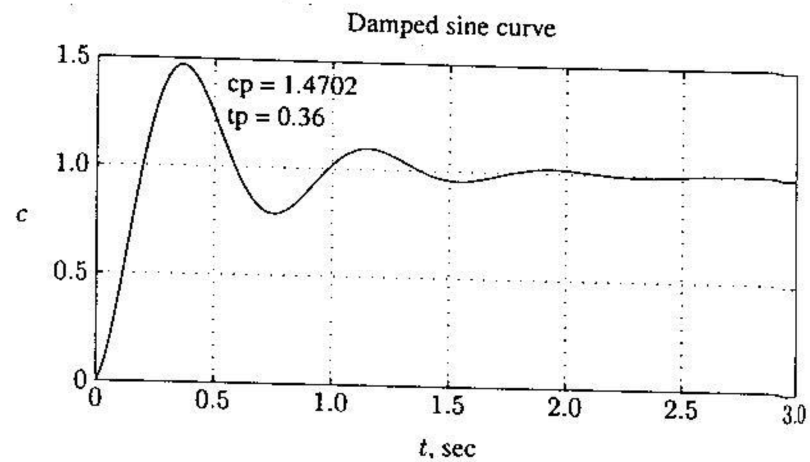


FIGURE A.4
Graph of Example A.15.

An interactive way to find the data points on the curve is by using the **ginput** command. Entering **[x, y] = ginput** will put a crosshair on the graph. Position the crosshair at the desired location on the curve, and click the mouse. You can repeat this procedure for extracting coordinates for as many points as required. When the return key is pressed, the input is terminated and the extracted data is printed on the command menu. For example, to find the peak value and the peak time for the function in Example A.15, try

```
[tp, cp] = ginput
```

A crosshair will appear. Move the crosshair to the peak position, and click the mouse. Press the return key to get

```
cp =
    1.47
tp =
    0.36
```

subplot splits the Figure window into multiple portions, in order to show several plots at the same time. The statement **subplot(m, n, p)** breaks the Figure window into an m-by-n box and uses the pth box for the subsequent plot. Thus, the command **subplot(2, 2, 3), plot(x,y)** divides the Figure window into four subwindows and plots **y** versus **x** in the third subwindow, which is the first subwindow in the second row. The command **subplot(111)** returns to the default Figure window. This is demonstrated in the next example.

Example A.16

Divide the Figure window into four partitions, and plot the following functions for ωt from 0 to 3π in steps of 0.05.

1. Plot $v = 120 \sin \omega t$ and $i = 100 \sin(\omega t - \pi/4)$ versus ωt on the upper left portion.

2. Plot $p = vi$ on the upper right portion.

3. Given $F_m = 3.0$, plot $f_a = F_m \sin \omega t$, $F_b = F_m \sin(\omega t - 2\pi/3)$, and $F_c = F_m \sin(\omega t - 4\pi/3)$ versus ωt on the lower left portion.

4. For $f_R = 3F_m$, construct a circle of radius f_R on the lower right portion.

```
wt = 0: 0.05: 3*pi; v=120*sin(wt);            %Sinusoidal voltage
i = 100*sin(wt - pi/4);                       %Sinusoidal current
p = v.*i;                                     %Instantaneous power
subplot(2, 2, 1), plot(wt, v, wt, i); %Plot of v & i versus wt
title('Voltage & current'), xlabel('wt, radians');
subplot(2, 2, 2), plot(wt, p);    % Instantaneous power vs. wt
title('Power'), xlabel(' wt,  radians ')
Fm=3.0;
fa = Fm*sin(wt);                     % Three-phase mmf's fa, fb, fc
fb = Fm*sin(wt - 2*pi/3); fc = Fm*sin(wt - 4*pi/3);
subplot(2, 2, 3), plot(wt, fa, wt, fb, wt, fc)
title('3-phase mmf'), xlabel(' wt, radians ')
fR = 3/2*Fm;
subplot(2, 2, 4), plot(-fR*cos(wt), fR*sin(wt))
title('Rotating mmf'), subplot(111)
```

Example A.16 results are shown in Figure A.5.

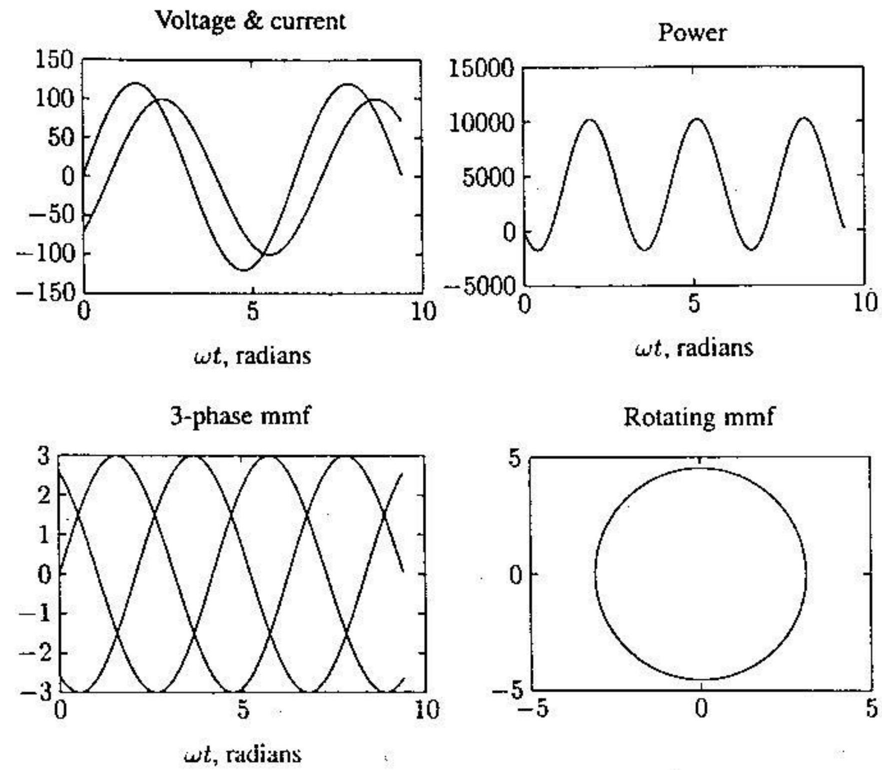


FIGURE A.5
Subplot demonstration.

A.12 THREE-DIMENSIONAL PLOTS

MATLAB provides extensive facilities for visualization of three-dimensional data. The most common are plots of curves in a three-dimensional space, mesh plots, surface plots, and contour plots. The command **plot3(x, y, z , 'style option')** produces a curve in the three-dimensional space. The viewing angle may be specified by the command view(*azimuth, elevation*). The arguments *azimuth*, and *elevation* specifies the horizontal and vertical rotation in degrees, respectively. The **title, xlabel, ylabel**, etc., may be used for three-dimensional plots. The **mesh** and **surf** commands have several optional arguments and are used for plotting meshes and surfaces. The **contour**(z) command creates a contour plot of matrix **z**, treating the values in **z** as heights above the plane. The statement **mesh(z)** creates a three-dimensional plot of the elements in matrix **z**. A mesh surface is defined by the **z** coordinates of points above a rectangular grid in the x-y plane. The plot is formed by joining adjacent points with straight lines. **meshgrid** transforms the domain specified by vector **x** and **y** into arrays **X** and **Y**. For a complete list and help on general purpose Graphic functions and three-dimensional graphics, see **help graphics** and **help plotxyz**. Also type **demo** to open the *MATLAB Expo Menu Map* and visit *MATLAB*. Select and observe the demos in the Visualization section.

Example A.17

Obtain the cartesian plot of the Bessel function $J_0\sqrt{x^2+y^2}$ over the range $-12 < x < 12$, $-12 < y < 12$.

Use the following commands

```
% Cartesian plot of Bessel function J0(sqrt(x^2+y^2))
[x, y] = meshgrid(-12:0.6:12, -12:.6:12);
                        % meshgrid transforms the specified domain
                             % into array x and y for evaluating z
r = sqrt(x.^2 + y.^2); z = bessel(0,r);
m = [-45 60];                                   % viewing angle
mesh(z, m)                                      % 3-D mesh plot
```

Enter **exa17** at the *MATLAB* prompt to see the result.

A.13 HANDLE GRAPHICS

It is often desirable to be able to customize the graphical output. *MATLAB* allows object-oriented programming, enabling the user to have complete control over the details of a graph. *MATLAB* provides many low-level commands known as *Handle Graphics*. These commands makes it possible to access individual objects and their properties and change any property of an object without affecting other properties or objects. Handle Graphics provides a graphical user interface (GUI) in which the user interface includes push buttons and menus. These topics are not discussed here; like *MATLAB* syntax, they are easy to follow, and we leave these topics for the interested reader to explore.

A.14 LOOPS AND LOGICAL STATEMENTS

MATLAB provides loops and logical statements for programming, like **for**, **while**, and **if** statements. The **for** statement instructs the computer to perform all subsequent expressions up to the **end** statement for a specified number of counted times. The expression may be a matrix. The following is an example of a nested loop.


```
for i = 1:n,  for j = 1:n
   expression
end, end
```

The **while** statement allows statements to be repeated an indefinite number of times under the control of a logic statement. The **if**, **else**, and **elseif** statements allow conditional execution of statements. *MATLAB* has six relational operators and four logical operators, which are defined in the following table.

Relational Operator		*Logical Operator*	
==	equal	&	logical AND
~=	not equal	|	logical OR
<	less than	~	logical complement
<=	less than or equal to	xor	exclusive OR
>	greater than		
>=	greater than or equal to		

A.15 SOLUTION OF DIFFERENTIAL EQUATIONS

Analytical solutions of linear time-invariant equations are obtained through the Laplace transform and its inversion. There are other techniques which use the state transition matrix $\phi(t)$ to provide a solution. These analytical methods are normally restricted to linear differential equations with constant coefficients. Numerical techniques solve differential equations directly in the time domain; they apply not only to linear time-invariant, but also to nonlinear and time varying differential equations. The value of the function obtained at any step is an approximation of the value which would have been obtained analytically; whereas, the analytical solution is exact. However, an analytical solution may be difficult, time consuming, or even impossible to find.

MATLAB provides two functions for numerical solutions of differential equations employing the Runge-Kutta method. These are **ode23** and **ode45**, based on the Fehlberg- second and third-order pair of formulas for medium accuracy and fourth- and fifth-order pair for high accuracy. The nth-order differential equation must be transformed into n first-order differential equations and must be placed in an M-file that returns the state derivatives of the equations. The following examples demonstrate the use of these functions.

Example A.18

Consider the simple mechanical system of Figure A.6. Three forces influence the motion of the mass, namely, the applied force, the frictional force, and the spring force.

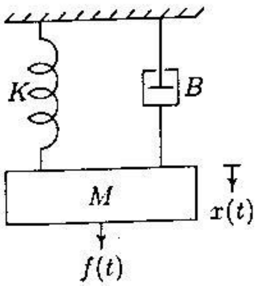


FIGURE A.6
Mechanical translational system.

Applying Newton's law of motion, the force equation of the system is

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

Let

$$x_1 = x$$

and

$$x_2 = \frac{dx}{dt}$$

then

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{1}{M}[f(t) - Bx_2 - Kx_1]$$

With the system initially at rest, a force of 25 newtons is applied at time $t = 0$. Assume that the mass $M = 1$ kg, frictional coefficient $B = 5$ N/m/sec, and the spring constant $K = 25$ N/m. The above equations are defined in an M-file **mechsys.m** as follows.

```
function xdot = mechsys(t, x); % returns the state derivatives
F = 25;                                           % Step input
M =1; B = 5; K = 25; xdot = [x(2); 1/M*(F - B*x(2)-K*x(1) ) ];
```

The following M-file, **exa18.m**, uses **ode23** to simulate the system over an interval of 0 to 3 sec., with zero initial conditions.

```
tspan = [0, 3];                              % time interval
x0 = [0, 0];                           % initial conditions
```

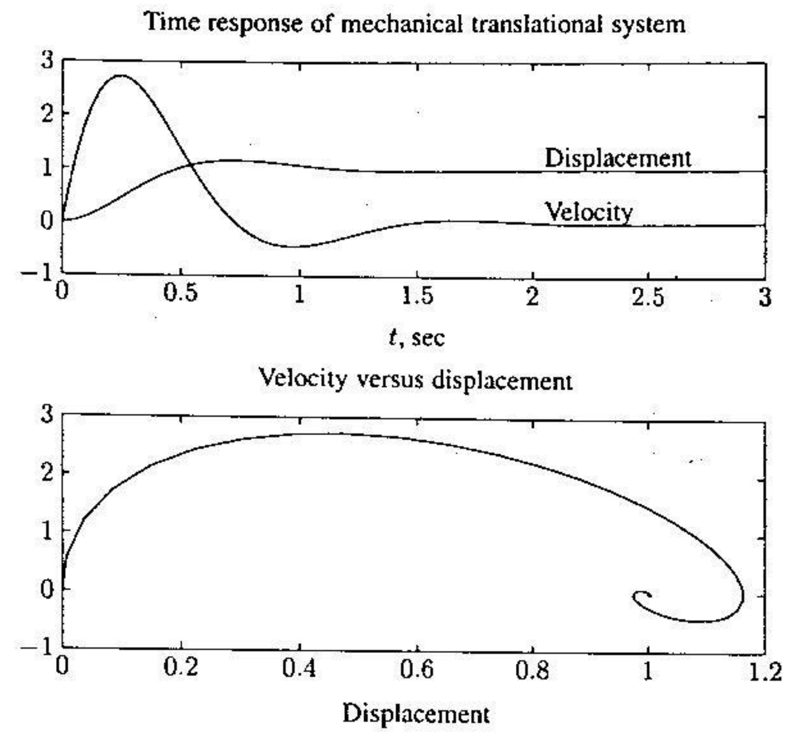



FIGURE A.7
Response of the mechanical system of Example A.18.

```
[t,x] = ode23('mechsys', tspan, x0);
subplot(2, 1, 1), plot(t, x), xlabel('t,  sec')
title('Time response of mechanical translational system')
text(2, 1.2, 'Displacement'), text(2, 0.2, 'Velocity')
d = x(:, 1);    v = x(:, 2);
subplot(2, 1, 2), plot(d, v)
title('Velocity versus displacement ')
xlabel('Displacement'), ylabel('Velocity'), subplot(111)
```

Results of the simulation are shown in Figure A.7.

Example A.19

The circuit elements in Figure A.8 are $R = 1.4\ \Omega$, $L = 2$ H, and $C = 0.32$ F. The initial inductor current is zero, and the initial capacitor voltage is 0.5 volts. A step voltage of 1 volt is applied at time $t = 0$. Determine $i(t)$ and $v(t)$ over the range $0 < t < 15$ sec. Also, obtain a plot of current versus capacitor voltage.

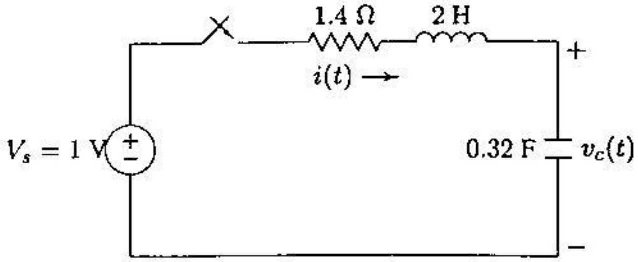


FIGURE A.8
RLC circuit for Example A.19.

Applying KVL

$$Ri + L\frac{di}{dt} + v_c = V_s$$

and

$$i = C\frac{dv_c}{dt}$$

Let

$$x_1 = v_c$$

and

$$x_2 = i$$

Then

$$\dot{x}_1 = \frac{1}{C}x_2$$

and

$$\dot{x}_2 = \frac{1}{L}(V_s - x_1 - Rx_2)$$

The above equations are defined in an M-file **electsys.m** as follows.

```
function xdot = electsys(t, x);
                              % returns the state derivatives
V = 1;                                          % Step input
R =1.4; L = 2; C = 0.32;
xdot = [x(2)/C; 1/L*( V - x(1) - R*x(2) )];
```

The following M-file, **exa19.m**, uses **ode23** to simulate the system over an interval of 0 to 15 sec.

```
tspan = [0, 15];                           % time interval
x0 = [0.5, 0];                        % initial conditions
```



```
[t,x] = ode23('electsys', tspan, x0);
subplot(2, 1, 1), plot(t, x)
title('Time response of an RLC series circuit')
xlabel('t,  sec')
text(8,1.05,'Capacitor voltage'), text(8, .05,'Current')
vc= x(:, 1);    i = x(:, 2);
subplot(2, 1, 2), plot(vc, i)
title('Current versus capacitor voltage ')
xlabel('Capacitor voltage'), ylabel('Current')
subplot(111)
```

Results of the simulation are shown in Figure A.9.

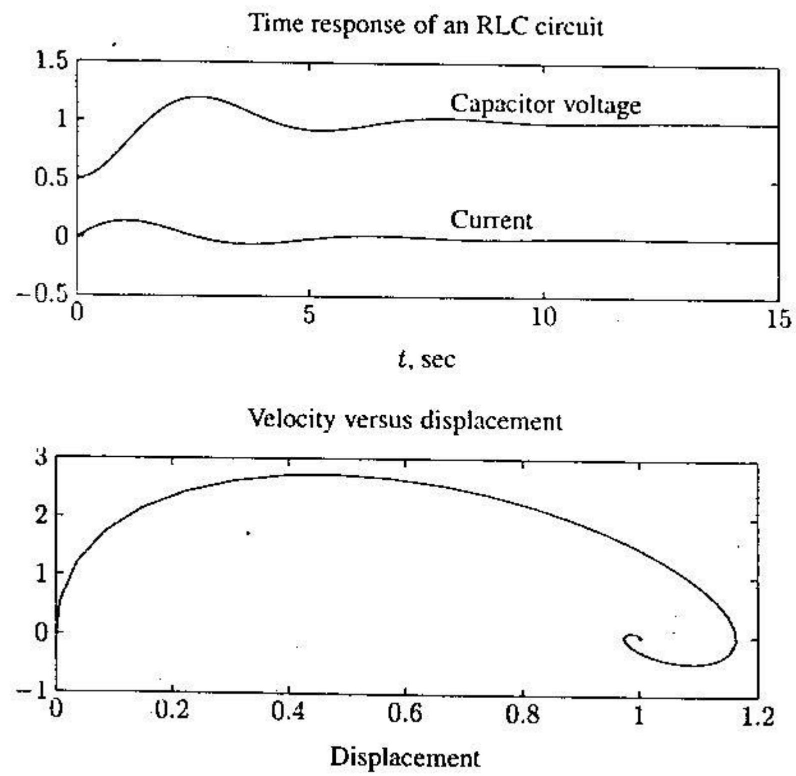


FIGURE A.9
Response of the series RLC circuit of Example A.19.

A.16 NONLINEAR SYSTEMS

A great majority of physical systems are linear within some range of the variables. However, all systems ultimately become nonlinear as the ranges are increased without limit. For the nonlinear systems, the principle of superposition does not apply. **ode23** and **ode45** simplify the task of solving a set of nonlinear differential equations, as demonstrated in Example A.20.

Example A.20

Consider the simple pendulum illustrated in Figure A.10, where a weight of $W = mg$ kg is hung from a support by a weightless rod of length L meters. While usually approximated by a linear differential equation, the system really is nonlinear and includes viscous damping with a damping coefficient of B kg/m/sec.

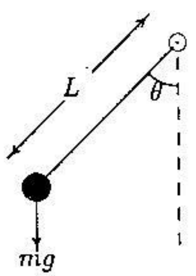


FIGURE A.10
Pendulum oscillator.

If θ in radians is the angle of deflection of the rod, the velocity of the weight at the end will be $L\dot{\theta}$ and the tangential force acting to increase the angle θ can be written as

$$F_T = -W \sin\theta - BL\dot{\theta}$$

From Newton's law

$$F_T = mL\ddot{\theta}$$

Combining the two equations for the force, we get

$$mL\ddot{\theta} + BL\dot{\theta} + W \sin\theta = 0$$

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$ (angular velocity), then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{B}{m}x_2 - \frac{W}{mL}\sin x_1$$

The above equations are defined in an M-file **pendulum.m** as follows.

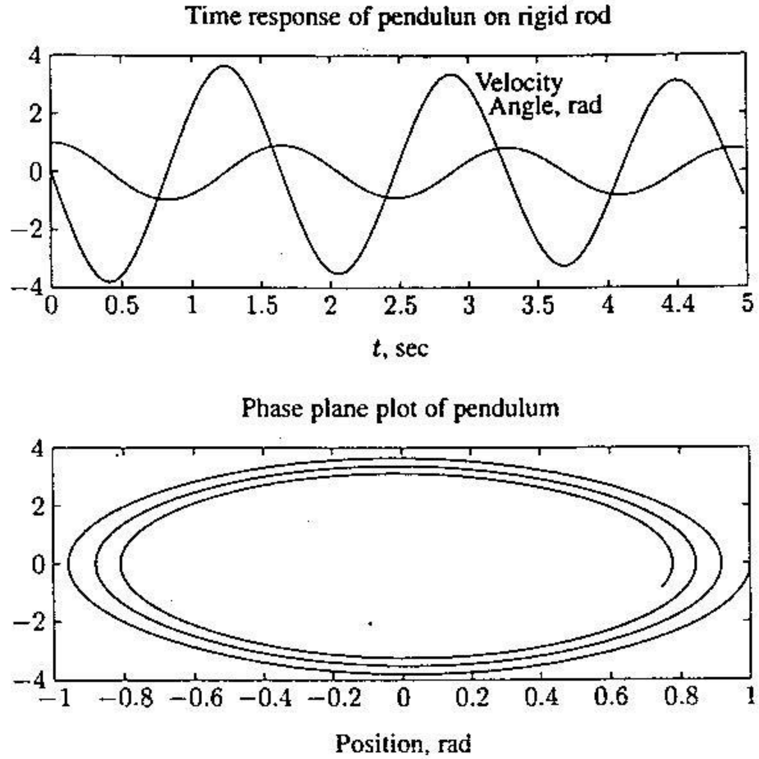


FIGURE A.11
Response of the pendulum described in Example A.20.

```
function xdot = pendulum(t,x);%returns the state derivatives
W = 2; L = .6; B = 0.02; g = 9.81;  m = W/g;
xdot = [x(2) ; -B/m*x(2)-W/(m*L)*sin(x(1)) ];
```

The following M-file, **exa20.m**, uses **ode23** to simulate the system over an interval of 0 to 5 sec. Results of the simulation are shown in Figure A.11.

```
tspan = [0, 5];                                    % time interval
x0 = [1, 0];                                   % initial conditions
[t,x] = ode23('pendulum', tspan, x0);
subplot(2, 1, 1), plot(t, x)
title('Time response of a rigid pendulum')
xlabel('t, sec')
text(3.2, 3.5, 'Velocity') , text(3.2, 1.2, 'Angle-rad.')
th= x(:, 1);   w = x(:, 2);
subplot(2, 1, 2), plot(th, w)
```

```
title('Phase plane plot of pendulum')
xlabel('Position, rad'), ylabel('Angular velocity')
subplot(111)
```

A.17 SIMULATION DIAGRAM

The differential equations of a lumped linear network can be written in the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned} \tag{A.1}$$

This system of first-order differential equations is known as the state equation of the system, and **x** is the state vector. One advantage of the state-space method is that the form lends itself easily to the digital and/or analog computer methods of solution. Further, the state-space method can be easily extended to analysis of nonlinear systems. State equations may be obtained from an nth-order differential equation or directly from the system model by identifying appropriate state variables.

To illustrate how we select a set of state variables, consider an nth-order linear plant model described by the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = u(t) \tag{A.2}$$

where $y(t)$ is the plant output and $u(t)$ is its input. A state model for this system is not unique, but depends on the choice of a set of state variables. A useful set of state variables, referred to as *phase variables*, is defined as

$$x_1 = y,\ x_2 = \dot{y},\ x_3 = \ddot{y},\ \ldots,\ x_n = y^{n-1}$$

We express $\dot{x}_k = x_{k+1}$ for $k = 1, 2, \ldots, n-1$, and then solve for $d^n y/dt^n$, and replace y and its derivatives by the corresponding state variables to give

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \ldots - a_{n-1}\,x_n + u(t)\end{aligned} \tag{A.3}$$

or in matrix form

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ 1\end{bmatrix}u(t) \tag{A.4}$$

and the output equation is

$$y = [\,1 \quad 0 \quad 0 \quad \ldots \quad 0\,]\,\mathbf{x} \tag{A.5}$$

The M-file **ode2phv.m** is developed which converts an nth-order ordinary differential equation to the state-space phase variable form. **[A, B, C] = ode2phv(ai, k)** returns the matrices **A, B, C**, where **ai** is a row vector containing coefficients of the equation in descending order, and **k** is the coefficient of the right-hand side.

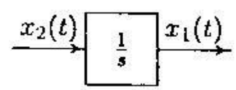
Equation (A.3) indicates that state variables are determined by integrating the corresponding state equation. A diagram known as the simulation diagram can be constructed to model the given differential equations. The basic element of the simulation diagram is the integrator. The first equation in (A.3) is

$$\dot{x}_1 = x_2$$

Integrating, we have

$$x_1 = \int x_2\, dx$$

The above integral is shown by the following time-domain symbol. The integrating block is identified by symbol $\frac{1}{s}$. Adding an integrator for the remaining state variables and completing the last equation in (A.3) via a summing point and feedback paths, a simulation diagram is obtained.



A.18 INTRODUCTION TO SIMULINK

SIMULINK is an interactive environment for modeling, analyzing, and simulating a wide variety of dynamic systems. *SIMULINK* provides a graphical user interface for constructing block diagram models using "drag-and-drop" operations. A system is configured in terms of block diagram representation from a library of standard components. *SIMULINK* is very easy to learn. A system in block diagram representation is built easily and the simulation results are displayed quickly.

Simulation algorithms and parameters can be changed in the middle of a simulation with intuitive results, thus providing the user with a ready access learnulation with intuitive results, thus providing the user with a ready access learning tool for simulating many of the operational problems found in the real world. *SIMULINK* is particularly useful for studying the effects of nonlinearities on the behavior of the system, and as such, it is also an ideal research tool. The key features of *SIMULINK* are

- Interactive simulations with live display.
- A comprehensive block library for creating linear, nonlinear, discrete or hybrid multi-input/output systems.
- Seven integration methods for fixed-step, variable-step, and stiff systems.
- Unlimited hierarchical model structure.
- Scalar and vector connections.
- Mask facility for creating custom blocks and block libraries.

SIMULINK provides an open architecture that allows you to extend the simulation environment:

- You can easily perform "what if" analyses by changing model parameters – either interactively or in batch mode – while your simulations are running.
- Creating custom blocks and block libraries with your own icons and user interfaces from *MATLAB*, Fortran, or C code.
- You can generate C code from *SIMULINK* models for embedded applications and for rapid prototyping of control systems.
- You can create hierarchical models by grouping blocks into subsystems. There are no limits on the number of blocks or connections.
- *SIMULINK* provides immediate access to the mathematical, graphical, and programming capabilities of *MATLAB*, you can analyze data, automate procedures, and optimize parameters directly from *SIMULINK*.
- The advanced design and analysis capabilities of the toolboxes can be executed from within a simulation using the mask facility in *SIMULINK*.

- The *SIMULINK* block library can be extended with special-purpose blocksets. The DSP Blockset can be used for DSP algorithm development, while the Fixed-Point Blockset extends *SIMULINK* for modeling and simulating digital control systems and digital filters.

A.18.1 SIMULATION PARAMETERS AND SOLVER

You set the simulation parameters and select the solver by choosing **Parameters** from the Simulation menu. *SIMULINK* displays the **Simulation Parameters** dialog box, which uses three "pages" to manage simulation parameters. **Solver**, **Workspace I/O**, and **Diagnostics**.

SOLVER PAGE

The Solver page appears when you first choose **Parameters** from the **Simulation menu** or when you select the Solver tab. The Solver page allows you to:

- Set the start and stop times – You can change the start time and stop time for the simulation by entering new values in the Start time and Stop time fields. The default start time is 0.0 seconds and the default stop time is 10.0 seconds.
- Choose the solver and specify solver parameters – The default solver provide accurate and efficient results for most problems. Some solvers may be more efficient that others at solving a particular problem; you can choose between variable-step and fixed-step solvers. Variable-step solvers can modify their step sizes during the simulation. These are **ode45**, **ode23**, **ode113**, **ode15s**, **ode23s**, and **discrete**. The default is **ode45**. For variable-step solvers, you can set the maximum and suggested initial step size parameters. By default, these parameters are automatically determined, indicated by the value auto. For fixed-step solvers, you can choose **ode5**, **ode4**, **ode3**, **ode2**, **ode1**, and **discrete**.
- Output Options – The Output options area of the dialog box enables you to control how much output the simulation generates. You can choose from three popup options. These are: Refine output, Produce additional output, and Produce specified output only.

WORKSPACE I/O PAGE

The Workspace I/O page manages the input from and the output to the *MATLAB* workspace, and allows:

- Loading input from the workspace – Input can be specified either as *MATLAB* command or as a matrix for the Import blocks.
- Saving the output to the workspace –You can specify return variables by selecting the Time, State, and/or Output check boxes in the Save to workspace area.

DIAGNOSTICS PAGE

The Diagnostics page allows you to select the level of warning messages displayed during a simulation.

A.18.2 THE SIMULATION PARAMETERS DIALOG BOX

Table below summarizes the actions performed by the dialog box buttons, which appear on the bottom of each dialog box page.

Button	Action
Apply	Applies the current parameter values and keeps the dialog box open. During a simulation, the parameter values are applied immediately.
Revert	Changes the parameter values back to the values they had when the Dialog box was most recently opened and applies the parameters.
Help	Displays help text for the dialog box page.
Close	Applies the parameter values and closes the dialog box. During a simulation, the parameter values are applied immediately.

To stop a simulation, choose Stop from the Simulation menu. The keyboard shortcut for stopping a simulation is Ctrl-T. You can suspend a running simulation by choosing Pause from the Simulation menu. When you select Pause, the menu item changes to Continue. You proceed with a suspended simulation by choosing Continue.

A.18.3 BLOCK DIAGRAM CONSTRUCTION

At the *MATLAB* prompt, type *SIMULINK*. The *SIMULINK* BLOCK LIBRARY, containing seven icons, and five pull-down menu heads, appears. Each icon contains various components in the titled category. To see the content of each category, double click on its icon. The easy-to-use pull-down menus allow you to create a *SIMULINK* block diagram, or open an existing file, perform the simulation, and make any modifications. Basically, one has to specify the model of the system

(state space, discrete, transfer functions, nonlinear ode's, etc), the input (source) to the system, and where the output (sink) of the simulation of the system will go. Generally when building a model, design it first on the paper, then build it using the computer. When you start putting the blocks together into a model, add the blocks to the model window before adding the lines that connect them. This way, you can reduce how often you need to open block libraries. An introduction to *SIMULINK* is presented by constructing the *SIMULINK* diagram for the following examples.

MODELING EQUATIONS

Here are some examples that may improve your understanding of how to model equations.

Example A.21

Model the equation that converts Celsius temperature to Fahrenheit. Obtain a display of Fahrenheit-Celsius temperature graph over a range of 0 to 100°C.

$$T_F = \frac{9}{5}T_C + 32 \tag{A.6}$$

First, consider the blocks needed to build the model. These are:

- A ramp block to input the temperature signal, from the source library.
- A constant block, to define the constant of 32, also from the source library.
- A gain block, to multiply the input signal by 9/5, from the Linear library.
- A sum block, to add the two quantities, also from the Linear library.
- A scope block to display the output, from the sink library.

To create a *SIMULINK* block diagram presentation select **new...** from the **File** menu. This provides an untitled blank window for designing and simulating a dynamic system. Copy the above blocks from the block libraries into the new window by depressing the mouse button and dragging. Assign the parameter values to the Gain and Constant blocks by opening (double clicking on) each block and entering the appropriate value. Then click on the close button to apply the value and close the dialog box. The next step is to connect these icons together by drawing lines connecting the icons using the left mouse button (hold the button down and drag the mouse to draw a line). You should now have the *SIMULINK* block diagram as shown in Figure A.12.

The Ramp block inputs Celsius temperature. Open this block, set the Slope to 1, Start time to 0, and the Initial output to 0. The Gain block multiplies that

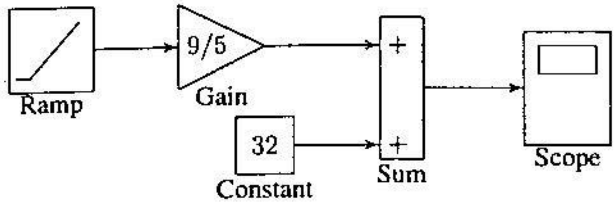


FIGURE A.12
Simulink diagram for the system of Example A.21.

temperature by the constant 9/5. The sum block adds the value 32 to the result and outputs the Fahrenheit temperature. Pull down the Simulation dialog box and select Parameters. Set the Start time to zero and the Stop Time to 100. Pull down the **File** menu and use **Save** to save the model under **simexa21** Start the simulation. Double click on the Scope, click on the **Auto Scale**, the result is displayed as shown in Figure A.13.

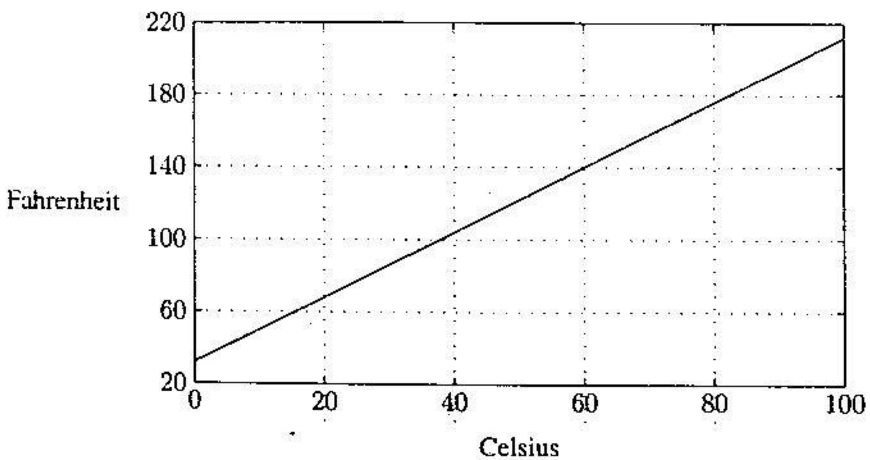


FIGURE A.13
Fahrenheit-Celsius temperature graph for Example A.21.

Example A.22

Construct a simulation diagram for the state equation described in Example A.18. Use *SIMULINK* to model and simulate the step response of this system, and display the results graphically.

State equation in Example A.18 for $M = 1$ kg, $B = 5$ N/m/sec, $K = 25$ N/m, and

$f(t) = 25u(t)$, is given by

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -25x_1 - 5x_2 + 25u(t)
\end{aligned}
$$

The simulation diagram is drawn from the above equations by inspection and is shown in Figure A.14.

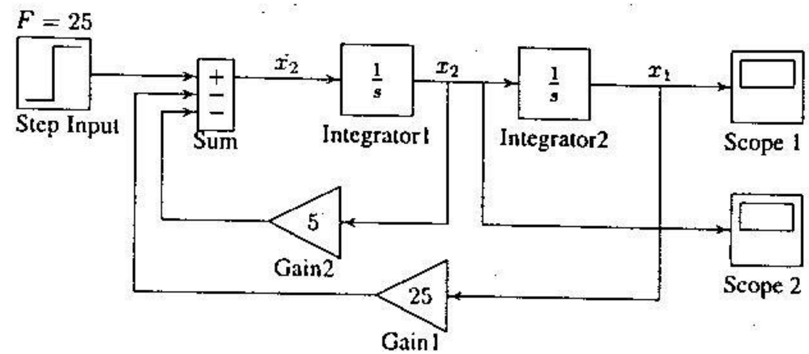


FIGURE A.14
Simulink diagram for the system of Example A.22.

To create a *SIMULINK* block diagram presentation select **new...** from the **File** menu. This provides an untitled blank window for designing and simulating a dynamic system. You can copy blocks from within any of the seven block libraries or other previously opened windows into the new window by depressing the mouse button and dragging. Open the **Source Library** and drag the Step Input block to your window. Double click on Step Input to open its dialog box. Set the step time to a large value, say 100, and set the Initial Value and the Final Value to 25 to represent the step input. Open the **Linear Library** and drag the **Sum** block to the right of the Step Input block. Open the Sum dialog box and enter + - - under List of Signs. Using the left mouse button, click and drag from the Step output port to the Summing block input port to connect them. Drag a copy of the Integrator block from the **Linear Library** and connect it to the output port of the Sum block. Click on the Integrator block once to highlight it. Use the Edit command from the menu bar to copy and paste a second Integrator. Next drag a copy of the Gain block from the **Linear Library**. Highlight the Gain block, and from the pull-down **Options** menu, click on the Flip Horizontal to rotate the **Gain block** by 180°. Double click on Gain block to open its dialog box and set the gain to 5. Make a copy of this block and set its gain to 25. Connect the output ports of the Gain blocks to the Sum block and their input ports to the locations shown in Figure A.14. Finally, get two Auto-Scale Graphs from the **Sink Library**, and connect them to the output of each Integrator. Before starting simulation, you must set the simulation parameters. Pull down the **Simulation** dialog box and select **Parameters.** Set the Start Time to zero, the Stop Time to 3, and for a more accurate integration, set the Maximum Step Size to 0.1. Leave the other parameters at their default values. Press OK to close the dialog box.

If you don't like some aspect of the diagram, you can change it in a variety of ways. You can move any of the icons by clicking on its center and dragging. You can move any of the lines by clicking on one of its corners and dragging. You can change the size and the shape of any of the icons by clicking and dragging on its corners. You can remove any line or icon by clicking on it to select it and using the **cut** command from the **edit** menu. You should now have exactly the same system as shown in Figure A.14. Pull down the **File** menu and use **Save as** to save the model under a file name **simexa22.** Start the simulation. *SIMULINK* will create the Figure windows and display the system responses. To see the second Figure window, click and drag the first one to a new location. The simulation results are shown in Figures A.15 and A.16, which are the same as the curves shown in Figure A.7.

SIMULINK enables you to construct and simulate many complex systems, such as control systems modeled by block diagram with transfer functions including the effect of nonlinearities. In addition, *SIMULINK* provides a number of built-in state variable models and subsystems that can be utilized easily.

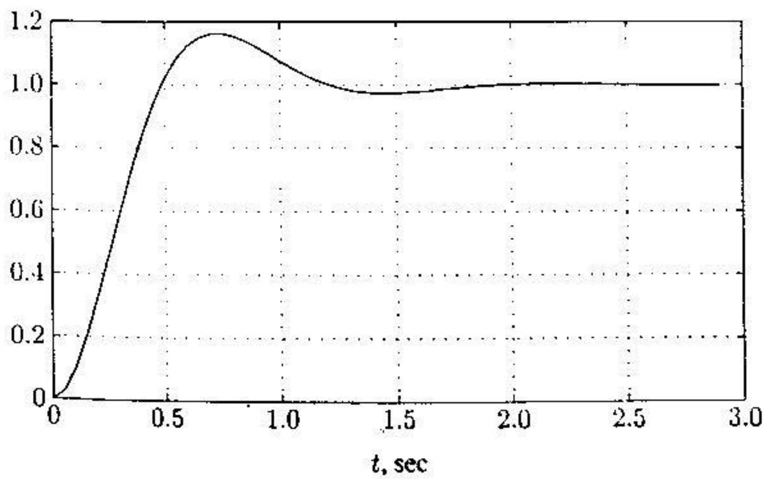


FIGURE A.15
Displacement response of the system described in Example A.22.

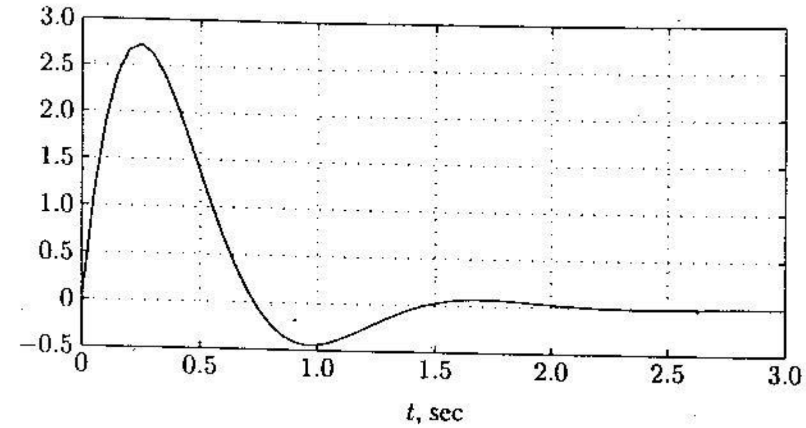


FIGURE A.16
Velocity response of the system described in Example A.22.

Example A.23

Consider the system defined by

$$2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 10y = 10u(t)$$

We have a third-order system; thus there are three state variables. Let us choose the state variables as

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}$$

Then we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_1 - 4x_2 - 2x_3 + 5u(t) \end{aligned}$$

The last of these three equations was obtained by solving the original differential equation for the highest derivative term $\dddot{y}$ and then substituting $y = x_1$, $\dot{y} = x_2$, and $\ddot{y} = x_3$ into the resulting equation. Using matrix notation, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

and the output equation is given by

$$y = [\,1 \quad 0 \quad 0\,]\,\mathbf{x}$$

The simulation diagram is obtained from the system differential equations and is given in Figure A.17.

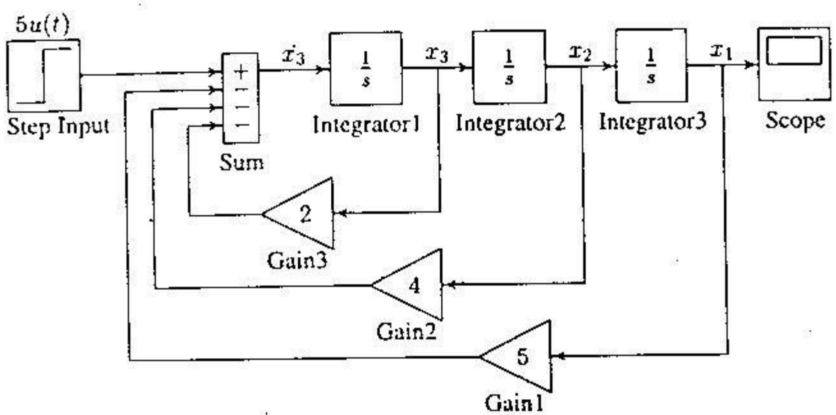


FIGURE A.17
Simulation diagram for the system of Example A.23.

A *SIMULINK* Block diagram is constructed and saved as **simexa23**. The simulation response is shown in Figure A.18.

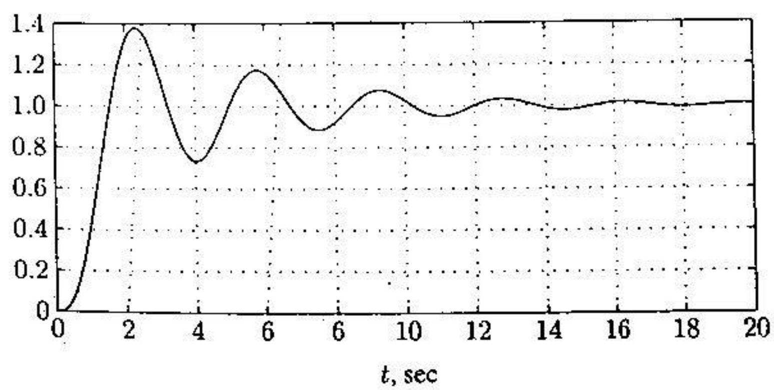


FIGURE A.18
Simulation result for the system in Example A.23.

Example A.24

Use the **state-space** model to simulate the state and output equations described in Example A.23.

The **State-Space** model provides a dialog box where the A, B, C, and D matrices can be entered in *MATLAB* matrix notation, or by variables defined in Workspace. A *SIMULINK* diagram using the **State-Space** model is constructed as shown in Figure A.19, and saved as **simexa24**. The simulation result is the same as in Figure A.18.

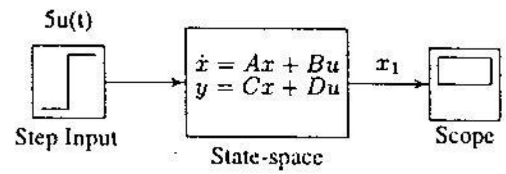


FIGURE A.19
State-space model for system in Example A.24.

Note that in this example, the output is given by $y = x_1$, and we define C as $C = [1 \quad 0 \quad 0]$. If it is desired to access all the states, then we can define C as an identity matrix, in this case a third order, i.e., C = eye(3), and D as D = zeros(3, 1). The output is a vector of state variables. A **DeMux** block may be added to produce individual states for graphing separately.

A.18.4 USING THE TO WORKSPACE BLOCK

The To Workspace block can be used to return output trajectories to the *MATLAB* Workspace. Example A.25 illustrates this use.

Example A.25

Obtain the step response of the following transfer function, and send the result to the *MATLAB* Workspace.

$$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 2s + 25}$$

where $r(t)$ is a unit step function. The *SIMULINK* block diagram is constructed and saved in a file named **simexa25** as shown in Figure A.20.

The To Workspace block can accept a vector input, with each input element's trajectories stored as a column vector in the resulting workspace variable. To specify the variables open the To Workspace block and for the variable name enter c.

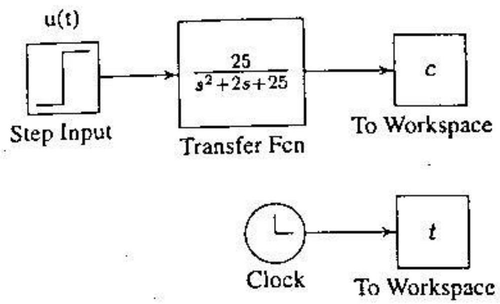


FIGURE A.20
Simulink model for system in Example A.25.

The time vector is stored by feeding a Clock block into To Workspace block. For this block variable name specify t. The vectors c and t are returned to *MATLAB* Workspace upon simulation.

A.18.5 LINEAR STATE-SPACE MODEL FROM SIMULINK DIAGRAM

SIMULINK provides the **linmod**, and **dlinmod** functions to extract linear models from the block diagram model in the form of the state-space matrices A, B, C, and D. State-space matrices describe the linear input-output relationship as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A.7}$$

$$y(t) = Cx(t) + Du(t) \tag{A.8}$$

The following Example illustrates the use of **linmod** function. The input and outputs of the *SIMULINK* diagram must be defined using **Inport** and **Outport** blocks in place of the **Source** and **Sink** blocks.

Example A.26

Obtain the state-space model for the system represented by the block diagram shown in Figure A.21. The model is saved with a filename **simexa26**. Run the simulation and to extract the linear model of this *SIMULINK* system, in the Command Window, enter the command

```
[A,B,C,D] = linmod('simexa26')
```

The result is

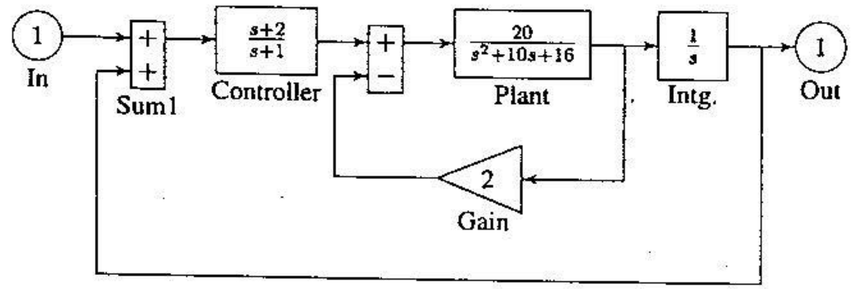


FIGURE A.21
Simulink model for system in Example A.26.

```
A =                                 B =
     0     0     0    20                 0
    -1    -1     0     0                 1
    -1     1   -10   -56                 1
     0     0     1     0                 0
C =

     1     0     0     0
D =
     0
```

In order to obtains the transfer function of the system from the state-space model, we use the command

```
[num, den]=ss2tf(A, B, C, D)
```

the result is

```
num =
      0.0000    0.0000    0.0000   20.0000   40.0000
den =
      1.0000   11.0000   66.0000   76.0000   40.0000
```

Thus, the transfer function model is

$$T(s) = \frac{20s + 40}{s^4 + 11s^3 + 66s^2 + 76s + 40}$$

Once the data is in the state-space form, or converted to a transfer function model, you can apply functions in Control System Toolbox for further analysis:

- Bode phase and magnitude frequency plot:

```
bode(A, B, C, D) or bode(num, den)
```

- Linearized time response:

```
step(A, B, C, D)    or  step(num, den)
lsim(A, B, C, D)    or  lsim(num, den)
impulse(A, B, C, D) or  impulse(num, den)
```

A.18.6 SUBSYSTEMS AND MASKING

SIMULINK subsystems, provide a capability within *SIMULINK* similar to subprograms in traditional programming languages.

Masking is a powerful *SIMULINK* feature that enables you to customize the dialog box and icon for a block or subsystem. With masking, you can simplify the use of your model by replacing many dialog boxes in a subsystem with a single one.

Example A.27

To encapsulate a portion of an existing *SIMULINK* model into a subsystem, consider the *SIMULINK* model of Example A.23 shown in Figure A.22, and proceed as follows:

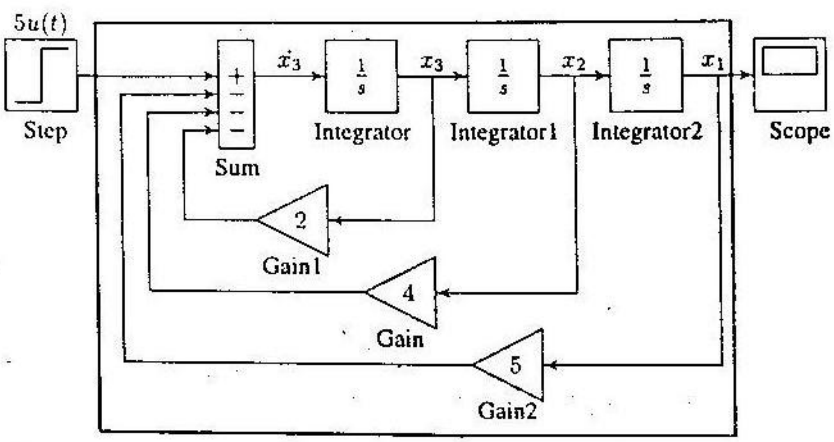


FIGURE A.22
Simulation diagram for the system of Example A.23.

1. Select all the blocks and signal lines to be included in the subsystem with the bounding box as shown.

2. Choose Edit and select Create Subsystem from the model window menu bar. *SIMULINK* will replace the select blocks with a subsystem block that has an input port for each signal entering the new subsystem and an output port for each signal leaving the new subsystem. *SIMULINK* will assign default names to the input and output ports.

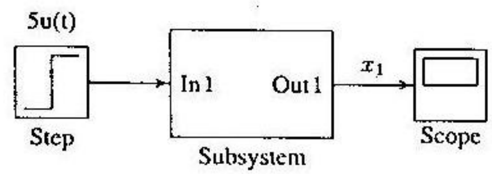


FIGURE A.23
Simulation diagram for the system of Example A.23.

To mask a block, select the block, then choose Create Mask from the Edit menu. The Mask Editor appears. The Mask Editor consists of three pages, each handling a different aspect of the mask.

- The Initialization page enables you to define and describe mask dialog box parameter prompts, name the variables associated with the parameters, and specify initialization commands.
- The Icon page enables you to define the block icon.
- The Documentation page enables you to define the mask type, and specify the block description and the block help.

In this example for icon the system transfer function is entered with command

```
dpoly([10], [2 4 8 10])
```

A short description of the system and relevant help topics can be entered in the Documentation page. The subsystem block is saved in a file named **simexa28**. Additional *SIMULINK* examples are found in Chapter 12. Also, many interesting examples are available in *SIMULINK* demo.

APPENDIX B

REVIEW OF FEEDBACK CONTROL SYSTEMS

B.1 THE CONTROL PROBLEM

The first step in the analysis and design of control systems is mathematical modeling of the system. The two most common methods are the transfer function approach and the state equation approach. The state equations can be applied to portray linear as well as nonlinear systems

All physical systems are nonlinear to some extent. In order to use the transfer function and linear state equations, the system must first be linearized. Thus, proper assumptions and approximations are made so that the system can be characterized by a linear mathematical model. The model may be validated by analyzing its performance for realistic input conditions and then by comparing with field test data taken from the dynamic system in its operating environment. Further analysis of the simulated model is usually necessary to obtain the model response for different feedback configurations and parameters settings. Once an acceptable controller has been designed and tested on the model, the feedback control strategy is then applied to the actual system to be controlled.

When we wish to develop a feedback control system for a specific purpose, the general procedure may be summarized as follows:

1. Choose a way to adjust the variable to be controlled; e.g., the mechanical load will be positioned with an electric motor or the temperature will be controlled by an electrical resistance heater.

2. Select suitable sensors, power supplies, amplifiers, etc., to complete the loop.

3. Determine what is required for the system to operate with the specified accuracy in steady-state and for the desired response time.

4. Analyze the resulting system to determine its stability.

5. Modify the system to provide stability and other desired operating conditions by redesigning the amplifier/controller, or by introducing additional control loops.

The objective of the control system is to control the output $c(t)$ in some prescribed manner by the input $r(t)$ through the elements of the control system. Some of the essential characteristics of feedback control systems are investigated in the following sections.

B.2 STABILITY

Consider the block diagram of a simple closed-loop control system as shown in Figure B.1 where $R(s)$ is the s-domain reference input, and $C(s)$ is the s-domain controlled output. $G(s)$ is the plant transfer function, K is a simple gain controller,

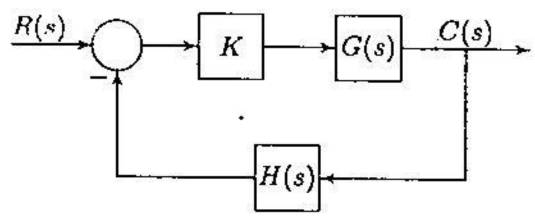


FIGURE B.1
A simple closed-loop control system.

and the feedback elements $H(s)$ represent the sensor transfer function. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \tag{B.1}$$

or the s-domain response is

$$C(s) = T(s)R(s) \tag{B.2}$$

The gain $KG(s)H(s)$ is commonly referred to as the *open-loop transfer function*. For a system to be usable, it must be stable. A linear time-invariant system is stable if every bounded input produces a bounded output. We call this characteristic *stability*. The denominator polynomial of the closed-loop transfer function set equal to zero is the system characteristic equation. That is, the characteristic equation is given by

$$1 + KG(s)H(s) = 0 \tag{B.3}$$

The roots of the characteristic equation are known as the poles of the closed-loop transfer function. The response is bounded if the poles of the closed-loop system are in the left-hand portion of the s-plane. Thus, a necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

The stability of a linear time-invariant system may be checked by using the *Control System Toolbox* function **impulse** to obtain the impulse response of the system. The system is stable if its impulse response approaches zero as time approaches infinity. One way to determine the stability of a system is by simulation. The function **lsim** can be used to observe the output for typical inputs. This is particularly useful for nonlinear systems. Alternatively, the *MATLAB* function **roots** can be utilized to obtain the roots of the characteristic equations. In the classical control theory, several techniques have been developed requiring little computation for stability analysis. One of these techniques is the *Routh-Hurwitz criterion*. Consideration of the *degree* of stability of a system often provides valuable information about its behavior. That is, if it is stable, how close is it to being unstable? This is the concept of *relative stability*. Usually, relative stability is expressed in terms of the speed of response and overshoot. Other methods frequently used for stability studies are the *Bode diagram*, *Root-locus plot*, *Nyquist criterion*, and *Lyapunov's stability criterion*.

B.2.1 THE ROUTH-HURWITZ STABILITY CRITERION

The Routh-Hurwitz criterion provides a quick method for determining absolute stability that can be applied to an nth-order characteristic equation of the form

$$a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0 \tag{B.4}$$

The criterion is applied through the use of a *Routh table* defined as

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$a_n, a_{n-1}, \ldots, a_0$ are the coefficients of the characteristic equation and

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \quad \text{etc.}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1}, \quad c_2 = \frac{b_1 a_{n-5} - a_{n-1}b_3}{b_1}, \quad \text{etc.}$$

Calculations in each row are continued until only zero elements remain. The necessary and sufficient condition that all roots of (B.4) lie in the left half of the s-plane is that the elements of the first column of the Routh-Hurwitz array have the same sign. If there are changes of signs in the elements of the first column, the number of sign changes indicates the number of roots with positive real parts.

A function called **routh(a)** is written that forms the Routh-Hurwitz array and determines if any roots have positive real parts. **a** is a row vector containing the coefficients of the characteristic equation.

If the first element in a row is zero, it is replaced by a very small positive number ϵ, and the calculation of the array is completed. If all elements in a row are zero, the system has poles on the imaginary axis, pairs of complex conjugate roots forming symmetry about the origin of the s-plane, or pairs of real roots with opposite signs. In this case, an auxiliary equation is formed from the preceding row. The all-zero row is then replaced with coefficients obtained by differentiating the auxiliary equation.

B.2.2 ROOT-LOCUS METHOD

The *root-locus method*, developed by W. R. Evans, enables us to find the closed-loop poles from the open-loop poles for all the values of the gain of the open-loop transfer function. The root locus of a system is a plot of the roots of the system characteristic equation as the gain factor K is varied. Therefore, the designer can select a suitable gain factor to achieve the desired performance criteria. If the required performance cannot be achieved, a controller can be added to the system to alter the root locus in the required manner.

Consider the feedback control system given in Figure B.1. In general, the open-loop transfer function is given by

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{B.5}$$

where m is the number of finite zeros, and n is the number of finite poles of the loop transfer function. If $n > m$, there are $(n-m)$ zeros at infinity. The characteristic equation of the closed-loop transfer function is given by (B.3); therefore

$$\frac{(s+p_1)(s+p_2)\cdots(s+p_n)}{(s+z_1)(s+z_2)\cdots(s+z_m)} = -K \tag{B.6}$$

From (B.6) it follows that for a point in the s-plane to be on the root locus, when $0 < K < \infty$, it must satisfy the following two conditions.

$$K = \frac{\text{product of vector lengths from finite poles}}{\text{product of vector lengths from finite zeros}} \tag{B.7}$$

and

$$\sum \text{angles of zeros of } GH(s) - \sum \text{angles of poles of } GH(s) = r(180)^\circ \tag{B.8}$$

$$\text{where} \quad r = \pm 1, \pm 3. \pm 5, \cdots$$

Given a transfer function of an open-loop control system, the *Control System Toolbox* function **rlocus(num, den)** produces a root-locus plot with the gain vector automatically determined. If the open-loop system is defined in state space, we use **rlocus(A, B, C, D)**. **rlocus(num, den, K)** or **rlocus(A, B, C, D, K)** uses the user-supplied gain vector **K**. If the above commands are invoked with the left hand arguments **[r, K]**, the matrix **r** and the gain vector **K** are returned, and we need to use **plot(r, ' . ')** to obtain the plot. **rlocus** function is accurate, and we use it to obtain the root-locus. A good knowledge of the characteristics of the root loci offers insights into the effects of adding poles and zeros to the system transfer function. It is important to know how to construct the root locus by hand, so we can design a simple system and be able to understand and develop the computer-generated loci. For the basic construction rules for sketching the root locus, refer to any text on feedback control systems.

B.3 STEADY-STATE ERROR

In addition to being stable, a control system is also expected to meet a specified performance requirement when it is commanded by a set-point change or disturbed

by an external force. The performance of the control system is judged not only by the transient response, but also by steady-state error. The steady-state error is the error as the transient response has decayed, leaving only the continuous response. High loop gains, in addition to sensitivity reduction, will also reduce the steady-state error. The steady-state error for a control system is classified according to its response characteristics to a polynomial input. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. This depends on the type of the open-loop transfer function.

Consider the system shown in Figure B.1. The closed-loop transfer function is given by (B.1). The error of the closed-loop system is

$$E(s) = R(s) - H(s)C(s) = \frac{1}{1+KG(s)H(s)}R(s) \tag{B.9}$$

Using the final-value theorem, we have

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+KG(s)H(s)} \tag{B.10}$$

For the polynomial inputs, such as step, ramp, and parabolas, the steady-state error from the above equation will be:

Unit step input

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} KG(s)H(s)} = \frac{1}{1+K_p} \tag{B.11}$$

Unit ramp input

$$e_{ss} = \frac{1}{\lim_{s\to 0} sKG(s)H(s)} = \frac{1}{K_v} \tag{B.12}$$

Unit parabolic input

$$e_{ss} = \frac{1}{\lim_{s\to 0} s^2KG(s)H(s)} = \frac{1}{K_a} \tag{B.13}$$

In order to define the *system type*, the general open-loop transfer function is written in the following form.

$$KG(s)H(s) = \frac{K(1+T_1S)(1+T_2s)\ldots(1+T_ms)}{s^j(1+T_as)(1+T_bs)\ldots(1+T_ns)} \tag{B.14}$$

The *type* of feedback control system refers to the *order* of the pole of $G(s)H(s)$ at $s = 0$.

Two functions, **errorzp(z,p,k)** and **errortf(num, den)**, are written for computation of system steady-state error due to typical inputs, namely unit step, unit ramp, and unit parabolic. **errorzp(z,p,k)** finds the steady-state error when the system is represented by the zeros, poles, and gain. **z** is a column vector containing the transfer function zeros, **p** is a column vector containing the poles, and **k** is the gain. If the numerator power m is less than the denominator power n, then there are $(n-m)$ zeros at infinity, and vector **z** must be padded with $(n-m)$ **inf. errortf(num, den)** finds the steady-state error when the transfer function is expressed as the ratio of two polynomials.

B.4 STEP RESPONSE

Assessing the time-domain performance of closed-loop system models is important, because control systems are inherently time-domain systems. The performance of dynamic systems in the time domain can be defined in terms of the time response to standard test inputs. One very common input to control systems is the step function. If the response to a step input is known, it is mathematically possible to compute the response to any input. The step response for a second-order system is obtained. The standard form of the second-order transfer function is given by

$$G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \tag{B.15}$$

where ω_n is the natural frequency. The natural frequency is the frequency of oscillation if all of the damping is removed. Its value gives us an indication of the speed of the response. ζ is the dimensionless damping ratio. The damping ratio gives us an idea about the nature of the transient response. It gives us a feel for the amount of overshoot and oscillation that the response undergoes.

The transient response of a practical control system often exhibits damped oscillations before reaching steady-state. The underdamped response $(\zeta < 1)$ to a unit step input, subject to zero initial condition, is given by

$$c(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\beta\omega_n t + \theta) \tag{B.16}$$

where $\beta = \sqrt{1-\zeta^2}$ and $\theta = \tan^{-1}(\beta/\zeta)$.

The performance criteria that are used to characterize the transient response to a unit step input include rise time, peak time, overshoot, and settling time. We define the rise time t_r as the time required for the response to rise from 10 percent of the final value to 90 percent of the final value. The time to reach the peak value is t_p. The swiftness of the response is measured by t_r and t_p. The similarity with which the actual response matches the step input is measured by the percent overshoot and settling time t_s. For underdamped systems, the percent overshoot $P.O.$ is defined as

$$P.O. = \frac{\text{maximum value} - \text{final value}}{\text{final value}} \tag{B.17}$$

The peak time is obtained by setting the derivative of (B.16) to zero.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{B.18}$$

The peak value of the step response occurs at this time, and evaluating the response in (B.16) at $t = t_p$ yields

$$C(t_p) = M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{B.19}$$

Therefore, from (B.17), the percent overshoot is

$$P.O. = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 \tag{B.20}$$

Settling time is the time required for the step response to settle within a small percent of its final value. Typically, this value may be assumed to be ± 2 percent of the final value. For the second-order system, the response remains within 2 percent after 4 time constants, that is

$$t_s = 4\tau = \frac{4}{\zeta\omega_n} \tag{B.21}$$

Given a transfer function of a closed-loop control system, the *Control System Toolbox* function **step(num, den)** produces the step response plot with the time vector automatically determined. If the closed-loop system is defined in state space, we use **step(A, B, C, D)**. **step(num, den, t)** or **step(A, B, C, D, iu, t)** uses the user-supplied time vector **t**. The scalar **iu** specifies which input is to be used for the step response. If the above commands are invoked with the left-hand arguments **[y, x, t]**, the output vector, the state response vectors, and the time vector **t** are returned, and we need to use **plot** function to obtain the plot. See also **initial** and **lsim** functions. A function called **timespec(num, den)** is written which obtains the time-domain performance specifications, $P.O.$, t_p, t_r, and t_s. **num** and **den** are the numerator and denominator of the system closed-loop transfer function.

B.5 ROOT-LOCUS DESIGN

The design specifications considered here are limited to those dealing with system accuracy and time-domain performance specifications. These performance specifications can be defined in terms of the desirable location of the dominant closed-loop poles.

The root locus can be used to determine the value of the loop gain K, which results in a satisfactory closed-loop behavior. This is called the proportional controller and provides gradual response to deviations from the set point. There are practical limits as to how large the gain can be made. In fact, very high gains lead to instabilities. If the root-locus plot is such that the desired performance cannot be achieved by the adjustment of the gain, then it is necessary to reshape the root loci by adding the additional controller $G_c(s)$ to the forward path, as shown in Figure B.2. $G_c(s)$ must be chosen so that the root locus will pass through the proper region of the s-plane.

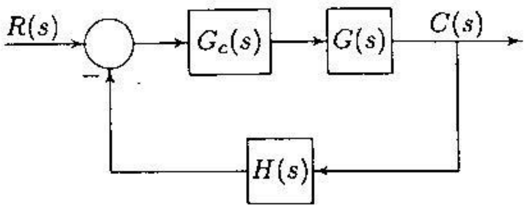


FIGURE B.2
A closed-loop control system with controller.

The proportional controller (P) has no sense of time, and its action is determined by the present value of the error. An appropriate controller must make corrections based on the past and future values. This can be accomplished by combining proportional with integral action (PI) or proportional with derivative action (PD). There is also a proportional-plus-integral-plus-derivative controller (PID).

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \tag{B.22}$$

The ideal integral and differential compensators require the use of active amplifiers.

Other compensators which can be realized with only passive network elements are lead, lag, and lead-lag compensators. A first-order compensator having a single zero and pole in its transfer function is

$$G_c(s) = \frac{K_c(s + z_0)}{s + p_0} \tag{B.23}$$

B.5.1 GAIN FACTOR COMPENSATION OR P CONTROLLER

The proportional controller is a pure gain controller. The design is accomplished by choosing a value of K_0 which results, in a satisfactory transient response.

B.5.2 PHASE-LEAD DESIGN

In (B.23) the compensator is a high-pass filter or phase-lead, if $p_0 > z_0$. The phase-lead network contributes a positive angle to the root-locus angle criterion of (B.8) and tends to shift the root locus of the plant toward the left in the s-plane. The lead network acts mainly to modify the dynamic response to raise bandwidth and to increase the speed of response. In a sense, a lead network approximates derivative control. If $p_0 < z_0$, the compensator is a low-pass filter or phase-lag. The phase-lag compensator adds a negative angle to the angle criterion and tends to shift the root locus to the right in the s-plane. The compensator angle must be small to maintain the stability of the system. The lag network is usually used to raise the low-frequency gain and thus to improve the steady-state accuracy of the system. The lag network is an approximate integral control. The DC gain of the compensator is

$$a_0 = G_c(0) = \frac{K_c z_0}{p_0} \tag{B.24}$$

For a given desired location of a closed-loop pole s_1, the design can be accomplished by trial and error. Select a proper value of z_0 and use the angle criterion of (B.8) to determine p_0. Then, the gain K_c is obtained by applying the magnitude criterion of (B.7). Alternatively, if the compensator DC gain, $a_0 = (K_c z_0)/p_0$, is specified, then for a given location of the closed-loop pole

$$s_1 = |s_1| \angle\beta \tag{B.25}$$

z_0 and p_0 are obtained such that the equation

$$1 + G_c(s_1)G_p(s_1) = 0 \tag{B.26}$$

is satisfied. It can be shown that the above parameters are found from the following equations.

$$z_0 = \frac{a_0}{a_1} \qquad p_0 = \frac{1}{b_1} \qquad \text{and} \qquad K_c = \frac{a_0 p_0}{z_0} \tag{B.27}$$

where

$$a_1 = \frac{\sin\beta + a_0 M \sin(\beta - \psi)}{|s_1|\, M \sin\psi}$$

$$b_1 = -\frac{\sin(\beta + \psi) + a_0 M \sin\beta}{|s_1|\, \sin\psi} \tag{B.28}$$

where M and ψ are the magnitude and phase angle of the open-loop plant transfer function evaluated at s_1, i.e.,

$$G_p(s_1) = M\angle\psi \tag{B.29}$$

For the case that ψ is either 0° or 180°, (B.28) is given by

$$a_1 |s_1| \cos\beta \pm \frac{b_1 |s_1|}{M} \cos\beta \pm \frac{1}{M} + a_0 = 0 \tag{B.30}$$

where the plus sign applies for $\psi = 0°$ and the minus sign applies for $\psi = 180°$. For this case, the zero of the compensator must also be assigned.

B.5.3 PHASE-LAG DESIGN

In the phase-lag control, the poles and zeros of the controller are placed very close together, and the combination is located relatively close to the origin of the s-plane. Thus, the root loci in the compensated system are shifted only slightly from their original locations. Hence, the phase-lag compensator is used when the system transient response is satisfactory but requires a reduction in the steady-state error. The function **[numo, deno, denc] = phlead(num, den, s_1)** can be used for phase-lag compensation by specifying the desired pole s_1 slightly to the right of the uncompensated pole location. Alternatively, phase-lag compensation can be obtained by assuming a DC gain of unity for the compensator based on the following approximate method.

$$a_0 = G_c(0) = \frac{K_c z_0}{p_0} = 1 \tag{B.31}$$

Therefore,

$$K_c = \frac{p_0}{z_0} \qquad \text{since } p_0 < z_0 \text{ then } K_c < 1 \tag{B.32}$$

If K_0 is the gain required for the desired closed-loop pole s_1, then from (B.3)

$$K_0 = -\frac{1}{G_p(s_1)} \tag{B.33}$$

If we place the pole and zero of the lag compensator very close to each other with their magnitude much smaller than s_1, then

$$G_c(s_1) = \frac{K_c(s+z_0)}{s+p_0} \simeq K_c \tag{B.34}$$

Now, the gain K required to place a closed-loop pole at approximately s_1 is given by

$$K = -\frac{1}{G_c(s_1)G_p(s_1)} \simeq -\frac{1}{K_c G_p(s_1)} \simeq \frac{K_0}{K_c} \tag{B.35}$$

Since $K_c < 1$, then $K > K_0$. Next, select the compensator zero z_0, arbitrarily small. Then from (B.31) the compensator pole is

$$p_0 = K_0 z_0 \tag{B.36}$$

The compensated system transfer function is then given by

$$KG_pG_c = KK_c\frac{s+z_0}{s+p_0}G_p \tag{B.37}$$

A lag-lead controller may be obtained by appropriately combining a lag and a lead network in series.

B.5.4 PID DESIGN

One of the most common controllers available commercially is the PID controller. Different processes are suited to different combinations of proportional, integral, and derivative control. The control engineer's task is to adjust the three gain factors to arrive at an acceptable degree of error reduction simultaneously with acceptable dynamic response. For a desired location of the closed-loop pole s_1, as given by (B.25), the following equations are obtained to satisfy (B.26).

$$K_P = \frac{-\sin(\beta+\psi)}{M\sin\beta} - \frac{2K_I\cos\beta}{|s_1|}$$

$$K_D = \frac{\sin\psi}{|s_1|\, M\sin\beta} + \frac{K_I}{|s_1|^2} \tag{B.38}$$

For PD or PI controllers, the appropriate gain is set to zero. The above equations can be used only for the complex pole s_1. For the case that s_1 is real, the zero of the PD controller ($z_0 = K_P/K_D$) and the zero of the PI controller ($z_0 = K_I/K_P$) are specified, and the corresponding gains to satisfy angle and magnitude criteria are obtained accordingly. For the PID design, the value of K_I to achieve a desired steady-state error is specified. Again, (B.38) is applied only for the complex pole s_1.

B.5.5 PD CONTROLLER

Here, both the error and its derivative are used for control, and the compensator transfer function is

$$G_c(s) = K_P + K_D s = K_D\left(s + \frac{K_P}{K_D}\right) \tag{B.39}$$

From above, it can be seen that the PD controller is equivalent to the addition of a simple zero at $s = -K_P/K_D$ to the open-loop transfer function, which improves the transient response. From a different point of view, the PD controller may also be used to improve the steady-state error, because it anticipates large errors and attempts corrective action before they occur. The function **[numo, deno, denc] = rldesign(num, den, s_1)** with option 4 is used for the PD controller design.

B.5.6 PI CONTROLLER

The integral of the error as well as the error itself is used for control, and the compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} = \frac{K_P(s + K_I/K_P)}{s} \tag{B.40}$$

The PI controller is common in process control or regulating systems. Integral control bases its corrective action on the cumulative error integrated over time. The controller increases the type of system by 1 and is used to reduce the steady-state errors. The function **[numo, deno, denc] = rldesign(num, den, s_1)** with option 5 is used for the PI controller design.

B.5.7 PID CONTROLLER

The PID controller is used to improve the dynamic response as well as to reduce or eliminate the steady-state error. The function **[numo, deno, denc] = rldesign(num, den, s_1)** with option 6 is used for the PID controller design.

Based on the above equations, several functions are developed for the root-locus design. These are

Function	Controller
[numo, deno, denc] = pcomp(num, den, ζ)	Proportional
[numo, deno, denc] = phlead(num, den, s_1)	Phase-Lead
[numo, deno, denc] = phlag(num, den, ζ)	Phase-Lag
[numo, deno, denc] = pdcomp(num, den, s_1)	PD
[numo, deno, denc] = picomp(num, den, s_1)	PI
[numo, deno, denc] = pidcomp(num, den, s_1)	PID

Alternatively, the function **[numo, deno, denc] = rldesign(num, den, s_1)** displays a menu with six options that allow the user to select any of the above controller designs. $s_1 = \sigma + j\omega$ is a desired pole of the closed-loop transfer function, except for the **pcomp** and **phlag** controllers, where ζ, the damping ratio of the dominant poles, is substituted for s_1. **num** and **den** are row vectors of polynomial coefficients of the uncompensated open-loop plant transfer function. The function

phlead(num, den, s1) may also be used to design phase-lag controllers. To do this, the desired pole location s_1 must be assumed slightly to the right of the uncompensated pole position. The function obtains the controller transfer function and roots of the compensated characteristic equation. Also, the function returns the open-loop and closed-loop numerators and denominators of the compensated system transfer function.

Example B.1

The block diagram of a control system is as shown in Figure B.3. $G_c(s)$ is a simple proportional controller of gain K.

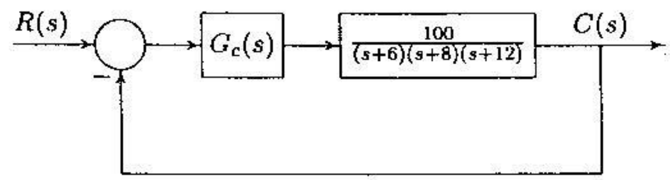


FIGURE B.3
Control system for Example B.1.

(a) Construct the Routh-Hurwitz array and determine the range of K for closed-loop stability.
(b) Find the value of K to yield a steady-state error of 0.15 for a unit step input.
(c) Use *MATLAB* **rlocus** function to obtain the root-locus plot.
(d) Use **rldesign** and option 1 to find the gain K_0 such that the dominant closed-loop poles damping ratio will be equal to 0.96. Obtain the step response curves for K_0, and the value of K found in (b).

(a) The closed-loop transfer function of the control system shown in Figure B.3 is

$$\frac{C(s)}{R(s)} = \frac{100K}{s^3 + 26s^2 + 216s + 576 + 100K}$$

The Routh-Hurwitz array for this polynomial is then (see Appendix B.2.1)

$$\begin{array}{l|ll} s^3 & 1 & 216 \\ s^2 & 26 & 576 + 100K \\ s^1 & 193.846 - 3.846K & 0 \\ s^0 & 576 + 10K & 0 \end{array}$$

From the s^1 row we see that, for control system stability, K must be less than 50.4, also from the s^0 row, K must be greater than -5.76. Thus, with positive values of

K, for control system stability, the gain must be

$$K < 50.4$$

For $K = 50.4$, the auxiliary equation from the s^2 row is

$$26s^2 + 5616 = 0$$

or $s = \pm j14.7$. That is, for $K = 50.4$, we have a pair of conjugate poles on the $j-\omega$ axis, and the control system is marginally stable.

(b) The position error constant given by (B.11) is

$$K_p = \lim_{s\to 0} G_c(s)G_p(s) = \lim_{s\to 0} \frac{100K}{(s+6)(s+8)(s+12)} = \frac{100K}{576}$$

For a unit step input

$$e_{ss} = \frac{1}{1+K_p} = 0.15$$

Thus

$$K_p = 5.667 = \frac{100K}{576}$$

or

$$K = 32.64$$

The closed-loop transfer function for this gain is

$$\frac{C(s)}{R(s)} = \frac{(100)(32.64)}{s^3 + 26s^2 + 216s + 3840}$$

(c) The *MATLAB Control Toolbox* function **rlocus** is used to obtain the root-locus plot.

(d) To find the gain for the step response damping ratio of $\zeta = 0.96$, and the step response plots, we use the following commands.

```
num = 100;
den = [1  26  216  576];
figure(1), rlocus(num, den), grid,  axis([-20 0 -15 15]);
zeta = 0.96;                                   % damping ratio
[numo, deno, denc]=rldesign(num,den,zeta);% Gain controller
```



```
t = 0:.005:4;
c1 = step(numo, denc, t); % Step response for zeta = 0.96
num2 = 100*32.64;  den2 = [1 26 216 3840];
c2 = step(num2, den2, t);   % Step response for K = 32.64
figure(2), plot(t, c1,  t, c2) , grid
xlabel('t, sec'), ylabel('c(t)')
text(3.1, 0.75, 'K = 32.64'), text(3.1, 0.1, 'K = 0.28')
timespec(num2, den2)     % Time-domain spec. for K = 32.64
```

The result is

Compensator type	Enter
Gain compensation	1
Phase-lead (or phase-lag)	2
Phase-lag (Approximate $K = K_0/K_c$)	3
PD Controller	4
PI Controller	5
PID Controller	6
To quit	0

```
Enter your choice → 1

Controller gain:          K0 = 0.28

Row vectors of polynomial coefficients of the compensated
system:

   Open-loop num.    28
   Open-loop den.     1   26   216   576
   Closed-loop den.   1   26   216   604

   Roots of the compensated characteristic equation:
   -12.8445
   -6.5778 + 1.9383i
   -6.5778 - 1.9383i

   Peak time = 0.289           Percent overshoot = 65.9
   Rise time = 0.096
   Settling time = 3.3
```

The root-locus plot is shown in Figure B.4, and the step response is shown in Figure B.5. The step response damping ratio of 0.96 resulted in a controller gain of $K_0 = 0.28$.

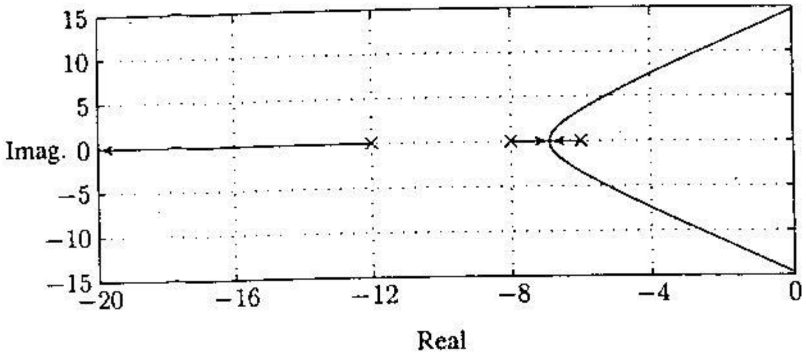


FIGURE B.4
Root-locus plot for Example B.1.

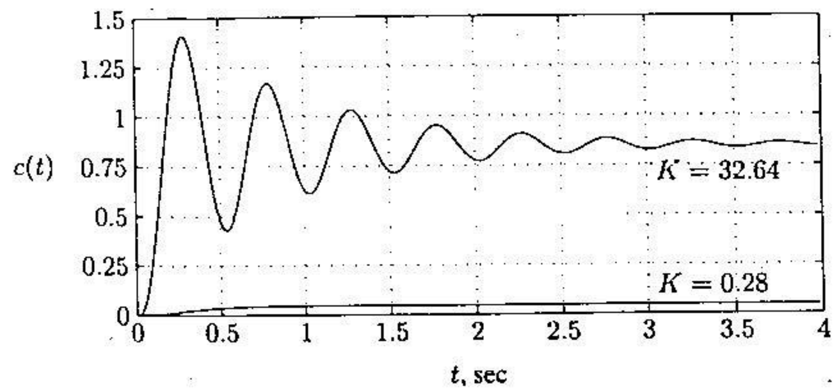


FIGURE B.5
Step response with proportional controller for Example B.1.

From Figure B.5, we see that the transient response is satisfactory, but the steady-state error given by

$$e_{ss} = \frac{1}{1 + \frac{28}{576}} = 0.9536$$

is very large, and the steady-state response is $1 - 0.9536 = 0.0464$. In order to reduce the steady-state error, the gain must be increased. The gain for a steady-state error of 0.15 was found to be 32.64, but the step response is highly oscillatory with an overshoot of 65.9 percent, which is not satisfactory.

Example B.2

For the control system in Example B.1, design a controller to meet the following specifications.

- Zero steady-state error for a step input
- Step response dominant poles damping ratio $\zeta = 0.995$
- Step response dominant pole time constant $\tau = 0.1$ second

The plant transfer function of the control system in Example B.1 is type zero. To reduce the steady-state error to zero, we must increase the system type by one. Thus, we select a PID controller, i.e.,

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

From the last two specifications, $\zeta\omega_n = \frac{1}{\tau} = 10$, and $\theta = \cos^{-1} 0.995 = 5.73°$. Thus, the required complex closed-loop poles are $-10 \pm j1$. The function **rldesign** with option 6 is used for a PID controller design. The user is prompted to enter a value for the integral gain K_I, and the program determines K_P and K_D. The process may be repeated for different values of K_I until a satisfactory response is obtained. For this example, use a value of 9.09 for K_I. The following commands

```
num = 100; den = [1  26  216  576];
s1= -10+j*1;         % Desired location of closed-loop poles
[numo, deno, denc]=rldesign(num, den, s1);     %PID design
t = 0:.01:4;
step(numo, denc, t), grid
xlabel('Time -sec.'), ylabel('c(t)')
```

result in

```
Compensator type                                  Enter
Gain compensation                                   1
Phase-lead (or phase-lag )                          2
Phase-lag (Approximate K = K0/Kc)                   3
PD Controller                                       4
PI Controller                                       5
PID Controller                                      6
To quit                                             0

Enter your choice → 6
Enter the integrator gain KI → 9.09
```

```
Gc = 2.1 + 9.09/s + 0.14s

Row vectors of polynomial coefficients of the compensated
system:

  Open-loop num.    14  210  909
  Open-loop den.     1   26  230  576    0
  Closed-loop den    1   26  230  786  909

  Roots of the compensated characteristic equation:
  -10 + 1i
  -10 - 1i
  -3
  -3
```

Thus, the compensated open-loop transfer function is

$$G_cG_p = \frac{14s^2 + 210s + 909}{s(s^3 + 26s^2 + 216s + 576)}$$

and the compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{14s^2 + 210s + 909}{s^4 + 26s^3 + 2230s^2 + 786s + 909}$$

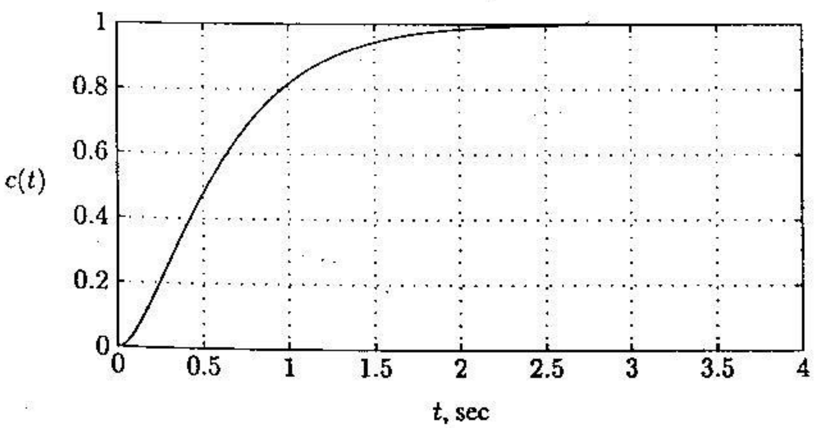


FIGURE B.6
Step response with PID controller for Example B.2.

The PID controller increases the system type by 1. That is, we have a type 1 system, and the steady-state error due to a unit step input is zero. The transient response is also improved as shown in Figure B.6.

B.6 FREQUENCY RESPONSE

The frequency response of a system is the steady-state response of the system to a sinusoidal input signal. The frequency response method and the root-locus method are simply two different ways of applying the same basic principles of analysis. These methods supplement each other, and in many practical design problems, both techniques are employed. One advantage of the frequency response method is that the transfer function of a system can be determined experimentally by frequency response tests. Furthermore, the design of a system in the frequency domain provides the designer with control over the system bandwidth and over the effect of noise and disturbance on the system response.

The response of a linear time-invariant system to sinusoidal input $r(t) = A\sin(\omega t)$ is given by

$$c(t) = A\,|G(j\omega)|\,\sin[\omega t + \theta(\omega)] \tag{B.41}$$

where the transfer function $G(j\omega)$ is obtained by substituting $j\omega$ for s in the expression for $G(s)$. The resulting transfer function may be written in *polar form* as

$$G(j\omega) = |G(j\omega)|\ \angle\theta(\omega) \tag{B.42}$$

Alternatively, the transfer function can be represented in rectangular complex form as

$$G(j\omega) = \Re\, G(j\omega) + j\Im\, G(j\omega) = R(j\omega) + jX(j\omega) \tag{B.43}$$

The most common graphical representation of a frequency response function is the *Bode plot*. Other representations of sinusoidal transfer functions are *polar plot* and *log-magnitude versus phase plot*.

B.6.1 BODE PLOT

The *Bode plot* consists of two graphs plotted on semi-log paper with linear vertical scales and logarithmic horizontal scales. The first graph is a plot of the magnitude of a frequency response function $G(j\omega)$ in decibels versus the logarithm of ω, the frequency. The second graph of a Bode plot shows the phase function $\theta(\omega)$ versus the logarithm of ω. The logarithmic representation is useful in that it shows both the low- and high-frequency characteristics of the transfer function in one diagram. Furthermore, the frequency response of a system may be approximated by a series of straight line segments.

Given a transfer function of a system, the *Control System Toolbox* function **bode(num, den)** produces the frequency response plot with the frequency vector automatically determined. If the system is defined in state space, we use **bode(A, B, C, D)**. **bode(num, den, ω)** or **bode(A, B, C, D, iu, ω)** uses the user-supplied frequency vector ω. The scalar **iu** specifies which input is to be used for the frequency response. If the above commands are invoked with the left-hand arguments **[mag, phase, ω]**, the frequency response of the system in the matrices **mag**, **phase**, and ω are returned, and we need to use **plot** or **semilogx** functions to obtain the plot.

B.6.2 POLAR PLOT

A *polar plot*, also called the *Nyquist plot*, is a graph of $\Im G(j\omega)$ versus $\Re G(j\omega)$ with ω varying from $-\infty$ to $+\infty$. The polar plot may be directly graphed from sinusoidal steady-state measurements on the components of the open-loop transfer function.

Given a transfer function of a system, the *Control System Toolbox* function **nyquist(num, den)** produces the Nyquist plot with the frequency vector automatically determined. If the system is defined in state space, we use **nyquist(A, B, C, D)**. **nyquist(num, den, ω)** or **nyquist(A, B, C, D, iu, ω)** uses the user-supplied frequency vector ω. The scalar **iu** specifies which input is to be used for the Nyquist response. If the above commands are invoked with the left-hand arguments **[re, im, ω]**, the frequency response of the system in the matrices **re**, **im**, and ω are returned, and we need to use **plot(re, im)** function to obtain the plot.

B.6.3 RELATIVE STABILITY

The closed-loop transfer function of a control system is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KGH(s)} \tag{B.44}$$

For *BIBO* stability, poles of $T(s)$ must lie in the left-half s-plane. Since zeros of $1 + KGH(s)$ are poles of $T(s)$, the system is *BIBO* stable when the roots of the characteristic equation $1 + KGH(s)$ lie in the left-half s-plane. All points on the root locus satisfy the following conditions.

$$|\,KGH(s)\,| = 1 \qquad \text{and} \qquad \angle GH(s) = -180^\circ \tag{B.45}$$

The intersection of the polar plot with the negative real axis has a phase angle of $-180°$. The frequency ω_{pc} corresponding to this point is known as the *phase crossover frequency*. In addition, as the loop gain is increased, the polar plot crossing $(-1, 0)$ point has the property described by

$$|\,K_c GH(j\omega_{pc})\,| = 1 \qquad \text{and} \qquad \angle GH(j\omega_{pc}) = -180^\circ \tag{B.46}$$

The closed-loop response becomes marginally stable when the frequency response magnitude is unity and its phase angle is $-180°$. The frequency at which the polar plot intersect $(-1, 0)$ point is the same frequency that the root locus crosses the $j\omega$-axis. For a still larger value of K, the polar plot will enclose the $(-1, 0)$ point, and the system is unstable.

Thus, the system is stable if

$$|\,KGH(j\omega)\,| < 1 \qquad \text{at} \qquad \angle GH(j\omega_{pc}) = -180^\circ \tag{B.47}$$

The proximity of the $KGH(j\omega)$ plot in the polar coordinates to the $(-1, 0)$ point gives an indication of the stability of the closed-loop system.

B.6.4 GAIN AND PHASE MARGINS

Gain margin and *phase margin* are two common design criteria related to the open-loop frequency response. The *gain margin* is the amount of gain by which the gain of a stable system must be increased for the polar plot to pass through the $(-1, 0)$ point. The gain margin is defined as

$$G.M. = \frac{K_c}{K} \tag{B.48}$$

where K_c is the critical loop gain for marginal stability and K is the actual loop gain. The above ratio can be written as

$$G.M. = \frac{K_c\,|GH(j\omega_{pc})|}{K\,|GH(j\omega_{pc})|} = \frac{1}{K\,|GH(\omega_{pc})|} = \frac{1}{a} \tag{B.49}$$

In terms of decibels, the gain margin is

$$G.M._{dB} = 20\log_{10}(G.M.) = -20\log_{10}|KGH(j\omega_{pc})| = -20\log_{10} a \tag{B.50}$$

The gain margin is simply the factor by which K must be changed in order to render the system unstable. The gain margin alone is inadequate to indicate relative stability when system parameters affecting the phase of $GH(j\omega)$ are subject to variation. Another measure, called *phase margin*, is required to indicate the degree of stability. Let ω_{gc}, known as the *gain crossover frequency*, be the frequency at which the open-loop frequency response magnitude is unity. The phase margin is the angle in degrees through which the polar plot must be rotated about the origin in order to intersect the $(-1, 0)$ point. The phase margin is given by

$$P.M. = \angle GH(j\omega_{gc}) - (-180°) \tag{B.51}$$

For satisfactory performance, the phase margin should be between 30° and 60°, and the gain margin should be greater than 6 dB. The *MATLAB Control System Toolbox* function **[Gm, Pm, ω_{pc}, ω_{gc}] = margin(mag, phase ω)** can be used with **bode** function for evaluation of gain and phase margins, ω_{pc} and ω_{gc}.

B.6.5 NYQUIST STABILITY CRITERION

The *Nyquist stability criterion* provides a convenient method for finding the number of zeros of $1+GH(s)$ in the right-half s-plane directly from the Nyquist plot of $GH(s)$. The Nyquist stability criterion is defined in terms of the $(-1, 0)$ point on the Nyquist plot or the *zero-dB*, 180° point on the Bode plot. The Nyquist criterion is based upon a theorem of complex variable mathematics developed by Cauchy. The *Nyquist diagram* is obtained by mapping the *Nyquist path* into the complex plane via the mapping function $GH(s)$. The Nyquist path is chosen so that it encircles the entire right-half s-plane. When the s-plane locus is the Nyquist path, the Nyquist stability criterion is given by

$$Z = N + P \tag{B.52}$$

where

P = number of poles of $GH(s)$ in the right-half s-plane,

N = number of clockwise encirclements of $(-1, 0)$ point by the Nyquist diagram,

Z = number of zeros of $1 + GH(s)$ in the right-half s-plane.

For the closed-loop system to be stable, Z must be zero, that is

$$N = -P \tag{B.53}$$

B.6.6 SIMPLIFIED NYQUIST CRITERION

If the open-loop transfer function $GH(s)$ does not have poles in the right-half s-plane ($P = 0$), it is not necessary to plot the complete Nyquist diagram; the polar plot for ω increasing from 0^+ to ∞ is sufficient. Such an open-loop transfer function is called *minimum-phase* transfer function . For minimum-phase open-loop transfer functions, the closed-loop system is stable if and only if the polar plot lies to the right of $(-1, 0)$ point. For a minimum-phase open-loop transfer function, the criterion is defined in terms of the polar plot crossing with respect to $(-1, 0)$ point, as follows.

Right of	-1	stable	$\omega_{pc} > \omega_{gc}$	$\lvert GH(j\omega_{pc})\rvert < 1,\ GM_{dB} > 0,\ PM > 0°$
On	-1	marg. stable	$\omega_{pc} = \omega_{gc}$	$\lvert GH(j\omega_{pc})\rvert = 1,\ GM_{dB} = 0,\ PM = 0°$
Left of	-1	not stable	$\omega_{pc} < \omega_{gc}$	$\lvert GH(j\omega_{pc})\rvert > 1,\ GM_{dB} < 0,\ PM < 0°$

If P is not zero, the closed-loop system is stable if and only if the number of counterclockwise encirclements of the Nyquist diagram about $(-1, 0)$ point is equal to P.

The *MATLAB Control System Toolbox* function **[re, im] = nyquist(num, den, ω)** can obtain the Nyquist diagram by mapping the Nyquist path. However, the argument ω is specified as a real number. In order to map a complex number $s = a+jb$, we must specify $\omega = -js$, since the above function automatically multiplies ω by the operator j. To avoid this, the developed function **[re, im] = cnyquist(num, den, s)** can be used, where the argument s must be specified as a complex number. In defining the Nyquist path, care must be taken for the path not to pass through any poles or zeros of $GH(s)$.

B.6.7 CLOSED-LOOP FREQUENCY RESPONSE

The *closed-loop frequency response* is the frequency response of the closed-loop transfer function $T(j\omega)$. The *Control System Toolbox* function **bode**, described in Section B.6.1, is used to obtained the closed-loop frequency.

The performance specifications in terms of closed-loop frequency response are the closed-loop system *bandwidth* ω_B and the closed-loop system *resonant peak magnitude* M_p. The bandwidth ω_B is defined as the frequency at which the $|T(j\omega)|$ drops to 70.7 percent of its zero frequency value, or 3 dB down from the zero frequency value. The bandwidth indicates how well the system tracks an input sinusoid and is a measure of the speed of response. If the bandwidth is small, only signals of relatively low frequency are passed, and the response is slow; whereas, a large bandwidth corresponds to a fast rise time. Therefore, the rise time and the bandwidth are inversely proportional to each other. The frequency at which the peak occurs, the *resonant frequency* , is denoted by ω_r, and the maximum amplitude, M_p, is called the resonant peak magnitude. M_p is a measure of the relative stability of the system. A large M_p corresponds to the presence of a pair of dominant closed-loop poles with small damping ratio, which results in a large maximum overshoot of the step response in the time domain. If the gain K is set so that the open-loop frequency response $GH(j\omega)$ passes through the $(-1, 0)$ point, M_p will be infinity. In general, if M_p is kept between 1.0 and 1.7, the transient response will be acceptable. The developed function **frqspec(w, mag)** calculates M_p, ω_r, and the bandwidth ω_B from the frequency response data.

B.6.8 FREQUENCY RESPONSE DESIGN

The frequency response design provides information on the steady-state response, stability margin, and system bandwidth. The transient response performance can be estimated indirectly in terms of the phase margin, gain margin, and resonant peak magnitude. Percent overshoot is reduced with an increase in the phase margin, and the speed of response is increased with an increase in the bandwidth. Thus, the gain crossover frequency, resonant frequency, and bandwidth give a rough estimate of the speed of transient response.

A common approach to the frequency response design is to adjust the open-loop gain so that the requirement on the steady-state accuracy is achieved. This is called the *proportional controller*. If the specifications on the phase margin and gain margin are not satisfied, then it is necessary to reshape the open-loop transfer function by adding the additional controller $G_c(s)$ to the open-loop transfer function. $G_c(s)$ must be chosen so that the system has certain specified characteristics. This can be accomplished by combining proportional with integral action (PI) or proportional with derivative action (PD). There are also proportional-plus-integral-plus-derivative (PID) controllers with the following transfer function.

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \tag{B.54}$$

The ideal integral and differential compensators require the use of active amplifiers. Other compensators which can be realized with only passive network elements are lead, lag, and lead-lag compensators. A first-order compensator having a single zero and pole in its transfer function is

$$G_c(s) = \frac{K_c(s + z_0)}{s + p_0} \tag{B.55}$$

Several functions have been developed for the selection of suitable controller parameters based on the satisfaction of frequency response criteria, such as gain margin and phase margin. These functions tabulated below.

Alternatively, the function **[numo, deno, denc] = frdesign(num, den)** allows the user to select any of the above controller designs where **num** and **den** are row vectors of polynomial coefficients of the uncompensated open-loop plant transfer function. The function returns the open-loop and closed-loop numerators and denominators of the compensated system transfer function.

Function	Controller
[numo, deno, denc] = frqp(num, den)	Proportional
[numo, deno, denc] = frqlead(num, den)	Phase-lead
[numo, deno, denc] = frqlag(num, den)	Phase-lag
[numo, deno, denc] = frqpd(num, den)	PD
[numo, deno, denc] = frqpi(num, den)	PI
[numo, deno, denc] = frqpid(num,den)	PID

Example B.3

Design a PID controller for the system of Example B.1 for a compensated system phase margin of 77.8°. Choose a value of 9.09 for K_I, and select the new phase crossover frequency of 1.53 rad/s. Also, obtain the Bode plot of the compensated open-loop transfer function. The following commands:

```
num = 100;   den = [1  26  216  576];
[numo, deno, denc]=frdesign(num, den); % PID design
w = .1:.1:20;
[mag, phase] =bode(numo, deno, w); dB = 20*log10(mag);
figure(1), plot(w, dB), grid
xlabel('w, rad/sec'), ylabel('dB')
figure(2), plot(w, phase), grid
xlabel('w, rad/sec'), ylabel('Degrees')
```

result in

```
      Compensator type                          Enter
      Gain compensation                           1
      Phase-lead                                  2
      Phase-lag                                   3
      PD Controller                               4
      PI Controller                               5
      PID Controller                              6
      To quit                                     0
Enter your choice → 6
Enter the integrator gain KI → 9.09
Enter desired Phase Margin → 77.8
Enter wgc → 1.53
Uncompensated control system
Gain Margin = 50.4 Gain crossover w = NaN
Phase Margin = Inf Phase crossover w = 14.7
Gc = 2.10655 + 9.09/s + 0.14074s
Row vectors of polynomial coefficients of the compensated
system:
 Open-loop num.    14.07  210.65     909
 Open-loop den.        1      26     216     576    0
 Closed-loop den       1      26  230.07  786.65  909
```

```
     Gain Margin = 30300  Gain crossover w = 1.53
     Phase Margin = 77.8  Phase crossover w = 653
     Bandwidth = 1.95
Roots of the compensated characteristic equation:
-9.9943 + 1.0097i
-9.9943 - 1.0097i
-3.1671
-2.8444
```

The PID controller increases the system type by 1. That is, we have a type 1 system, and the steady-state error due to a unit step input is zero, and the step response is similar to Figure B.6. The compensated open-loop Bode plot is shown in Figure B.7.

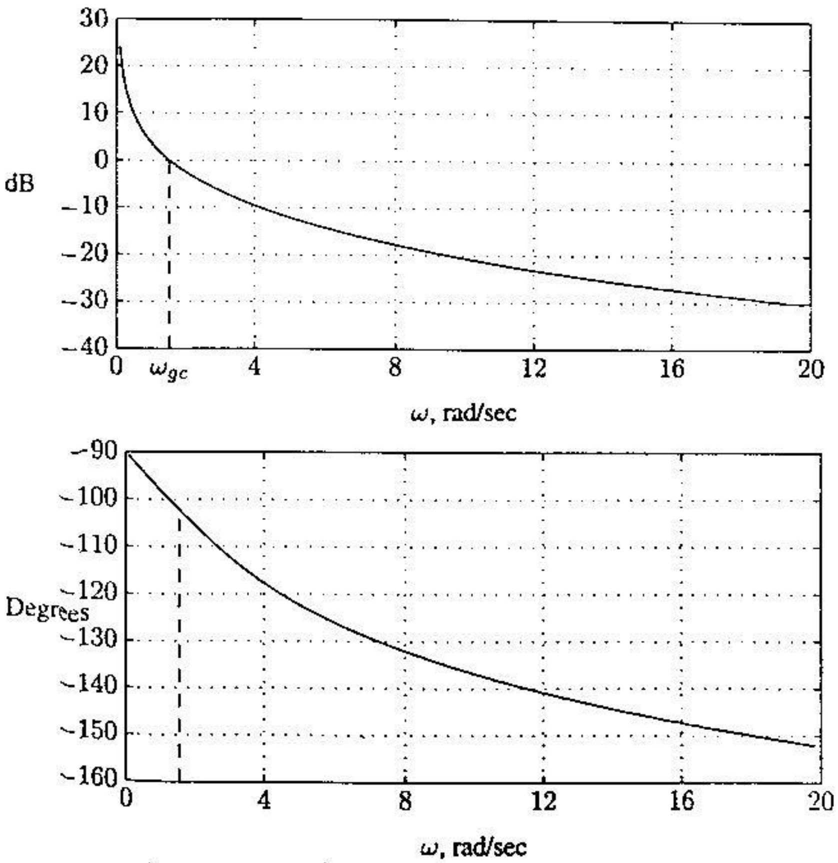


FIGURE B.7
The compensated open-loop Bode plot for Example B.3.

APPENDIX C

POWER SYSTEM TOOLBOX

The Power System Toolbox, containing a set of M-files, has been developed by the author to assist in typical power system analysis. Some of the programs, such as power flow, optimization, short-circuit and stability analysis, were originally developed by the author for a mainframe computer when working for power system consulting firms many years ago. These programs have been refined and modularized for interactive use with *MATLAB* for many problems related to the operation and analysis of power systems. The software modules are structured in such a way that the user may mix them for other power system analyses. The M-files for typical power system analyses are designed to work in synergy and communicate with each other through the use of some global variables.

The software diskette included with this book contains all the developed functions and chapter examples. Instructions for installing the Power System Toolbox can be found with the Installing the Text Toolbox described in Appendix A. We recommend that you store the files from this toolbox in a directory named *power*, where the *MATLAB* toolbox resides. Add the necessary search path to the *MATLAB* Path Browser. This appendix contains a list of all functions and script files in the Power System Toolbox developed by the author. The file names for the chapter examples are also included.

LIST OF FUNCTIONS, SCRIPT FILES, AND EXAMPLES IN THE POWER SYSTEM TOOLBOX

Load Cycle	
barcycle(data)	Plot load cycle for a given load interval

Transmission Line Parameters	
[GMD, GMRL, GMRC] = gmd	Multicircuits GMD and GMR
[L, C] = gmd2lc	Multicircuit GMD, GMRL, L, and C
acsr	Displays the ACSR characteristics

Transmission Line Performance	
lineperf	Line performance program
[r, L, C, f] = abcd2rlc(ABCD)	ABCD to rLC conversion
[Z, Y, ABCD] = abcd2pi(A, B, C)	ABCD to π model conversion
[Z, Y, ABCD] = pi2abcd(Z, Y)	π model to ABCD conversion
[Z, Y, ABCD] = rlc2abcd(r, L, C, g, f, Ln)	rLC to ABCD conversion
[Z, Y, ABCD] = zy2abcd(z, y, Ln)	zy to ABCD conversion
listmenu	Displays 8 options for analysis
givensr(ABCD)	Sending end values from receiving end power
givenss(ABCD)	Receiving end values from sending end power
givenzl(ABCD)	Sending end values from load impedance
loadabil(L, C, f)	Line loadability curves
openline((ABCD)	Open line analysis and reactor compensation
shcktlin(ABCD)	Receiving end short circuit
compmenu	Displays 3 options for capacitive compensation
sercomp(ABCD)	Series capacitor compensation
shntcomp(ABCD)	Shunt capacitor compensation
srshcomp(ABCD)	Shunt and series capacitors compensation
profmenu	Displays two options for loadabil and vprofile
pwrcirc(ABCD)	Receiving end power circle diagram
vprofile(ABCD)	Voltage curves for various loading

Optimal Dispatch of Generation	
bloss	Returns loss coefficients when followed by power flow program
dispatch	Obtains optimum dispatch of generation
gencost	Computes the total generation cost $/hr

Transformer and Induction Motor	
trans	Transformer characteristics
tperf	This script file is called by trans
[Rc, Xm] = troct(Vo, Io, Po)	Shunt branch from OC test
[Ze] = trsct(Vsc, Isc, Psc)	Obtains the series branch from SC test
[Ze1, Ze2]= trsct(E1, E2, Z1, Z2)	Winding impedances to Eq. impedance
rotfield	Revolving field demonstration
im	Equivalent circuit analysis
imchar	Torque/speed curve (called by im)
imsol	Motor performance (called by im)

Power Flow Analysis	
ybus	Obtains Y_{bus}, given R and X values
lfybus	Obtains Y_{bus}, given π model with specified *linedata* file
lfgauss	Power flow solution by the Gauss-Seidel method
lfnewton	Power flow solution by the Newton-Raphson method
decouple	Power flow solution by the Fast Decoupled method
busout	Returns the bus output result in tabular form
lineflow	Returns the line flow and losses in tabular form

Symmetrical Components	
sctm	Symmetrical Components Transformation Matrix
phasor(F)	Plots phasors expressed in rectangular or polar
F012 = abc2sc(Fabc)	Phasors to symmetrical components conversion
Fabc = sc2abc(F012)	Symmetrical components to phasors conversion
Z012 = zabc2sc(Zabc)	Impedance matrix to symmetrical components
Fr = pol2rec(Fp)	Polar phasor to rectangular phasor conversion
Fp = rec2pol(Fr)	Rectangular phasor to polar phasor conversion

Fault Analysis	
dlgfault(Z0, Zbus0, Z1, Zbus1, Z2, Zbus2, V)	Double line-to-ground fault
lgfault(Z0, Zbus0, Z1, Zbus1, Z2, Zbus2, V)	Line-to-ground fault
llfault(Z1, Zbus1, Z2, Zbus2, V)	Line-to-line fault
symfault(Z1, Zbus1, V)	Line-to-ground fault
Zbus = zbuild(zdata)	Builds the Bus Impedance Matrix
Zbus = zbuildpi(linedata, gendata, load)	Builds the Bus Impedance Matrix, compatible with load flow data

Synchronous Machine Transients	
lgshort(t,i)	Returns state derivatives of current for L-G short circuit
llshort(t, i)	Returns state derivatives of current for line-line short circuit
symshort(t, i)	Returns state derivatives of current for 3-phase short circuit

Power System Stability	
cctime	Obtains the critical clearing time for fault
eacfault(P0, E, V, X1, X2, X3)	Displays equal area criterion & finds critical clearing time of fault
eacpower(P0, E, V, X)	Displays equal area criterion & max. steady-state power
xdot = afpower(t, x)	One-machine system state derivative after fault
xdot = pfpower(t, x)	One-machine system state derivative during fault
swingmeu(Pm, E, V, X1, X2, X3, H, f, tc, tf)	One-machine swing curve, modified Euler
swingrk2(Pm, E, V, X1, X2, X3, H, f, tc, tf)	One-machine swing curve, *MATLAB* ode23
swingrk4(Pm, E, V, X1, X2, X3, H, f, tc, tf)	One-machine swing curve, *MATLAB* ode34
xot = afpek(t, x)	Multimachine system state derivative after fault
xdot = dfpek(t, x)	Multimachine system state derivative during fault
trstab	Stability analysis works in synergy with load flow
[Ybus, Ybf] = ybusbf(linedata, yload, nbus1, nbust)	Multimachine system reduced Y_{bus} before fault
Ypf = ybusbf(Ybus, nbus1, nbust, nf)	Multimachine system reduced Y_{bus} during fault
Yaf = ybusaf(linedata, yload, nbus1, nbust, nbrt)	Multimachine system reduced Y_{bus} after fault

Control System Functions	
electsys	Returns the state derivatives for Example A.19
errortf	Steady-state error, transfer function in polynomial form
errorzp	Steady-state error, transfer function in zero pole form
frcntrl	Frequency response design equations
frdesign	Frequency response design program
frqlag	Frequency response design phase-lag controller
frqlead	Frequency response design phase-lead controller
frqp	Frequency response design P controller
frqpd	Frequency response design PD controller
frqpi	Frequency response design PI controller
frqpid	Frequency response design PID controller
frqspec	Frequency response performance specifications
ghs	Returns magnitude and phase of a complex function $GH(s)$
ltstm	Laplace transform of state transition matrix
mechsys	Returns the state derivatives for Example A.18
pcomp	Root-locus design P controller
pdcomp	Root-locus design PD controller
pdlead	Root-locus design phase-lead controller
pendulum	Returns the state derivatives for Example A.20
phlag	Root-locus design phase-lag controller
picomp	Root-locus design PI controller
pidcomp	Root-locus design PID controller
placepol	Pole-placement design
pnetfdbk	Feedback compensation using passive elements
riccasim	Returns state derivative of Riccati equation
riccati	Optimal regulator design
rldesign	Root-locus design program
routh	Routh-Hurwitz array
ss2phv	Transformation to control canonical form
statesim	Returns state derivatives for use in Riccati equation
stm	Determines the state transition matrix $\phi(t)$
system	System matrices defined for use in Riccati equation
tachfdbk	Tachometer feedback control
timespec	Time-domain performance specifications

List of M-Files for Chapter Examples

CHP1EX1	CHP5EX5	CHP7EX7	CHP10EX3	EXA1
CHP2EX1	CHP5EX6	CHP7EX8	CHP10EX4	EXA2
CHP2EX2	CHP5EX7	CHP7EX9	CHP10EX5	EXA3
CHP2EX3	CHP5EX8	CHP7EX10	CHP10EX6	EXA4
CHP2EX4	CHP5EX9	CHP7EX11	CHP10EX7	EXA5
CHP2EX5	CHP6EX1	CHP8EX1	CHP10EX8	EXA6
CHP2EX6	CHP6EX2	CHP8EX2	CHP11EX1	EXA7
CHP2EX7	CHP6EX3	CHP8EX3	CHP11EX2	EXA8
CHP2EX8	CHP6EX4	CHP8EX4	CHP11EX3	EXA9
CHP3EX1	CHP6EX5	CHP8EX5	CHP11EX4	EXA10
CHP3EX2	CHP6EX6	CHP8EX6	CHP11EX5	EXA11
CHP3EX3	CHP6EX7	CHP8EX7	CHP11EX6	EXA12
CHP3EX4	CHP6EX8	CHP8EX8	CHP11EX7	EXA13
CHP3EX5	CHP6EX9	CHP8EX9	CHP12EX1	EXA14
CHP3EX6	CHP6EX10	CHP8EX10	CHP12EX2	EXA15
CHP4EX1	CHP6EX11	CHP9EX1	CHP12EX3	EXA16
CHP4EX2	CHP6EX12	CHP9EX2	CHP12EX4	EXA17
CHP4EX3	CHP6EX13	CHP9EX3	CHP12EX5	EXA18
CHP4EX4	CHP6EX14	CHP9EX4	CHP12EX6	EXA19
CHP4EX5	CHP6EX15	CHP9EX5	CHP12EX7	EXA20
CHP4EX6	CHP7EX1	CHP9EX6	CHP12EX8	EXB1
CHP4EX7	CHP7EX2	CHP9EX7	CHP12EX9	EXB2
CHP5EX1	CHP7EX3	CHP9EX8	CHP12XX	EXB3
CHP5EX2	CHP7EX4	CHP9EX9	CHP12XX1	
CHP5EX3	CHP7EX5	CHP10EX1		
CHP5EX4	CHP7EX6	CHP10EX2		

List of SIMULINK-Files for Chapter Examples

SIM11EX3	SIM11EX6	SIM12EX1	SIM12EX3	SIM12EX4
SIM12EX5	SIM12EX6	SIM12EX7	SIM12EX8	SIM12EX9
SIM12XXB	SIM12XXD	SIM12XX1	SIMEXA21	SIMEXA22
SIMEXA23	SIMEXA24	SIMEXA25	SIMEXA26	SIMEXA27
SIMEXA28	SIMEXB1			

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ANSWERS TO PROBLEMS

Chapter 1

1.1. 799.34 GW

1.2. 6.93%

1.3. 8 MW, 50%

Chapter 2

2.1. $S_1 = 2000$ W $+ j3464.1$ var, $S_2 = 2165.1$ W $- j1250$ var, $S_3 = 2000$ W $+ j0$ var, $S = 6165.1$ W $+ j2214.1$ var

2.2. (a) 800 W $+ j600$ var, (b) $10\cos(377t - 36.87°)$ A, $7.071\angle -36.87°$ A, (c) $20\angle 36.87°\ \Omega$

2.3. 12.8 Ω, 9.6 Ω

2.4. 20Ω, 26.67Ω

2.5. 280 kW $+ j335$ kvar

2.6. (a) 60 Ω, 80 Ω, (b) $1250\angle 16.26°$ V

2.7. (a) $S_1 = 1$ kW $+ j7$ kvar, $S_2 = 1$ kW $- j2$ kvar, $S_3 = 4$ kW $+ j3$ kvar, (b) $S = 6$ kW $+ j8$ Kvar, $50\angle -53.13°$ A, 0.6 lagging, (c) 8 kvar, 530.5 μF, 30 A

2.8. Source 1 delivers 28 kW and receives 21 kvar, Source 2 receives 24.57 kW and delivers 32.76 kvar, 3.43 kW, 11.76 kvar

2.10. (b) 30 kW

2.11. (a) $50\angle -36.87°$ A, $50\angle -156.87°$ A, $50\angle -276.87°$ A, (b) 288 kW, 216 kvar

2.12. (a) $150\angle -66.87°$ A, $150\angle -186.87°$ A, $50\angle 53.13°$ A, (b) 864 kW, 648 kvar

2.13. (a) $12\angle -53.13°$ A, (b) 2592 W + j 3456 var, (c) 162.33 V

2.14. (a) 18 kW, 0 kvar, unity power factor, 50 A, (b) $66.9\angle -41.63°$ A,0.7474 lagging

2.15. (a) 360 kW $+ j480$ kvar, 0.6 lagging, $27.78\angle -53.13°$ A, (b) 210 kvar, 3.58 μF, $20.835\angle -36.87°$ A

2.16. (a) $40\angle -36.87°$ A, 19.2 kW $+ j14.4$ kvar, (b) 160 V, 277.1 V

Chapter 3

3.1. (a) 2440.80 V, 6.12 %, (b) 2200.4 V, −4.33%

3.2. (a) 36 kV, 9.59°, (b) 288 MW, (c) 547.47 A, 0.7306 lagging

3.3. (a) 12806 V, (b) 80.4 MW, (c) $3344\angle 36.73°$

3.4. (b) 16.26°, 30 kV, (c) 138.712 MW at 75°

3.5. (a) $0.4 + j0.9\ \Omega$, 1000 Ω, $j1500\Omega$, (b) 2453.9 V, 2.247%, (c) 2387 V, −0.541%

3.6. (a) $28 + j96\ \Omega$, 6666.67 Ω, $j5000\ \Omega$, (b) 21.839%, 85.97%, (c) 53.237 kVA, 86.057%, (d) 85.88%

3.7. (a) 21 kVA, (b) 96%

3.8. 13.346 kV

3.9. (a) 247.69 kV, (b) 249.72 kV

3.10. (a) 1.03205 pu, 247.69 kV, (b) 1.0405 pu, 249.72 kV

3.11. 0.926 pu, 1.0 pu

3.12. $0.122 + j0.252$ pu

3.13. $X_{G_1} = j0.1$, $X_{T_1} = j0.2$, $X_{T_2} = j0.25$, $X_{G_2} = j0.081$, $X_{Line} = j0.3$, $X_{Load} = 0.75 + j1.0$,

3.14. $X_G = j0.3$, $X_{T_1} = j0.2$, $X_{T_2} = j0.15$, $X_{T_3} = j0.16$ $X_{Line_1} = j0.25$, $X_{Line_2} = j0.35$ $X_M = j0.27$, $X_{Load} = -j10$, $Z_P = j0.06$, $Z_S = j0.18$, $Z_T = j0.12$

3.15. (a) $X_{G_1} = j0.15$, $X_{T_1} = j0.2$, $X_{T_2} = j0.2$, $X_{Line} = 0.3 + j.5$ $X_M = j0.15$, (b) 26.359 kV, 27.5 kV

3.16. 440 kV , 480 kV

3.17. 126.5 kV, 27.6 kV

Chapter 4

4.1. (a) 4.1 Ω, (b) 4.6 Ω

4.2. 0.3774 Ω/km

4.3. 1.894 cm, 556000 cmil

4.4. 0.35 cm, 46 mH

4.5. (a) $1.46r$, (b) $1.723r$

4.6. 1.486 mH/km

4.7. 10 m

4.8. (a) 1.3 mH/km, (b) 1.15 cm

4.9. 27.5% decrease, 35.25% increase

4.10. 0.88929 mH/km, 0.012658 μF/km

4.11. 0.4752 mH/km, 0.0240035 μF/km, 0.517453 mH/km, 0.0219974 μF/km

4.12. $\frac{4\pi\varepsilon}{\ln 0.866D/r}$

4.13. 5 V/km

4.14. 90 V

Chapter 5

5.1. (a) 70.508 kV, 10.17%, 58.39 MW $+j50.37$ Mvar, 95.90%,
(b) 69.0 kV, 7.83%, 127 MW $+j24.61$ Mvar, 94.465%

5.2. (a) 14.117 Mvar, 9.14μF, (b) 61.24 Mvar, 39.66μF

5.3. (a) $0.9951 + j0.000544$, $4 + j36$ Ω, $j0.0002713$ S
(b) 242.67 kV, $502.38\angle-33.69°$ A, 10.847%, 163.18 MW $+j134.02$ Mvar, 98.052%
(c) 230.03 kV, $799.86\angle 2.5°$ A, 5.073%, 313.74 MW $+j55.9$ Mvar, 97.53%

5.4. 141.123 Mvar, 7.734μF

5.5. $0.98182 + j0.0012447$, $4.035 + j58.947$, $j0.00061137$

5.6. 387.025 kV, $592.29\angle-27.325°$ A, 324.87 MW, $+j228.25$ Mvar, 14.259%, 98.5%

5.7. 345 kV, $672.54\angle-9.633°$ A, 401.884 MW $+j0.0$ Mvar, 2.73%, 98.743%

5.8. (a) 264.702 Ω, 29°, 4965.2 km, 2210.88 MW, 0.8746, $j128.34$ Ω, $j0.0018316$ S
(b) 896.982 kV, $1100.23\angle-2.456°$ A, 1600 MW $+j601.508$ Mvar, 39.536%
(c) 653.33 kV, $1748.78\angle-43.556°$ A, 1920 MW $+j479.33$ Mvar, 33.88%
(d) 735.13 kV, $1604.07\angle 28.98°$ A, 2042.44 MW $+j1.32$ Mvar, 14.358%

5.9. 874.68 kV, (b) 772.13 Ω, 699.658 Mvar

5.10. $3441.47\angle-90°$ A, $3009.92\angle-90°$ A

5.11. 802.95 Mvar, 3.943 μF, $1209.46\angle 24.653°$ A, 1600 MW $-j90.38$ Mvar, 19%

5.12. 822.677 kV, $1164.59\angle-3.625°$ A, 1600 MW $+j440.16$ Mvar, 21.035%

5.13. 81.464 Mvar, 51.65 μF, 563.25 Mvar, 2.765 μF, 765 kV, $1209.72\angle 16.1°$ A, 1600 MW $-j96.32$ Mvar, 12.55%

5.14. Use **lineperf** to obtain the transmission line performance. Present a summary of the calculations along with your recommendations.

5.15. (a) 622.153 kV, $794.649\angle-1.33°$ A, 800 MW $+j305.408$ Mvar, 44.687%
(b) 0.96, $j39.2$, $j0.002$
(c) 530.759 kV, $891.142\angle-5.65°$ A, 800 MW $+j176.448$ Mvar, 10.575%

5.16. (a) 0.002 Rad/km, 500 Ω, (b) 1000 Ω, 176.4 Mvar

5.17. 400 kV

Chapter 6

6.1. $$Y_{bus} = \begin{bmatrix} 0.0 - j20.25 & 0.0 + j4.00 & 0.0 + j10.00 & 0.0 + j2.50 \\ 0.0 + j4.00 & 0.0 - j15.00 & 0.0 + j0.00 & 0.0 + j6.25 \\ 0.0 + j10.00 & 0.0 + j0.0 & 1.0 - j15.00 & 0.0 + j5.00 \\ 0.0 + j2.50 & 0.0 + j6.25 & 0.0 + j5.00 & 2.0 - j14.00 \end{bmatrix}$$

6.2. $$V_{bus} = \begin{bmatrix} 1.0293\angle 1.46° \\ 1.0217\angle 0.99° \\ 1.0001\angle -0.015° \end{bmatrix}$$

6.3. (a) $x_1 = 5.0000 \quad x_2 = 1.0000$, $x_1 = 2.0006 \quad x_2 = 3.9994$

6.4. (a) 1, (b) 1, 4, 7, 9

6.5. $$X^{(1)} = \begin{bmatrix} 4.3929 \\ 4.9286 \end{bmatrix} \quad X^{(2)} = \begin{bmatrix} 4.0222 \\ 4.9964 \end{bmatrix} \quad X^{(3)} = \begin{bmatrix} 4.0001 \\ 5.0000 \end{bmatrix}$$

6.6. (a) $V_2^{(1)} = 0.9200 - j0.1000 \quad V_2^{(2)} = 0.9024 - j0.0981$
$V_2^{(3)} = 0.9005 - j0.1000 \quad V_2^{(4)} = 0.9001 - j0.1000$
(b) $S_{12} = 300$ MW $+j100$ Mvar
$S_{21} = -280$ MW $-j60$ Mvar
$S_L = 20$ MW $+j40$ Mvar

6.7. (a) $V_2^{(1)} = 0.9360 - j0.0800 \quad V_3^{(1)} = 0.9602 - j0.0460$
$V_2^{(2)} = 0.9089 - j0.0974 \quad V_3^{(2)} = 0.9522 - j0.0493$
(b) $S_{12} = 300$ MW $+ j300$ Mvar
$S_{21} = -300$ MW $- j240$ Mvar
$S_{L_{12}} = 0$ MW $+ j60$ Mvar
$S_{13} = 400$ MW $+ j400$ Mvar
$S_{31} = -400$ MW $- j360$ Mvar
$S_{L_{13}} = 0$ MW $+ j40$ Mvar
$S_{23} = -100$ MW $- j80$ Mvar
$S_{32} = 100$ MW $+ j90$ Mvar
$S_{L_{23}} = 0$ MW $+ j10$ Mvar
$S_1 = 700$ MW $+ j700$ Mvar

6.8. (a) $V_2^{(1)} = 1.0025 - j0.0500 \quad Q_3^{(1)} = 1.2360$
$V_3^{(1)} = 1.0299 + j0.0152$
$V_2^{(2)} = 1.0001 - j0.0409 \quad Q_3^{(2)} = 1.3671$
$V_3^{(2)} = 1.0298 + j0.0216$
(b) $S_{12} = 150.428$ MW $+ j100.159$ Mvar
$S_{21} = -150.428$ MW $- j92.387$ Mvar
$S_{L_{12}} = 0$ MW $+ j7.772$ Mvar
$S_{13} = -50.428$ MW $- j9.648$ Mvar
$S_{31} = 50.428$ MW $+ j10.902$ Mvar
$S_{L_{13}} = 0$ MW $+ j1.255$ Mvar
$S_{23} = -249.572$ MW $- j107.613$ Mvar
$S_{32} = 249.572$ MW $+ j126.034$ Mvar
$S_{L_{23}} = 0$ MW $+ j18.421$ Mvar
$S_1 = 100$ MW $+ j90.51$ Mvar

6.9. $$Y_{bus} = \begin{bmatrix} -j125 & 0 & j100 & 0 \\ 0 & -j6.25 & 0 & j5 \\ j100 & 0 & -j89 & j9 \\ 0 & j5 & j9 & -j13 \end{bmatrix}$$

6.10. $|V_2^{(1)}| = 0.9100 \quad \delta_2^{(1)} = -0.1300$ rad
$|V_2^{(2)}| = 0.8886 \quad \delta_2^{(2)} = -0.1464$ rad

6.11. $|V_2^{(1)}| = 0.8000 \quad \delta_2^{(1)} = -0.1000$ rad
$|V_2^{(2)}| = 0.7227 \quad \delta_2^{(2)} = -0.1350$ rad

6.12. The bus admittance matrix in polar form is
$$Y_{bus} = \begin{bmatrix} 60\angle-\frac{\pi}{2} & 40\angle\frac{\pi}{2} & 20\angle\frac{\pi}{2} \\ 40\angle\frac{\pi}{2} & 60\angle-\frac{\pi}{2} & 20\angle\frac{\pi}{2} \\ 20\angle\frac{\pi}{2} & 20\angle\frac{\pi}{2} & 40\angle-\frac{\pi}{2} \end{bmatrix}$$

(a) Substituting for the elements of the bus admittance matrix in (6.52) and (6.53) result in the given equations.
(b)
$\delta_2^{(1)} = 0.0275$ radian $= 1.5782°$ $\quad \delta_2^{(2)} = 0.0285$ radian $= 1.6327°$
$\delta_3^{(1)} = -0.1078$ radian $= -6.179°$ $\quad \delta_3^{(2)} = -0.1189$ radian $= -6.816°$
$|V_3^{(1)}| = 0.9231$ pu $\quad |V_3^{(2)}| = 0.9072$ pu
(c) The power flow program **lfnewton** is used to obtain the solution (See Example 6.9).

6.13. (a)
$\delta_2^{(1)} = 0.0262$ radian $= 1.5006°$ $\quad \delta_2^{(2)} = 0.0277$ radian $= 1.5863°$
$\delta_3^{(1)} = -0.1119$ radian $= -6.412°$ $\quad \delta_3^{(2)} = -0.1182$ radian $= -6.772°$
$|V_2^{(1)}| = 0.9250$ pu $\quad |V_3^{(2)}| = 0.9088$ pu
(b) The power flow program **decouple** is used to obtain the solution (See Example 6.11).

6.14. Follow the Instruction for Data Preparation (Section 6.9) and Example 6.9.

Chapter 7

7.1. A square of side length = 1.4142, perimeter = 5.6568
For $\lambda = -2.828$, $\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 L}{\partial y^2} = -5.6568$. Second derivatives are negative. Thus, objective function is maximized.

7.2. $x = y = -\frac{4}{3}$, $\lambda = \frac{4}{3}$
$\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 L}{\partial y^2} = 2$. Second derivatives are positive. Thus, objective function is minimized.

7.3. Base = 1.732, Height = 1.5, Area = 1.299

7.4. $\zeta = 0.5$, $\omega_n = 10,000$ rad/sec, $M_{p\omega} = 1.1547$

7.5. Minimum value of the function = 12.5, at $x = 1.5$, $y = 3.2$, $\lambda = 1$

7.6. Minimum value of the function = 17, at $x = 1$, $y = 4$, $\lambda = \frac{14}{13}$

7.7. (a) $P_1 = 250$ MW, $P_2 = 300$ MW
(b) $P_1 = 500$ MW, $P_2 = 800$ MW
(c) $\beta = 6.8$, $\gamma = 0.002$

7.8. (i) $C_t = 4,849.75$ \$/h (ii) $C_t = 7,310.46$ \$/h (iii) $C_t = 12,783.04$ \$/h

7.9. (i) $P_1 = 100$ MW, $P_2 = 140$ MW, $P_3 = 210$ MW, $\lambda = 8.0$
$C_t = 4,828.70$ \$/h
(ii) $P_1 = 175$ MW, $P_2 = 260$ MW, $P_3 = 310$ MW, $\lambda = 8.6$

$C_t = 7,277.20$ \$/h
(iii) $P_1 = 325$ MW, $P_2 = 500$ MW, $P_3 = 510$ MW, $\lambda = 9.8$
$C_t = 12,705.20$ \$/h
(c) Savings: (i) 21.05 \$/h (ii) 33.26 \$/h (iii) 77.84 \$/h

7.10. (i) $P_1 = 122$ MW, $P_2 = 260$ MW, $P_3 = 68$ MW, $\lambda = 7.148$
$C_t = 4,927.13$ \$/h
(ii) $P_1 = 175$ MW, $P_2 = 260$ MW, $P_3 = 310$ MW, $\lambda = 8.6$
$C_t = 7,277.20$ \$/h
(iii) $P_1 = 350$ MW, $P_2 = 540$ MW, $P_3 = 445$ MW, $\lambda = 10$
$C_t = 12,724.38$ \$/h

7.11. $P_1 = 161.1765$ MW, $P_2 = 258.6003$ MW, $\lambda = 7.8038$ $C_t = 3,375.43$ \$/h

7.12. $P_1 = 70.360$ MW, $P_2 = 181.557$ MW, $P_7 = 97.111$ MW, $\lambda = 8.1513$
$C_t = 3,194.85$ \$/h

Chapter 8

8.1. (a) $\alpha = 75.75°$, $i(t) = 3\sin 315t$
(b) $\alpha = -14.25°$, $i(t) = 3\sin(315t - \pi/2) + 3e^{-80t}$
(c) In *MATLAB* using [Imax, k] = max(i), tmax= t(k) result in
imax = 4.37 A, tmax = 0.0096 sec.

8.2. In the file chp8ex2.m set $d = 30°$, rename the file and run the program.

8.3. In the file chp8ex3.m set $d = 30°$, rename the file and run the program.

8.5. In the file chp8ex4.m set $d = 30°$, rename the file and run the program.

8.6. (a) 0.6667 pu, 2.2222 pu, 4.0 pu
(b) $i_{ac}(t) = (2.5142e^{-25t} + 2.2e^{-0.7143t} + 0.9428)\sin\omega t$

8.7. $i_{asy}(t) = (2.5142e^{-25t}+2.2e^{-0.7143t}+0.9428)\sin(\omega t+\pi/2)+5.6568e^{-3.3333t}$

8.8. $X'_d = 0.449$, $\tau'_d = 1.382$ sec, $X''_d = 0.2498$, $\tau''_d = 1.0397$ sec,

8.9. $i_{asy}(t) = (2.357e^{-25t} + 3.5355e^{-t} + 1.1785)\sin(\omega t + \pi/2) + 7.071e^{-4t}$
$I_{ac} = 5.0$ rms, $I_{dcmax} = 7.071$, $Iasy = 8.66$ rms

8.10. (a) $I''_d = 2.5$ pu, 7,216.88 A, 360.84 A
$I'_d = 2.0$ pu, 5,773.50 A, 288.68 A
$I_d = 0.6667$ pu, 1,924.5 A, 96.23 A
(b) $I_{asy} = 4.3333$ pu, 12,500 A, 625 A
(c) $i_{asy}(t) = (0.7071e^{-28.57t} + 1.8856e^{-2t} + 0.9428)\sin(\omega t + \pi/2) + 3.5355e^{-3.333t}$

8.11. $I'_g = 2.56\angle-75.53°$ pu, or $7393.69\angle-75.53°$ A

8.12. $I'_g = 3.545\angle-78.6°$ pu, $I'_m = 3.599\angle-95.3°$ pu, $I'_f = 7.068\angle-87.03°$ pu

Chapter 9

9.1. $2.0\angle-90°$ pu = $288.675\angle-90°$ A, 200 MVA

9.2. 1.8 Ω

9.3. (a) $j0.2$ pu, $5.0\angle-90°$ pu, (b) $V_1 = 0.4$ pu, $V_2 = 0.8$ pu, $V_3 = 0.7$ pu

9.4. (a) $j0.4$ pu, $2.5\angle-90°$ pu
(b) $V_1 = 0.925$ pu, $V_2 = 0.925$ pu, $V_3 = 0.475$ pu
$I_{12} = 0$ pu, $I_{13} = 1.5\angle-90°$ pu, $I_{23} = 1.0\angle-90°$ pu

9.5. (a) $j0.5$ pu, $2.0\angle-90°$ pu
(b) $V_1 = 0.60$ pu, $V_2 = 0.65$ pu, $V_3 = 0.38$ pu, $V_4 = 0$ pu
$I_{13} = 1.1\angle-90°$ pu, $I_{21} = 0.1\angle-90°$ pu, $I_{23} = 0.9\angle-90°$ pu
$I_{34} = 2.0\angle-90°$ pu
(c): (a) $j0.25$ pu, $4.0\angle-90°$ pu
(b) $V_1 = 0.44$ pu, $V_2 = 0.09$ pu, $V_3 = 0.3$ pu, $V_4 = 0.3$ pu
$I_{12} = 0.7\angle-90°$ pu, $I_{13} = 0.7\angle-90°$ pu, $I_{32} = 0.7\angle-90°$ pu

9.6. $$Z_{bus} = \begin{bmatrix} j0.2400 & j0.1400 & j0.2000 & j0.1400 \\ j0.1400 & j0.2275 & j0.1750 & j0.2275 \\ j0.2000 & j0.1750 & j0.3100 & j0.1750 \\ j0.1400 & j0.2275 & j0.1750 & j0.4175 \end{bmatrix}$$

9.7. $$Z_{bus} = \begin{bmatrix} j0.12 & j0.04 & j0.06 \\ j0.04 & j0.08 & j0.02 \\ j0.06 & j0.02 & j0.08 \end{bmatrix}$$

9.8. $$Z_{bus} = \begin{bmatrix} j0.0450 & j0.00750 & j0.0300 \\ j0.0075 & j0.06375 & j0.0300 \\ j0.0300 & j0.03000 & j0.2100 \end{bmatrix}$$

9.9. $$Z_{bus} = \begin{bmatrix} j0.32 & j0.16 & j0.28 \\ j0.16 & j0.48 & j0.24 \\ j0.28 & j0.24 & j0.42 \end{bmatrix}$$

9.10. Same as Problem 9.4

9.11. Same as Problem 9.5

9.12. $4.0\angle-90°$ pu
$V_1 = 0.46$ pu, $V_2 = 0.61$ pu, $V_3 = 0.16$ pu, $V_4 = 0.01$ pu
$I_{13} = 1.5\angle-90°$ pu, $I_{14} = 1.5\angle-90°$ pu, $I_{21} = 0.3\angle-90°$ pu,
$I_{24} = 1.0\angle-90°$ pu, $I_{34} = 1.5\angle-90°$ pu

9.13. Run chp9ex7 for a bolted fault at bus 9.

9.14. Run chp9ex8 for a bolted fault at bus 9.

9.15. Run chp9ex9 for a bolted fault at bus 9.

9.16–9.18. Make data similar to Examples 9.7–9.9.

Chapter 10

10.1. $V_a^0 = 42.265\angle -120^\circ$, $V_a^1 = 193.185\angle -135^\circ$, $V_a^2 = 86.947\angle -84.896^\circ$

10.2. $I_a = 8.185\angle 42.216^\circ$, $I_b = 4.0\angle -30^\circ$, $I_c = 8.185\angle -102.216^\circ$

10.4.

$$V_L^{012} = \begin{bmatrix} 0 \\ 763.763\angle -10.93^\circ \\ 288.675\angle 30^\circ \end{bmatrix} \quad V_{an}^{012} = \begin{bmatrix} 0 \\ 440.958\angle -40.89^\circ \\ 166.667\angle 60^\circ \end{bmatrix}$$

$$V^{abc} = \begin{bmatrix} 440.958\angle -19.106^\circ \\ 600.925\angle -166.102^\circ \\ 333.333\angle 60^\circ \end{bmatrix}$$

10.5. $$I_a^{012} = \begin{bmatrix} 20\angle 90^\circ \\ 60\angle -90^\circ \\ 40\angle 90^\circ \end{bmatrix}$$

10.6. $$I^{abc} = \begin{bmatrix} 20\angle -90^\circ \\ 20\angle 150^\circ \\ 20\angle 30^\circ \end{bmatrix}$$

10.7. (a)

$$Z^{012} = \begin{bmatrix} 10 + j50 & 0 & 0 \\ 0 & 10 + j35 & 0 \end{bmatrix} \quad (b) \quad V_a^{012} = \begin{bmatrix} 42.265\angle -120^\circ \\ 193.185\angle -135^\circ \\ 86.947\angle -84.896^\circ \end{bmatrix}$$

$$(c) \quad I_a^{012} = \begin{bmatrix} 0.829\angle 161.31^\circ \\ 5.307\angle 150.95^\circ \\ 2.388\angle -158.95^\circ \end{bmatrix} \quad (d) \quad I^{abc} = \begin{bmatrix} 7.907\angle 165.46^\circ \\ 5.819\angle 14.867^\circ \\ 2.701\angle -96.93^\circ \end{bmatrix}$$

(e) $S_{3\phi} = 1,036.8 + j3,659.6$

(f) Same as (e)

10.8.

$$V_{an}^{012} = \begin{bmatrix} 0 \\ 136.879\angle 139.933^\circ \\ 451.105\angle 54.603^\circ \end{bmatrix} \quad V^{abc} = \begin{bmatrix} 480.754\angle 70.560^\circ \\ 333.338\angle 163.741^\circ \\ 569.611\angle -73.685^\circ \end{bmatrix}$$

$$I^{abc} = \begin{bmatrix} 12.993\angle 70.561^\circ \\ 900.9\angle 163.741^\circ \\ 15.395\angle -73.686^\circ \end{bmatrix}$$

10.9. 1.8Ω

10.10. 0.825Ω

10.11. $I_a = 12\angle -90^\circ pu$

10.12. $I_b = -9.116$ pu

10.13. $I_f = I_a + I_b = 12.5\angle 90^\circ$ pu

10.14. (a) $5\angle -90^\circ$ pu, (b) $6\angle -90^\circ$ pu, (c) $-4.33\angle -90^\circ$ pu, (d) $7.5\angle 90^\circ$ pu

10.15. (a) $4.395\angle -90^\circ$ pu, (b) $4.669\angle -90^\circ$ pu, (c) $-3.807\angle -90^\circ$ pu, (d) $4.979\angle 90^\circ$ pu

10.16. $I_f = 4.6693\angle -90^\circ$ pu

Bus	Voltage Magnitude		
No.	Phase a	Phase b	Phase c
1	0.0000	0.9704	0.9704
2	0.5214	0.9567	0.9567
3	0.7977	0.9535	0.9535
4	0.8911	0.9739	0.9739

From	To	Line current magnitude		
Bus	Bus	Phase a	Phase b	Phase c
1	F	4.6693	0.0000	0.0000
2	1	1.4786	0.1556	0.1556
3	1	2.0234	1.0117	1.0117
4	2	1.0895	0.5447	0.5447

10.17. $I_f = -3.8067$ pu

Bus	Voltage Magnitude		
No.	Phase a	Phase b	Phase c
1	1.0000	0.5000	0.5000
2	1.0000	0.6401	0.6401
3	1.0000	0.7954	0.7954
4	1.0000	0.8871	0.8871

From	To	Line current magnitude		
Bus	Bus	Phase a	Phase b	Phase c
1	F	0.0000	3.8067	3.8067
2	1	0.0000	1.3323	1.3323
3	1	0.0000	2.4744	2.4744
4	2	0.0000	1.3323	1.3323

10.18. $I_f = 4.9793\angle 90°$ pu

Bus No.	Voltage Magnitude Phase a	Phase b	Phase c
1	0.9336	0.0000	0.0000
2	0.9004	0.4965	0.4965
3	0.8921	0.7626	0.7626
4	0.9419	0.8711	0.8711

From Bus	To Bus	Line current magnitude Phase a	Phase b	Phase c
1	F	0.0000	4.5486	4.5485
2	1	0.1660	1.5076	1.5076
3	1	1.0788	2.5325	2.5325
4	2	0.5809	1.3636	1.3636

10.19.

$$Z_{bus}^{(1)} = \begin{bmatrix} 0.120 & 0.040 & 0.030 & 0.020 \\ 0.040 & 0.080 & 0.010 & 0.040 \\ 0.030 & 0.010 & 0.045 & 0.005 \\ 0.020 & 0.04070.005 & 0.045 & \end{bmatrix}$$

Total fault current = 10.8253 per unit

Bus No.	Voltage Magnitude Phase a	Phase b	Phase c
1	1.0000	0.6614	0.6614
2	1.0000	0.5000	0.5000
3	1.0000	0.9079	0.9079
4	1.0000	0.6614	0.6614

From Bus	To Bus	Line current magnitude Phase a	Phase b	Phase c
1	2	0.0000	0.7217	0.7217
1	2	0.0000	1.4434	1.4434
2	F	0.0000	10.8253	10.8253
3	1	0.0000	2.1651	2.1651
4	2	0.0000	8.6603	8.6603

10.20 Run chp10ex8 for a bolted fault at bus 9.

10.21. Make data similar to Example 10.8.

Chapter 11

11.1. $M = 8.5$ MJ.rad/sec, $H = 4.0$ MJ/MVA

11.2. $600°/\text{sec}^2 = 100$rpm/sec, 20°, 3620 rpm

11.3. $376°/\text{sec}^2 = 62.667$rpm/sec, 28.2°, 3609.4 rpm

11.5. $H = 2.4$ MJ/MVA, $P_m = 0.5$ pu, $P_{max} = 2$,
$\frac{d^2\delta}{dt^2} = 4500(1 - 2\sin\delta)$, (δ is in degrees)

11.6. $E' = 1.25\angle 27.189°$, $0.03\frac{d^2\delta}{dt^2} = 0.77 - 1.65\sin\delta)$, (δ is in radians)

11.7. $0.03\frac{d^2\delta}{dt^2} = 0.77 - 0.5\sin\delta)$, (δ is in radians)

11.8. $\frac{H}{\pi f_0}\frac{d^2\Delta\delta}{dt^2} + P_s\Delta\delta = 0$, where $P_s = \frac{dP}{d\delta}\Big|_{\delta_0} = P_{max}\cos\delta_0 + 2P_k\cos 2\delta_0$

11.9. $\zeta = 0.6$, $\omega_d = 4.0$ rad/sec

11.10. $\delta = 27.835° + 16.0675e^{-2.4977t}\sin(6.5059t + 69°)$

$f = 60 - 0.311e^{-2.4977t}\sin 6.5059t$ Hz

11.11.

$$A = \begin{bmatrix} 0 & 1 \\ -48.5649 & -4.9955 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

11.12. $\delta = 27.835° + 5.8935[1 - 1.0712e^{-2.4977t}\sin(6.5059t + 69°)$

$f = 60 + 0.1222e^{-2.4977t}\sin 6.5059t$ Hz

11.13. A, B, D are the same as in Problem 11.11, $B = \begin{bmatrix} 0 \\ 4.9955 \end{bmatrix}$

11.14. (a) 0.649 pu, (b) 1.195 pu

11.15. (a) $\delta_c = 82.593°$, $t_c = 0.273$ sec (b) $\delta_c = 77.82°$

11.16. $t_c = 0.37$ sec

11.17. (a) Stable (b) Unstable (c) $t_c = 0.29$ sec

11.18. (a) Stable (b) Unstable (c) $t_c = 0.72$ sec

Chapter 12

12.1. 25 MW

12.2. $P_1 = 200$ MW, $P_2 = 300$ MW

12.3. (a) $\Delta f = -0.3$ Hz, $P_1 = 250$ MW, $P_2 = 580$ MW

(b) $\Delta f = -0.291$ Hz, $P_1 = 248.5002$ MW, $P_2 = 577.6004$ MW,
$\Delta P_L = -3.8994$

12.4. (a) $R > 0.009678$

(b) Use rlocus for $KG(s)H(s) = \frac{k}{2s^3+12.2s^2+17.2s+1.6}$, where $K = \frac{1}{R}$

12.5. (a) -0.5563 Hz

(b) $T(s) = \frac{0.0625s^2+0.375s+0.5}{s^3+6.1s^2+8.6s+13.3}$, use **step** function with value of -0.25 pu.

(c) Simulation response same as the response in (b)

12.6. (b) $T(s) = \frac{0.0625s^3+0.375s^2+0.5s}{s^4+6.1s^3+8.6s^2+13.3s+0.5K_I}$, use **step** function with value of -0.25

12.7. Modify sim12ex4.mdl

12.8. Modify sim12ex5.mdl

12.9. (a) $K_A < 43.3125$

(b) b) Use **rlocus** for $KG(s)H(s) = \frac{32k_A}{s^3+23s^2+62s+1320}$

(c) $T(s) = \frac{1280}{s^3+23s^2+62s+1320}$, use **step** to obtain the response

(d) Simulation response same as the response in (c)

12.10. (a) $T(s) = \frac{1280(s+25)}{s^4+48s^3+687s^2+6870s+37000}$, use **step** to obtain the response

(b) Simulation response same as the response in (a)

12.12. (b) $0, 0, \pm 4.5388$, (c) Unstable

(d) $K = [\ -170.3666 \quad -38.0540 \quad -17.3293 \quad -24.1081\ ]$

(d) $A_f = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -64.5823 & -19.0270 & -8.6646 & -12.0540 \\ 0 & 0 & 0 & 1.0000 \\ 42.1012 & 9.5135 & 4.3323 & 6.0270 \end{bmatrix}$

12.14. (a) $K = [\ -125.0988 \quad -28.6369 \quad -5.0000 \quad -10.5129\ ]$

(b) $A_f = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -41.9484 & -14.3185 & -2.5000 & -5.2565 \\ 0 & 0 & 0 & 1.0000 \\ 30.7842 & 7.1592 & 1.2500 & 2.6282 \end{bmatrix}$

12.16. (a) $A = \begin{bmatrix} -4 & 0 & -100 \\ 2 & -2 & 0 \\ 0 & 0.0625 & -0.1 \end{bmatrix}$

$B = [\ 0 \quad 0 \quad -0.0625\ ], BPL = B * PL, C = [\ 0 \quad 0 \quad 1\ ], D = 0$

(c) $K = [\ 6.4 \quad 5.4 \quad -94.4\ ]$ $\quad A - BK = \begin{bmatrix} -4.0 & 0.0 & -100 \\ 2.0 & -2.0 & 0 \\ 0.4 & 0.4 & -6 \end{bmatrix}$

12.17. (a) $K = [\ 8.3477 \quad 1.4637 \quad -162.0007\ ]$

$A_f = \begin{bmatrix} -4.0000 & 0 & -100.0000 \\ 2.0000 & -2.0000 & 0 \\ 0.5217 & 0.1540 & -10.2250 \end{bmatrix}$

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